This is Volume 2 of the five-volume book Mathematical Inequalities, which introduces and develops the main types of elementary inequalities. The first three volumes are a great opportunity to look into many old and new inequalities, as well as elementary procedures for solving them: Volume 1 - Symmetric Polynomial Inequalities, Volume 2 - Symmetric Rational and Nonrational Inequalities, Volume 3 - Cyclic and Noncyclic Inequalities. As a rule, the inequalities in these volumes are increasingly ordered according to the number of variables: two, three, four, ... , n-variables. The last two volumes (Volume 4 – Extensions and Refinements of Jensen’s Inequality, Volume 5 – Other Recent Methods for Creating and Solving Inequalities) present beautiful and original methods for solving inequalities, such as Half/Partial convex function method, Equal variables method, Arithmetic compensation method, Highest coefficient cancellation method, pqr method etc. The book is intended for a wide audience: advanced middle school students, high school students, college and university students, and teachers. Many problems and methods can be used as group projects for advanced high school students.

The author, Vasile Cirtoaje, is a Professor at the Department of Automatic Control and Computers from the University of Ploiesti, Romania. He is the author of many well-known interesting and delightful inequalities, as well as strong methods for creating and proving mathematical inequalities.

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Cirtoaje

Vasile Cirtoaje

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Chapter 1

Symmetric Rational Inequalities

1.1 Applications

1.1. If a, b are nonnegative real numbers, then

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2} \geq \frac{1}{1+ab}.$$

1.2. Let a, b, c be positive real numbers. Prove that

(a) if $abc \leq 1$, then

$$\frac{1}{2a+1}+\frac{1}{2b+1}+\frac{1}{2c+1} \geq 1;$$

(b) if $abc \geq 1$, then

$$\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2} \leq 1.$$

1.3. If $0 \leq a, b, c \leq 1$, then

$$2\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right) \geq 3\left(\frac{1}{2a+1}+\frac{1}{2b+1}+\frac{1}{2c+1}\right).$$

1.4. If a, b, c are nonnegative real numbers such that $a+b+c \leq 3$, then

$$2\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right) \geq 5\left(\frac{1}{2a+3}+\frac{1}{2b+3}+\frac{1}{2c+3}\right).$$

1.5. If a, b, c are nonnegative real numbers, then

$$\frac{a^2-bc}{3a+b+c}+\frac{b^2-ca}{3b+c+a}+\frac{c^2-ab}{3c+a+b}\ge 0.$$

1.6. If a, b, c are positive real numbers, then

$$\frac{4a^2-b^2-c^2}{a(b+c)}+\frac{4b^2-c^2-a^2}{b(c+a)}+\frac{4c^2-a^2-b^2}{c(a+b)}\le 3.$$

1.7. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

(a) $$\frac{1}{a^2+bc}+\frac{1}{b^2+ca}+\frac{1}{c^2+ab}\ge\frac{3}{ab+bc+ca};$$

(b) $$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab}\ge\frac{2}{ab+bc+ca}.$$

(c) $$\frac{1}{a^2+2bc}+\frac{1}{b^2+2ca}+\frac{1}{c^2+2ab}>\frac{2}{ab+bc+ca}.$$

1.8. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a(b+c)}{a^2+bc}+\frac{b(c+a)}{b^2+ca}+\frac{c(a+b)}{c^2+ab}\ge 2.$$

1.9. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}\ge\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}.$$

1.10. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\ge\frac{a}{a^2+bc}+\frac{b}{b^2+ca}+\frac{c}{c^2+ab}.$$

1.11. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\ge\frac{2a}{3a^2+bc}+\frac{2b}{3b^2+ca}+\frac{2c}{3c^2+ab}.$$

1.12. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

(a) $$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \ge \frac{13}{6}-\frac{2(ab+bc+ca)}{3(a^2+b^2+c^2)};$$

(b) $$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2} \ge (\sqrt{3}-1)\left(1-\frac{ab+bc+ca}{a^2+b^2+c^2}\right).$$

1.13. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a^2+2bc}+\frac{1}{b^2+2ca}+\frac{1}{c^2+2ab} \le \left(\frac{a+b+c}{ab+bc+ca}\right)^2.$$

1.14. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a^2(b+c)}{b^2+c^2}+\frac{b^2(c+a)}{c^2+a^2}+\frac{c^2(a+b)}{a^2+b^2} \ge a+b+c.$$

1.15. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a} \le \frac{3(a^2+b^2+c^2)}{a+b+c)}.$$

1.16. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a^2+ab+b^2}+\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2} \ge \frac{9}{(a+b+c)^2}.$$

1.17. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a^2}{(2a+b)(2a+c)}+\frac{b^2}{(2b+c)(2b+a)}+\frac{c^2}{(2c+a)(2c+b)} \le \frac{1}{3}.$$

1.18. Let a, b, c be positive real numbers. Prove that

(a) $$\sum \frac{a}{(2a+b)(2a+c)} \le \frac{1}{a+b+c};$$

(b) $$\sum \frac{a^3}{(2a^2+b^2)(2a^2+c^2)} \le \frac{1}{a+b+c}.$$

1.19. If a, b, c are positive real numbers, then

$$\sum \frac{1}{(a+2b)(a+2c)} \geq \frac{1}{(a+b+c)^2} + \frac{2}{3(ab+bc+ca)}.$$

1.20. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

(a) $$\frac{1}{(a-b)^2} + \frac{1}{(b-c)^2} + \frac{1}{(c-a)^2} \geq \frac{4}{ab+bc+ca};$$

(b) $$\frac{1}{a^2-ab+b^2} + \frac{1}{b^2-bc+c^2} + \frac{1}{c^2-ca+a^2} \geq \frac{3}{ab+bc+ca};$$

(c) $$\frac{1}{a^2+b^2} + \frac{1}{b^2+c^2} + \frac{1}{c^2+a^2} \geq \frac{5}{2(ab+bc+ca)}.$$

1.21. If a, b, c are positive real numbers, then

$$\frac{(a^2+b^2)(a^2+c^2)}{(a+b)(a+c)} + \frac{(b^2+c^2)(b^2+a^2)}{(b+c)(b+a)} + \frac{(c^2+a^2)(c^2+b^2)}{(c+a)(c+b)} \geq a^2+b^2+c^2.$$

1.22. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{1}{a^2+b+c} + \frac{1}{b^2+c+a} + \frac{1}{c^2+a+b} \leq 1.$$

1.23. Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$\frac{a^2-bc}{a^2+3} + \frac{b^2-ca}{b^2+3} + \frac{c^2-ab}{c^2+3} \geq 0.$$

1.24. Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$\frac{1-bc}{5+2a} + \frac{1-ca}{5+2b} + \frac{1-ab}{5+2c} \geq 0.$$

1.25. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{1}{a^2+b^2+2} + \frac{1}{b^2+c^2+2} + \frac{1}{c^2+a^2+2} \leq \frac{3}{4}.$$

1.26. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{1}{4a^2+b^2+c^2}+\frac{1}{4b^2+c^2+a^2}+\frac{1}{4c^2+a^2+b^2} \leq \frac{1}{2}.$$

1.27. Let a, b, c be nonnegative real numbers such that $a+b+c=2$. Prove that

$$\frac{bc}{a^2+1}+\frac{ca}{b^2+1}+\frac{ab}{c^2+1} \leq 1.$$

1.28. Let a, b, c be nonnegative real numbers such that $a+b+c=1$. Prove that

$$\frac{bc}{a+1}+\frac{ca}{b+1}+\frac{ab}{c+1} \leq \frac{1}{4}.$$

1.29. Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\frac{1}{a(2a^2+1)}+\frac{1}{b(2b^2+1)}+\frac{1}{c(2c^2+1)} \leq \frac{3}{11abc}.$$

1.30. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{1}{a^3+b+c}+\frac{1}{b^3+c+a}+\frac{1}{c^3+a+b} \leq 1.$$

1.31. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{a^2}{1+b^3+c^3}+\frac{b^2}{1+c^3+a^3}+\frac{c^2}{1+a^3+b^3} \geq 1.$$

1.32. Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$\frac{1}{6-ab}+\frac{1}{6-bc}+\frac{1}{6-ca} \leq \frac{3}{5}.$$

1.33. Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$\frac{1}{2a^2+7}+\frac{1}{2b^2+7}+\frac{1}{2c^2+7} \leq \frac{1}{3}.$$

1.34. Let a,b,c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$\frac{1}{2a^2+3}+\frac{1}{2b^2+3}+\frac{1}{2c^2+3}\geq\frac{3}{5}.$$

1.35. Let a,b,c be nonnegative real numbers such that $ab+bc+ca=3$. Prove that

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq\frac{a+b+c}{6}+\frac{3}{a+b+c}.$$

1.36. Let a,b,c be nonnegative real numbers such that $ab+bc+ca=3$. Prove that

$$\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}\geq\frac{3}{2}.$$

1.37. Let a,b,c be positive real numbers such that $ab+bc+ca=3$. Prove that

$$\frac{a^2}{a^2+b+c}+\frac{b^2}{b^2+c+a}+\frac{c^2}{c^2+a+b}\geq 1.$$

1.38. Let a,b,c be positive real numbers such that $ab+bc+ca=3$. Prove that

$$\frac{bc+4}{a^2+4}+\frac{ca+4}{b^2+4}+\frac{ab+4}{c^2+4}\leq 3\leq\frac{bc+2}{a^2+2}+\frac{ca+2}{b^2+2}+\frac{ab+2}{c^2+2}.$$

1.39. Let a,b,c be nonnegative real numbers such that $ab+bc+ca=3$. If

$$k\geq 2+\sqrt{3},$$

then

$$\frac{1}{a+k}+\frac{1}{b+k}+\frac{1}{c+k}\leq\frac{3}{1+k}.$$

1.40. Let a,b,c be nonnegative real numbers such that $a^2+b^2+c^2=3$. Prove that

$$\frac{a(b+c)}{1+bc}+\frac{b(c+a)}{1+ca}+\frac{c(a+b)}{1+ab}\leq 3.$$

1.41. Let a, b, c be positive real numbers such that $a^2+b^2+c^2=3$. Prove that

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}\geq 3.$$

1.42. Let a, b, c be positive real numbers such that $a^2+b^2+c^2=3$. Prove that

$$\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}+2\leq \frac{7}{6}(a+b+c).$$

1.43. Let a, b, c be positive real numbers such that $a^2+b^2+c^2=3$. Prove that

(a) $$\frac{1}{3-ab}+\frac{1}{3-bc}+\frac{1}{3-ca}\leq \frac{3}{2};$$

(b) $$\frac{1}{5-2ab}+\frac{1}{5-2bc}+\frac{1}{5-2ca}\leq 1;$$

(c) $$\frac{1}{\sqrt{6}-ab}+\frac{1}{\sqrt{6}-bc}+\frac{1}{\sqrt{6}-ca}\leq \frac{3}{\sqrt{6}-1}.$$

1.44. Let a, b, c be positive real numbers such that $a^2+b^2+c^2=3$. Prove that

$$\frac{1}{1+a^5}+\frac{1}{1+b^5}+\frac{1}{1+c^5}\geq \frac{3}{2}.$$

1.45. Let a, b, c be positive real numbers such that $abc=1$. Prove that

$$\frac{1}{a^2+a+1}+\frac{1}{b^2+b+1}+\frac{1}{c^2+c+1}\geq 1.$$

1.46. Let a, b, c be positive real numbers such that $abc=1$. Prove that

$$\frac{1}{a^2-a+1}+\frac{1}{b^2-b+1}+\frac{1}{c^2-c+1}\leq 3.$$

1.47. Let a, b, c be positive real numbers such that $abc=1$. Prove that

$$\frac{3+a}{(1+a)^2}+\frac{3+b}{(1+b)^2}+\frac{3+c}{(1+c)^2}\geq 3.$$

1.48. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{7-6a}{2+a^2} + \frac{7-6b}{2+b^2} + \frac{7-6c}{2+c^2} \ge 1.$$

1.49. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{a^6}{1+2a^5} + \frac{b^6}{1+2b^5} + \frac{c^6}{1+2c^5} \ge 1.$$

1.50. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{a}{a^2+5} + \frac{b}{b^2+5} + \frac{c}{c^2+5} \le \frac{1}{2}.$$

1.51. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{2}{(1+a)(1+b)(1+c)} \ge 1.$$

1.52. Let a, b, c be nonnegative real numbers such that

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} = \frac{3}{2}.$$

Prove that

$$\frac{3}{a+b+c} \ge \frac{2}{ab+bc+ca} + \frac{1}{a^2+b^2+c^2}.$$

1.53. Let a, b, c be nonnegative real numbers such that

$$7(a^2+b^2+c^2) = 11(ab+bc+ca).$$

Prove that

$$\frac{51}{28} \le \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \le 2.$$

1.54. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{1}{a^2+b^2} + \frac{1}{b^2+c^2} + \frac{1}{c^2+a^2} \ge \frac{10}{(a+b+c)^2}.$$

1.55. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{1}{a^2-ab+b^2}+\frac{1}{b^2-bc+c^2}+\frac{1}{c^2-ca+a^2}\geq\frac{3}{\max\{ab,bc,ca\}}.$$

1.56. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a(2a+b+c)}{b^2+c^2}+\frac{b(2b+c+a)}{c^2+a^2}+\frac{c(2c+a+b)}{a^2+b^2}\geq 6.$$

1.57. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a^2(b+c)^2}{b^2+c^2}+\frac{b^2(c+a)^2}{c^2+a^2}+\frac{c^2(a+b)^2}{a^2+b^2}\geq 2(ab+bc+ca).$$

1.58. If a, b, c are positive real numbers, then

$$3\sum\frac{a}{b^2-bc+c^2}+5\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right)\geq 8\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

1.59. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

(a) $$2abc\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)+a^2+b^2+c^2\geq 2(ab+bc+ca);$$

(b) $$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}\leq\frac{3(a^2+b^2+c^2)}{2(a+b+c)}.$$

1.60. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

(a) $$\frac{a^2-bc}{b^2+c^2}+\frac{b^2-ca}{c^2+a^2}+\frac{c^2-ab}{a^2+b^2}+\frac{3(ab+bc+ca)}{a^2+b^2+c^2}\geq 3;$$

(b) $$\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}+\frac{ab+bc+ca}{a^2+b^2+c^2}\geq\frac{5}{2};$$

(c) $$\frac{a^2+bc}{b^2+c^2}+\frac{b^2+ca}{c^2+a^2}+\frac{c^2+ab}{a^2+b^2}\geq\frac{ab+bc+ca}{a^2+b^2+c^2}+2.$$

1.61. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}\geq\frac{(a+b+c)^2}{2(ab+bc+ca)}.$$

1.62. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{2ab}{(a+b)^2}+\frac{2bc}{(b+c)^2}+\frac{2ca}{(c+a)^2}+\frac{a^2+b^2+c^2}{ab+bc+ca}\geq\frac{5}{2}.$$

1.63. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{ab}{(a+b)^2}+\frac{bc}{(b+c)^2}+\frac{ca}{(c+a)^2}+\frac{1}{4}\geq\frac{ab+bc+ca}{a^2+b^2+c^2}.$$

1.64. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{3ab}{(a+b)^2}+\frac{3bc}{(b+c)^2}+\frac{3ca}{(c+a)^2}\leq\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{5}{4}.$$

1.65. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

(a) $$\frac{a^3+abc}{b+c}+\frac{b^3+abc}{c+a}+\frac{c^3+abc}{a+b}\geq a^2+b^2+c^2;$$

(b) $$\frac{a^3+2abc}{b+c}+\frac{b^3+2abc}{c+a}+\frac{c^3+2abc}{a+b}\geq\frac{1}{2}(a+b+c)^2;$$

(c) $$\frac{a^3+3abc}{b+c}+\frac{b^3+3abc}{c+a}+\frac{c^3+3abc}{a+b}\geq 2(ab+bc+ca).$$

1.66. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a^3+3abc}{(b+c)^2}+\frac{b^3+3abc}{(c+a)^2}+\frac{c^3+3abc}{(a+b)^2}\geq a+b+c.$$

1.67. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

(a) $$\frac{a^3+3abc}{(b+c)^3}+\frac{b^3+3abc}{(c+a)^3}+\frac{c^3+3abc}{(a+b)^3}\geq\frac{3}{2};$$

(b) $$\frac{3a^3+13abc}{(b+c)^3}+\frac{3b^3+13abc}{(c+a)^3}+\frac{3c^3+13abc}{(a+b)^3}\geq 6.$$

1.68. Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that

(a) $$\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b}+ab+bc+ca \ge \frac{3}{2}(a^2+b^2+c^2);$$

(b) $$\frac{2a^2+bc}{b+c}+\frac{2b^2+ca}{c+a}+\frac{2c^2+ab}{a+b} \ge \frac{9(a^2+b^2+c^2)}{2(a+b+c)}.$$

1.69. Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a(b+c)}{b^2+bc+c^2}+\frac{b(c+a)}{c^2+ca+a^2}+\frac{c(a+b)}{a^2+ab+b^2} \ge 2.$$

1.70. Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a(b+c)}{b^2+bc+c^2}+\frac{b(c+a)}{c^2+ca+a^2}+\frac{c(a+b)}{a^2+ab+b^2} \ge 2+4\prod\left(\frac{a-b}{a+b}\right)^2.$$

1.71. Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{ab-bc+ca}{b^2+c^2}+\frac{bc-ca+ab}{c^2+a^2}+\frac{ca-ab+bc}{a^2+b^2} \ge \frac{3}{2}.$$

1.72. Let a,b,c be nonnegative real numbers, no two of which are zero. If $k>-2$, then

$$\sum \frac{ab+(k-1)bc+ca}{b^2+kbc+c^2} \ge \frac{3(k+1)}{k+2}.$$

1.73. Let a,b,c be nonnegative real numbers, no two of which are zero. If $k>-2$, then

$$\sum \frac{3bc-a(b+c)}{b^2+kbc+c^2} \le \frac{3}{k+2}.$$

1.74. Let a,b,c be nonnegative real numbers such that $ab+bc+ca=3$. Prove that

$$\frac{ab+1}{a^2+b^2}+\frac{bc+1}{b^2+c^2}+\frac{ca+1}{c^2+a^2} \ge \frac{4}{3}.$$

1.75. Let a, b, c be nonnegative real numbers such that $ab + bc + ca = 3$. Prove that

$$\frac{5ab+1}{(a+b)^2} + \frac{5bc+1}{(b+c)^2} + \frac{5ca+1}{(c+a)^2} \ge 2.$$

1.76. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a^2-bc}{2b^2-3bc+2c^2} + \frac{b^2-ca}{2c^2-3ca+2a^2} + \frac{c^2-ab}{2a^2-3ab+2b^2} \ge 0.$$

1.77. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{2a^2-bc}{b^2-bc+c^2} + \frac{2b^2-ca}{c^2-ca+a^2} + \frac{2c^2-ab}{a^2-ab+b^2} \ge 3.$$

1.78. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a^2}{2b^2-bc+2c^2} + \frac{b^2}{2c^2-ca+2a^2} + \frac{c^2}{2a^2-ab+2b^2} \ge 1.$$

1.79. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{1}{4b^2-bc+4c^2} + \frac{1}{4c^2-ca+4a^2} + \frac{1}{4a^2-ab+4b^2} \ge \frac{9}{7(a^2+b^2+c^2)}.$$

1.80. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{2a^2+bc}{b^2+c^2} + \frac{2b^2+ca}{c^2+a^2} + \frac{2c^2+ab}{a^2+b^2} \ge \frac{9}{2}.$$

1.81. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{2a^2+3bc}{b^2+bc+c^2} + \frac{2b^2+3ca}{c^2+ca+a^2} + \frac{2c^2+3ab}{a^2+ab+b^2} \ge 5.$$

1.82. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{2a^2+5bc}{(b+c)^2} + \frac{2b^2+5ca}{(c+a)^2} + \frac{2c^2+5ab}{(a+b)^2} \ge \frac{21}{4}.$$

1.83. Let a, b, c be nonnegative real numbers, no two of which are zero. If $k > -2$, then

$$\sum \frac{2a^2 + (2k+1)bc}{b^2 + kbc + c^2} \geq \frac{3(2k+3)}{k+2}.$$

1.84. Let a, b, c be nonnegative real numbers, no two of which are zero. If $k > -2$, then

$$\sum \frac{3bc - 2a^2}{b^2 + kbc + c^2} \leq \frac{3}{k+2}.$$

1.85. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{a^2 + 16bc}{b^2 + c^2} + \frac{b^2 + 16ca}{c^2 + a^2} + \frac{c^2 + 16ab}{a^2 + b^2} \geq 10.$$

1.86. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{a^2 + 128bc}{b^2 + c^2} + \frac{b^2 + 128ca}{c^2 + a^2} + \frac{c^2 + 128ab}{a^2 + b^2} \geq 46.$$

1.87. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{a^2 + 64bc}{(b+c)^2} + \frac{b^2 + 64ca}{(c+a)^2} + \frac{c^2 + 64ab}{(a+b)^2} \geq 18.$$

1.88. Let a, b, c be nonnegative real numbers, no two of which are zero. If $k \geq -1$, then

$$\sum \frac{a^2(b+c) + kabc}{b^2 + kbc + c^2} \geq a + b + c.$$

1.89. Let a, b, c be nonnegative real numbers, no two of which are zero. If $k \geq \frac{-3}{2}$, then

$$\sum \frac{a^3 + (k+1)abc}{b^2 + kbc + c^2} \geq a + b + c.$$

1.90. Let a, b, c be nonnegative real numbers, no two of which are zero. If $k > 0$, then

$$\frac{2a^k - b^k - c^k}{b^2 - bc + c^2} + \frac{2b^k - c^k - a^k}{c^2 - ca + a^2} + \frac{2c^k - a^k - b^k}{a^2 - ab + b^2} \geq 0.$$

1.91. If a,b,c are the lengths of the sides of a triangle, then

(a) $$\frac{b+c-a}{b^2-bc+c^2}+\frac{c+a-b}{c^2-ca+a^2}+\frac{a+b-c}{a^2-ab+b^2}\ge\frac{2(a+b+c)}{a^2+b^2+c^2};$$

(b) $$\frac{2bc-a^2}{b^2-bc+c^2}+\frac{2ca-b^2}{c^2-ca+a^2}+\frac{2ab-c^2}{a^2-ab+b^2}\ge 0.$$

1.92. If a,b,c are nonnegative real numbers, then

(a) $$\frac{a^2}{5a^2+(b+c)^2}+\frac{b^2}{5b^2+(c+a)^2}+\frac{c^2}{5c^2+(a+b)^2}\le\frac{1}{3};$$

(b) $$\frac{a^3}{13a^3+(b+c)^3}+\frac{b^3}{13b^3+(c+a)^3}+\frac{c^3}{13c^3+(a+b)^3}\le\frac{1}{7}.$$

1.93. If a,b,c are nonnegative real numbers, then

$$\frac{b^2+c^2-a^2}{2a^2+(b+c)^2}+\frac{c^2+a^2-b^2}{2b^2+(c+a)^2}+\frac{a^2+b^2-c^2}{2c^2+(a+b)^2}\ge\frac{1}{2}.$$

1.94. Let a,b,c be positive real numbers. If $k>0$, then

$$\frac{3a^2-2bc}{ka^2+(b-c)^2}+\frac{3b^2-2ca}{kb^2+(c-a)^2}+\frac{3c^2-2ab}{kc^2+(a-b)^2}\le\frac{3}{k}.$$

1.95. Let a,b,c be nonnegative real numbers, no two of which are zero. If $k\ge 3+\sqrt{7}$, then

(a) $$\frac{a}{a^2+kbc}+\frac{b}{b^2+kca}+\frac{c}{c^2+kab}\ge\frac{9}{(1+k)(a+b+c)};$$

(b) $$\frac{1}{ka^2+bc}+\frac{1}{kb^2+ca}+\frac{1}{kc^2+ab}\ge\frac{9}{(k+1)(ab+bc+ca)}.$$

1.96. Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab}\ge\frac{6}{a^2+b^2+c^2+ab+bc+ca}.$$

1.97. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{1}{22a^2+5bc}+\frac{1}{22b^2+5ca}+\frac{1}{22c^2+5ab}\ge\frac{1}{(a+b+c)^2}.$$

1.98. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab}\ge\frac{8}{(a+b+c)^2}.$$

1.99. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{1}{a^2+bc}+\frac{1}{b^2+ca}+\frac{1}{c^2+ab}\ge\frac{12}{(a+b+c)^2}.$$

1.100. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

(a) $$\frac{1}{a^2+2bc}+\frac{1}{b^2+2ca}+\frac{1}{c^2+2ab}\ge\frac{1}{a^2+b^2+c^2}+\frac{2}{ab+bc+ca};$$

(b) $$\frac{a(b+c)}{a^2+2bc}+\frac{b(c+a)}{b^2+2ca}+\frac{c(a+b)}{c^2+2ab}\ge 1+\frac{ab+bc+ca}{a^2+b^2+c^2}.$$

1.101. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

(a) $$\frac{a}{a^2+2bc}+\frac{b}{b^2+2ca}+\frac{c}{c^2+2ab}\le\frac{a+b+c}{ab+bc+ca};$$

(b) $$\frac{a(b+c)}{a^2+2bc}+\frac{b(c+a)}{b^2+2ca}+\frac{c(a+b)}{c^2+2ab}\le 1+\frac{a^2+b^2+c^2}{ab+bc+ca}.$$

1.102. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

(a) $$\frac{a}{2a^2+bc}+\frac{b}{2b^2+ca}+\frac{c}{2c^2+ab}\ge\frac{a+b+c}{a^2+b^2+c^2};$$

(b) $$\frac{b+c}{2a^2+bc}+\frac{c+a}{2b^2+ca}+\frac{a+b}{2c^2+ab}\ge\frac{6}{a+b+c}.$$

1.103. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a(b+c)}{a^2+bc}+\frac{b(c+a)}{b^2+ca}+\frac{c(a+b)}{c^2+ab}\ge\frac{(a+b+c)^2}{a^2+b^2+c^2}.$$

1.104. Let a, b, c be nonnegative real numbers, no two of which are zero. If $k>0$, then

$$\frac{b^2+c^2+\sqrt{3}bc}{a^2+kbc}+\frac{c^2+a^2+\sqrt{3}ca}{b^2+kca}+\frac{a^2+b^2+\sqrt{3}ab}{c^2+kab}\ge\frac{3(2+\sqrt{3})}{1+k}.$$

1.105. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}+\frac{8}{a^2+b^2+c^2}\ge\frac{6}{ab+bc+ca}.$$

1.106. If a, b, c are the lengths of the sides of a triangle, then

$$\frac{a(b+c)}{a^2+2bc}+\frac{b(c+a)}{b^2+2ca}+\frac{c(a+b)}{c^2+2ab}\le 2.$$

1.107. If a, b, c are real numbers, then

$$\frac{a^2-bc}{2a^2+b^2+c^2}+\frac{b^2-ca}{2b^2+c^2+a^2}+\frac{c^2-ab}{2c^2+a^2+b^2}\ge 0.$$

1.108. If a, b, c are nonnegative real numbers, then

$$\frac{3a^2-bc}{2a^2+b^2+c^2}+\frac{3b^2-ca}{2b^2+c^2+a^2}+\frac{3c^2-ab}{2c^2+a^2+b^2}\le\frac{3}{2}.$$

1.109. If a, b, c are nonnegative real numbers, then

$$\frac{(b+c)^2}{4a^2+b^2+c^2}+\frac{(c+a)^2}{4b^2+c^2+a^2}+\frac{(a+b)^2}{4c^2+a^2+b^2}\ge 2.$$

1.110. If a, b, c are positive real numbers, then

(a) $$\sum\frac{1}{11a^2+2b^2+2c^2}\le\frac{3}{5(ab+bc+ca)};$$

(b) $$\sum\frac{1}{4a^2+b^2+c^2}\le\frac{1}{2(a^2+b^2+c^2)}+\frac{1}{ab+bc+ca}.$$

1.111. If $a,\ b,\ c$ are nonnegative real numbers such that $ab+bc+ca=3$, then

$$\frac{\sqrt{a}}{b+c}+\frac{\sqrt{b}}{c+a}+\frac{\sqrt{c}}{a+b}\ge\frac{3}{2}.$$

1.112. If $a,\ b,\ c$ are nonnegative real numbers such that $ab+bc+ca\ge 3$, then

$$\frac{1}{2+a}+\frac{1}{2+b}+\frac{1}{2+c}\ge\frac{1}{1+b+c}+\frac{1}{1+c+a}+\frac{1}{1+a+b}.$$

1.113. If a,b,c are the lengths of the sides of a triangle, then

(a) $$\frac{a^2-bc}{3a^2+b^2+c^2}+\frac{b^2-ca}{3b^2+c^2+a^2}+\frac{c^2-ab}{3c^2+a^2+b^2}\le 0;$$

(b) $$\frac{a^4-b^2c^2}{3a^4+b^4+c^4}+\frac{b^4-c^2a^2}{3b^4+c^4+a^4}+\frac{c^4-a^2b^2}{3c^4+a^4+b^4}\le 0.$$

1.114. If a,b,c are the lengths of the sides of a triangle, then

$$\frac{bc}{4a^2+b^2+c^2}+\frac{ca}{4b^2+c^2+a^2}+\frac{ab}{4c^2+a^2+b^2}\ge\frac{1}{2}.$$

1.115. If a,b,c are the lengths of the sides of a triangle, then

$$\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}+\frac{1}{a^2+b^2}\le\frac{9}{2(ab+bc+ca)}.$$

1.116. If a,b,c are the lengths of the sides of a triangle, then

(a) $$\left|\frac{a+b}{a-b}+\frac{b+c}{b-c}+\frac{c+a}{c-a}\right|>5;$$

(b) $$\left|\frac{a^2+b^2}{a^2-b^2}+\frac{b^2+c^2}{b^2-c^2}+\frac{c^2+a^2}{c^2-a^2}\right|\ge 3.$$

1.117. If a,b,c are the lengths of the sides of a triangle, then

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}+3\ge 6\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right).$$

1.118. Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that
$$\sum \frac{3a(b+c)-2bc}{(b+c)(2a+b+c)} \ge \frac{3}{2}.$$

1.119. Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that
$$\sum \frac{a(b+c)-2bc}{(b+c)(3a+b+c)} \ge 0.$$

1.120. Let a,b,c be positive real numbers such that $a^2+b^2+c^2 \ge 3$. Prove that
$$\frac{a^5-a^2}{a^5+b^2+c^2}+\frac{b^5-b^2}{b^5+c^2+a^2}+\frac{c^5-c^2}{c^5+a^2+b^2} \ge 0.$$

1.121. Let a,b,c be positive real numbers such that $a^2+b^2+c^2=a^3+b^3+c^3$. Prove that
$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} \ge \frac{3}{2}.$$

1.122. If $a,b,c \in [0,1]$, then
$$\frac{a}{bc+2}+\frac{b}{ca+2}+\frac{c}{ab+2} \le 1.$$

1.123. Let a,b,c be positive real numbers such that $a+b+c=2$. Prove that
$$5(1-ab-bc-ca)\left(\frac{1}{1-ab}+\frac{1}{1-bc}+\frac{1}{1-ca}\right)+9 \ge 0.$$

1.124. Let a,b,c be nonnegative real numbers such that $a+b+c=2$. Prove that
$$\frac{2-a^2}{2-bc}+\frac{2-b^2}{2-ca}+\frac{2-c^2}{2-ab} \le 3.$$

1.125. Let a,b,c be nonnegative real numbers such that $a+b+c=3$. Prove that
$$\frac{3+5a^2}{3-bc}+\frac{3+5b^2}{3-ca}+\frac{3+5c^2}{3-ab} \ge 12.$$

1.126. Let a,b,c be nonnegative real numbers such that $a+b+c=2$. If

$$\frac{-1}{7} \le m \le \frac{7}{8},$$

then

$$\frac{a^2+m}{3-2bc}+\frac{b^2+m}{3-2ca}+\frac{c^2+m}{3-2ab} \ge \frac{3(4+9m)}{19}.$$

1.127. Let a,b,c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$\frac{47-7a^2}{1+bc}+\frac{47-7b^2}{1+ca}+\frac{47-7c^2}{1+ab} \ge 60.$$

1.128. Let a,b,c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$\frac{26-7a^2}{1+bc}+\frac{26-7b^2}{1+ca}+\frac{26-7c^2}{1+ab} \le \frac{57}{2}.$$

1.129. If a,b,c are nonnegative real numbers, then

$$\sum \frac{5a(b+c)-6bc}{a^2+b^2+c^2+bc} \le 3.$$

1.130. Let a,b,c be nonnegative real numbers, no two of which are zero, and let

$$x=\frac{a^2+b^2+c^2}{ab+bc+ca}.$$

Prove that

(a) $$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{1}{2} \ge x+\frac{1}{x};$$

(b) $$6\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right) \ge 5x+\frac{4}{x};$$

(c) $$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2} \ge \frac{1}{3}\left(x-\frac{1}{x}\right).$$

1.131. If a,b,c are real numbers, then

$$\frac{1}{a^2+7(b^2+c^2)}+\frac{1}{b^2+7(c^2+a^2)}+\frac{1}{c^2+7(a^2+b^2)} \le \frac{9}{5(a+b+c)^2}.$$

1.132. If a,b,c are real numbers, then

$$\frac{bc}{3a^2+b^2+c^2}+\frac{ca}{3b^2+c^2+a^2}+\frac{ab}{3c^2+a^2+b^2}\le\frac{3}{5}.$$

1.133. If a,b,c are real numbers such that $a+b+c=3$, then

$$\frac{1}{8+5(b^2+c^2)}+\frac{1}{8+5(c^2+a^2)}+\frac{1}{8+5(a^2+b^2)}\le\frac{1}{6}.$$

1.134. If a,b,c are real numbers, then

$$\frac{(a+b)(a+c)}{a^2+4(b^2+c^2)}+\frac{(b+c)(b+a)}{b^2+4(c^2+a^2)}+\frac{(c+a)(c+b)}{c^2+4(a^2+b^2)}\le\frac{4}{3}.$$

1.135. Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that

$$\sum\frac{1}{(b+c)(7a+b+c)}\le\frac{1}{2(ab+bc+ca)}.$$

1.136. Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that

$$\sum\frac{1}{b^2+c^2+4a(b+c)}\le\frac{9}{10(ab+bc+ca)}.$$

1.137. Let a,b,c be nonnegative real numbers, no two of which are zero. If $a+b+c=3$, then

$$\frac{1}{3-ab}+\frac{1}{3-bc}+\frac{1}{3-ca}\le\frac{9}{2(ab+bc+ca)}.$$

1.138. If a,b,c are nonnegative real numbers such that $a+b+c=3$, then

$$\frac{bc}{a^2+a+6}+\frac{ca}{b^2+b+6}+\frac{ab}{c^2+c+6}\le\frac{3}{8}.$$

1.139. If a,b,c are nonnegative real numbers such that $ab+bc+ca=3$, then

$$\frac{1}{8a^2-2bc+21}+\frac{1}{8b^2-2ca+21}+\frac{1}{8c^2-2ab+21}\ge\frac{1}{9}.$$

1.140. Let a, b, c be real numbers, no two of which are zero. Prove that

(a) $$\frac{a^2+bc}{b^2+c^2}+\frac{b^2+ca}{c^2+a^2}+\frac{c^2+ab}{a^2+b^2} \ge \frac{(a+b+c)^2}{a^2+b^2+c^2};$$

(b) $$\frac{a^2+3bc}{b^2+c^2}+\frac{b^2+3ca}{c^2+a^2}+\frac{c^2+3ab}{a^2+b^2} \ge \frac{6(ab+bc+ca)}{a^2+b^2+c^2}.$$

1.141. Let a, b, c be real numbers, no two of which are zero. If $ab+bc+ca \ge 0$, then

$$\frac{a(b+c)}{b^2+c^2}+\frac{b(c+a)}{c^2+a^2}+\frac{c(a+b)}{a^2+b^2} \ge \frac{3}{10}.$$

1.142. If a, b, c are positive real numbers such that $abc > 1$, then

$$\frac{1}{a+b+c-3}+\frac{1}{abc-1} \ge \frac{4}{ab+bc+ca-3}.$$

1.143. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\sum \frac{(4b^2-ac)(4c^2-ab)}{b+c} \le \frac{27}{2}abc.$$

1.144. Let a, b, c be nonnegative real numbers, no two of which are zero, such that

$$a+b+c=3.$$

Prove that

$$\frac{a}{3a+bc}+\frac{b}{3b+ca}+\frac{c}{3c+ab} \ge \frac{2}{3}.$$

1.145. Let a, b, c be positive real numbers such that

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=10.$$

Prove that

$$\frac{19}{12} \le \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \le \frac{5}{3}.$$

1.146. Let a,b,c be nonnegative real numbers, no two of which are zero, such that $a+b+c=3$. Prove that

$$\frac{9}{10} < \frac{a}{2a+bc}+\frac{b}{2b+ca}+\frac{c}{2c+ab} \le 1.$$

1.147. Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a^3}{2a^2+bc}+\frac{b^3}{2b^2+ca}+\frac{c^3}{2c^2+ab} \le \frac{a^3+b^3+c^3}{a^2+b^2+c^2}.$$

1.148. If a,b,c are positive real numbers, then

$$\frac{a^3}{4a^2+bc}+\frac{b^3}{4b^2+ca}+\frac{c^3}{4c^2+ab} \ge \frac{a+b+c}{5}.$$

1.149. If a,b,c are positive real numbers, then

$$\frac{1}{(2+a)^2}+\frac{1}{(2+b)^2}+\frac{1}{(2+c)^2} \ge \frac{3}{6+ab+bc+ca}.$$

1.150. If a,b,c are positive real numbers, then

$$\frac{1}{1+3a}+\frac{1}{1+3b}+\frac{1}{1+3c} \ge \frac{3}{3+abc}.$$

1.151. Let a,b,c be real numbers, no two of which are zero. If $1<k\le 3$, then

$$\left(k+\frac{2ab}{a^2+b^2}\right)\left(k+\frac{2bc}{b^2+c^2}\right)\left(k+\frac{2ca}{c^2+a^2}\right) \ge (k-1)(k^2-1).$$

1.152. If a,b,c are non-zero and distinct real numbers, then

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+3\left[\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}\right] \ge 4\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right).$$

1.153. Let a, b, c be positive real numbers, and let

$$A = \frac{a}{b} + \frac{b}{a} + k, \quad B = \frac{b}{c} + \frac{c}{b} + k, \quad C = \frac{c}{a} + \frac{a}{b} + k,$$

where $-2 < k \le 4$. Prove that

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C} \le \frac{1}{k+2} + \frac{4}{A+B+C-k-2}.$$

1.154. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{b^2+bc+c^2} + \frac{1}{c^2+ca+a^2} + \frac{1}{a^2+ab+b^2} \ge \frac{1}{2a^2+bc} + \frac{1}{2b^2+ca} + \frac{1}{2c^2+ab}.$$

1.155. If a, b, c are nonnegative real numbers such that $a+b+c \le 3$, then

(a) $$\frac{1}{2a+1} + \frac{1}{2b+1} + \frac{1}{2c+1} \ge \frac{1}{a+2} + \frac{1}{b+2} + \frac{1}{c+2};$$

(b) $$\frac{1}{2ab+1} + \frac{1}{2bc+1} + \frac{1}{2ca+1} \ge \frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2}.$$

1.156. If a, b, c are nonnegative real numbers such that $a+b+c = 4$, then

$$\frac{1}{ab+2} + \frac{1}{bc+2} + \frac{1}{ca+2} \ge \frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2}.$$

1.157. If a, b, c are nonnegative real numbers, no two of which are zero, then

(a) $$\frac{ab+bc+ca}{a^2+b^2+c^2} + \frac{(a-b)^2(b-c)^2(c-a)^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)} \le 1;$$

(b) $$\frac{ab+bc+ca}{a^2+b^2+c^2} + \frac{(a-b)^2(b-c)^2(c-a)^2}{(a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2)} \le 1.$$

1.158. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{a^2+b^2+c^2}{ab+bc+ca} \ge 1 + \frac{9(a-b)^2(b-c)^2(c-a)^2}{(a+b)^2(b+c)^2(c+a)^2}.$$

1.159. If a,b,c are nonnegative real numbers, no two of which are zero, then

$$\frac{a^2+b^2+c^2}{ab+bc+ca} \ge 1+(1+\sqrt{2})^2\frac{(a-b)^2(b-c)^2(c-a)^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}.$$

1.160. If a,b,c are nonnegative real numbers, no two of which are zero, then

$$\frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a} \ge \frac{5}{3a+b+c}+\frac{5}{3b+c+a}+\frac{5}{3c+a+b}.$$

1.161. If a,b,c are real numbers, no two of which are zero, then

(a) $$\frac{8a^2+3bc}{b^2+bc+c^2}+\frac{8b^2+3ca}{c^2+ca+a^2}+\frac{8c^2+3ab}{a^2+ab+b^2} \ge 11;$$

(b) $$\frac{8a^2-5bc}{b^2-bc+c^2}+\frac{8b^2-5ca}{c^2-ca+a^2}+\frac{8c^2-5ab}{a^2-ab+b^2} \ge 9.$$

1.162. If a,b,c are real numbers, no two of which are zero, then

$$\frac{4a^2+bc}{4b^2+7bc+4c^2}+\frac{4b^2+ca}{4c^2+7ca+4a^2}+\frac{4c^2+ab}{4a^2+7ab+4b^2} \ge 1.$$

1.163. If a,b,c are real numbers, no two of which are equal, then

$$\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2} \ge \frac{27}{4(a^2+b^2+c^2-ab-bc-ca)}.$$

1.164. If a,b,c are real numbers, no two of which are zero, then

$$\frac{1}{a^2-ab+b^2}+\frac{1}{b^2-bc+c^2}+\frac{1}{c^2-ca+a^2} \ge \frac{14}{3(a^2+b^2+c^2)}.$$

1.165. If a,b,c are real numbers, then

$$\frac{a^2+bc}{2a^2+b^2+c^2}+\frac{b^2+ca}{a^2+2b^2+c^2}+\frac{c^2+ab}{a^2+b^2+2c^2} \ge \frac{1}{6}.$$

1.166. If a, b, c are real numbers, then

$$\frac{2b^2+2c^2+3bc}{(a+3b+3c)^2}+\frac{2c^2+2a^2+3ca}{(b+3c+3a)^2}+\frac{2a^2+2b^2+3ab}{(c+3a+3b)^2}\ge\frac{3}{7}.$$

1.167. If a, b, c are nonnegative real numbers, then

$$\frac{6b^2+6c^2+13bc}{(a+2b+2c)^2}+\frac{6c^2+6a^2+13ca}{(b+2c+2a)^2}+\frac{6a^2+6b^2+13ab}{(c+2a+2b)^2}\le 3.$$

1.168. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

$$\frac{3a^2+8bc}{9+b^2+c^2}+\frac{3b^2+8ca}{9+c^2+a^2}+\frac{3c^2+8ab}{9+a^2+b^2}\le 3.$$

1.169. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

$$\frac{5a^2+6bc}{9+b^2+c^2}+\frac{5b^2+6ca}{9+c^2+a^2}+\frac{5c^2+6ab}{9+a^2+b^2}\ge 3.$$

1.170. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

$$\frac{1}{a^2+bc+12}+\frac{1}{b^2+ca+12}+\frac{1}{c^2+ab+12}\le\frac{3}{14}.$$

1.171. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}\ge\frac{45}{8(a^2+b^2+c^2)+2(ab+bc+ca)}.$$

1.172. If a, b, c are real numbers, no two of which are zero, then

$$\frac{a^2-7bc}{b^2+c^2}+\frac{b^2-7ca}{a^2+b^2}+\frac{c^2-7ab}{a^2+b^2}+\frac{9(ab+bc+ca)}{a^2+b^2+c^2}\ge 0.$$

1.173. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{a^2-4bc}{b^2+c^2}+\frac{b^2-4ca}{c^2+a^2}+\frac{c^2-4ab}{a^2+b^2}+\frac{9(ab+bc+ca)}{a^2+b^2+c^2}\ge\frac{9}{2}.$$

1.174. If a,b,c are real numbers such that $abc \neq 0$, then

$$\frac{(b+c)^2}{a^2}+\frac{(c+a)^2}{b^2}+\frac{(a+b)^2}{c^2} \geq 2+\frac{10(a+b+c)^2}{3(a^2+b^2+c^2)}.$$

1.175. Let a,b,c be real numbers, no two of which are zero. If $ab+bc+ca \geq 0$, then

(a)
$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2};$$

(b) $if\ ab \leq 0$, then

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq 2.$$

1.176. If a,b,c are nonnegative real numbers, then

$$\frac{a}{7a+b+c}+\frac{b}{7b+c+a}+\frac{c}{7c+a+b} \geq \frac{ab+bc+ca}{(a+b+c)^2}.$$

1.177. If a,b,c are positive real numbers such that $abc=1$, then

$$\frac{a+b+c}{30}+\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1} \geq \frac{8}{5}.$$

1.178. Let f be a real function defined on an interval $\mathbb{I}$, and let $x,y,s \in \mathbb{I}$ such that $x+my=(1+m)s$, where $m>0$. Prove that the inequality

$$f(x)+mf(y) \geq (1+m)f(s)$$

holds if and only if

$$h(x,y) \geq 0,$$

where

$$h(x,y)=\frac{g(x)-g(y)}{x-y}, \qquad g(u)=\frac{f(u)-f(s)}{u-s}.$$

1.179. Let $a,b,c \leq 8$ be real numbers such that $a+b+c=3$. Prove that

$$\frac{13a-1}{a^2+23}+\frac{13b-1}{b^2+23}+\frac{13c-1}{c^2+23} \leq \frac{3}{2}.$$

1.180. Let $a, b, c \neq \frac{3}{4}$ be nonnegative real numbers such that $a + b + c = 3$. Prove that

$$\frac{1-a}{(4a-3)^2} + \frac{1-b}{(4b-3)^2} + \frac{1-c}{(4c-3)^2} \geq 0.$$

1.181. If a, b, c are the lengths of the sides of a triangle, then

$$\frac{a^2}{4a^2+5bc} + \frac{b^2}{4b^2+5ca} + \frac{c^2}{4c^2+5ab} \geq \frac{1}{3}.$$

1.182. If a, b, c are the lengths of the sides of a triangle, then

$$\frac{1}{7a^2+b^2+c^2} + \frac{1}{7b^2+c^2+a^2} + \frac{1}{7c^2+a^2+b^2} \geq \frac{3}{(a+b+c)^2}.$$

1.183. Let a, b, c be the lengths of the sides of a triangle. If $k > -2$, then

$$\sum \frac{a(b+c)+(k+1)bc}{b^2+kbc+c^2} \leq \frac{3(k+3)}{k+2}.$$

1.184. Let a, b, c be the lengths of the sides of a triangle. If $k > -2$, then

$$\sum \frac{2a^2+(4k+9)bc}{b^2+kbc+c^2} \leq \frac{3(4k+11)}{k+2}.$$

1.185. If a, b, c are nnonnegative numbers such that $abc = 1$, then

$$\frac{1}{(a+1)^2} + \frac{1}{(b+1)^2} + \frac{1}{(c+1)^2} + \frac{1}{2(a+b+c-1)} \geq 1.$$

1.186. If a, b, c are positive real numbers such that

$$a \leq b \leq c, \qquad a^2bc \geq 1,$$

then

$$\frac{1}{1+a^3} + \frac{1}{1+b^3} + \frac{1}{1+c^3} \geq \frac{3}{1+abc}.$$

1.187. If a, b, c are positive real numbers such that

$$a \le b \le c, \qquad a^2c \ge 1,$$

then

$$\frac{1}{1+a^3}+\frac{1}{1+b^3}+\frac{1}{1+c^3} \ge \frac{3}{1+abc}.$$

1.188. If a, b, c are positive real numbers such that

$$a \le b \le c, \qquad 2a+c \ge 3,$$

then

$$\frac{1}{3+a^2}+\frac{1}{3+b^2}+\frac{1}{3+c^2} \ge \frac{3}{3+\left(\frac{a+b+c}{3}\right)^2}.$$

1.189. If a, b, c are positive real numbers such that

$$a \le b \le c, \qquad 9a+8b \ge 17,$$

then

$$\frac{1}{3+a^2}+\frac{1}{3+b^2}+\frac{1}{3+c^2} \ge \frac{3}{3+\left(\frac{a+b+c}{3}\right)^2}.$$

1.190. Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that

$$\sum \frac{1}{1+ab+bc+ca} \le 1.$$

1.191. Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}+\frac{1}{(1+c)^2}+\frac{1}{(1+d)^2} \ge 1.$$

1.192. Let $a, b, c, d \neq \dfrac{1}{3}$ be positive real numbers such that $abcd = 1$. Prove that

$$\frac{1}{(3a-1)^2}+\frac{1}{(3b-1)^2}+\frac{1}{(3c-1)^2}+\frac{1}{(3d-1)^2} \ge 1.$$

1.193. Let a,b,c,d be positive real numbers such that $abcd=1$. Prove that

$$\frac{1}{1+a+a^2+a^3}+\frac{1}{1+b+b^2+b^3}+\frac{1}{1+c+c^2+c^3}+\frac{1}{1+d+d^2+d^3}\ge 1.$$

1.194. Let a,b,c,d be positive real numbers such that $abcd=1$. Prove that

$$\frac{1}{1+a+2a^2}+\frac{1}{1+b+2b^2}+\frac{1}{1+c+2c^2}+\frac{1}{1+d+2d^2}\ge 1.$$

1.195. Let a,b,c,d be positive real numbers such that $abcd=1$. Prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{9}{a+b+c+d}\ge\frac{25}{4}.$$

1.196. If a,b,c,d are real numbers such that $a+b+c+d=0$, then

$$\frac{(a-1)^2}{3a^2+1}+\frac{(b-1)^2}{3b^2+1}+\frac{(c-1)^2}{3c^2+1}+\frac{(d-1)^2}{3d^2+1}\le 4.$$

1.197. If $a,b,c,d\ge -5$ such that $a+b+c+d=4$, then

$$\frac{1-a}{(1+a)^2}+\frac{1-b}{(1+b)^2}+\frac{1-c}{(1+c)^2}+\frac{1-d}{(1+d)^2}\ge 0.$$

1.198. Let $a_1,a_2,\ldots,a_n$ be positive real numbers such that $a_1+a_2+\cdots+a_n=n$. Prove that

$$\sum\frac{1}{(n+1)a_1^2+a_2^2+\cdots+a_n^2}\le\frac{1}{2}.$$

1.199. Let $a_1,a_2,\ldots,a_n$ be real numbers such that $a_1+a_2+\cdots+a_n=0$. Prove that

$$\frac{(a_1+1)^2}{a_1^2+n-1}+\frac{(a_2+1)^2}{a_2^2+n-1}+\cdots+\frac{(a_n+1)^2}{a_n^2+n-1}\ge\frac{n}{n-1}.$$

1.200. Let $a_1,a_2,\ldots,a_n$ be positive real numbers such that $a_1a_2\cdots a_n=1$. Prove that

(a) $$\frac{1}{1+(n-1)a_1}+\frac{1}{1+(n-1)a_2}+\cdots+\frac{1}{1+(n-1)a_n}\ge 1;$$

(b) $$\frac{1}{a_1+n-1}+\frac{1}{a_2+n-1}+\cdots+\frac{1}{a_n+n-1}\le 1.$$

1.201. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2\cdots a_n = 1$. Prove that

$$\frac{1}{1-a_1+na_1^2}+\frac{1}{1-a_2+na_2^2}+\cdots+\frac{1}{1-a_n+na_n^2}\ge 1.$$

1.202. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that

$$a_1, a_2, \ldots, a_n \ge \frac{k(n-k-1)}{kn-k-1}, \quad k>1$$

and

$$a_1a_2\cdots a_n = 1.$$

Prove that

$$\frac{1}{a_1+k}+\frac{1}{a_2+k}+\cdots+\frac{1}{a_n+k}\le \frac{n}{1+k}.$$

1.203. If $a_1, a_2, \ldots, a_n \ge 0$, then

$$\frac{1}{1+na_1}+\frac{1}{1+na_2}+\cdots+\frac{1}{1+na_n}\ge \frac{n}{n+a_1a_2\cdots a_n}.$$

1.2 Solutions

P 1.1. *If a, b are nonnegative real numbers, then*

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}\ge\frac{1}{1+ab}.$$

First Solution. Use the Cauchy-Schwarz inequality as follows:

$$\begin{aligned}\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}-\frac{1}{1+ab}&\ge\frac{(b+a)^2}{b^2(1+a)^2+a^2(1+b)^2}-\frac{1}{1+ab}\\&=\frac{ab[a^2+b^2-2(a+b)+2]}{(1+ab)[b^2(1+a)^2+a^2(1+b)^2]}\\&=\frac{ab[(a-1)^2+(b-1)^2]}{(1+ab)[b^2(1+a)^2+a^2(1+b)^2]}\ge 0.\end{aligned}$$

The equality holds for $a=b=1$.

Second Solution. By the Cauchy-Schwarz inequality, we have

$$(a+b)\left(a+\frac{1}{b}\right)\ge(a+1)^2,\qquad (a+b)\left(\frac{1}{a}+b\right)\ge(1+b)^2,$$

hence

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}\ge\frac{1}{(a+b)(a+1/b)}+\frac{1}{(a+b)(1/a+b)}=\frac{1}{1+ab}.$$

Third Solution. The desired inequality follows from the identity

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}-\frac{1}{1+ab}=\frac{ab(a-b)^2+(1-ab)^2}{(1+a)^2(1+b)^2(1+ab)}.$$

Remark. Replacing a by a/x and b by and b/x, where x is a positive number, we get the inequality

$$\frac{1}{(x+a)^2}+\frac{1}{(x+b)^2}\ge\frac{1}{x^2+ab},$$

which is valid for any $x, a, b \ge 0$.

□

P 1.2. *Let a, b, c be positive real numbers. Prove that*

(a) if $abc \le 1$, then

$$\frac{1}{2a+1}+\frac{1}{2b+1}+\frac{1}{2c+1}\ge 1;$$

(b) if $abc \ge 1$, then

$$\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}\le 1.$$

Solution. (a) Use the substitution

$$a=\frac{kx^2}{yz},\quad b=\frac{ky^2}{zx},\quad c=\frac{kz^2}{xy},$$

where $x,y,z>0$ and $0<k\le 1$. Applying the Cauchy-Schwarz inequality, we have

$$\sum\frac{1}{a+2}=\sum\frac{yz}{2kx^2+yz}\ge\sum\frac{yz}{2x^2+yz}\ge\frac{\left(\sum yz\right)^2}{\sum yz(2x^2+yz)}=1.$$

The equality holds for $a=b=c=1$.

(b) The desired inequality follows from the inequality in (a) by replacing a,b,c with $1/a,1/b,1/c$, respectively. The equality holds for $a=b=c=1$.

□

P 1.3. *If* $0\le a,b,c\le 1$, *then*

$$2\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)\ge 3\left(\frac{1}{2a+1}+\frac{1}{2b+1}+\frac{1}{2c+1}\right).$$

Solution. Write the inequality as $E(a,b,c)\ge 0$, assume that $0\le a\le b\le c\le 1$ and show that

$$E(a,b,c)\ge E(a,b,1)\ge E(a,1,1)\ge 0.$$

The inequality $E(a,b,c)\ge E(a,b,1)$ is equivalent to

$$2\left(\frac{1}{b+c}-\frac{1}{b+1}\right)+2\left(\frac{1}{c+a}-\frac{1}{1+a}\right)-3\left(\frac{1}{2c+1}-\frac{1}{3}\right)\ge 0,$$

$$(1-c)\left[\frac{1}{(b+c)(b+1)}+\frac{1}{(c+a)(1+a)}-\frac{1}{2c+1}\right]\ge 0.$$

We have

$$\frac{1}{(b+c)(b+1)}+\frac{1}{(c+a)(1+a)}-\frac{1}{2c+1}\ge\frac{1}{(1+c)(1+1)}+\frac{1}{(c+1)(1+1)}-\frac{1}{2c+1}$$
$$=\frac{c}{(c+1)(2c+1)}>0.$$

The inequality $E(a,b,1)\ge E(a,1,1)$ is equivalent to

$$2\left(\frac{1}{a+b}-\frac{1}{a+1}\right)+2\left(\frac{1}{1+b}-\frac{1}{2}\right)-3\left(\frac{1}{2b+1}-\frac{1}{3}\right),$$

$$(1-b)\left[\frac{2}{(a+b)(a+1)}+\frac{1}{1+b}-\frac{2}{2b+1}\right]\ge 0.$$

We have

$$\frac{2}{(a+b)(a+1)}+\frac{1}{1+b}-\frac{2}{2b+1} \geq \frac{2}{(1+b)(1+1)}+\frac{1}{1+b}-\frac{2}{2b+1}$$
$$=\frac{2b}{(1+b)(2b+1)}>0.$$

Finally,

$$E(a,1,1)=\frac{2a(1-a)}{(a+1)(2a+1)} \geq 0.$$

The equality holds for $a=b=c=1$, and also for $a=0$ and $b=c=1$ (or any cyclic permutation).

□

P 1.4. *If a,b,c are nonnegative real numbers such that $a+b+c \leq 3$, then*

$$2\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right) \geq 5\left(\frac{1}{2a+3}+\frac{1}{2b+3}+\frac{1}{2c+3}\right).$$

Solution. It suffices to prove the homogeneous inequality

$$\sum\left(\frac{2}{b+c}-\frac{5}{3a+b+c}\right) \geq 0.$$

We use the SOS (sum-of-squares) method. Without loss of generality, assume that

$$a \geq b \geq c.$$

Write the inequality as follows:

$$\sum \frac{2a-b-c}{(b+c)(3a+b+c)} \geq 0,$$

$$\sum \frac{a-b}{(b+c)(3a+b+c)}+\sum \frac{a-c}{(b+c)(3a+b+c)} \geq 0,$$

$$\sum \frac{a-b}{(b+c)(3a+b+c)}+\sum \frac{b-a}{(c+a)(3b+c+a)} \geq 0,$$

$$\sum(a-b)\left(\frac{1}{(b+c)(3a+b+c)}-\frac{1}{(c+a)(3b+c+a)}\right) \geq 0,$$

$$\sum(a-b)^2(a+b-c)(a+b)(3c+a+b) \geq 0.$$

Consider the nontrivial case $a>b+c$. Since $a+b-c>0$, it suffices to show that

$$(a-c)^2(a+c-b)(a+c)(3b+c+a) \geq (b-c)^2(a-b-c)(b+c)(3a+b+c).$$

This inequality is true since

$$(a-c)^2 \ge (b-c)^2, \qquad a+c-b \ge a-b-c$$

and

$$(a+c)(3b+c+a) \ge (b+c)(3a+b+c).$$

The last inequality is equivalent to

$$(a-b)(a+b-c) \ge 0.$$

The equality holds for $a=b=c=1$, and also for $a=b=3/2$ and $c=0$ (or any cyclic permutation).

□

P 1.5. *If a,b,c are nonnegative real numbers, then*

$$\frac{a^2-bc}{3a+b+c}+\frac{b^2-ca}{3b+c+a}+\frac{c^2-ab}{3c+a+b} \ge 0.$$

Solution. We use the SOS method. Without loss of generality, assume that

$$a \ge b \ge c.$$

We have

$$\begin{aligned}
2\sum \frac{a^2-bc}{3a+b+c} &= \sum \frac{(a-b)(a+c)+(a-c)(a+b)}{3a+b+c} \\
&= \sum \frac{(a-b)(a+c)}{3a+b+c} + \sum \frac{(b-a)(b+c)}{3b+c+a} \\
&= \sum \frac{(a-b)^2(a+b-c)}{(3a+b+c)(3b+c+a)}
\end{aligned}$$

Since $a+b-c \ge 0$, it suffices to show that

$$(b-c)^2(b+c-a)(3a+b+c)+(c-a)^2(c+a-b)(3b+c+a) \ge 0;$$

that is,

$$(a-c)^2(c+a-b)(3b+c+a) \ge (b-c)^2(a-b-c)(3a+b+c).$$

For the nontrivial case $a > b+c$, we can get this inequality by multiplying the obvious inequalities

$$c+a-b \ge a-b-c,$$

$$b^2(a-c)^2 \ge a^2(b-c)^2,$$

$$a(3b+c+a) \ge b(3a+b+c),$$

$$a \ge b.$$

The equality holds for $a=b=c$, and also for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 1.6. *If a,b,c are positive real numbers, then*

$$\frac{4a^2-b^2-c^2}{a(b+c)}+\frac{4b^2-c^2-a^2}{b(c+a)}+\frac{4c^2-a^2-b^2}{c(a+b)}\le 3.$$

(Vasile Cîrtoaje, 2006)

Solution. We use the SOS method. Write the inequality as follows:

$$\sum\left[1-\frac{4a^2-b^2-c^2}{a(b+c)}\right]\ge 0,$$

$$\sum\frac{b^2+c^2-4a^2+a(b+c)}{a(b+c)}\ge 0,$$

$$\sum\frac{(b^2-a^2)+a(b-a)+(c^2-a^2)+a(c-a)}{a(b+c)}\ge 0,$$

$$\sum\frac{(b-a)(2a+b)+(c-a)(2a+c)}{a(b+c)}\ge 0,$$

$$\sum\frac{(b-a)(2a+b)}{a(b+c)}+\sum\frac{(a-b)(2b+a)}{b(c+a)}\ge 0,$$

$$\sum c(a+b)(a-b)^2(bc+ca-ab)\ge 0.$$

Without loss of generality, assume that

$$a\ge b\ge c.$$

Since $ca+ab-bc>0$, it suffices to show that

$$b(c+a)(c-a)^2(ab+bc-ca)+c(a+b)(a-b)^2(bc+ca-ab)\ge 0,$$

that is,

$$b(c+a)(a-c)^2(ab+bc-ca)\ge c(a+b)(a-b)^2(ab-bc-ca).$$

For the nontrivial case $ab-bc-ca>0$, this inequality follows by multiplying the inequalities

$$ab+bc-ca>ab-bc-ca,$$

$$(a-c)^2\ge(a-b)^2,$$

$$b(c+a)\ge c(a+b).$$

The equality holds for $a=b=c$

□

P 1.7. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

(a) $$\frac{1}{a^2+bc}+\frac{1}{b^2+ca}+\frac{1}{c^2+ab}\ge \frac{3}{ab+bc+ca};$$

(b) $$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab}\ge \frac{2}{ab+bc+ca}.$$

(c) $$\frac{1}{a^2+2bc}+\frac{1}{b^2+2ca}+\frac{1}{c^2+2ab}> \frac{2}{ab+bc+ca}.$$

(Vasile Cîrtoaje, 2005)

Solution. (a) Since
$$\frac{ab+bc+ca}{a^2+bc}=1+\frac{a(b+c-a)}{a^2+bc},$$
we can write the inequality as
$$\frac{a(b+c-a)}{a^2+bc}+\frac{b(c+a-b)}{b^2+ca}+\frac{c(a+b-c)}{c^2+ab}\ge 0.$$
Without loss of generality, assume that
$$a=\min\{a,b,c\}.$$
Since $b+c-a>0$, it suffices to show that
$$\frac{b(c+a-b)}{b^2+ca}+\frac{c(a+b-c)}{c^2+ab}\ge 0.$$
This is equivalent to each of the following inequalities
$$(b^2+c^2)a^2-(b+c)(b^2-3bc+c^2)a+bc(b-c)^2\ge 0,$$
$$(b-c)^2a^2-(b+c)(b-c)^2a+bc(b-c)^2+abc(2a+b+c)\ge 0,$$
$$(b-c)^2(a-b)(a-c)+abc(2a+b+c)\ge 0.$$
The last inequality is obviously true. The equality holds for $a=0$ and $b=c$ (or any cyclic permutation thereof).

(b) Using the identities
$$2a^2+bc=a(2a-b-c)+ab+bc+ca,$$
$$2b^2+ca=b(2b-c-a)+ab+bc+ca,$$
$$2c^2+ab=c(2c-a-b)+ab+bc+ca,$$
we can write the inequality as
$$\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}\ge 2,$$

where
$$x = \frac{a(2a-b-c)}{ab+bc+ca}, \quad y = \frac{b(2b-c-a)}{ab+bc+ca}, \quad z = \frac{c(2c-a-b)}{ab+bc+ca}.$$
Without loss of generality, assume that $a = \min\{a, b, c\}$. Since
$$x \le 0, \qquad \frac{1}{1+x} \ge 1,$$
it suffices to show that
$$\frac{1}{1+y} + \frac{1}{1+z} \ge 1.$$
This is equivalent to
$$1 \ge yz,$$
$$(ab+bc+ca)^2 \ge bc(2b-c-a)(2c-a-b),$$
$$a^2(b^2+bc+c^2) + 3abc(b+c) + 2bc(b-c)^2 \ge 0.$$
The last inequality is obviously true. The equality holds for $a = 0$ and $b = c$ (or any cyclic permutation thereof).

(c) According to the identities
$$a^2 + 2bc = (a-b)(a-c) + ab + bc + ca,$$
$$b^2 + 2ca = (b-c)(b-a) + ab + bc + ca,$$
$$c^2 + 2ab = (c-a)(c-b) + ab + bc + ca,$$
we can write the inequality as
$$\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} > 2,$$
where
$$x = \frac{(a-b)(a-c)}{ab+bc+ca}, \quad y = \frac{(b-c)(b-a)}{ab+bc+ca}, \quad z = \frac{(c-a)(c-b)}{ab+bc+ca}.$$
Since
$$xy + yz + zx = 0$$
and
$$xyz = \frac{-(a-b)^2(b-c)^2(c-a)^2}{(ab+bc+ca)^3} \le 0,$$
we have
$$\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} - 2 = \frac{1-2xyz}{(1+x)(1+y)(1+z)} > 0.$$
□

P 1.8. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a(b+c)}{a^2+bc}+\frac{b(c+a)}{b^2+ca}+\frac{c(a+b)}{c^2+ab} \ge 2.$$

(Pham Kim Hung, 2006)

Solution. Without loss of generality, assume that $a \ge b \ge c$ and write the inequality as

$$\frac{b(c+a)}{b^2+ca} \ge \frac{(a-b)(a-c)}{a^2+bc}+\frac{(a-c)(b-c)}{c^2+ab}.$$

Since

$$\frac{(a-b)(a-c)}{a^2+bc} \le \frac{(a-b)a}{a^2+bc} \le \frac{a-b}{a}$$

and

$$\frac{(a-c)(b-c)}{c^2+ab} \le \frac{a(b-c)}{c^2+ab} \le \frac{b-c}{b},$$

it suffices to show that

$$\frac{b(c+a)}{b^2+ca} \ge \frac{a-b}{a}+\frac{b-c}{b}.$$

This inequality is equivalent to

$$b^2(a-b)^2-2abc(a-b)+a^2c^2+ab^2c \ge 0,$$

$$(ab-b^2-ac)^2+ab^2c \ge 0.$$

The equality holds for for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 1.9. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2} \ge \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}.$$

(Vasile Cîrtoaje, 2002)

Solution. Use the SOS method. We have

$$\sum\left(\frac{a^2}{b^2+c^2}-\frac{a}{b+c}\right)=\sum\frac{ab(a-b)+ac(a-c)}{(b^2+c^2)(b+c)}$$

$$=\sum\frac{ab(a-b)}{(b^2+c^2)(b+c)}+\sum\frac{ba(b-a)}{(c^2+a^2)(c+a)}$$

$$=(a^2+b^2+c^2+ab+bc+ca)\sum\frac{ab(a-b)^2}{(b^2+c^2)(c^2+a^2)(b+c)(c+a)} \ge 0.$$

The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.10. *Let a, b, c be positive real numbers. Prove that*

$$\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b} \geq \frac{a}{a^2+bc}+\frac{b}{b^2+ca}+\frac{c}{c^2+ab}.$$

First Solution. Without loss of generality, assume that $a = \min\{a, b, c\}$. Since

$$\begin{aligned}\sum \frac{1}{b+c}-\sum \frac{a}{a^2+bc} &= \sum\left(\frac{1}{b+c}-\frac{a}{a^2+bc}\right)\\ &= \sum \frac{(a-b)(a-c)}{(b+c)(a^2+bc)}\end{aligned}$$

and $(a-b)(a-c) \geq 0$, it suffices to show that

$$\frac{(b-c)(b-a)}{(c+a)(b^2+ca)}+\frac{(c-a)(c-b)}{(a+b)(c^2+ab)} \geq 0.$$

This inequality is equivalent to

$$(b-c)[(b^2-a^2)(c^2+ab)+(a^2-c^2)(b^2+ca)] \geq 0,$$

$$a(b-c)^2(b^2+c^2-a^2+ab+bc+ca) \geq 0.$$

The last inequality is clearly true. The equality holds for $a = b = c$.

Second Solution. Since

$$\sum \frac{1}{b+c} = \sum\left[\frac{b}{(b+c)^2}+\frac{c}{(b+c)^2}\right] = \sum a\left[\frac{1}{(a+b)^2}+\frac{1}{(a+c)^2}\right],$$

we can write the inequality as

$$\sum a\left[\frac{1}{(a+b)^2}+\frac{1}{(a+c)^2}-\frac{1}{a^2+bc}\right] \geq 0.$$

This is true since, according to Remark from P 1.1, we have

$$\frac{1}{(a+b)^2}+\frac{1}{(a+c)^2}-\frac{1}{a^2+bc} \geq 0.$$

□

P 1.11. *Let a, b, c be positive real numbers. Prove that*

$$\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b} \geq \frac{2a}{3a^2+bc}+\frac{2b}{3b^2+ca}+\frac{2c}{3c^2+ab}.$$

(Vasile Cîrtoaje, 2005)

Solution. Since

$$\sum \frac{1}{b+c} - \sum \frac{2a}{3a^2+bc} = \sum \left(\frac{1}{b+c} - \frac{2a}{3a^2+bc} \right)$$
$$= \sum \frac{(a-b)(a-c)+a(2a-b-c)}{(b+c)(3a^2+bc)},$$

it suffices to show that

$$\sum \frac{(a-b)(a-c)}{(b+c)(3a^2+bc)} \ge 0$$

and

$$\sum \frac{a(2a-b-c)}{(b+c)(3a^2+bc)} \ge 0.$$

In order to prove the first inequality, assume that $a = \min\{a,b,c\}$. Since

$$(a-b)(a-c) \ge 0,$$

it is enough to show that

$$\frac{(b-c)(b-a)}{(c+a)(3b^2+ca)} + \frac{(c-a)(c-b)}{(a+b)(3c^2+ab)} \ge 0.$$

This is equivalent to the obvious inequality

$$a(b-c)^2(b^2+c^2-a^2+3ab+bc+3ca) \ge 0.$$

The second inequality can be proved by the SOS method. We have

$$\sum \frac{a(2a-b-c)}{(b+c)(3a^2+bc)} = \sum \frac{a(a-b)+a(a-c)}{(b+c)(3a^2+bc)}$$
$$= \sum \frac{a(a-b)}{(b+c)(3a^2+bc)} + \sum \frac{b(b-a)}{(c+a)(3b^2+ca)}$$
$$= \sum (a-b)\left[\frac{a}{(b+c)(3a^2+bc)} - \frac{b}{(c+a)(3b^2+ca)} \right]$$
$$= \sum \frac{c(a-b)^2[(a-b)^2+c(a+b)]}{(b+c)(c+a)(3a^2+bc)(3b^2+ca)} \ge 0.$$

The equality holds for $a = b = c$.

□

P 1.12. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

(a) $$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{13}{6} - \frac{2(ab+bc+ca)}{3(a^2+b^2+c^2)};$$

(b) $$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - \frac{3}{2} \ge (\sqrt{3}-1)\left(1 - \frac{ab+bc+ca}{a^2+b^2+c^2}\right).$$

(Vasile Cîrtoaje, 2006)

Solution. (a) We use the SOS method. Rewrite the inequality as

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2}\ge\frac{2}{3}\left(1-\frac{ab+bc+ca}{a^2+b^2+c^2}\right).$$

Since

$$\begin{aligned}\sum\left(\frac{a}{b+c}-\frac{1}{2}\right)&=\sum\frac{(a-b)+(a-c)}{2(b+c)}\\&=\sum\frac{a-b}{2(b+c)}+\sum\frac{b-a}{2(c+a)}\\&=\sum\frac{a-b}{2}\left(\frac{1}{b+c}-\frac{1}{c+a}\right)\\&=\sum\frac{(a-b)^2}{2(b+c)(c+a)}\end{aligned}$$

and

$$\frac{2}{3}\left(1-\frac{ab+bc+ca}{a^2+b^2+c^2}\right)=\sum\frac{(a-b)^2}{3(a^2+b^2+c^2)},$$

the inequality can be restated as

$$\sum(a-b)^2\left[\frac{1}{2(b+c)(c+a)}-\frac{1}{3(a^2+b^2+c^2)}\right]\ge 0.$$

This is true since

$$3(a^2+b^2+c^2)-2(b+c)(c+a)=(a+b-c)^2+2(a-b)^2\ge 0.$$

The equality holds for $a=b=c$.

(b) Let

$$p=a+b+c,\quad q=ab+bc+ca,\quad r=abc.$$

We have

$$\begin{aligned}\sum\frac{a}{b+c}&=\sum\left(\frac{a}{b+c}+1\right)-3=p\sum\frac{1}{b+c}-3\\&=\frac{p(p^2+q)}{pq-r}-3.\end{aligned}$$

According to P 3.57-(a) in Volume 1, for fixed p and q, the product r is minimum when $a=0$ or $b=c$. Therefore, it suffices to prove the inequality for $a=0$ and for $b=c=1$.

Case 1: $a=0$. The original inequality can be written as

$$\frac{b}{c}+\frac{c}{b}-\frac{3}{2}\ge(\sqrt{3}-1)\left(1-\frac{bc}{b^2+c^2}\right).$$

It suffices to show that
$$\frac{b}{c}+\frac{c}{b}-\frac{3}{2}\ge 1-\frac{bc}{b^2+c^2}.$$
Denoting
$$t=\frac{b^2+c^2}{bc},\quad t\ge 2,$$
this inequality becomes
$$t-\frac{3}{2}\ge 1-\frac{1}{t},$$
$$(t-2)(2t-1)\ge 0.$$

Case 2: $b=c=1$. The original inequality becomes as follows:
$$\frac{a}{2}+\frac{2}{a+1}-\frac{3}{2}\ge(\sqrt{3}-1)\left(1-\frac{2a+1}{a^2+2}\right),$$
$$\frac{(a-1)^2}{2(a+1)}\ge\frac{(\sqrt{3}-1)(a-1)^2}{a^2+2},$$
$$(a-1)^2(a-\sqrt{3}+1)^2\ge 0.$$
The equality holds for $a=b=c$, and for $\frac{a}{\sqrt{3}-1}=b=c$ (or any cyclic permutation).

□

P 1.13. *Let a,b,c be positive real numbers. Prove that*
$$\frac{1}{a^2+2bc}+\frac{1}{b^2+2ca}+\frac{1}{c^2+2ab}\le\left(\frac{a+b+c}{ab+bc+ca}\right)^2.$$

(Vasile Cîrtoaje, 2006)

First Solution. Assume that $a\ge b\ge c$ and write the inequality as
$$\frac{(a+b+c)^2}{ab+bc+ca}-3\ge\sum\left(\frac{ab+bc+ca}{a^2+2bc}-1\right),$$
$$\frac{(a-b)^2+(b-c)^2+(a-b)(b-c)}{ab+bc+ca}+\sum\frac{(a-b)(a-c)}{a^2+2bc}\ge 0.$$
Since
$$(a-b)(a-c)\ge 0,\qquad (c-a)(c-b)\ge 0,$$
it suffices to show that
$$(a-b)^2+(b-c)^2+(a-b)(b-c)-\frac{(ab+bc+ca)(a-b)(b-c)}{b^2+2ca}\ge 0.$$

This inequality is equivalent to

$$(a-b)^2+(b-c)^2-\frac{(a-b)^2(b-c)^2}{b^2+2ca}\geq 0,$$

$$(b-c)^2+\frac{c(a-b)^2(2a+2b-c)}{b^2+2ca}\geq 0.$$

Clearly, the last inequality is true. The equality holds for $a=b=c$.

Second Solution. Assume that $a\geq b\geq c$ and write the desired inequality as

$$\frac{(a+b+c)^2}{ab+bc+ca}-3\geq\sum\left(\frac{ab+bc+ca}{a^2+2bc}-1\right),$$

$$\frac{1}{ab+bc+ca}\sum(a-b)(a-c)+\sum\frac{(a-b)(a-c)}{a^2+2bc}\geq 0,$$

$$\sum\left(1+\frac{ab+bc+ca}{a^2+2bc}\right)(a-b)(a-c)\geq 0.$$

Since $(c-a)(c-b)\geq 0$ and $a-b\geq 0$, it suffices to prove that

$$\left(1+\frac{ab+bc+ca}{a^2+2bc}\right)(a-c)+\left(1+\frac{ab+bc+ca}{b^2+2ca}\right)(c-b)\geq 0.$$

Write this inequality as

$$a-b+(ab+bc+ca)\left(\frac{a-c}{a^2+2bc}+\frac{c-b}{b^2+2ca}\right)\geq 0,$$

$$(a-b)\left[1+\frac{(ab+bc+ca)(3ac+3bc-ab-2c^2}{(a^2+2bc)(b^2+2ca)}\right]\geq 0.$$

Since $a-b\geq 0$ and $2ac+3bc-2c^2>0$, it is enough to show that

$$1+\frac{(ab+bc+ca)(ac-ab)}{(a^2+2bc)(b^2+2ca)}\geq 0.$$

We have

$$1+\frac{(ab+bc+ca)(ac-ab)}{(a^2+2bc)(b^2+2ca)}\geq 1+\frac{(ab+bc+ca)(ac-ab)}{a^2(b^2+ca)}$$

$$=\frac{(a+b)c^2+(a^2-b^2)c}{a(b^2+ca)}>0.$$

□

P 1.14. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a^2(b+c)}{b^2+c^2}+\frac{b^2(c+a)}{c^2+a^2}+\frac{c^2(a+b)}{a^2+b^2}\geq a+b+c.$$

(Darij Grinberg, 2004)

First Solution. Use the SOS method. We have

$$\begin{aligned}\sum \frac{a^2(b+c)}{b^2+c^2}-\sum a &= \sum\left[\frac{a^2(b+c)}{b^2+c^2}-a\right]\\ &=\sum \frac{ab(a-b)+ac(a-c)}{b^2+c^2}\\ &=\sum \frac{ab(a-b)}{b^2+c^2}+\sum \frac{ba(b-a)}{c^2+a^2}\\ &=\sum \frac{ab(a+b)(a-b)^2}{(b^2+c^2)(c^2+a^2)}\ge 0.\end{aligned}$$

The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

Second Solution. By virtue of the Cauchy-Schwarz inequality, we have

$$\sum \frac{a^2(b+c)}{b^2+c^2} \ge \frac{\left[\sum a^2(b+c)\right]^2}{\sum a^2(b+c)(b^2+c^2)}.$$

Then, it suffices to show that

$$\left[\sum a^2(b+c)\right]^2 \ge \left(\sum a\right)\left[\sum a^2(b+c)(b^2+c^2)\right].$$

Let $p=a+b+c$ and $q=ab+bc+ca$. Since

$$\begin{aligned}\left[\sum a^2(b+c)\right]^2 &= (pq-3abc)^2\\ &= p^2q^2-6abcpq+9a^2b^2c^2\end{aligned}$$

and

$$\begin{aligned}\sum a^2(b+c)(b^2+c^2) &= \sum(b+c)[(a^2b^2+b^2c^2+c^2a^2)-b^2c^2]\\ &=2p(a^2b^2+b^2c^2+c^2a^2)-\sum b^2c^2(p-a)\\ &=p(a^2b^2+b^2c^2+c^2a^2)+abcq=p(q^2-2abcp)+abcq,\end{aligned}$$

the inequality can be written as

$$p^2q^2-6abcpq+9a^2b^2c^2 \ge p^2(q^2-2abcp)+abcpq,$$

$$abc(2p^3+9abc-7pq)\ge 0.$$

Using Schur's inequality

$$p^3+9abc-4pq\ge 0,$$

we have

$$2p^3+9abc-7pq \ge p(p^2-3q)\ge 0.$$

□

P 1.15. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}\le\frac{3(a^2+b^2+c^2)}{a+b+c}.$$

Solution. Use the SOS method.

First Solution. Multiplying by $2(a+b+c)$, the inequality successively becomes:

$$\sum\left(1+\frac{a}{b+c}\right)(b^2+c^2)\le 3(a^2+b^2+c^2),$$

$$\sum\frac{a}{b+c}(b^2+c^2)\le\sum a^2,$$

$$\sum a\left(a-\frac{b^2+c^2}{b+c}\right)\ge 0,$$

$$\sum\frac{ab(a-b)-ac(c-a)}{b+c}\ge 0,$$

$$\sum\frac{ab(a-b)}{b+c}-\sum\frac{ba(a-b)}{c+a}\ge 0,$$

$$\sum\frac{ab(a-b)^2}{(b+c)(c+a)}\ge 0.$$

The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

Second Solution. Subtracting $a+b+c$ from the both sides, the desired inequality becomes as follows:

$$\frac{3(a^2+b^2+c^2)}{a+b+c}-(a+b+c)\ge\sum\left(\frac{a^2+b^2}{a+b}-\frac{a+b}{2}\right),$$

$$\sum\frac{(a-b)^2}{a+b+c}\ge\sum\frac{(a-b)^2}{2(a+b)},$$

$$\sum\frac{(a+b-c)(a-b)^2}{a+b}\ge 0.$$

Without loss of generality, assume that $a\ge b\ge c$. Since $a+b-c\ge 0$, it suffices to prove that

$$\frac{(a+c-b)(a-c)^2}{a+c}\ge\frac{(a-b-c)(b-c)^2}{b+c}.$$

This inequality is true because

$$a+c-b\ge a-b-c,\quad a-c\ge b-c,\quad \frac{a-c}{a+c}\ge\frac{b-c}{b+c}.$$

The last inequality reduces to $c(a-b)\ge 0$.

Third Solution. Write the inequality as follows:

$$\sum\left[\frac{3(a^2+b^2)}{2(a+b+c)}-\frac{a^2+b^2}{a+b}\right]\ge 0,$$

$$\sum\frac{(a^2+b^2)(a+b-2c)}{a+b}\ge 0,$$

$$\sum\frac{(a^2+b^2)(a-c)}{a+b}+\sum\frac{(a^2+b^2)(b-c)}{a+b}\ge 0,$$

$$\sum\frac{(a^2+b^2)(a-c)}{a+b}+\sum\frac{(b^2+c^2)(c-a)}{b+c}\ge 0,$$

$$\sum\frac{(a-c)^2(ab+bc+ca-b^2)}{(a+b)(b+c)}\ge 0.$$

It suffices to prove that

$$\sum\frac{(a-c)^2(ab+bc-ca-b^2)}{(a+b)(b+c)}\ge 0.$$

Since

$$ab+bc-ca-b^2=(a-b)(b-c),$$

this inequality is equivalent to

$$(a-b)(b-c)(c-a)\sum\frac{c-a}{(a+b)(b+c)}\ge 0,$$

which is true because

$$\sum\frac{c-a}{(a+b)(b+c)}=0.$$

□

P 1.16. *Let a,b,c be positive real numbers. Prove that*

$$\frac{1}{a^2+ab+b^2}+\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2}\ge\frac{9}{(a+b+c)^2}.$$

(Vasile Cîrtoaje, 2000)

First Solution. Due to homogeneity, we may assume that

$$a+b+c=1.$$

Let $q=ab+bc+ca$. Since

$$b^2+bc+c^2=(a+b+c)^2-a(a+b+c)-(ab+bc+ca)=1-a-q,$$

we can write the inequality as

$$\sum \frac{1}{1-a-q} \geq 9,$$

$$9q^3 - 6q^2 - 3q + 1 + 9abc \geq 0.$$

From Schur's inequality

$$(a+b+c)^3 + 9abc \geq 4(a+b+c)(ab+bc+ca),$$

we get

$$1 + 9abc - 4q \geq 0.$$

Therefore,

$$9q^3 - 6q^2 - 3q + 1 + 9abc = (1 + 9abc - 4q) + q(3q-1)^2 \geq 0.$$

The equality holds for $a = b = c$.

Second Solution. Multiplying by $a^2 + b^2 + c^2 + ab + bc + ca$, the inequality can be written as

$$(a+b+c)\sum \frac{a}{b^2+bc+c^2} + \frac{9(ab+bc+ca)}{(a+b+c)^2} \geq 6.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a}{b^2+bc+c^2} \geq \frac{(a+b+c)^2}{\sum a(b^2+bc+c^2)} = \frac{a+b+c}{ab+bc+ca}.$$

Then, it suffices to show that

$$\frac{(a+b+c)^2}{ab+bc+ca} + \frac{9(ab+bc+ca)}{(a+b+c)^2} \geq 6.$$

This follows immediately from the AM-GM inequality.

□

P 1.17. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a^2}{(2a+b)(2a+c)} + \frac{b^2}{(2b+c)(2b+a)} + \frac{c^2}{(2c+a)(2c+b)} \leq \frac{1}{3}.$$

(Tigran Sloyan, 2005)

First Solution. The inequality is equivalent to each of the inequalities

$$\sum\left[\frac{a^2}{(2a+b)(2a+c)}-\frac{a}{3(a+b+c)}\right]\le 0,$$

$$\sum\frac{a(a-b)(a-c)}{(2a+b)(2a+c)}\ge 0.$$

Due to symmetry, we may consider

$$a\ge b\ge c.$$

Since $c(c-a)(c-b)\ge 0$, it suffices to prove that

$$\frac{a(a-b)(a-c)}{(2a+b)(2a+c)}+\frac{b(b-c)(b-a)}{(2b+c)(2b+a)}\ge 0.$$

This is equivalent to the obvious inequality

$$(a-b)^2[(a+b)(2ab-c^2)+c(a^2+b^2+5ab)]\ge 0.$$

The equality holds for $a=b=c$, and also for $a=b$ and $c=0$ (or any cyclic permutation).

Second Solution (by *Vo Quoc Ba Can*). Apply the Cauchy-Schwarz inequality in the following manner

$$\frac{9a^2}{(2a+b)(2a+c)}=\frac{(2a+a)^2}{2a(a+b+c)+(2a^2+bc)}\le\frac{2a}{a+b+c}+\frac{a^2}{2a^2+bc}.$$

Then,

$$\sum\frac{9a^2}{(2a+b)(2a+c)}\le 2+\sum\frac{a^2}{2a^2+bc}\le 3.$$

For the nontrivial case $a,b,c>0$, the right inequality is equivalent to

$$\sum\frac{1}{2+bc/a^2}\le 1,$$

which follows immediately from P 1.2-(b).

Remark. From the inequality in P 1.17 and Hölder's inequality

$$\left[\sum\frac{a^2}{(2a+b)(2a+c)}\right]\left[\sum\sqrt{a(2a+b)(2a+c)}\right]^2\ge(a+b+c)^3,$$

we get the following result:

- *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

$$\sqrt{a(2a+b)(2a+c)}+\sqrt{b(2b+c)(2b+a)}+\sqrt{c(2c+a)(2c+bc)}\ge 9,$$

with equality for $a=b=c=1$, and for $(a,b,c)=\left(0,\frac{3}{2},\frac{3}{2}\right)$ (or any cyclic permutation).

□

P 1.18. *Let a, b, c be positive real numbers. Prove that*

(a) $$\sum \frac{a}{(2a+b)(2a+c)} \le \frac{1}{a+b+c};$$

(b) $$\sum \frac{a^3}{(2a^2+b^2)(2a^2+c^2)} \le \frac{1}{a+b+c}.$$

(Vasile Cîrtoaje, 2005)

Solution. (a) Write the inequality as

$$\sum \left[\frac{1}{3} - \frac{a(a+b+c)}{(2a+b)(2a+c)}\right] \ge 0,$$

$$\sum \frac{(a-b)(a-c)}{(2a+b)(2a+c)} \ge 0.$$

Assume that

$$a \ge b \ge c.$$

Since $(a-b)(a-c) \ge 0$, it suffices to prove that

$$\frac{(b-c)(b-a)}{(2b+c)(2b+a)} + \frac{(a-c)(b-c)}{(2c+a)(2c+b)} \ge 0.$$

In addition, since $b-c \ge 0$ and $a-c \ge a-b \ge 0$, it is enough to show that

$$\frac{1}{(2c+a)(2c+b)} \ge \frac{1}{(2b+c)(2b+a)}.$$

This is equivalent to the obvious inequality

$$(b-c)(a+4b+4c) \ge 0.$$

The equality holds for $a = b = c$.

(b) We obtain the desired inequality by summing the inequalities

$$\frac{a^3}{(2a^2+b^2)(2a^2+c^2)} \le \frac{a}{(a+b+c)^2},$$

$$\frac{b^3}{(2b^2+c^2)(2b^2+a^2)} \le \frac{b}{(a+b+c)^2},$$

$$\frac{c^3}{(2c^2+a^2)(2c^2+b^2)} \le \frac{c}{(a+b+c)^2},$$

which are consequences of the Cauchy-Schwarz inequality. For example, from

$$(a^2+a^2+b^2)(c^2+a^2+a^2) \ge (ac+a^2+ba)^2,$$

the first inequality follows. The equality holds for $a = b = c$.

□

P 1.19. *If a,b,c are positive real numbers, then*

$$\sum \frac{1}{(a+2b)(a+2c)} \ge \frac{1}{(a+b+c)^2} + \frac{2}{3(ab+bc+ca)}.$$

Solution. Write the inequality as follows:

$$\sum \left[\frac{1}{(a+2b)(a+2c)} - \frac{1}{(a+b+c)^2} \right] \ge \frac{2}{3(ab+bc+ca)} - \frac{2}{(a+b+c)^2},$$

$$\sum \frac{(b-c)^2}{(a+2b)(a+2c)} \ge \sum \frac{(b-c)^2}{3(ab+bc+ca)},$$

$$(a-b)(b-c)(c-a)\sum \frac{b-c}{(a+2b)(a+2c)} \ge 0.$$

Since

$$\begin{aligned}\sum \frac{b-c}{(a+2b)(a+2c)} &= \sum \left[\frac{b-c}{(a+2b)(a+2c)} - \frac{b-c}{3(ab+bc+ca)} \right] \\ &= \frac{(a-b)(b-c)(c-a)}{3(ab+bc+ca)} \sum \frac{1}{(a+2b)(a+2c)},\end{aligned}$$

the desired inequality is equivalent to the obvious inequality

$$(a-b)^2(b-c)^2(c-a)^2 \sum \frac{1}{(a+2b)(a+2c)} \ge 0.$$

The equality holds for $a=b$, or $b=c$, or $c=a$.

□

P 1.20. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

(a) $$\frac{1}{(a-b)^2} + \frac{1}{(b-c)^2} + \frac{1}{(c-a)^2} \ge \frac{4}{ab+bc+ca};$$

(b) $$\frac{1}{a^2-ab+b^2} + \frac{1}{b^2-bc+c^2} + \frac{1}{c^2-ca+a^2} \ge \frac{3}{ab+bc+ca};$$

(c) $$\frac{1}{a^2+b^2} + \frac{1}{b^2+c^2} + \frac{1}{c^2+a^2} \ge \frac{5}{2(ab+bc+ca)}.$$

Solution. Let

$$E_k(a,b,c) = \frac{ab+bc+ca}{a^2-kab+b^2} + \frac{ab+bc+ca}{b^2-kbc+c^2} + \frac{ab+bc+ca}{c^2-kca+a^2},$$

where $k \in [0,2]$. We will prove that

$$E_k(a,b,c) \ge \alpha_k,$$

where

$$\alpha_k = \begin{cases} \dfrac{5-2k}{2-k}, & 0 \le k \le 1 \\ 2+k, & 1 \le k \le 2 \end{cases}.$$

Assume that $a \le b \le c$ and show that

$$E_k(a,b,c) \ge E_k(0,b,c) \ge \alpha_k.$$

The left inequality is true because

$$\begin{aligned} &\frac{E_k(a,b,c)-E_k(0,b,c)}{a} = \\ &= \frac{b^2+(1+k)bc-ac}{b(a^2-kab+b^2)} + \frac{b+c}{b^2-kbc+c^2} + \frac{c^2+(1+k)bc-ab}{c(c^2-kca+a^2)} \\ &> \frac{bc-ac}{b(a^2-kab+b^2)} + \frac{b+c}{b^2-kbc+c^2} + \frac{bc-ab}{c(c^2-kca+a^2)} > 0. \end{aligned}$$

In order to prove the right inequality, $E_k(0,b,c) \ge \alpha_k$, where

$$E_k(0,b,c) = \frac{bc}{b^2-kbc+c^2} + \frac{b}{c} + \frac{c}{b},$$

we well use the AM-GM inequality. Thus, for $k \in [1,2]$, we have

$$E_k(0,b,c) = \frac{bc}{b^2-kbc+c^2} + \frac{b^2-kbc+c^2}{bc} + k \ge 2+k.$$

Also, for $k \in [0,1]$, we have

$$\begin{aligned} E_k(0,b,c) =& \frac{bc}{b^2-kbc+c^2} + \frac{b^2-kbc+c^2}{(2-k)^2bc} \\ &+ \left[1-\frac{1}{(2-k)^2}\right]\left(\frac{b}{c}+\frac{c}{b}\right) + \frac{k}{(2-k)^2} \\ &\ge \frac{2}{2-k} + 2\left[1-\frac{1}{(2-k)^2}\right] + \frac{k}{(2-k)^2} = \frac{5-2k}{2-k}. \end{aligned}$$

For $k \in [1,2]$, the equality holds when $a = 0$ and $\dfrac{b}{c} + \dfrac{c}{b} = 1+k$ (or any cyclic permutation). For $k \in [0,1]$, the equality holds when $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 1.21. *If a,b,c are positive real numbers, then*

$$\frac{(a^2+b^2)(a^2+c^2)}{(a+b)(a+c)}+\frac{(b^2+c^2)(b^2+a^2)}{(b+c)(b+a)}+\frac{(c^2+a^2)(c^2+b^2)}{(c+a)(c+b)}\ge a^2+b^2+c^2.$$

(Vasile Cîrtoaje, 2011)

Solution. Using the identity

$$(a^2+b^2)(a^2+c^2)=b^2c^2+a^2(a^2+b^2+c^2),$$

we can write the inequality as follows:

$$\sum\frac{b^2c^2}{(a+b)(a+c)}\ge(a^2+b^2+c^2)\left[1-\sum\frac{a^2}{(a+b)(a+c)}\right],$$

$$\sum b^2c^2(b+c)\ge 2abc(a^2+b^2+c^2),$$

$$\sum a^3(b^2+c^2)\ge 2\sum a^3bc,$$

$$\sum a^3(b-c)^2\ge 0.$$

The equality holds for $a=b=c$.

□

P 1.22. *Let a,b,c be positive real numbers such that $a+b+c=3$. Prove that*

$$\frac{1}{a^2+b+c}+\frac{1}{b^2+c+a}+\frac{1}{c^2+a+b}\le 1.$$

First Solution. By virtue of the Cauchy-Schwarz inequality, we have

$$(a^2+b+c)(1+b+c)\ge(a+b+c)^2.$$

Therefore,

$$\sum\frac{1}{a^2+b+c}\le\sum\frac{1+b+c}{(a+b+c)^2}=\frac{3+2(a+b+c)}{(a+b+c)^2}=1.$$

The equality occurs for $a=b=c=1$.

Second Solution. Rewrite the inequality as

$$\frac{1}{a^2-a+3}+\frac{1}{b^2-b+3}+\frac{1}{c^2-c+3}\le 1.$$

We see that the equality holds for $a=b=c=1$. Thus, if there exists a real number k such that

$$\frac{1}{a^2-a+3}\le k+\left(\frac{1}{3}-k\right)a$$

for all $a \in [0,3]$, then

$$\sum \frac{1}{a^2-a+3} \le \sum \left[k + \left(\frac{1}{3}-k\right)a\right] = 3k + \left(\frac{1}{3}-k\right)\sum a = 1.$$

We have

$$k + \left(\frac{1}{3}-k\right)a - \frac{1}{a^2-a+3} = \frac{(a-1)f(a)}{3(a^2-a+3)},$$

where

$$f(a) = (1-3k)a^2 + 3ka + 3(1-3k).$$

From $f(1) = 0$, we get $k = 4/9$. Thus, setting $k = 4/9$, we get

$$k + \left(\frac{1}{3}-k\right)a - \frac{1}{a^2-a+3} = \frac{(a-1)^2(3-a)}{9(a^2-a+3)} \ge 0.$$

□

P 1.23. *Let a,b,c be real numbers such that $a+b+c=3$. Prove that*

$$\frac{a^2-bc}{a^2+3} + \frac{b^2-ca}{b^2+3} + \frac{c^2-ab}{c^2+3} \ge 0.$$

(Vasile Cîrtoaje, 2005)

Solution. Apply the SOS method. We have

$$\begin{aligned}
2\sum \frac{a^2-bc}{a^2+3} &= \sum \frac{(a-b)(a+c)+(a-c)(a+b)}{a^2+3} \\
&= \sum \frac{(a-b)(a+c)}{a^2+3} + \sum \frac{(b-a)(b+c)}{b^2+3} \\
&= \sum (a-b)\left(\frac{a+c}{a^2+3} - \frac{b+c}{b^2+3}\right) \\
&= (3-ab-bc-ca)\sum \frac{(a-b)^2}{(a^2+3)(b^2+3)} \ge 0.
\end{aligned}$$

Thus, it suffices to show that

$$3-ab-bc-ca \ge 0.$$

This follows immediately from the known inequality

$$(a+b+c)^2 \ge 3(ab+bc+ca),$$

which is equivalent to

$$(a-b)^2+(b-c)^2+(c-a)^2 \ge 0.$$

The equality holds for $a=b=c=1$.

□

P 1.24. *Let a,b,c be nonnegative real numbers such that $a+b+c=3$. Prove that*

$$\frac{1-bc}{5+2a}+\frac{1-ca}{5+2b}+\frac{1-ab}{5+2c}\ge 0.$$

Solution. We apply the SOS method. Since

$$9(1-bc)=(a+b+c)^2-9bc,$$

we can write the inequality as

$$\sum\frac{a^2+b^2+c^2+2a(b+c)-7bc}{5+2a}\ge 0.$$

From

$$(a-b)(a+kb+mc)+(a-c)(a+kc+mb)=$$
$$=2a^2-k(b^2+c^2)+(k+m-1)a(b+c)-2mbc,$$

choosing $k=-2$ and $m=7$, we get

$$(a-b)(a-2b+7c)+(a-c)(a-2c+7b)=2[a^2+b^2+c^2+2a(b+c)-7bc].$$

Therefore, the desired inequality becomes as follows:

$$\sum\frac{(a-b)(a-2b+7c)}{5+2a}+\sum\frac{(a-c)(a-2c+7b)}{5+2a}\ge 0,$$

$$\sum\frac{(a-b)(a-2b+7c)}{5+2a}+\sum\frac{(b-a)(b-2a+7c)}{5+2b}\ge 0,$$

$$\sum(a-b)(5+2c)[(5+2b)(a-2b+7c)-(5+2a)(b-2a+7c)]\ge 0,$$

$$\sum(a-b)^2(5+2c)(15+4a+4b-14c)\ge 0,$$

$$\sum(a-b)^2(5+2c)(a+b-c)\ge 0.$$

Without loss of generality, assume that $a\ge b\ge c$. Clearly, it suffices to show that

$$(a-c)^2(5+2b)(a+c-b)\ge(b-c)^2(5+2a)(a-b-c).$$

Since $a-c\ge b-c\ge 0$ and $a+c-b\ge a-b-c$, we only need to show that

$$(a-c)(5+2b)\ge(b-c)(5+2a).$$

Indeed,

$$(a-c)(5+2b)-(b-c)(5+2a)=(a-b)(5+2c)\ge 0.$$

The equality holds for $a=b=c=1$, and for $a=b=3/2$ and $c=0$ (or any cyclic permutation).

□

P 1.25. *Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that*

$$\frac{1}{a^2+b^2+2}+\frac{1}{b^2+c^2+2}+\frac{1}{c^2+a^2+2}\le\frac{3}{4}.$$

(Vasile Cîrtoaje, 2006)

Solution. Since

$$\frac{2}{a^2+b^2+2}=1-\frac{a^2+b^2}{a^2+b^2+2},$$

we may write the inequality as

$$\frac{a^2+b^2}{a^2+b^2+2}+\frac{b^2+c^2}{b^2+c^2+2}+\frac{c^2+a^2}{c^2+a^2+2}\ge\frac{3}{2}.$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\sum\frac{a^2+b^2}{a^2+b^2+2}&\ge\frac{\left(\sum\sqrt{a^2+b^2}\right)^2}{\sum(a^2+b^2+2)}\\&=\frac{2\sum a^2+2\sum\sqrt{(a^2+b^2)(a^2+c^2)}}{2\sum a^2+6}\\&\ge\frac{2\sum a^2+2\sum(a^2+bc)}{2\sum a^2+6}\\&=\frac{3\sum a^2+9}{2\sum a^2+6}=\frac{3}{2}.\end{aligned}$$

The equality holds for $a=b=c=1$.

□

P 1.26. *Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that*

$$\frac{1}{4a^2+b^2+c^2}+\frac{1}{4b^2+c^2+a^2}+\frac{1}{4c^2+a^2+b^2}\le\frac{1}{2}.$$

(Vasile Cîrtoaje, 2007)

Solution. According to the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\frac{9}{4a^2+b^2+c^2}&=\frac{(a+b+c)^2}{2a^2+(a^2+b^2)+(a^2+c^2)}\\&\le\frac{1}{2}+\frac{b^2}{a^2+b^2}+\frac{c^2}{a^2+c^2}.\end{aligned}$$

Therefore,

$$\sum \frac{9}{4a^2+b^2+c^2} \le \frac{3}{2} + \sum \left(\frac{b^2}{a^2+b^2} + \frac{c^2}{a^2+c^2} \right)$$
$$= \frac{3}{2} + \sum \left(\frac{b^2}{a^2+b^2} + \frac{a^2}{b^2+a^2} \right) = \frac{3}{2} + 3 = \frac{9}{2}.$$

The equality holds for $a = b = c = 1$.

□

P 1.27. *Let a, b, c be nonnegative real numbers such that $a + b + c = 2$. Prove that*

$$\frac{bc}{a^2+1} + \frac{ca}{b^2+1} + \frac{ab}{c^2+1} \le 1.$$

(Pham Kim Hung, 2005)

Solution. Let

$$p = a + b + c = 2, \quad q = ab + bc + ca, \qquad q \le p^2/3 = 4/3.$$

If $a = 0$, then the inequality reduces to $4ab \le (a + b)^2$. Otherwise, for $a, b, c > 0$, write the inequality as

$$\sum \frac{1}{a(a^2+1)} \le \frac{1}{abc},$$

$$\sum \left(\frac{1}{a} - \frac{a}{a^2+1} \right) \le \frac{1}{abc},$$

$$\sum \frac{a}{a^2+1} \ge \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{1}{abc},$$

$$\sum \frac{a}{a^2+1} \ge \frac{q-1}{abc},$$

Using the inequality

$$\frac{2}{a^2+1} \ge 2 - a,$$

which is equivalent to

$$a(a-1)^2 \ge 0,$$

we get

$$\sum \frac{a}{a^2+1} \ge \sum \frac{a(2-a)}{2} = \sum \frac{a(b+c)}{2} = q.$$

Therefore, it suffices to prove that

$$1 + abcq \ge q.$$

By Schur's inequality of degree four, we have

$$abc \ge \frac{(p^2-q)(4q-p^2)}{6p} = \frac{(4-q)(q-1)}{3}.$$

Thus,

$$1+abcq-q \ge 1+\frac{q(4-q)(q-1)}{3}-q = \frac{(3-q)(q-1)^2}{3} \ge 0.$$

The equality holds if $a=0$ and $b=c=1$ (or any cyclic permutation).

□

P 1.28. *Let a,b,c be nonnegative real numbers such that $a+b+c=1$. Prove that*

$$\frac{bc}{a+1}+\frac{ca}{b+1}+\frac{ab}{c+1} \le \frac{1}{4}.$$

(Vasile Cîrtoaje, 2009)

First Solution. We have

$$\begin{aligned}
\sum \frac{bc}{a+1} &= \sum \frac{bc}{(a+b)+(c+a)} \\
&\le \frac{1}{4}\sum bc\left(\frac{1}{a+b}+\frac{1}{c+a}\right) \\
&= \frac{1}{4}\sum \frac{bc}{a+b}+\frac{1}{4}\sum \frac{bc}{c+a} \\
&= \frac{1}{4}\sum \frac{bc}{a+b}+\frac{1}{4}\sum \frac{ca}{a+b} \\
&= \frac{1}{4}\sum \frac{bc+ca}{a+b} = \frac{1}{4}\sum c = \frac{1}{4}.
\end{aligned}$$

The equality holds for $a=b=c=1/3$, and for $a=0$ and $b=c=1/2$ (or any cyclic permutation).

Second Solution. It is easy to check that the inequality is true if one of a,b,c is zero. Otherwise, write the inequality as

$$\frac{1}{a(a+1)}+\frac{1}{b(b+1)}+\frac{1}{c(c+1)} \le \frac{1}{4abc}.$$

Since

$$\frac{1}{a(a+1)} = \frac{1}{a}-\frac{1}{a+1},$$

we may write the required inequality as

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1} \ge \frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{1}{4abc}.$$

In virtue of the Cauchy-Schwarz inequality, we have

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1} \ge \frac{9}{(a+1)+(b+1)+(c+1)} = \frac{9}{4}.$$

Therefore, it suffices to prove that

$$\frac{9}{4} \ge \frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{1}{4abc}.$$

This is equivalent to Schur's inequality

$$(a+b+c)^3+9abc \ge 4(a+b+c)(ab+bc+ca).$$

□

P 1.29. *Let a,b,c be positive real numbers such that $a+b+c=1$. Prove that*

$$\frac{1}{a(2a^2+1)}+\frac{1}{b(2b^2+1)}+\frac{1}{c(2c^2+1)} \le \frac{3}{11abc}.$$

(Vasile Cîrtoaje, 2009)

Solution. Since

$$\frac{1}{a(2a^2+1)} = \frac{1}{a}-\frac{2a}{2a^2+1},$$

we can write the inequality as

$$\sum \frac{2a}{2a^2+1} \ge \frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{3}{11abc}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a}{2a^2+1} \ge \frac{(\sum a)^2}{\sum a(2a^2+1)} = \frac{1}{2(a^3+b^3+c^3)+1}.$$

Therefore, it suffices to show that

$$\frac{2}{2(a^3+b^3+c^3)+1} \ge \frac{11q-3}{11abc},$$

where

$$q = ab+bc+ca, \qquad q \le \frac{1}{3}(a+b+c)^2 = \frac{1}{3}.$$

Since

$$a^3+b^3+c^3 = 3abc+(a+b+c)^3-3(a+b+c)(ab+bc+ca) = 3abc+1-3q,$$

we need to prove that

$$22abc \ge (11q-3)(6abc+3-6q),$$

or, equivalently,

$$2(20-33q)abc \ge 3(11q-3)(1-2q).$$

From Schur's inequality

$$(a+b+c)^3 + 9abc \ge 4(a+b+c)(ab+bc+ca),$$

we get

$$9abc \ge 4q-1.$$

Thus,

$$\begin{aligned} &2(20-33q)abc - 3(11q-3)(1-2q) \ge \\ &\ge \frac{2(20-33q)(4q-1)}{9} - 3(11q-3)(1-2q) \\ &= \frac{330q^2-233q+41}{9} = \frac{(1-3q)(41-110q)}{9} \ge 0. \end{aligned}$$

This completes the proof. The equality holds for $a=b=c=1/3$.

□

P 1.30. *Let a,b,c be positive real numbers such that $a+b+c=3$. Prove that*

$$\frac{1}{a^3+b+c} + \frac{1}{b^3+c+a} + \frac{1}{c^3+a+b} \le 1.$$

(Vasile Cîrtoaje, 2009)

Solution. Write the inequality in the form

$$\frac{1}{a^3-a+3} + \frac{1}{b^3-b+3} + \frac{1}{c^3-c+3} \le 1.$$

Assume that $a \ge b \ge c$. There are two cases to consider.

Case 1: $2 \ge a \ge b \ge c$. The desired inequality follows by adding the inequalities

$$\frac{1}{a^3-a+3} \le \frac{5-2a}{9}, \quad \frac{1}{b^3-b+3} \le \frac{5-2b}{9}, \quad \frac{1}{c^3-c+3} \le \frac{5-2c}{9}.$$

These inequalities are true since

$$\frac{1}{a^3-a+3} - \frac{5-2a}{9} = \frac{(a-1)^2(a-2)(2a+3)}{9(a^3-a+3)} \le 0.$$

Case 2: $a > 2$. From $a+b+c=3$, we get $b+c<1$. Since

$$\sum \frac{1}{a^3-a+3} < \frac{1}{a^3-a+3} + \frac{1}{3-b} + \frac{1}{3-c} < \frac{1}{9} + \frac{1}{3-b} + \frac{1}{3-c},$$

it suffices to prove that

$$\frac{1}{3-b} + \frac{1}{3-c} \le \frac{8}{9}.$$

We have

$$\frac{1}{3-b} + \frac{1}{3-c} - \frac{8}{9} = \frac{-3-15(1-b-c)-8bc}{9(3-b)(3-c)} < 0.$$

The equality holds for $a=b=c=1$.

□

P 1.31. *Let a,b,c be positive real numbers such that $a+b+c=3$. Prove that*

$$\frac{a^2}{1+b^3+c^3} + \frac{b^2}{1+c^3+a^3} + \frac{c^2}{1+a^3+b^3} \ge 1.$$

Solution. Using the Cauchy-Schwarz inequality, we have

$$\sum \frac{a^2}{1+b^3+c^3} \ge \frac{\left(\sum a^2\right)^2}{\sum a^2(1+b^3+c^3)},$$

and it remains to show that

$$(a^2+b^2+c^2)^2 \ge (a^2+b^2+c^2) + \sum a^2b^2(a+b).$$

Let

$$p = a+b+c, \qquad q = ab+bc+ca, \qquad q \le 3.$$

Since $a^2+b^2+c^2 = 9-2q$ and

$$\sum a^2b^2(a+b) = \sum a^2b^2(3-c) = 3\sum a^2b^2 - qabc = 3q^2-(q+18)abc,$$

the desired inequality can be written as

$$(9-2q)^2 \ge (9-2q) + 3q^2 - (q+18)abc,$$

$$q^2 - 34q + 72 + (q+18)abc \ge 0.$$

This inequality is clearly true for $q \le 2$. Consider further that $2 < q \le 3$. By Schur's inequality of degree four, we get

$$abc \ge \frac{(p^2-q)(4q-p^2)}{6p} = \frac{(9-q)(4q-9)}{18}.$$

Therefore

$$q^2 - 34q + 72 + (q+18)abc \geq q^2 - 34q + 72 + \frac{(q+18)(9-q)(4q-9)}{18}$$
$$= \frac{(3-q)(4q^2+21q-54)}{18} \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 1.32. *Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. Prove that*

$$\frac{1}{6-ab} + \frac{1}{6-bc} + \frac{1}{6-ca} \leq \frac{3}{5}.$$

Solution. Rewrite the inequality as

$$108 - 48(ab + bc + ca) + 13abc(a + b + c) - 3a^2b^2c^2 \geq 0,$$

$$4[9 - 4(ab + bc + ca) + 3abc] + abc(1 - abc) \geq 0.$$

By the AM-GM inequality,

$$1 = \left(\frac{a+b+c}{3}\right)^3 \geq abc.$$

Consequently, it suffices to show that

$$9 - 4(ab + bc + ca) + 3abc \geq 0.$$

We see that the homogeneous form of this inequality is just Schur's inequality of third degree

$$(a + b + c)^3 + 9abc \geq 4(a + b + c)(ab + bc + ca).$$

The equality holds for $a = b = c = 1$, as well as for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

□

P 1.33. *Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. Prove that*

$$\frac{1}{2a^2+7} + \frac{1}{2b^2+7} + \frac{1}{2c^2+7} \leq \frac{1}{3}.$$

(Vasile Cîrtoaje, 2005)

Solution. Use the mixing variables method. Assume that $a = \max\{a, b, c\}$ and prove that

$$E(a,b,c) \le E(a,s,s) \le \frac{1}{3},$$

where

$$s = \frac{b+c}{2}, \quad 0 \le s \le 1,$$

$$E(a,b,c) = \frac{1}{2a^2+7} + \frac{1}{2b^2+7} + \frac{1}{2c^2+7}.$$

We have

$$\begin{aligned} E(a,s,s) - E(a,b,c) &= \left(\frac{1}{2s^2+7} - \frac{1}{2b^2+7}\right) + \left(\frac{1}{2s^2+7} - \frac{1}{2c^2+7}\right) \\ &= \frac{1}{2s^2+7}\left[\frac{(b-c)(b+s)}{2b^2+7} + \frac{(c-b)(c+s)}{2c^2+7}\right] \\ &= \frac{(b-c)^2(7-4s^2-2bc)}{(2s^2+7)(2b^2+7)(2c^2+7)}. \end{aligned}$$

Since $bc \le s^2 \le 1$, it follows that

$$7 - 4s^2 - 2bc = 1 + 4(1-s^2) + 2(1-bc) > 0,$$

hence $E(a,s,s) \ge E(a,b,c)$. Also,

$$\frac{1}{3} - E(a,s,s) = \frac{1}{3} - E(3-2s,s,s) = \frac{4(s-1)^2(2s-1)^2}{3(2a^2+7)(2s^2+7)} \ge 0.$$

The equality holds for $a = b = c = 1$, as well as for $a = 2$ and $b = c = 1/2$ (or any cyclic permutation).

□

P 1.34. *Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. Prove that*

$$\frac{1}{2a^2+3} + \frac{1}{2b^2+3} + \frac{1}{2c^2+3} \ge \frac{3}{5}.$$

(Vasile Cîrtoaje, 2005)

First Solution *(by Nguyen Van Quy)*. Write the inequality as

$$\sum \left(\frac{1}{3} - \frac{1}{2a^2+3}\right) \le \frac{2}{5},$$

$$\sum \frac{a^2}{2a^2+5} \le \frac{3}{5}.$$

Using the Cauchy-Schwarz inequality gives

$$\begin{aligned}\frac{25}{3(2a^2+3)} &= \frac{25}{6a^2+(a+b+c)^2} \\ &= \frac{(2+2+1)^2}{2(2a^2+bc)+2a(a+b+c)+a^2+b^2+c^2} \\ &\le \frac{2^2}{2(2a^2+bc)}+\frac{2^2}{2a(a+b+c)}+\frac{1}{a^2+b^2+c^2},\end{aligned}$$

hence

$$\begin{aligned}\sum \frac{25a^2}{3(2a^2+3)} &\le \sum \frac{2a^2}{2a^2+bc}+\sum \frac{2a}{a+b+c}+\sum \frac{a^2}{a^2+b^2+c^2} \\ &= \sum \frac{2a^2}{2a^2+bc}+3.\end{aligned}$$

Therefore, it suffices to show that

$$\sum \frac{a^2}{2a^2+bc} \le 1.$$

For the nontrivial case $a, b, c > 0$, this is equivalent to

$$\sum \frac{1}{2+bc/a^2} \le 1,$$

which follows immediately from P 1.2-(b). The equality holds for $a = b = c = 1$, as well as for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

Second Solution. First, we can check that the desired inequality becomes an equality for $a = b = c = 1$, and for $a = 0$ and $b = c = 3/2$. Consider then the inequality $f(x) \ge 0$, where

$$f(x) = \frac{1}{2x^2+3} - A - Bx, \qquad f'(x) = \frac{-4x}{(2x^2+3)^2} - B.$$

The conditions $f(1) = 0$ and $f'(1) = 0$ involve $A = 9/25$ and $B = -4/25$. Also, the conditions $f(3/2) = 0$ and $f'(3/2) = 0$ involve $A = 22/75$ and $B = -8/75$. Using these values of A and B, we obtain the identities

$$\frac{1}{2x^2+3} - \frac{9-4x}{25} = \frac{2(x-1)^2(4x-1)}{25(2x^2+3)},$$

$$\frac{1}{2x^2+3} - \frac{22-8x}{75} = \frac{(2x-3)^2(4x+1)}{75(2x^2+3)},$$

and the inequalities

$$\frac{1}{2x^2+3} \ge \frac{9-4x}{25}, \qquad x \ge \frac{1}{4},$$

$$\frac{1}{2x^2+3} \ge \frac{22-8x}{75}, \quad x \ge 0.$$

Without loss of generality, assume that $a \ge b \ge c$.

Case 1: $a \ge b \ge c \ge \frac{1}{4}$. By summing the inequalities

$$\frac{1}{2a^2+3} \ge \frac{9-4a}{25}, \quad \frac{1}{2b^2+3} \ge \frac{9-4b}{25}, \quad \frac{1}{2c^2+3} \ge \frac{9-4c}{25},$$

we get

$$\frac{1}{2a^2+3} + \frac{1}{2b^2+3} + \frac{1}{2c^2+3} \ge \frac{27-4(a+b+c)}{25} = \frac{3}{5}.$$

Case 2: $a \ge b \ge \frac{1}{4} \ge c$. We have

$$\begin{aligned}\sum \frac{1}{2a^2+3} &\ge \frac{22-8a}{75} + \frac{22-8b}{75} + \frac{1}{2c^2+3} \\ &= \frac{44-8(a+b)}{75} + \frac{1}{2c^2+3} = \frac{20+8c}{75} + \frac{1}{2c^2+3}.\end{aligned}$$

Therefore, it suffices to show that

$$\frac{20+8c}{75} + \frac{1}{2c^2+3} \ge \frac{3}{5},$$

which is equivalent to the obvious inequality

$$c(8c^2-25c+12) \ge 0.$$

Case 3: $a \ge \frac{1}{4} \ge b \ge c$. We have

$$\sum \frac{1}{2a^2+3} > \frac{1}{2b^2+3} + \frac{1}{2c^2+3} \ge \frac{2}{1/8+3} > \frac{3}{5}.$$

□

P 1.35. *Let a, b, c be nonnegative real numbers such that $ab+bc+ca = 3$. Prove that*

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \ge \frac{a+b+c}{6} + \frac{3}{a+b+c}.$$

(Vasile Cîrtoaje, 2007)

First Solution. Denoting

$$x = a + b + c, \qquad x \geq 3,$$

we have

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} = \frac{(a+b+c)^2 + ab + bc + ca}{(a+b+c)(ab+bc+ca) - abc} = \frac{x^2+3}{3x - abc}.$$

Then, the inequality becomes

$$\frac{x^2+3}{3x-abc} \geq \frac{x}{6} + \frac{3}{x},$$

$$3(x^3 + 9abc - 12x) + abc(x^2 - 9) \geq 0.$$

This inequality is true since

$$x^2 - 9 \geq 0, \quad x^3 + 9abc - 12x \geq 0.$$

The last inequality is just Schur's inequality of degree three

$$(a+b+c)^3 + 9abc \geq 4(a+b+c)(ab+bc+ca).$$

The equality holds for $a = b = c = 1$, and for $a = 0$ and $b = c = \sqrt{3}$ (or any cyclic permutation).

Second Solution. We apply the SOS method. Write the inequality as follows:

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{a+b+c}{2(ab+bc+ca)} + \frac{3}{a+b+c},$$

$$2(a+b+c)\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) \geq \frac{(a+b+c)^2}{ab+bc+ca} + 6,$$

$$[(a+b)+(b+c)+(c+a)]\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) - 9 \geq \frac{(a+b+c)^2}{ab+bc+ca} - 3,$$

$$\sum \frac{(b-c)^2}{(a+b)(c+a)} \geq \frac{1}{2(ab+bc+ca)} \sum (b-c)^2,$$

$$\sum \frac{ab+bc+ca-a^2}{(a+b)(c+a)}(b-c)^2 \geq 0,$$

$$\sum \frac{3-a^2}{3+a^2}(b-c)^2 \geq 0,$$

Without loss of generality, assume that $a \geq b \geq c$. Since $3 - c^2 \geq 0$, it suffices to show that

$$\frac{3-a^2}{3+a^2}(b-c)^2 + \frac{3-b^2}{3+b^2}(c-a)^2 \geq 0.$$

Having in view that

$$3-b^2=ab+bc+ca-b^2\geq b(a-b)\geq 0,\qquad (c-a)^2\geq(b-c)^2,$$

it is enough to prove that

$$\frac{3-a^2}{3+a^2}+\frac{3-b^2}{3+b^2}\geq 0.$$

This is true since

$$\frac{3-a^2}{3+a^2}+\frac{3-b^2}{3+b^2}=\frac{2(9-a^2b^2)}{(3+a^2)(3+b^2)}=\frac{2c(a+b)(3+ab)}{(3+a^2)(3+b^2)}\geq 0.$$

□

P 1.36. *Let a,b,c be nonnegative real numbers such that $ab+bc+ca=3$. Prove that*

$$\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}\geq\frac{3}{2}.$$

(Vasile Cîrtoaje, 2005)

First Solution. After expanding, the inequality can be restated as

$$a^2+b^2+c^2+3\geq a^2b^2+b^2c^2+c^2a^2+3a^2b^2c^2.$$

From

$$(a+b+c)(ab+bc+ca)-9abc=a(b-c)^2+b(c-a)^2+c(a-b)^2\geq 0,$$

we get

$$a+b+c\geq 3abc.$$

So, it suffices to show that

$$a^2+b^2+c^2+3\geq a^2b^2+b^2c^2+c^2a^2+abc(a+b+c).$$

This is equivalent to the homogeneous inequalities

$$(ab+bc+ca)(a^2+b^2+c^2)+(ab+bc+ca)^2\geq 3(a^2b^2+b^2c^2+c^2a^2)+3abc(a+b+c),$$

$$ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)\geq 2(a^2b^2+b^2c^2+c^2a^2),$$

$$ab(a-b)^2+bc(b-c)^2+ca(c-a)^2\geq 0.$$

The equality holds for $a=b=c=1$, and for $a=0$ and $b=c=\sqrt{3}$ (or any cyclic permutation).

Second Solution. Without loss of generality, assume that

$$a=\min\{a,b,c\},\qquad bc\geq 1.$$

From

$$(a+b+c)(ab+bc+ca)-9abc = a(b-c)^2 + b(c-a)^2 + c(a-b)^2 \geq 0,$$

we get

$$a+b+c \geq 3abc.$$

The desired inequality follows by summing the inequalities

$$\frac{1}{b^2+1} + \frac{1}{c^2+1} \geq \frac{2}{bc+1},$$

$$\frac{1}{a^2+1} + \frac{2}{bc+1} \geq \frac{3}{2}.$$

We have

$$\frac{1}{b^2+1} + \frac{1}{c^2+1} - \frac{2}{bc+1} = \frac{b(c-b)}{(b^2+1)(bc+1)} + \frac{c(b-c)}{(c^2+1)(bc+1)}$$
$$= \frac{(b-c)^2(bc-1)}{(b^2+1)(c^2+1)(bc+1)} \geq 0$$

and

$$\frac{1}{a^2+1} + \frac{2}{bc+1} - \frac{3}{2} = \frac{a^2-bc+3-3a^2bc}{2(a^2+1)(bc+1)} = \frac{a(a+b+c-3abc)}{2(a^2+1)(bc+1)} \geq 0.$$

Third Solution. Since

$$\frac{1}{a^2+1} = 1 - \frac{a^2}{a^2+1}, \quad \frac{1}{b^2+1} = 1 - \frac{b^2}{b^2+1}, \quad \frac{1}{c^2+1} = 1 - \frac{c^2}{c^2+1},$$

we can rewrite the inequality as

$$\frac{a^2}{a^2+1} + \frac{b^2}{b^2+1} + \frac{c^2}{c^2+1} \leq \frac{3}{2},$$

or, in the homogeneous form,

$$\sum \frac{a^2}{3a^2+ab+bc+ca} \leq \frac{1}{2}.$$

According to the Cauchy-Schwarz inequality, we have

$$\frac{4a^2}{3a^2+ab+bc+ca} = \frac{(a+a)^2}{a(a+b+c)+(2a^2+bc)} \leq \frac{a}{a+b+c} + \frac{a^2}{2a^2+bc},$$

hence

$$\sum \frac{4a^2}{3a^2+ab+bc+ca} \leq 1 + \sum \frac{a^2}{2a^2+bc}.$$

It suffices to show that

$$\sum \frac{a^2}{2a^2+bc} \le 1.$$

For the nontrivial case $a,b,c>0$, this is equivalent to

$$\sum \frac{1}{2+bc/a^2} \le 1,$$

which follows immediately from P 1.2-(b).

Remark. We can write the inequality in P 1.36 in the homogeneous form

$$\frac{1}{1+\dfrac{3a^2}{ab+bc+ca}}+\frac{1}{1+\dfrac{3b^2}{ab+bc+ca}}+\frac{1}{1+\dfrac{3c^2}{ab+bc+ca}} \ge \frac{3}{2}.$$

Substituting a,b,c by $\frac{1}{x},\frac{1}{y},\frac{1}{z}$, respectively, we get

$$\frac{x}{x+\dfrac{3yz}{x+y+z}}+\frac{y}{y+\dfrac{3zx}{x+y+z}}+\frac{z}{z+\dfrac{3xy}{x+y+z}} \ge \frac{3}{2}.$$

So, we find the following result.

- *If x,y,z are positive real numbers such that $x+y+z=3$, then*

$$\frac{x}{x+yz}+\frac{y}{y+zx}+\frac{z}{z+xy} \ge \frac{3}{2}.$$

□

P 1.37. *Let a,b,c be positive real numbers such that $ab+bc+ca=3$. Prove that*

$$\frac{a^2}{a^2+b+c}+\frac{b^2}{b^2+c+a}+\frac{c^2}{c^2+a+b} \ge 1.$$

(Vasile Cîrtoaje, 2005)

Solution. We apply the Cauchy-Schwarz inequality in the following way

$$\sum \frac{a^2}{a^2+b+c} \ge \frac{\left(a^{3/2}+b^{3/2}+c^{3/2}\right)^2}{\sum a(a^2+b+c)} = \frac{\sum a^3+2\sum (ab)^{3/2}}{\sum a^3+6}.$$

Then, we still have to show that

$$(ab)^{3/2}+(bc)^{3/2}+(ca)^{3/2} \ge 3.$$

By the AM-GM inequality, we have

$$(ab)^{3/2}=\frac{(ab)^{3/2}+(ab)^{3/2}+1}{2}-\frac{1}{2}\ge\frac{3ab}{2}-\frac{1}{2},$$

hence

$$(ab)^{3/2}+(bc)^{3/2}+(ca)^{3/2}\ge\frac{3}{2}(ab+bc+ca)-\frac{3}{2}=3.$$

The equality holds for $a=b=c=1$.

□

P 1.38. *Let a,b,c be positive real numbers such that $ab+bc+ca=3$. Prove that*

$$\frac{bc+4}{a^2+4}+\frac{ca+4}{b^2+4}+\frac{ab+4}{c^2+4}\le 3\le\frac{bc+2}{a^2+2}+\frac{ca+2}{b^2+2}+\frac{ab+2}{c^2+2}.$$

(Vasile Cîrtoaje, 2007)

Solution. More general, using the SOS method, we will show that

$$(k-3)\left(\frac{bc+k}{a^2+k}+\frac{ca+k}{b^2+k}+\frac{ab+k}{c^2+k}-3\right)\le 0$$

for $k>0$. This inequality is equivalent to

$$(k-3)\sum\frac{a^2-bc}{a^2+k}\ge 0.$$

Since

$$\begin{aligned}2\sum\frac{a^2-bc}{a^2+k}&=\sum\frac{(a-b)(a+c)+(a-c)(a+b)}{a^2+k}\\&=\sum\frac{(a-b)(a+c)}{a^2+k}+\sum\frac{(b-a)(b+c)}{b^2+k}\\&=(k-ab-bc-ca)\sum\frac{(a-b)^2}{(a^2+k)(b^2+k)}\\&=(k-3)\sum\frac{(a-b)^2}{(a^2+k)(b^2+k)},\end{aligned}$$

we have

$$2(k-3)\sum\frac{a^2-bc}{a^2+k}=(k-3)^2\sum\frac{(a-b)^2}{(a^2+k)(b^2+k)}\ge 0.$$

The equality in both inequalities holds for $a=b=c=1$.

□

P 1.39. *Let a, b, c be nonnegative real numbers such that $ab+bc+ca=3$. If*

$$k \ge 2+\sqrt{3},$$

then

$$\frac{1}{a+k}+\frac{1}{b+k}+\frac{1}{c+k} \le \frac{3}{1+k}.$$

(Vasile Cîrtoaje, 2007)

Solution. Let us denote

$$p=a+b+c, \qquad p \ge 3.$$

By expanding, the inequality becomes

$$k(k-2)p+3abc \ge 3(k-1)^2.$$

Since this inequality is true for $p \ge 3(k-1)^2/(k^2-2k)$, consider further that

$$p \le \frac{3(k-1)^2}{k(k-2)}.$$

From Schur's inequality

$$(a+b+c)^3+9abc \ge 4(ab+bc+ca)(a+b+c),$$

we get

$$9abc \ge 12p-p^3.$$

Therefore, it suffices to prove that

$$3k(k-2)p+12p-p^3 \ge 9(k-1)^2,$$

or, equivalently,

$$(p-3)[(3(k-1)^2-p^2-3p] \ge 0.$$

Thus, it remains to prove that

$$3(k-1)^2-p^2-3p \ge 0.$$

Since $p \le 3(k-1)^2/(k^2-2k)$ and $k \ge 2+\sqrt{3}$, we have

$$\begin{aligned} 3(k-1)^2-p^2-3p &\ge 3(k-1)^2-\frac{9(k-1)^4}{k^2(k-2)^2}-\frac{9(k-1)^2}{k(k-2)} \\ &= \frac{3(k-1)^2(k^2-3)(k^2-4k+1)}{k^2(k-2)^2} \ge 0. \end{aligned}$$

The equality holds for $a=b=c=1$. In the case $k=2+\sqrt{3}$, the equality holds also for $a=0$ and $b=c=\sqrt{3}$ (or any cyclic permutation).

□

P 1.40. *Let a, b, c be nonnegative real numbers such that $a^2+b^2+c^2=3$. Prove that*

$$\frac{a(b+c)}{1+bc}+\frac{b(c+a)}{1+ca}+\frac{c(a+b)}{1+ab}\le 3.$$

(Vasile Cîrtoaje, 2010)

Solution. Write the inequality in the homogeneous forms

$$\sum \frac{a(b+c)}{a^2+b^2+c^2+3bc}\le 1,$$

$$\sum\left[\frac{a(b+c)}{a^2+b^2+c^2+3bc}-\frac{a}{a+b+c}\right]\le 0,$$

$$\sum \frac{a(a-b)(a-c)}{a^2+b^2+c^2+3bc}\ge 0.$$

Without loss of generality, assume that $a\ge b\ge c$. Then, it suffices to prove that

$$\frac{a(a-b)(a-c)}{a^2+b^2+c^2+3bc}+\frac{b(b-c)(b-a)}{a^2+b^2+c^2+3ca}\ge 0,$$

which is true if

$$\frac{a(a-c)}{a^2+b^2+c^2+3bc}\ge \frac{b(b-c)}{a^2+b^2+c^2+3ca}.$$

Since

$$a(a-c)\ge b(b-c)$$

and

$$\frac{1}{a^2+b^2+c^2+3bc}\ge \frac{1}{a^2+b^2+c^2+3ca},$$

the conclusion follows. The equality holds for $a=b=c=1$, and for $a=b=\sqrt{3/2}$ and $c=0$ (or any cyclic permutation).

□

P 1.41. *Let a, b, c be positive real numbers such that $a^2+b^2+c^2=3$. Prove that*

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}\ge 3.$$

(Cezar Lupu, 2005)

First Solution. We apply the SOS method. Write the inequality in the homogeneous forms

$$\sum\left(\frac{b^2+c^2}{b+c}-\frac{b+c}{2}\right)\ge \sqrt{3(a^2+b^2+c^2)}-a-b-c,$$

$$\sum \frac{(b-c)^2}{2(b+c)} \ge \frac{\sum(b-c)^2}{\sqrt{3(a^2+b^2+c^2)}+a+b+c}.$$

Since

$$\sqrt{3(a^2+b^2+c^2)}+a+b+c \ge 2(a+b+c) > 2(b+c),$$

the conclusion follows. The equality holds for $a=b=c=1$.

Second Solution. By virtue of the Cauchy-Schwarz inequality, we have

$$\sum \frac{a^2+b^2}{a+b} \ge \frac{\left(\sum \sqrt{a^2+b^2}\right)^2}{\sum(a+b)} = \frac{2\sum a^2 + 2\sum \sqrt{(a^2+b^2)(a^2+c^2)}}{2\sum a}$$

$$\ge \frac{2\sum a^2 + 2\sum(a^2+bc)}{2\sum a} = \frac{3\sum a^2 + \left(\sum a\right)^2}{2\sum a}$$

$$= \frac{9+\left(\sum a\right)^2}{2\sum a} = 3 + \frac{\left(\sum a - 3\right)^2}{2\sum a} \ge 3.$$

□

P 1.42. *Let a,b,c be positive real numbers such that $a^2+b^2+c^2=3$. Prove that*

$$\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a} + 2 \le \frac{7}{6}(a+b+c).$$

(Vasile Cîrtoaje, 2011)

Solution. We apply the SOS method. Write the inequality as

$$3\sum\left(b+c-\frac{4bc}{b+c}\right) \ge 8(3-a-b-c).$$

Since

$$b+c-\frac{4bc}{b+c} = \frac{(b-c)^2}{b+c}$$

and

$$3-a-b-c = \frac{9-(a+b+c)^2}{3+a+b+c} = \frac{3(a^2+b^2+c^2)-(a+b+c)^2}{3+a+b+c}$$

$$= \frac{1}{3+a+b+c}\sum(b-c)^2,$$

we can write the inequality as

$$S_a(b-c)^2 + S_b(c-a)^2 + S_c(a-b)^2 \ge 0,$$

where

$$S_a = \frac{3}{b+c} - \frac{8}{3+a+b+c}.$$

Without loss of generality, assume that $a \ge b \ge c$, which involves $S_a \ge S_b \ge S_c$. If

$$S_b + S_c \ge 0,$$

then

$$S_a \ge S_b \ge 0,$$

hence

$$\begin{aligned} S_a(b-c)^2 + S_b(c-a)^2 + S_c(a-b)^2 &\ge S_b(c-a)^2 + S_c(a-b)^2 \\ &\ge (S_b + S_c)(a-b)^2 \ge 0. \end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} S_b + S_c &= 3\left(\frac{1}{a+c} + \frac{1}{a+b}\right) - \frac{16}{3+a+b+c} \\ &\ge \frac{12}{(a+c)+(a+b)} - \frac{16}{3+a+b+c} \\ &= \frac{4(9-5a-b-c)}{(2a+b+c)(3+a+b+c)}. \end{aligned}$$

Therefore,we only need to show that

$$9 \ge 5a + b + c.$$

This follows immediately from the Cauchy-Schwarz inequality

$$(25+1+1)(a^2+b^2+c^2) \ge (5a+b+c)^2.$$

Thus, the proof is completed. The equality holds for $a = b = c = 1$, and also for $a = 5/3$ and $b = c = 1/3$ (or any cyclic permutation).

□

P 1.43. *Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that*

(a) $$\frac{1}{3-ab} + \frac{1}{3-bc} + \frac{1}{3-ca} \le \frac{3}{2};$$

(b) $$\frac{1}{5-2ab} + \frac{1}{5-2bc} + \frac{1}{5-2ca} \le 1;$$

(c) $$\frac{1}{\sqrt{6}-ab} + \frac{1}{\sqrt{6}-bc} + \frac{1}{\sqrt{6}-ca} \le \frac{3}{\sqrt{6}-1}.$$

(Vasile Cîrtoaje, 2005)

Solution. (a) Since

$$\frac{3}{3-ab}=1+\frac{ab}{3-ab}=1+\frac{2ab}{a^2+b^2+2c^2+(a-b)^2}$$
$$\le 1+\frac{2ab}{a^2+b^2+2c^2}\le 1+\frac{(a+b)^2}{2(a^2+b^2+2c^2)},$$

it suffices to prove that

$$\frac{(a+b)^2}{a^2+b^2+2c^2}+\frac{(b+c)^2}{b^2+c^2+2a^2}+\frac{(c+a)^2}{c^2+a^2+2b^2}\le 3.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{(a+b)^2}{a^2+b^2+2c^2}=\frac{(a+b)^2}{(a^2+c^2)+(b^2+c^2)}\le\frac{a^2}{a^2+c^2}+\frac{b^2}{b^2+c^2}.$$

Thus,

$$\sum\frac{(a+b)^2}{a^2+b^2+2c^2}\le\sum\frac{a^2}{a^2+c^2}+\sum\frac{b^2}{b^2+c^2}=\sum\frac{a^2}{a^2+c^2}+\sum\frac{c^2}{c^2+a^2}=3.$$

The equality holds for $a=b=c=1$.

(b) Write the inequality in the homogeneous form

$$\sum\frac{a^2+b^2+c^2}{5(a^2+b^2+c^2)-6bc}\le 1.$$

Since

$$\frac{2(a^2+b^2+c^2)}{5(a^2+b^2+c^2)-6bc}=1-\frac{3a^2+3(b-c)^2}{5(a^2+b^2+c^2)-6bc},$$

the inequality is equivalent to

$$\sum\frac{a^2+(b-c)^2}{5(a^2+b^2+c^2)-6bc}\ge\frac{1}{3}.$$

Assume that

$$a\ge b\ge c.$$

By the Cauchy-Schwarz inequality, we have

$$\sum\frac{a^2}{5(a^2+b^2+c^2)-6bc}\ge\frac{\left(\sum a\right)^2}{\sum[5(a^2+b^2+c^2)-6bc]}=\frac{\sum a^2+2\sum ab}{15\sum a^2-6\sum ab}.$$

$$\sum\frac{(b-c)^2}{5(a^2+b^2+c^2)-6bc}\ge\frac{[(b-c)+(a-c)+(a-b)]^2}{\sum[5(a^2+b^2+c^2)-6bc]}=\frac{4(a-c)^2}{15\sum a^2-6\sum ab}.$$

Therefore, it suffices to show that

$$\frac{\sum a^2+2\sum ab+4(a-c)^2}{15\sum a^2-6\sum ab}\ge\frac{1}{3},$$

which is equivalent to

$$\sum ab+(a-c)^2\ge\sum a^2,$$

$$(a-b)(b-c)\ge 0.$$

(c) According to P 1.32, the following inequality holds

$$\frac{1}{6-a^2b^2}+\frac{1}{6-b^2c^2}+\frac{1}{6-c^2a^2}\le\frac{3}{5}.$$

Since

$$\frac{2\sqrt{6}}{6-a^2b^2}=\frac{1}{\sqrt{6}-ab}+\frac{1}{\sqrt{6}+ab},$$

this inequality becomes

$$\sum\frac{1}{\sqrt{6}-ab}+\sum\frac{1}{\sqrt{6}+ab}\le\frac{6\sqrt{6}}{5}.$$

Thus, it suffices to show that

$$\sum\frac{1}{\sqrt{6}+ab}\ge\frac{3}{\sqrt{6}+1}.$$

Since $ab+bc+ca\le a^2+b^2+c^2=3$, by the Cauchy-Schwarz inequality, we have

$$\sum\frac{1}{\sqrt{6}+ab}\ge\frac{9}{\sum(\sqrt{6}+ab)}=\frac{9}{3\sqrt{6}+ab+bc+ca}\ge\frac{3}{\sqrt{6}+1}.$$

The equality holds for $a=b=c=1$.

□

P 1.44. *Let a,b,c be positive real numbers such that $a^2+b^2+c^2=3$. Prove that*

$$\frac{1}{1+a^5}+\frac{1}{1+b^5}+\frac{1}{1+c^5}\ge\frac{3}{2}.$$

(Vasile Cîrtoaje, 2007)

Solution. Let $a = \min\{a, b, c\}$. There are two cases to consider.

Case 1: $a \ge \frac{1}{2}$. The desired inequality follows by summing the inequalities

$$\frac{8}{1+a^5} \ge 9-5a^2, \quad \frac{8}{1+b^5} \ge 9-5b^2, \quad \frac{8}{1+c^5} \ge 9-5c^2.$$

To obtain these inequalities, we consider the inequality

$$\frac{8}{1+x^5} \ge p+qx^2,$$

where the real coefficients p and q will be determined such that $(x-1)^2$ is a factor of the polynomial

$$P(x) = 8-(1+x^5)(p+qx^2).$$

It is easy to check that $P(1) = 0$ involves $p+q = 4$, hence

$$P(x) = 4(2-x^2-x^7)-p(1-x^2+x^5-x^7) = (1-x)Q(x),$$

where

$$Q(x) = 4(2+2x+x^2+x^3+x^4+x^5+x^6)-p(1+x+x^5+x^6).$$

In addition, $Q(1) = 0$ involves $p = 9$, hence

$$\begin{aligned} P(x) &= (1-x)^2(5x^5+10x^4+6x^3+2x^2-2x-1) \\ &= (1-x)^2[x^5+(2x-1)(2x^4+6x^3+6x^2+4x+1)]. \end{aligned}$$

Clearly, we have $P(x) \ge 0$ for $x \ge \frac{1}{2}$.

Case 2: $a \le \frac{1}{2}$. Write the desired inequality as

$$\frac{1}{1+a^5} - \frac{1}{2} \ge \frac{b^5c^5-1}{(1+b^5)(1+c^5)}.$$

Since

$$\frac{1}{1+a^5} - \frac{1}{2} \ge \frac{32}{33} - \frac{1}{2} = \frac{31}{66}$$

and

$$(1+b^5)(1+c^5) \ge (1+\sqrt{b^5c^5}\,)^2,$$

it suffices to show that

$$31(1+\sqrt{b^5c^5}\,)^2 \ge 66(b^5c^5-1).$$

For the nontrivial case $bc > 1$, this inequality is equivalent to

$$31(1+\sqrt{b^5c^5}\,) \ge 66(\sqrt{b^5c^5}-1),$$

$$bc \le (97/35)^{2/5}.$$

Indeed, from

$$3 = a^2 + b^2 + c^2 > b^2 + c^2 \ge 2bc,$$

we get

$$bc < 3/2 < (97/35)^{2/5}.$$

This completes the proof. The equality holds for $a = b = c = 1$.

□

P 1.45. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\frac{1}{a^2+a+1} + \frac{1}{b^2+b+1} + \frac{1}{c^2+c+1} \ge 1.$$

First Solution. Using the substitution

$$a = \frac{yz}{x^2}, \qquad b = \frac{zx}{y^2}, \qquad c = \frac{xy}{z^2},$$

where x, y, z are positive real numbers, the inequality becomes

$$\sum \frac{x^4}{x^4 + x^2yz + y^2z^2} \ge 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{x^4}{x^4 + x^2yz + y^2z^2} \ge \frac{\left(\sum x^2\right)^2}{\sum(x^4 + x^2yz + y^2z^2)} = \frac{\sum x^4 + 2\sum y^2z^2}{\sum x^4 + xyz\sum x + \sum y^2z^2}.$$

Therefore, it suffices to show that

$$\sum y^2z^2 \ge xyz \sum x,$$

which is equivalent to $\sum x^2(y-z)^2 \ge 0$. The equality holds for $a = b = c = 1$.

Second Solution. Using the substitution

$$a = \frac{y}{x}, \qquad b = \frac{z}{y}, \qquad c = \frac{x}{z},$$

where $x, y, z > 0$, we need to prove that

$$\frac{x^2}{x^2 + xy + y^2} + \frac{y^2}{y^2 + yz + z^2} + \frac{z^2}{z^2 + zx + z^2} \ge 1.$$

Since
$$\frac{x^2(x^2+y^2+z^2+xy+yz+zx)}{x^2+xy+y^2}=x^2+\frac{x^2z(x+y+z)}{x^2+xy+y^2},$$
multiplying by $x^2+y^2+z^2+xy+yz+zx$, the inequality can be written as
$$\sum\frac{x^2z}{x^2+xy+y^2}\ge\frac{xy+yz+zx}{x+y+z}.$$
By the Cauchy-Schwarz inequality, we have
$$\sum\frac{x^2z}{x^2+xy+y^2}\ge\frac{\left(\sum xz\right)^2}{\sum z(x^2+xy+y^2)}=\frac{xy+yz+zx}{x+y+z}.$$

Remark. The inequality in P 1.45 is a particular case of the following more general inequality (*Vasile Cîrtoaje*, 2009).

- *Let $a_1,a_2,\ldots,a_n$ $(n\ge 3)$ be positive real numbers such that $a_1a_2\cdots a_n=1$. If $p,q\ge 0$ such that $p+q=n-1$, then*
$$\sum_{i=1}^{i=n}\frac{1}{1+pa_i+qa_i^2}\ge 1.$$

$\square$

P 1.46. *Let a,b,c be positive real numbers such that $abc=1$. Prove that*
$$\frac{1}{a^2-a+1}+\frac{1}{b^2-b+1}+\frac{1}{c^2-c+1}\le 3.$$

First Solution. Since
$$\frac{1}{a^2-a+1}+\frac{1}{a^2+a+1}=\frac{2(a^2+1)}{a^4+a^2+1}=2-\frac{2a^4}{a^4+a^2+1},$$
we can rewrite the inequality as
$$\sum\frac{1}{a^2+a+1}+2\sum\frac{a^4}{a^4+a^2+1}\ge 3.$$
Thus, it suffices to show that
$$\sum\frac{1}{a^2+a+1}\ge 1$$
and
$$\sum\frac{a^4}{a^4+a^2+1}\ge 1.$$

The first inequality is just the inequality in P 1.45, while the second follows from the first by substituting a,b,c with a^{-2},b^{-2},c^{-2}, respectively. The equality holds for $a=b=c=1$.

Second Solution. Write the inequality as

$$\sum\left(\frac{4}{3}-\frac{1}{a^2-a+1}\right)\geq 1,$$

$$\sum\frac{(2a-1)^2}{a^2-a+1}\geq 3.$$

Let $p=a+b+c$ and $q=ab+bc+ca$. By the Cauchy-Schwarz inequality, we have

$$\sum\frac{(2a-1)^2}{a^2-a+1}\geq\frac{\left(2\sum a-3\right)^2}{\sum(a^2-a+1)}=\frac{(2p-3)^2}{p^2-2q-p+3}.$$

Thus, it suffices to show that

$$(2p-3)^2\geq 3(p^2-2q-p+3),$$

which is equivalent to

$$p^2+6q-9p\geq 0.$$

From the known inequality

$$(ab+bc+ca)^2\geq 3abc(a+b+c),$$

we get $q^2\geq 3p$. Using this inequality and the AM-GM inequality, we find

$$p^2+6q=p^2+3q+3q\geq 3\sqrt[3]{9p^2q^2}\geq 3\sqrt[3]{9p^2(3p)}=9p.$$

□

P 1.47. *Let a,b,c be positive real numbers such that $abc=1$. Prove that*

$$\frac{3+a}{(1+a)^2}+\frac{3+b}{(1+b)^2}+\frac{3+c}{(1+c)^2}\geq 3.$$

Solution. Using the inequality in P 1.1, we have

$$\begin{aligned}\sum\frac{3+a}{(1+a)^2}&=\sum\frac{2}{(1+a)^2}+\sum\frac{1}{1+a}\\&=\sum\left[\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}\right]+\sum\frac{1}{1+c}\\&\geq\sum\frac{1}{1+ab}+\sum\frac{ab}{1+ab}=3.\end{aligned}$$

The equality holds for $a=b=c=1$.

□

P 1.48. *Let a,b,c be positive real numbers such that $abc=1$. Prove that*

$$\frac{7-6a}{2+a^2}+\frac{7-6b}{2+b^2}+\frac{7-6c}{2+c^2}\ge 1.$$

(Vasile Cîrtoaje, 2008)

Solution. Write the inequality as

$$\left(\frac{7-6a}{2+a^2}+1\right)+\left(\frac{7-6b}{2+b^2}+1\right)+\left(\frac{7-6c}{2+c^2}+1\right)\ge 4,$$

$$\frac{(3-a)^2}{2+a^2}+\frac{(3-b)^2}{2+b^2}+\frac{(3-c)^2}{2+c^2}\ge 4.$$

Substituting a,b,c by $1/a,1/b,1/c$, respectively, we need to prove that $abc=1$ involves

$$\frac{(3a-1)^2}{2a^2+1}+\frac{(3b-1)^2}{2b^2+1}+\frac{(3c-1)^2}{2c^2+1}\ge 4.$$

By the Cauchy-Schwarz inequality, we have

$$\sum\frac{(3a-1)^2}{2a^2+1}\ge\frac{\left(3\sum a-3\right)^2}{\sum(2a^2+1)}=\frac{9\sum a^2+18\sum ab-18\sum a+9}{2\sum a^2+3}.$$

Thus, it suffices to prove that

$$9\sum a^2+18\sum ab-18\sum a+9\ge 4\left(2\sum a^2+3\right),$$

which is equivalent to

$$f(a)+f(b)+f(c)\ge 3,$$

where

$$f(x)=x^2+18\left(\frac{1}{x}-x\right).$$

We use the mixing variables technique. Without loss of generality, assume that

$$a=\max\{a,b,c\},\qquad a\ge 1,\quad bc\le 1.$$

Since

$$f(b)+f(c)-2f(\sqrt{bc})=(b-c)^2+18(\sqrt{b}-\sqrt{c})^2\left(\frac{1}{bc}-1\right)\ge 0,$$

it suffices to show that

$$f(a)+2f(\sqrt{bc})\ge 3,$$

which is equivalent to

$$f(x^2)+2f\left(\frac{1}{x}\right)\ge 3,\qquad x=\sqrt{a},$$

$$x^6-18x^4+36x^3-3x^2-36x+20\ge 0,$$
$$(x-1)^2(x-2)^2(x+1)(x+5)\ge 0.$$

The equality holds for $a=b=c=1$, and also for $a=1/4$ and $b=c=2$ (or any cyclic permutation).

□

P 1.49. *Let a,b,c be positive real numbers such that $abc=1$. Prove that*

$$\frac{a^6}{1+2a^5}+\frac{b^6}{1+2b^5}+\frac{c^6}{1+2c^5}\ge 1.$$

(Vasile Cîrtoaje, 2008)

Solution. Using the substitutions

$$a=\sqrt[3]{\frac{x^2}{yz}},\quad b=\sqrt[3]{\frac{y^2}{zx}},\quad c=\sqrt[3]{\frac{z^2}{xy}},$$

the inequality becomes

$$\sum\frac{x^4}{y^2z^2+2x^3\sqrt[3]{xyz}}\ge 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum\frac{x^4}{y^2z^2+2x^3\sqrt[3]{xyz}}\ge\frac{(\sum x^2)^2}{\sum(y^2z^2+2x^3\sqrt[3]{xyz})}=\frac{(\sum x^2)^2}{\sum x^2y^2+2\sqrt[3]{xyz}\sum x^3}.$$

Therefore, we only need to show that

$$\left(\sum x^2\right)^2\ge\sum x^2y^2+2\sqrt[3]{xyz}\sum x^3.$$

Since, by the AM-GM inequality,

$$x+y+z\ge 3\sqrt[3]{xyz},$$

it suffices to prove that

$$3(\sum x^2)^2\ge 3\sum x^2y^2+2(\sum x)(\sum x^3);$$

that is,

$$\sum x^4+3\sum x^2y^2\ge 2\sum xy(x^2+y^2),$$
$$\sum(x-y)^4\ge 0.$$

The equality holds for $a=b=c=1$.

□

P 1.50. *Let a,b,c be positive real numbers such that $abc=1$. Prove that*

$$\frac{a}{a^2+5}+\frac{b}{b^2+5}+\frac{c}{c^2+5}\le\frac{1}{2}.$$

(Vasile Cîrtoaje, 2008)

Solution. Let

$$F(a,b,c)=\frac{a}{a^2+5}+\frac{b}{b^2+5}+\frac{c}{c^2+5}.$$

Without loss of generality, assume that $a=\min\{a,b,c\}$.

Case 1: $a\le 1/5$. We have

$$F(a,b,c)<\frac{a}{5}+\frac{b}{2\sqrt{5b^2}}+\frac{c}{2\sqrt{5c^2}}\le\frac{1}{25}+\frac{1}{\sqrt{5}}<\frac{1}{2}.$$

Case 2: $a>1/5$. Use the mixing variables method. We will show that

$$F(a,b,c)\le F(a,x,x)\le\frac{1}{2},$$

where

$$x=\sqrt{bc},\qquad a=1/x^2,\quad x<\sqrt{5}.$$

The left inequality, $F(a,b,c)\le F(a,x,x)$, is equivalent to

$$(\sqrt{b}-\sqrt{c})^2[10x(b+c)+10x^2-25-x^4]\ge 0.$$

This is true since

$$10x(b+c)+10x^2-25-x^4\ge 20x^2+10x^2-25x^2-x^4=x^2(5-x^2)>0.$$

The right inequality, $F(a,x,x)\le\dfrac{1}{2}$, is equivalent to

$$(x-1)^2(5x^4-10x^3-2x^2+6x+5)\ge 0.$$

It is also true since

$$5x^4-10x^3-2x^2+6x+5=5(x-1)^4+2x(5x^2-16x+13)$$

and

$$5x^2+13\ge 2\sqrt{65x^2}>16x.$$

The equality holds for $a=b=c=1$.

□

P 1.51. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}+\frac{1}{(1+c)^2}+\frac{2}{(1+a)(1+b)(1+c)} \ge 1.$$

(Pham Van Thuan, 2006)

First Solution. There are two of a, b, c either greater than or equal to 1, or less than or equal to 1. Let b and c be these numbers; that is, $(1-b)(1-c) \ge 0$. Since

$$\frac{1}{(1+b)^2}+\frac{1}{(1+c)^2} \ge \frac{1}{1+bc}$$

(see P 1.1), it suffices to show that

$$\frac{1}{(1+a)^2}+\frac{1}{1+bc}+\frac{2}{(1+a)(1+b)(1+c)} \ge 1.$$

This inequality is equivalent to

$$\frac{b^2c^2}{(1+bc)^2}+\frac{1}{1+bc}+\frac{2bc}{(1+bc)(1+b)(1+c)} \ge 1,$$

which can be written in the obvious form

$$\frac{bc(1-b)(1-c)}{(1+bc)(1+b)(1+c)} \ge 0.$$

The equality holds for $a = b = c = 1$.

Second Solution. Setting

$$a = yz/x^2, \quad b = zx/y^2, \quad c = xy/z^2,$$

where $x, y, z > 0$, the inequality becomes

$$\sum \frac{x^4}{(x^2+yz)^2}+\frac{2x^2y^2z^2}{(x^2+yz)(y^2+zx)(z^2+xy)} \ge 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{x^4}{(x^2+yz)^2} \ge \sum \frac{x^4}{(x^2+y^2)(x^2+z^2)} = 1-\frac{2x^2y^2z^2}{(x^2+y^2)(y^2+z^2)(z^2+x^2)}.$$

Then, it suffices to show that

$$(x^2+y^2)(y^2+z^2)(z^2+x^2) \ge (x^2+yz)(y^2+zx)(z^2+xy).$$

This inequality follows by multiplying the inequalities

$$(x^2+y^2)(x^2+z^2) \ge (x^2+yz)^2,$$

$$(y^2+z^2)(y^2+x^2) \ge (y^2+zx)^2,$$

$$(z^2+x^2)(z^2+y^2) \ge (z^2+xy)^2.$$

Third Solution. We make the substitution

$$\frac{1}{1+a} = \frac{1+x}{2}, \quad \frac{1}{1+b} = \frac{1+y}{2}, \quad \frac{1}{1+c} = \frac{1+z}{2},$$

which is equivalent to

$$a = \frac{1-x}{1+x}, \quad b = \frac{1-y}{1+y}, \quad c = \frac{1-z}{1+z},$$

where

$$-1 < x, y, z < 1, \qquad x+y+z+xyz = 0.$$

The desired inequality becomes

$$(1+x)^2 + (1+y)^2 + (1+z)^2 + (1+x)(1+y)(1+z) \ge 4,$$

$$x^2+y^2+z^2+(x+y+z)^2+4(x+y+z) \ge 0.$$

By virtue of the AM-GM inequality, we have

$$x^2+y^2+z^2+(x+y+z)^2+4(x+y+z) = x^2+y^2+z^2+x^2y^2z^2-4xyz$$

$$\ge 4\sqrt[4]{x^4y^4z^4} - 4xyz = 4|xyz| - 4xyz \ge 0.$$

□

P 1.52. *Let a, b, c be nonnegative real numbers such that*

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} = \frac{3}{2}.$$

Prove that

$$\frac{3}{a+b+c} \ge \frac{2}{ab+bc+ca} + \frac{1}{a^2+b^2+c^2}.$$

Solution. Write the inequality in the homogeneous form

$$\frac{2}{a+b+c}\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) \ge \frac{2}{ab+bc+ca} + \frac{1}{a^2+b^2+c^2}.$$

Due to homogeneity, we may assume that

$$a+b+c = 1, \qquad 0 \le a, b, c < 1.$$

Denote $q = ab + bc + ca$. From the known inequality $(a+b+c)^2 \geq 3(ab+bc+ca)$, we get

$$1 - 3q \geq 0.$$

Rewrite the desired inequality as follows:

$$2\left(\frac{1}{1-c} + \frac{1}{1-a} + \frac{1}{1-b}\right) \geq \frac{2}{q} + \frac{1}{1-2q},$$

$$\frac{2(q+1)}{q-abc} \geq \frac{2-3q}{q(1-2q)},$$

$$q^2(1-4q) + (2-3q)abc \geq 0.$$

By Schur's inequality, we have

$$(a+b+c)^3 + 9abc \geq 4(a+b+c)(ab+bc+ca),$$

$$1 - 4q \geq -9abc.$$

Then,

$$\begin{aligned} q^2(1-4q) + (2-3q)abc &\geq -9q^2abc + (2-3q)abc \\ &= (1-3q)(2+3q)abc \geq 0. \end{aligned}$$

The equality holds for $a = b = c = 1$, and for $a = 0$ and $b = c = \frac{5}{3}$ (or any cyclic permutation).

□

P 1.53. *Let a, b, c be nonnegative real numbers such that*

$$7(a^2 + b^2 + c^2) = 11(ab + bc + ca).$$

Prove that

$$\frac{51}{28} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq 2.$$

Solution. Due to homogeneity, we may assume that $b + c = 2$. Let us denote

$$x = bc, \quad 0 \leq x \leq 1.$$

By the hypothesis $7(a^2 + b^2 + c^2) = 11(ab + bc + ca)$, we get

$$x = \frac{7a^2 - 22a + 28}{25}.$$

Notice that the condition $x \le 1$ involves

$$\frac{1}{7} \le a \le 3.$$

Since

$$\begin{aligned}\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} &= \frac{a}{b+c}+\frac{a(b+c)+(b+c)^2-2bc}{a^2+(b+c)a+bc}\\ &= \frac{a}{2}+\frac{2(a+2-x)}{a^2+2a+x} = \frac{4a^3+27a+11}{8a^2+7a+7},\end{aligned}$$

the required inequalities become

$$\frac{51}{28} \le \frac{4a^3+27a+11}{8a^2+7a+7} \le 2.$$

We have

$$\frac{4a^3+27a+11}{8a^2+7a+7}-\frac{51}{28}=\frac{(7a-1)(4a-7)^2}{28(8a^2+7a+7)} \ge 0$$

and

$$2-\frac{4a^3+27a+11}{8a^2+7a+7}=\frac{(3-a)(2a-1)^2}{8a^2+7a+7} \ge 0.$$

This completes the proof. The left inequality becomes an equality for $7a=b=c$ (or any cyclic permutation), while the right inequality is an equality for $\frac{a}{3}=b=c$ (or any cyclic permutation).

□

P 1.54. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2} \ge \frac{10}{(a+b+c)^2}.$$

Solution. Assume that $a=\min\{a,b,c\}$, and denote

$$x=b+\frac{a}{2}, \quad y=c+\frac{a}{2}.$$

Since

$$a^2+b^2 \le x^2, \quad b^2+c^2 \le x^2+y^2, \quad c^2+a^2 \le y^2,$$

$$(a+b+c)^2=(x+y)^2 \ge 4xy,$$

it suffices to show that

$$\frac{1}{x^2}+\frac{1}{x^2+y^2}+\frac{1}{y^2} \ge \frac{5}{2xy}.$$

We have

$$\frac{1}{x^2}+\frac{1}{x^2+y^2}+\frac{1}{y^2}-\frac{5}{2xy}=\left(\frac{1}{x^2}+\frac{1}{y^2}-\frac{2}{xy}\right)+\left(\frac{1}{x^2+y^2}-\frac{1}{2xy}\right)$$
$$=\frac{(x-y)^2}{x^2y^2}-\frac{(x-y)^2}{2xy(x^2+y^2)}$$
$$=\frac{(x-y)^2(2x^2-xy+2y^2)}{2x^2y^2(x^2+y^2)}\geq 0.$$

The equality holds for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.55. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{1}{a^2-ab+b^2}+\frac{1}{b^2-bc+c^2}+\frac{1}{c^2-ca+a^2}\geq\frac{3}{\max\{ab,bc,ca\}}.$$

Solution. Assume that

$$a=\min\{a,b,c\},\qquad bc=\max\{ab,bc,ca\}.$$

Since

$$\frac{1}{a^2-ab+b^2}+\frac{1}{b^2-bc+c^2}+\frac{1}{c^2-ca+a^2}\geq\frac{1}{b^2}+\frac{1}{b^2-bc+c^2}+\frac{1}{c^2},$$

it suffices to show that

$$\frac{1}{b^2}+\frac{1}{b^2-bc+c^2}+\frac{1}{c^2}\geq\frac{3}{bc}.$$

We have

$$\frac{1}{b^2}+\frac{1}{b^2-bc+c^2}+\frac{1}{c^2}-\frac{3}{bc}=\frac{(b-c)^4}{b^2c^2(b^2-bc+c^2)}\geq 0.$$

The equality holds for $a=b=c$, and also $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.56. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a(2a+b+c)}{b^2+c^2}+\frac{b(2b+c+a)}{c^2+a^2}+\frac{c(2c+a+b)}{a^2+b^2}\geq 6.$$

Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a(2a+b+c)}{b^2+c^2} \geq \frac{\left[\sum a(2a+b+c)\right]^2}{\sum a(2a+b+c)(b^2+c^2)}.$$

Thus, we still need to show that

$$2\left(\sum a^2 + \sum ab\right)^2 \geq 3\sum a(2a+b+c)(b^2+c^2),$$

which is equivalent to

$$2\sum a^4 + 2abc\sum a + \sum ab(a^2+b^2) \geq 6\sum a^2b^2.$$

We can obtain this inequality by adding Schur's inequality of degree four

$$\sum a^4 + abc\sum a \geq \sum ab(a^2+b^2)$$

and

$$\sum ab(a^2+b^2) \geq 2\sum a^2b^2,$$

multiplied by 2 and 3, respectively. The equality occurs for $a = b = c$, and for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 1.57. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a^2(b+c)^2}{b^2+c^2} + \frac{b^2(c+a)^2}{c^2+a^2} + \frac{c^2(a+b)^2}{a^2+b^2} \geq 2(ab+bc+ca).$$

Solution. We apply the SOS method. Since

$$\frac{a^2(b+c)^2)}{b^2+c^2} = a^2 + \frac{2a^2bc}{b^2+c^2},$$

we can write the inequality as

$$2\left(\sum a^2 - \sum ab\right) - \sum a^2\left(1 - \frac{2bc}{b^2+c^2}\right) \geq 0,$$

$$\sum (b-c)^2 - \sum \frac{a^2(b-c)^2}{b^2+c^2} \geq 0,$$

$$\sum \left(1 - \frac{a^2}{b^2+c^2}\right)(b-c)^2 \geq 0.$$

Without loss of generality, assume that $a \ge b \ge c$. Since $1 - \dfrac{c^2}{a^2+b^2} > 0$, it suffices to prove that

$$\left(1-\frac{a^2}{b^2+c^2}\right)(b-c)^2+\left(1-\frac{b^2}{c^2+a^2}\right)(a-c)^2 \ge 0,$$

which is equivalent to

$$\frac{(a^2-b^2+c^2)(a-c)^2}{a^2+c^2} \ge \frac{(a^2-b^2-c^2)(b-c)^2}{b^2+c^2}.$$

This inequality follows by multiplying the inequalities

$$a^2-b^2+c^2 \ge a^2-b^2-c^2, \quad \frac{(a-c)^2}{a^2+c^2} \ge \frac{(b-c)^2}{b^2+c^2}.$$

The latter inequality is true since

$$\frac{(a-c)^2}{a^2+c^2}-\frac{(b-c)^2}{b^2+c^2}=\frac{2bc}{b^2+c^2}-\frac{2ac}{a^2+c^2}=\frac{2c(a-b)(ab-c^2)}{(b^2+c^2)(a^2+c^2)} \ge 0.$$

The equality occurs for $a=b=c$, and for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 1.58. *If a,b,c are positive real numbers, then*

$$3\sum \frac{a}{b^2-bc+c^2}+5\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right) \ge 8\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

(Vasile Cîrtoaje, 2011)

Solution. In order to apply the SOS method, we multiply the inequality by abc and write it as follows:

$$8\left(\sum a^2-\sum bc\right)-3\sum a^2\left(1-\frac{bc}{b^2-bc+c^2}\right) \ge 0,$$

$$4\sum(b-c)^2-3\sum\frac{a^2(b-c)^2}{b^2-bc+c^2} \ge 0,$$

$$\sum\frac{(b-c)^2(4b^2-4bc+4c^2-3a^2)}{b^2-bc+c^2} \ge 0.$$

Without loss of generality, assume that $a \ge b \ge c$. Since

$$4a^2-4ab+4b^2-3c^2=(2a-b)^2+3(b^2-c^2) \ge 0,$$

it suffices to prove that

$$\frac{(a-c)^2(4a^2-4ac+4c^2-3b^2)}{a^2-ac+c^2} \ge \frac{(b-c)^2(3a^2-4b^2+4bc-4c^2)}{b^2-bc+c^2}.$$

Notice that

$$4a^2-4ac+4c^2-3b^2 = (a-2c)^2+3(a^2-b^2) \ge 0.$$

Thus, the desired inequality follows by multiplying the inequalities

$$4a^2-4ac+4c^2-3b^2 \ge 3a^2-4b^2+4bc-4c^2$$

and

$$\frac{(a-c)^2}{a^2-ac+c^2} \ge \frac{(b-c)^2}{b^2-bc+c^2}.$$

The first inequality is equivalent to

$$(a-2c)^2+(b-2c)^2 \ge 0.$$

Also, we have

$$\begin{aligned}\frac{(a-c)^2}{a^2-ac+c^2} - \frac{(b-c)^2}{b^2-bc+c^2} &= \frac{bc}{b^2-bc+c^2} - \frac{ac}{a^2-ac+c^2} \\ &= \frac{c(a-b)(ab-c^2)}{(b^2-bc+c^2)(a^2-ac+c^2)} \ge 0.\end{aligned}$$

The equality occurs for $a=b=c$, and for $2a=b=c$ (or any cyclic permutation).

□

P 1.59. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

(a) $$2abc\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)+a^2+b^2+c^2 \ge 2(ab+bc+ca);$$

(b) $$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a} \le \frac{3(a^2+b^2+c^2)}{2(a+b+c)}.$$

Solution. (a) **First Solution.** We have

$$\begin{aligned}2abc\sum\frac{1}{b+c}+\sum a^2 &= \sum\frac{a(2bc+ab+ac)}{b+c} \\ &= \sum\frac{ab(a+c)}{b+c}+\sum\frac{ac(a+b)}{b+c} \\ &= \sum\frac{ab(a+c)}{b+c}+\sum\frac{ba(b+c)}{c+a} \\ &= \sum ab\left(\frac{a+c}{b+c}+\frac{b+c}{a+c}\right) \ge 2\sum ab.\end{aligned}$$

The equality occurs for $a = b = c$, and for $a = 0$ and $b = c$ (or any cyclic permutation).

Second Solution. Write the inequality as

$$\sum\left(\frac{2abc}{b+c}+a^2-ab-ac\right)\geq 0.$$

We have

$$\begin{aligned}\sum\left(\frac{2abc}{b+c}+a^2-ab-ac\right)&=\sum\frac{ab(a-b)+ac(a-c)}{b+c}\\&=\sum\frac{ab(a-b)}{b+c}+\sum\frac{ba(b-a)}{c+a}\\&=\sum\frac{ab(a-b)^2}{(b+c)(c+a)}\geq 0.\end{aligned}$$

(b) Since

$$\sum\frac{a^2}{a+b}=\sum\left(a-\frac{ab}{a+b}\right)=a+b+c-\sum\frac{ab}{a+b},$$

we can write the desired inequality as

$$\sum\frac{ab}{a+b}+\frac{3(a^2+b^2+c^2)}{2(a+b+c)}\geq a+b+c.$$

Multiplying by $2(a+b+c)$, the inequality can be written as

$$2\sum\left(1+\frac{a}{b+c}\right)bc+3(a^2+b^2+c^2)\geq 2(a+b+c)^2,$$

or

$$2abc\sum\frac{1}{b+c}+a^2+b^2+c^2\geq 2(ab+bc+ca),$$

which is just the inequality in (a).

□

P 1.60. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

(a) $$\frac{a^2-bc}{b^2+c^2}+\frac{b^2-ca}{c^2+a^2}+\frac{c^2-ab}{a^2+b^2}+\frac{3(ab+bc+ca)}{a^2+b^2+c^2}\geq 3;$$

(b) $$\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}+\frac{ab+bc+ca}{a^2+b^2+c^2}\geq\frac{5}{2};$$

(c) $$\frac{a^2+bc}{b^2+c^2}+\frac{b^2+ca}{c^2+a^2}+\frac{c^2+ab}{a^2+b^2}\geq\frac{ab+bc+ca}{a^2+b^2+c^2}+2.$$

(Vasile Cîrtoaje, 2014)

Solution. (a) Use the SOS method. Write the inequality as follows:

$$\sum\left(\frac{2a^2}{b^2+c^2}-1\right)+\sum\left(1-\frac{2bc}{b^2+c^2}\right)-6\left(1-\frac{ab+bc+ca}{a^2+b^2+c^2}\right)\ge 0,$$

$$\sum\frac{2a^2-b^2-c^2}{b^2+c^2}+\sum\frac{(b-c)^2}{b^2+c^2}-3\sum\frac{(b-c)^2}{a^2+b^2+c^2}\ge 0.$$

Since

$$\sum\frac{2a^2-b^2-c^2}{b^2+c^2}=\sum\frac{a^2-b^2}{b^2+c^2}+\sum\frac{a^2-c^2}{b^2+c^2}=\sum\frac{a^2-b^2}{b^2+c^2}+\sum\frac{b^2-a^2}{c^2+a^2}$$
$$=\sum\frac{(a^2-b^2)^2}{(b^2+c^2)(c^2+a^2)}=\sum\frac{(b^2-c^2)^2}{(a^2+b^2)(a^2+c^2)},$$

we can write the inequality as

$$\sum(b-c)^2S_a\ge 0,$$

where

$$S_a=\frac{(b+c)^2}{(a^2+b^2)(a^2+c^2)}+\frac{1}{b^2+c^2}-\frac{3}{a^2+b^2+c^2}.$$

It suffices to show that $S_a,S_b,S_c\ge 0$ for all nonnegative real numbers a,b,c, no two of which are zero. Denoting $x^2=b^2+c^2$, we have

$$S_a=\frac{x^2+2bc}{a^4+a^2x^2+b^2c^2}+\frac{1}{x^2}-\frac{3}{a^2+x^2},$$

and the inequality $S_a\ge 0$ becomes

$$(a^2-2x^2)b^2c^2+2x^2(a^2+x^2)bc+(a^2+x^2)(a^2-x^2)^2\ge 0.$$

Clearly, this is true if

$$-2x^2b^2c^2+2x^4bc\ge 0.$$

Indeed,

$$-2x^2b^2c^2+2x^4bc=2x^2bc(x^2-bc)=2bc(b^2+c^2)(b^2+c^2-bc)\ge 0.$$

The equality occurs for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

(b) ***First Solution.*** We get the desired inequality by summing the inequality in (a) and the inequality

$$\frac{bc}{b^2+c^2}+\frac{ca}{c^2+a^2}+\frac{ab}{a^2+b^2}+\frac{1}{2}\ge\frac{2(ab+bc+ca)}{a^2+b^2+c^2}.$$

This inequality is equivalent to

$$\sum\left(\frac{2bc}{b^2+c^2}+1\right) \ge \frac{4(ab+bc+ca)}{a^2+b^2+c^2}+2,$$

$$\sum\frac{(b+c)^2}{b^2+c^2} \ge \frac{2(a+b+c)^2}{a^2+b^2+c^2}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum\frac{(b+c)^2}{b^2+c^2} \ge \frac{\left[\sum(b+c)\right]^2}{\sum(b^2+c^2)} = \frac{2(a+b+c)^2}{a^2+b^2+c^2}.$$

The equality occurs for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

Second Solution. Let

$$p=a+b+c, \quad q=ab+bc+ca, \quad r=abc.$$

By the Cauchy-Schwarz inequality, we have

$$\sum\frac{a^2}{b^2+c^2} \ge \frac{\left(\sum a^2\right)^2}{\sum a^2(b^2+c^2)} = \frac{(p^2-2q)^2}{2(q^2-2pr)}.$$

Therefore, it suffices to show that

$$\frac{(p^2-2q)^2}{q^2-2pr}+\frac{2q}{p^2-2q} \ge 5. \tag{*}$$

Consider the following cases: $p^2 \ge 4q$ and $3q \le p^2 < 4q$.

Case 1: $p^2 \ge 4q$. The inequality (*) is true if

$$\frac{(p^2-2q)^2}{q^2}+\frac{2q}{p^2-2q} \ge 5,$$

which is equivalent to the obvious inequality

$$(p^2-4q)\left[(p^2-q)^2-2q^2\right] \ge 0.$$

Case 2: $3q \le p^2 < 4q$. Using Schur's inequality of degree four

$$6pr \ge (p^2-q)(4q-p^2),$$

the inequality (*) is true if

$$\frac{3(p^2-2q)^2}{3q^2-(p^2-q)(4q-p^2)}+\frac{2q}{p^2-2q} \ge 5,$$

which is equivalent to the obvious inequality

$$(p^2-3q)(p^2-4q)(2p^2-5q) \le 0.$$

Third Solution (by *Nguyen Van Quy*). Write the inequality (*) from the preceding solution as follows:

$$\frac{(a^2+b^2+c^2)^2}{a^2b^2+b^2c^2+c^2a^2}+\frac{2(ab+bc+ca)}{a^2+b^2+c^2} \ge 5,$$

$$\frac{(a^2+b^2+c^2)^2}{a^2b^2+b^2c^2+c^2a^2}-3 \ge 2-\frac{2(ab+bc+ca)}{a^2+b^2+c^2},$$

$$\frac{a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2}{a^2b^2+b^2c^2+c^2a^2} \ge \frac{2(a^2+b^2+c^2-ab-bc-ca)}{a^2+b^2+c^2}.$$

Since

$$2(a^2b^2+b^2c^2+c^2a^2) \le \sum ab(a^2+b^2) \le (ab+bc+ca)(a^2+b^2+c^2),$$

it suffices to prove that

$$\frac{a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2}{ab+bc+ca} \ge a^2+b^2+c^2-ab-bc-ca,$$

which is just Schur's inequality of degree four

$$a^4+b^4+c^4+abc(a+b+c) \ge ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2).$$

(c) We get the desired inequality by summing the inequality in (a) and the inequality

$$\frac{2bc}{b^2+c^2}+\frac{2ca}{c^2+a^2}+\frac{2ab}{a^2+b^2}+1 \ge \frac{4(ab+bc+ca)}{a^2+b^2+c^2},$$

which was proved at the first solution of (b). The equality occurs for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.61. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2} \ge \frac{(a+b+c)^2}{2(ab+bc+ca)}.$$

Solution. Applying the Cauchy-Schwarz inequality, we get

$$\sum \frac{a^2}{b^2+c^2} \ge \frac{\left(\sum a^2\right)^2}{\sum a^2(b^2+c^2)} = \frac{(a^2+b^2+c^2)^2}{2(a^2b^2+b^2c^2+c^2a^2)}.$$

Therefore, it suffices to show that

$$\frac{(a^2+b^2+c^2)^2}{2(a^2b^2+b^2c^2+c^2a^2)} \ge \frac{(a+b+c)^2}{2(ab+bc+ca)},$$

which is equivalent to

$$\frac{(a^2+b^2+c^2)^2}{a^2b^2+b^2c^2+c^2a^2} - 3 \ge \frac{(a+b+c)^2}{ab+bc+ca} - 3,$$

$$\frac{a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2}{a^2b^2+b^2c^2+c^2a^2} \ge \frac{a^2+b^2+c^2-ab-bc-ca}{ab+bc+ca}.$$

Since $a^2b^2+b^2c^2+c^2a^2 \le (ab+bc+ca)^2$, it suffices to show that

$$a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2 \ge (a^2+b^2+c^2-ab-bc-ca)(ab+bc+ca),$$

which is just Schur's inequality of degree four

$$a^4+b^4+c^4+abc(a+b+c) \ge ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2).$$

The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.62. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{2ab}{(a+b)^2}+\frac{2bc}{(b+c)^2}+\frac{2ca}{(c+a)^2}+\frac{a^2+b^2+c^2}{ab+bc+ca} \ge \frac{5}{2}.$$

(Vasile Cîrtoaje, 2006)

First Solution. We use the SOS method. Write the inequality as follows:

$$\frac{a^2+b^2+c^2}{ab+bc+ca} - 1 \ge \sum \left[\frac{1}{2} - \frac{2bc}{(b+c)^2}\right],$$

$$\sum \frac{(b-c)^2}{ab+bc+ca} \ge \sum \frac{(b-c)^2}{(b+c)^2},$$

$$(b-c)^2 S_a + (c-a)^2 S_b + (a-b)^2 S_c \ge 0,$$

where

$$S_a = 1 - \frac{ab+bc+ca}{(b+c)^2}, \quad S_b = 1 - \frac{ab+bc+ca}{(c+a)^2}, \quad S_c = 1 - \frac{ab+bc+ca}{(a+b)^2}.$$

Without loss of generality, assume that $a \ge b \ge c$. We have $S_c > 0$ and

$$S_b \ge 1 - \frac{(c+a)(c+b)}{(c+a)^2} = \frac{a-b}{c+a} \ge 0.$$

If $b^2S_a + a^2S_b \ge 0$, then

$$\sum(b-c)^2S_a \ge (b-c)^2S_a + (c-a)^2S_b \ge (b-c)^2S_a + \frac{a^2}{b^2}(b-c)^2S_b$$
$$= \frac{(b-c)^2(b^2S_a + a^2S_b)}{b^2} \ge 0.$$

We have

$$b^2S_a + a^2S_b = a^2 + b^2 - (ab+bc+ca)\left[\left(\frac{b}{b+c}\right)^2 + \left(\frac{a}{c+a}\right)^2\right]$$
$$\ge a^2 + b^2 - (b+c)(c+a))\left[\left(\frac{b}{b+c}\right)^2 + \left(\frac{a}{c+a}\right)^2\right]$$
$$= a^2\left(1 - \frac{b+c}{c+a}\right) + b^2\left(1 - \frac{c+a}{b+c}\right)$$
$$= \frac{(a-b)^2(ab+bc+ca)}{(b+c)(c+a)} \ge 0.$$

The equality occurs for $a = b = c$, and for $a = b$ and $c = 0$ (or any cyclic permutation).

Second Solution. Multiplying by $ab + bc + ca$, the inequality becomes

$$\sum \frac{2a^2b^2}{(a+b)^2} + 2abc\sum\frac{1}{a+b} + a^2 + b^2 + c^2 \ge \frac{5}{2}(ab+bc+ca),$$

$$2abc\sum\frac{1}{a+b} + a^2 + b^2 + c^2 - 2(ab+bc+ca) - \sum\frac{1}{2}ab\left[1 - \sum\frac{4ab}{(a+b)^2}\right] \ge 0.$$

According to the second solution of P 1.59-(a), we can write the inequality as follows:

$$\sum\frac{ab(a-b)^2}{(b+c)(c+a)} - \sum\frac{ab(a-b)^2}{2(a+b)^2} \ge 0,$$
$$(b-c)^2S_a + (c-a)^2S_b + (a-b)^2S_c \ge 0,$$

where

$$S_a = \frac{bc}{b+c}[2(b+c)^2 - (a+b)(a+c)].$$

Without loss of generality, assume that $a \ge b \ge c$. We have $S_c > 0$ and

$$S_b = \frac{ac}{a+c}[2(a+c)^2 - (a+b)(b+c)] \ge \frac{ac}{a+c}[2(a+c)^2 - (2a)(a+c)]$$
$$= \frac{2ac^2(a+c)}{a+c} \ge 0.$$

If $S_a + S_b \ge 0$, then

$$\sum(b-c)^2S_a \ge (b-c)^2S_a + (a-c)^2S_b \ge (b-c)^2(S_a + S_b) \ge 0.$$

The inequality $S_a + S_b \ge 0$ is equivalent to

$$\frac{ac}{a+c}[2(a+c)^2-(a+b)(b+c)] \ge \frac{bc}{b+c}[(a+b)(a+c)-2(b+c)^2].$$

Since

$$\frac{ac}{a+c} \ge \frac{bc}{b+c},$$

it suffices to show that

$$2(a+c)^2-(a+b)(b+c) \ge (a+b)(a+c)-2(b+c)^2.$$

This is true since is equivalent to

$$(a-b)^2+2c(a+b)+4c^2 \ge 0.$$

□

P 1.63. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{ab}{(a+b)^2}+\frac{bc}{(b+c)^2}+\frac{ca}{(c+a)^2}+\frac{1}{4} \ge \frac{ab+bc+ca}{a^2+b^2+c^2}.$$

(Vasile Cîrtoaje, 2011)

First Solution. We use the SOS method. Write the inequality as follows:

$$1-\frac{ab+bc+ca}{a^2+b^2+c^2} \ge \sum\left[\frac{1}{4}-\frac{bc}{(b+c)^2}\right],$$

$$2\sum\frac{(b-c)^2}{a^2+b^2+c^2} \ge \sum\frac{(b-c)^2}{(b+c)^2},$$

$$\sum(b-c)^2\left[2-\frac{a^2+b^2+c^2}{(b+c)^2}\right] \ge 0.$$

Since

$$2-\frac{a^2+b^2+c^2}{(b+c)^2} = 1+\frac{2bc-a^2}{(b+c)^2} \ge 1-\left(\frac{a}{b+c}\right)^2,$$

it suffices to show that

$$(b-c)^2S_a+(c-a)^2S_b+(a-b)^2S_c \ge 0,$$

where

$$S_a = 1-\left(\frac{a}{b+c}\right)^2,\quad S_b = 1-\left(\frac{b}{c+a}\right)^2,\quad S_c = 1-\left(\frac{c}{a+b}\right)^2.$$

Without loss of generality, assume that $a \ge b \ge c$. Since $S_b \ge 0$ and $S_c > 0$, if $b^2S_a + a^2S_b \ge 0$, then

$$\sum(b-c)^2S_a \ge (b-c)^2S_a + (c-a)^2S_b \ge (b-c)^2S_a + \frac{a^2}{b^2}(b-c)^2S_b$$
$$= \frac{(b-c)^2(b^2S_a + a^2S_b)}{b^2} \ge 0.$$

We have

$$b^2S_a + a^2S_b = a^2 + b^2 - \left(\frac{ab}{b+c}\right)^2 - \left(\frac{ab}{c+a}\right)^2$$
$$= a^2\left[1-\left(\frac{b}{b+c}\right)^2\right] + b^2\left[1-\left(\frac{a}{c+a}\right)^2\right] \ge 0.$$

The equality occurs for $a = b = c$, and for $a = b$ and $c = 0$ (or any cyclic permutation).

Second Solution. Since $(a+b)^2 \le 2(a^2+b^2)$, it suffices to prove that

$$\sum \frac{ab}{2(a^2+b^2)} + \frac{1}{4} \ge \frac{ab+bc+ca}{a^2+b^2+c^2},$$

which is equivalent to

$$\sum \frac{2ab}{a^2+b^2} + 1 \ge \frac{4(ab+bc+ca)}{a^2+b^2+c^2},$$

$$\sum \frac{(a+b)^2}{a^2+b^2} \ge 2 + \frac{4(ab+bc+ca)}{a^2+b^2+c^2},$$

$$\sum \frac{(a+b)^2}{a^2+b^2} \ge \frac{2(a+b+c)^2}{a^2+b^2+c^2}.$$

The last inequality follows immediately by the Cauchy-Schwarz inequality

$$\sum \frac{(a+b)^2}{a^2+b^2} \ge \frac{[\sum(a+b)]^2}{\sum(a^2+b^2)}.$$

Remark. The following generalization of the inequalities in P 1.62 and P 1.63 holds:

• *Let a, b, c be nonnegative real numbers, no two of which are zero. If $0 \le k \le 2$, then*

$$\sum \frac{4ab}{(a+b)^2} + k\frac{a^2+b^2+c^2}{ab+bc+ca} \ge 3k - 1 + 2(2-k)\frac{ab+bc+ca}{a^2+b^2+c^2}.$$

with equality for $a = b = c$, and for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 1.64. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{3ab}{(a+b)^2}+\frac{3bc}{(b+c)^2}+\frac{3ca}{(c+a)^2}\le \frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{5}{4}.$$

(Vasile Cîrtoaje, 2011)

Solution. We use the SOS method. Write the inequality as follows:

$$3\sum\left[\frac{1}{4}-\frac{bc}{(b+c)^2}\right]\ge 1-\frac{ab+bc+ca}{a^2+b^2+c^2},$$

$$3\sum\frac{(b-c)^2}{(b+c)^2}\ge 2\sum\frac{(b-c)^2}{a^2+b^2+c^2},$$

$$(b-c)^2S_a+(c-a)^2S_b+(a-b)^2S_c\ge 0,$$

where

$$S_a=\frac{3(a^2+b^2+c^2)}{(b+c)^2}-2,\quad S_b=\frac{3(a^2+b^2+c^2)}{(c+a)^2}-2,\quad S_c=\frac{3(a^2+b^2+c^2)}{(a+b)^2}-2.$$

Without loss of generality, assume that $a\ge b\ge c$. Since $S_a>0$ and

$$S_b=\frac{a^2+3b^2+c^2-4ac}{(c+a)^2}=\frac{(a-2c)^2+3(b^2-c^2)}{(c+a)^2}\ge 0,$$

if $S_b+S_c\ge 0$, then

$$\sum(b-c)^2S_a\ge (c-a)^2S_b+(a-b)^2S_c\ge (a-b)^2(S_b+S_c)\ge 0.$$

Using the Cauchy-Schwarz Inequality, we have

$$\begin{aligned}S_b+S_c&=3(a^2+b^2+c^2)\left[\frac{1}{(c+a)^2}+\frac{1}{(a+b)^2}\right]-4\\&\ge\frac{12(a^2+b^2+c^2)}{(c+a)^2+(a+b)^2}-4=\frac{4(a-b-c)^2+4(b-c)^2}{(c+a)^2+(a+b)^2}\ge 0.\end{aligned}$$

The equality occurs for $a=b=c$, and for $\frac{a}{2}=b=c$ (or any cyclic permutation).

□

P 1.65. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

(a) $$\frac{a^3+abc}{b+c}+\frac{b^3+abc}{c+a}+\frac{c^3+abc}{a+b}\ge a^2+b^2+c^2;$$

(b) $$\frac{a^3+2abc}{b+c}+\frac{b^3+2abc}{c+a}+\frac{c^3+2abc}{a+b}\ge \frac{1}{2}(a+b+c)^2;$$

(c) $$\frac{a^3+3abc}{b+c}+\frac{b^3+3abc}{c+a}+\frac{c^3+3abc}{a+b}\ge 2(ab+bc+ca).$$

Solution. (a) **First Solution.** Write the inequality as

$$\sum\left(\frac{a^3+abc}{b+c}-a^2\right)\ge 0,$$

$$\sum\frac{a(a-b)(a-c)}{b+c}\ge 0.$$

Assume that $a\ge b\ge c$. Since $(c-a)(c-b)\ge 0$ and

$$\frac{a(a-b)(a-c)}{b+c}+\frac{b(b-c)(b-a)}{b+c}=\frac{(a-b)^2(a^2+b^2+c^2+ab)}{(b+c)(c+a)}\ge 0,$$

the conclusion follows. The equality occurs for $a=b=c$, and for $a=b$ and $c=0$ (or any cyclic permutation).

(b) Taking into account the inequality in (a), it suffices to show that

$$\frac{abc}{b+c}+\frac{abc}{c+a}+\frac{abc}{a+b}+a^2+b^2+c^2\ge\frac{1}{2}(a+b+c)^2,$$

which is just the inequality (a) from P 1.59. The equality occurs for $a=b=c$, and for $a=b$ and $c=0$ (or any cyclic permutation).

(c) The desired inequality follows by adding the inequality in (a) and the inequality (a) from P 1.59. The equality occurs for $a=b=c$, and for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 1.66. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a^3+3abc}{(b+c)^2}+\frac{b^3+3abc}{(c+a)^2}+\frac{c^3+3abc}{(a+b)^2}\ge a+b+c.$$

(Vasile Cîrtoaje, 2005)

Solution. We use the SOS method. We have

$$\sum\frac{a^3+3abc}{(b+c)^2}-\sum a=\sum\left[\frac{a^3+3abc}{(b+c)^2}-a\right]=\sum\frac{a^3-a(b^2-bc+c^2)}{(b+c)^2}$$

$$=\sum\frac{a^3(b+c)-a(b^3+c^3)}{(b+c)^3}=\sum\frac{ab(a^2-b^2)+ac(a^2-c^2)}{(b+c)^3}$$

$$=\sum\frac{ab(a^2-b^2)}{(b+c)^3}+\sum\frac{ba(b^2-a^2)}{(c+a)^3}=\sum\frac{ab(a^2-b^2)[(c+a)^3-(b+c)^3]}{(b+c)^3(c+a)^3}$$

$$=\sum\frac{ab(a+b)(a-b)^2[(c+a)^2+(c+a)(b+c)+(b+c)^2]}{(b+c)^3(c+a)^3}\ge 0.$$

The equality occurs for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.67. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

(a)
$$\frac{a^3+3abc}{(b+c)^3}+\frac{b^3+3abc}{(c+a)^3}+\frac{c^3+3abc}{(a+b)^3}\ge\frac{3}{2};$$

(b)
$$\frac{3a^3+13abc}{(b+c)^3}+\frac{3b^3+13abc}{(c+a)^3}+\frac{3c^3+13abc}{(a+b)^3}\ge 6.$$

(Vasile Cîrtoaje and Ji Chen, 2005)

Solution. (a) **First Solution.** Use the SOS method. We have

$$\begin{aligned}\sum\frac{a^3+3abc}{(b+c)^3}&=\sum\frac{a(b+c)^2+a(a^2+bc-b^2-c^2)}{(b+c)^3}\\&=\sum\frac{a}{b+c}+\sum\frac{a^3-a(b^2-bc+c^2)}{(b+c)^3}\\&\ge\frac{3}{2}+\sum\frac{a^3(b+c)-a(b^3+c^3)}{(b+c)^4}\\&=\frac{3}{2}+\sum\frac{ab(a^2-b^2)+ac(a^2-c^2)}{(b+c)^4}\\&=\frac{3}{2}+\sum\frac{ab(a^2-b^2)}{(b+c)^4}+\sum\frac{ba(b^2-a^2)}{(c+a)^4}\\&=\frac{3}{2}+\sum\frac{ab(a+b)(a-b)[(c+a)^4-(b+c)^4]}{(b+c)^4(c+a)^4}\ge 0.\end{aligned}$$

The equality occurs for $a=b=c$.

Second Solution. Assume that $a\ge b\ge c$. Since

$$\frac{a^3+3abc}{b+c}\ge\frac{b^3+3abc}{c+a}\ge\frac{c^3+3abc}{a+b}$$

and

$$\frac{1}{(b+c)^2}\ge\frac{1}{(c+a)^2}\ge\frac{1}{(a+b)^2},$$

by Chebyshev's inequality, we get

$$\sum\frac{a^3+3abc}{(b+c)^3}\ge\frac{1}{3}\left(\sum\frac{a^3+3abc}{b+c}\right)\sum\frac{1}{(b+c)^2}.$$

Thus, it suffices to show that

$$\left(\sum\frac{a^3+3abc}{b+c}\right)\sum\frac{1}{(b+c)^2}\ge\frac{9}{2}.$$

We can obtain this inequality by multiplying the known inequality (Iran-1996)

$$\sum \frac{1}{(b+c)^2} \ge \frac{9}{4(ab+bc+ca)}$$

and the inequality (c) from P 1.65.

(b) We have

$$\sum \frac{3a^3+13abc}{(b+c)^3} = \sum \frac{3a(b+c)^2+4abc+3a(a^2+bc-b^2-c^2)}{(b+c)^3}$$

$$= \sum \frac{3a}{b+c} + 4abc \sum \frac{1}{(b+c)^3} + 3\sum \frac{a^3-a(b^2-bc+c^2)}{(b+c)^3}.$$

Since

$$\sum \frac{1}{(b+c)^3} \ge \frac{3}{(a+b)(b+c)(c+a)}$$

(by the AM-GM inequality) and

$$\sum \frac{a^3-a(b^2-bc+c^2)}{(b+c)^3} = \sum \frac{a^3(b+c)-a(b^3+c^3)}{(b+c)^4}$$

$$= \sum \frac{ab(a^2-b^2)+ac(a^2-c^2)}{(b+c)^4} = \sum \frac{ab(a^2-b^2)}{(b+c)^4} + \sum \frac{ba(b^2-a^2)}{(c+a)^4}$$

$$= \sum \frac{ab(a+b)(a-b)[(c+a)^4-(b+c)^4]}{(b+c)^4(c+a)^4} \ge 0,$$

it suffices to prove that

$$\sum \frac{3a}{b+c} + \frac{12abc}{(a+b)(b+c)(c+a)} \ge 6.$$

This inequality is equivalent to the third degree Schur's inequality

$$a^3+b^3+c^3+3abc \ge \sum ab(a+b).$$

The equality occurs for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.68. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

(a) $$\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b}+ab+bc+ca \ge \frac{3}{2}(a^2+b^2+c^2);$$

(b) $$\frac{2a^2+bc}{b+c}+\frac{2b^2+ca}{c+a}+\frac{2c^2+ab}{a+b} \ge \frac{9(a^2+b^2+c^2)}{2(a+b+c)}.$$

(Vasile Cîrtoaje, 2006)

Solution. (a) We apply the SOS method. Write the inequality as

$$\sum\left(\frac{2a^3}{b+c}-a^2\right)\ge\sum(a-b)^2.$$

Since

$$\sum\left(\frac{2a^3}{b+c}-a^2\right)=\sum\frac{a^2(a-b)+a^2(a-c)}{b+c}$$

$$=\sum\frac{a^2(a-b)}{b+c}+\sum\frac{b^2(b-a)}{c+a}=\sum\frac{(a-b)^2(a^2+b^2+ab+bc+ca)}{(b+c)(c+a)},$$

we can write the inequality as

$$(b-c)^2S_a+(c-a)^2S_b+(a-b)^2S_c\ge 0,$$

where

$$S_a=(b+c)(b^2+c^2-a^2),\quad S_b=(c+a)(c^2+a^2-b^2),\quad S_c=(a+b)(a^2+b^2-c^2).$$

Without loss of generality, assume that $a\ge b\ge c$. Since $S_b\ge 0$, $S_c\ge 0$ and

$$S_a+S_b=(a+b)(a-b)^2+c^2(a+b+2c)\ge 0,$$

we have

$$\sum(b-c)^2S_a\ge(b-c)^2S_a+(a-c)^2S_b\ge(b-c)^2(S_a+S_b)\ge 0.$$

The equality occurs for $a=b=c$, and for $a=b$ and $c=0$ (or any cyclic permutation).

(b) Multiplying by $a+b+c$, the inequality can be written as

$$\sum\left(1+\frac{a}{b+c}\right)(2a^2+bc)\ge\frac{9}{2}(a^2+b^2+c^2),$$

$$\sum\frac{2a^3+abc}{b+c}+ab+bc+ca\ge\frac{5}{2}(a^2+b^2+c^2).$$

This inequality follows using the inequality in (a) and the first inequality from P 1.59. The equality occurs for $a=b=c$, and for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 1.69. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a(b+c)}{b^2+bc+c^2}+\frac{b(c+a)}{c^2+ca+a^2}+\frac{c(a+b)}{a^2+ab+b^2}\ge 2.$$

First Solution. Apply the SOS method. We have

$$(a+b+c)\left[\sum \frac{a(b+c)}{b^2+bc+c^2}-2\right]=\sum \left[\frac{a(b+c)(a+b+c)}{b^2+bc+c^2}-2a\right]$$

$$=\sum \frac{a(ab+ac-b^2-c^2)}{b^2+bc+c^2}=\sum \frac{ab(a-b)-ca(c-a)}{b^2+bc+c^2}$$

$$=\sum \frac{ab(a-b)}{b^2+bc+c^2}-\sum \frac{ab(a-b)}{c^2+ca+a^2}$$

$$=(a+b+c)\sum \frac{ab(a-b)^2}{(b^2+bc+c^2)(c^2+ca+a^2)}\geq 0.$$

The equality occurs for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

Second Solution. By the AM-GM inequality, we have

$$\begin{aligned}4(b^2+bc+c^2)(ab+bc+ca) &\leq (b^2+bc+c^2+ab+bc+ca)^2\\ &=(b+c)^2(a+b+c)^2.\end{aligned}$$

Thus,

$$\sum \frac{a(b+c)}{b^2+bc+c^2}=\sum \frac{a(b+c)(ab+bc+ca)}{(b^2+bc+c^2)(ab+bc+ca)}$$

$$\geq \sum \frac{4a(ab+bc+ca)}{(b+c)(a+b+c)^2}=\frac{4(ab+bc+ca)}{(a+b+c)^2}\sum \frac{a}{b+c},$$

and it suffices to show that

$$\sum \frac{a}{b+c}\geq \frac{(a+b+c)^2}{2(ab+bc+ca)}.$$

This follows immediately from the Cauchy-Schwarz inequality

$$\sum \frac{a}{b+c}\geq \frac{(a+b+c)^2}{\sum a(b+c)}.$$

Third Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a(b+c)}{b^2+bc+c^2}\geq \frac{(a+b+c)^2}{\sum \frac{a(b^2+bc+c^2)}{b+c}}.$$

Thus, it is enough to show that

$$(a+b+c)^2\geq 2\sum \frac{a(b^2+bc+c^2)}{b+c}.$$

Since

$$\frac{a(b^2+bc+c^2)}{b+c} = a\left(b+c-\frac{bc}{b+c}\right) = ab+ca-\frac{abc}{b+c},$$

$$\sum \frac{a(b^2+bc+c^2)}{b+c} = 2(ab+bc+ca)-abc\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right),$$

this inequality is equivalent to

$$2abc\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)+a^2+b^2+c^2 \ge 2(ab+bc+ca),$$

which is just the inequality (a) from P 1.59.

Fourth Solution. By direct calculation, we can write the inequality as

$$\sum ab(a^4+b^4) \ge \sum a^2b^2(a^2+b^2),$$

which is equivalent to the obvious inequality

$$\sum ab(a-b)(a^3-b^3) \ge 0.$$

□

P 1.70. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a(b+c)}{b^2+bc+c^2}+\frac{b(c+a)}{c^2+ca+a^2}+\frac{c(a+b)}{a^2+ab+b^2} \ge 2+4\prod\left(\frac{a-b}{a+b}\right)^2.$$

(Vasile Cîrtoaje, 2011)

Solution. For $b=c=1$, the inequality reduces to $a(a-1)^2 \ge 0$. Assume further that

$$a > b > c.$$

As we have shown in the first solution of the preceding P 1.69,

$$\sum \frac{a(b+c)}{b^2+bc+c^2} - 2 = \sum \frac{bc(b-c)^2}{(a^2+ab+b^2)(a^2+ac+c^2)}.$$

Therefore, it remains to show that

$$\sum \frac{bc(b-c)^2}{(a^2+ab+b^2)(a^2+ac+c^2)} \ge 4\prod\left(\frac{a-b}{a+b}\right)^2.$$

Since

$$(a^2+ab+b^2)(a^2+ac+c^2) \le (a+b)^2(a+c)^2,$$

it suffices to show that

$$\sum \frac{bc(b-c)^2}{(a+b)^2(a+c)^2} \ge 4\prod\left(\frac{a-b}{a+b}\right)^2,$$

which is equivalent to

$$\sum \frac{bc(b+c)^2}{(a-b)^2(a-c)^2} \ge 4.$$

We have

$$\begin{aligned}\sum \frac{bc(b+c)^2}{(a-b)^2(a-c)^2} &\ge \frac{ab(a+b)^2}{(a-c)^2(b-c)^2}\\ &\ge \frac{ab(a+b)^2}{a^2b^2} = \frac{(a+b)^2}{ab} \ge 4.\end{aligned}$$

The equality occurs for $a=b=c$, and for $a=b$ and $c=0$ (or any cyclic permutation). □

P 1.71. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{ab-bc+ca}{b^2+c^2}+\frac{bc-ca+ab}{c^2+a^2}+\frac{ca-ab+bc}{a^2+b^2} \ge \frac{3}{2}.$$

Solution. Use the SOS method. We have

$$\begin{aligned}\sum\left(\frac{ab-bc+ca}{b^2+c^2}-\frac{1}{2}\right) &= \sum \frac{(b+c)(2a-b-c)}{2(b^2+c^2)}\\ &= \sum \frac{(b+c)(a-b)}{2(b^2+c^2)}+\sum \frac{(b+c)(a-c)}{2(b^2+c^2)}\\ &= \sum \frac{(b+c)(a-b)}{2(b^2+c^2)}+\sum \frac{(c+a)(b-a)}{2(c^2+a^2)}\\ &= \sum \frac{(a-b)^2(ab+bc+ca-c^2)}{2(b^2+c^2)(c^2+a^2)}.\end{aligned}$$

Since

$$ab+bc+ca-c^2 = (b-c)(c-a)+2ab \ge (b-c)(c-a),$$

it suffices to show that

$$\sum(a^2+b^2)(a-b)^2(b-c)(c-a) \ge 0.$$

This inequality is equivalent to

$$(a-b)(b-c)(c-a)\sum(a-b)(a^2+b^2) \ge 0,$$

$$(a-b)^2(b-c)^2(c-a)^2 \ge 0.$$

The equality occurs for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.72. *Let a,b,c be nonnegative real numbers, no two of which are zero. If $k>-2$, then*

$$\sum \frac{ab+(k-1)bc+ca}{b^2+kbc+c^2} \ge \frac{3(k+1)}{k+2}.$$

(Vasile Cîrtoaje, 2005)

First Solution. Apply the SOS method. Write the inequality as

$$\sum \left[\frac{ab+(k-1)bc+ca}{b^2+kbc+c^2} - \frac{k+1}{k+2} \right] \ge 0,$$

$$\sum \frac{A}{b^2+kbc+c^2} \ge 0,$$

where

$$A=(b+c)(2a-b-c)+k(ab+ac-b^2-c^2).$$

Since

$$\begin{aligned} A =& (b+c)[(a-b)+(a-c)]+k[b(a-b)+c(a-c)] \\ =& (a-b)[(k+1)b+c]+(a-c)[(k+1)c+b], \end{aligned}$$

the inequality is equivalent to

$$\sum \frac{(a-b)[(k+1)b+c]}{b^2+kbc+c^2} + \sum \frac{(a-c)[(k+1)c+b]}{b^2+kbc+c^2} \ge 0,$$

$$\sum \frac{(a-b)[(k+1)b+c]}{b^2+kbc+c^2} + \sum \frac{(b-a)[(k+1)a+c]}{c^2+kca+a^2} \ge 0,$$

$$\sum (b-c)^2 R_a S_a \ge 0,$$

where

$$R_a = b^2+kbc+c^2, \quad S_a = a(b+c-a)+(k+1)bc.$$

Without loss of generality, assume that

$$a \ge b \ge c.$$

Case 1: $k \ge -1$. Since $S_a \ge a(b+c-a)$, it suffices to show that

$$\sum a(b+c-a)(b-c)^2 R_a \ge 0.$$

We have

$$\begin{aligned} \sum a(b+c-a)(b-c)^2 R_a &\ge a(b+c-a)(b-c)^2 R_a + b(c+a-b)(c-a)^2 R_b \\ &\ge (b-c)^2[a(b+c-a)R_a + b(c+a-b)R_b]. \end{aligned}$$

Thus, it is enough to prove that

$$a(b+c-a)R_a + b(c+a-b)R_b \ge 0.$$

Since $b+c-a \ge -(c+a-b)$, we have

$$\begin{aligned} a(b+c-a)R_a + b(c+a-b)R_b &\ge (c+a-b)(bR_b - aR_a) \\ &= (c+a-b)(a-b)(ab-c^2) \ge 0. \end{aligned}$$

Case 2: $-2 < k \le 1$. Since

$$S_a = (a-b)(c-a) + (k+2)bc \ge (a-b)(c-a),$$

we have

$$\sum (b-c)^2 R_a S_a \ge (a-b)(b-c)(c-a) \sum (b-c)R_a.$$

From

$$\begin{aligned} \sum (b-c)R_a &= \sum (b-c)[b^2+bc+c^2-(1-k)bc] \\ &= \sum (b^3-c^3) - (1-k)\sum bc(b-c) \\ &= (1-k)(a-b)(b-c)(c-a), \end{aligned}$$

we get

$$(a-b)(b-c)(c-a)\sum (b-c)R_a = (1-k)(a-b)^2(b-c)^2(c-a)^2 \ge 0.$$

This completes the proof. The equality occurs for $a=b=c$, and also for $a=b$ and $c=0$ (or any cyclic permutation).

Second Solution. Use the *highest coefficient method* (see P 3.76 in Volume 1). Let

$$p = a+b+c, \quad q = ab+bc+ca.$$

Write the inequality in the form $f_6(a,b,c) \ge 0$, where

$$\begin{aligned} f_6(a,b,c) =& (k+2)\sum [a(b+c)+(k-1)bc](a^2+kab+b^2)(a^2+kac+c^2) \\ &- 3(k+1)\prod (b^2+kbc+c^2). \end{aligned}$$

Since

$$a(b+c)+(k-1)bc = (k-2)bc+q,$$

$$(a^2+kab+b^2)(a^2+kac+c^2) = (p^2-2q+kab-c^2)(p^2-2q+kac-b^2),$$

$f_6(a,b,c)$ has the same highest coefficient A as

$$(k+2)(k-2)P_2(a,b,c) - 3(k+1)P_4(a,b,c),$$

where

$$P_2(a,b,c) = \sum bc(kab-c^2)(kac-b^2), \qquad P_4(a,b,c) = \prod (b^2+kbc+c^2).$$

According to Remark 2 from the proof of P 2.75 in Volume 1,

$$A=(k+2)(k-2)P_2(1,1,1)-3(k+1)(k-1)^3=-9(k-1)^2.$$

Since $A\le 0$, according to P 3.76-(a) in Volume 1, it suffices to prove the original inequality for $b=c=1$, and for $a=0$.

For $b=c=1$, the inequality becomes as follows:

$$\frac{2a+k-1}{k+2}+\frac{2(ka+1)}{a^2+ka+1}\ge\frac{3(k+1)}{k+2},$$

$$\frac{a-k-2}{k+2}+\frac{ka+1}{a^2+ka+1}\ge 0,$$

$$\frac{a(a-1)^2}{(k+2)(a^2+ka+1)}\ge 0.$$

For $a=0$, the inequality becomes:

$$\frac{(k-1)bc}{b^2+c^2+kbc}+\frac{b}{c}+\frac{c}{b}\ge\frac{3(k+1)}{k+2},$$

$$\frac{k-1}{x+k}+x\ge\frac{3(k+1)}{k+2},\qquad x=\frac{b}{c}+\frac{c}{b},\quad x\ge 2,$$

$$\frac{(x-2)[(k+2)x+k^2+k+1]}{(k+2)(x+k)}\ge 0,$$

$$(b-c)^2[(k+2)(b^2+c^2)+(k^2+k+1)bc]\ge 0.$$

Remark. For $k=1$ and $k=0$, from P 1.72, we get the inequalities in P 1.69 and P 1.71, respectively. Besides, for $k=2$, we get the well-known inequality (Iran 1996):

$$\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\ge\frac{9}{4(ab+bc+ca)}.$$

□

P 1.73. *Let a,b,c be nonnegative real numbers, no two of which are zero. If $k>-2$, then*

$$\sum\frac{3bc-a(b+c)}{b^2+kbc+c^2}\le\frac{3}{k+2}.$$

(Vasile Cîrtoaje, 2011)

Solution. Write the inequality in P 1.72 as

$$\sum\left[1-\frac{ab+(k-1)bc+ca}{b^2+kbc+c^2}\right]\le\frac{3}{k+2},$$

$$\sum\frac{b^2+c^2+bc-a(b+c)}{b^2+kbc+c^2}\le\frac{3}{k+2}.$$

Since $b^2+c^2 \geq 2bc$, we get

$$\sum \frac{3bc-a(b+c)}{b^2+kbc+c^2} \leq \frac{3}{k+2},$$

which is just the desired inequality. The equality occurs for $a=b=c$.

□

P 1.74. *Let a,b,c be nonnegative real numbers such that $ab+bc+ca=3$. Prove that*

$$\frac{ab+1}{a^2+b^2}+\frac{bc+1}{b^2+c^2}+\frac{ca+1}{c^2+a^2} \geq \frac{4}{3}.$$

Solution. Write the inequality in the homogeneous form $E(a,b,c) \geq 4$, where

$$E(a,b,c)=\frac{4ab+bc+ca}{a^2+b^2}+\frac{4bc+ca+ab}{b^2+c^2}+\frac{4ca+ab+bc}{c^2+a^2}.$$

Without loss of generality, assume that $a=\min\{a,b,c\}$. We will show that

$$E(a,b,c) \geq E(0,b,c) \geq 4.$$

We have

$$\frac{E(a,b,c)-E(0,b,c)}{a}=\frac{4b^2+c(b-a)}{b(a^2+b^2)}+\frac{b+c}{b^2+c^2}+\frac{4c^2+b(c-a)}{c(c^2+a^2)}>0,$$

$$E(0,b,c)-4=\frac{b}{c}+\frac{4bc}{b^2+c^2}+\frac{c}{b}-4=\frac{(b-c)^4}{bc(b^2+c^2)} \geq 0.$$

The equality holds for $a=0$ and $b=c=\sqrt{3}$ (or any cyclic permutation).

□

P 1.75. *Let a,b,c be nonnegative real numbers such that $ab+bc+ca=3$. Prove that*

$$\frac{5ab+1}{(a+b)^2}+\frac{5bc+1}{(b+c)^2}+\frac{5ca+1}{(c+a)^2} \geq 2.$$

Solution. Write the inequality as $E(a,b,c) \geq 6$, where

$$E(a,b,c)=\frac{16ab+bc+ca}{(a+b)^2}+\frac{16bc+ca+ab}{(b+c)^2}+\frac{16ca+ab+bc}{(c+a)^2}.$$

Without loss of generality, assume that

$$a \leq b \leq c.$$

Case 1: $16b^2 \ge c(a+b)$. We will show that

$$E(a,b,c) \ge E(0,b,c) \ge 6.$$

Indeed,

$$\frac{E(a,b,c)-E(0,b,c)}{a} = \frac{16b^2-c(a+b)}{b(a+b)^2} + \frac{1}{b+c} + \frac{16c^2-b(a+c)}{c(c+a)^2} > 0,$$

$$E(0,b,c)-6 = \frac{b}{c} + \frac{16bc}{(b+c)^2} + \frac{c}{b} - 6 = \frac{(b-c)^4}{bc(b+c)^2} \ge 0.$$

Case 2: $16b^2 < c(a+b)$. We have

$$E(a,b,c)-6 > \frac{16ab+bc+ca}{(a+b)^2} - 6 > \frac{16ab+16b^2}{(a+b)^2} - 6 = \frac{2(5b-3a)}{a+b} > 0.$$

The equality holds for $a=0$ and $b=c=\sqrt{3}$ (or any cyclic permutation).

□

P 1.76. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a^2-bc}{2b^2-3bc+2c^2} + \frac{b^2-ca}{2c^2-3ca+2a^2} + \frac{c^2-ab}{2a^2-3ab+2b^2} \ge 0.$$

(Vasile Cîrtoaje, 2005)

Solution. The hint is applying the Cauchy-Schwarz inequality after we made the numerators of the fractions to be nonnegative and as small as possible. Thus, we write the inequality as

$$\sum \left(\frac{a^2-bc}{2b^2-3bc+2c^2} + 1 \right) \ge 3,$$

$$\sum \frac{a^2+2(b-c)^2}{2b^2-3bc+2c^2} \ge 3.$$

Without loss of generality, assume that

$$a \ge b \ge c.$$

Using the Cauchy-Schwarz inequality gives

$$\sum \frac{a^2}{2b^2-3bc+2c^2} \ge \frac{\left(\sum a^2\right)^2}{\sum a^2(2b^2-3bc+2c^2)} = \frac{\sum a^4 + 2\sum a^2b^2}{4\sum a^2b^2 - 3abc\sum a}$$

and

$$\sum \frac{(b-c)^2}{2b^2-3bc+2c^2} \ge \frac{[a(b-c)+b(a-c)+c(a-b)]^2}{\sum a^2(2b^2-3bc+2c^2)} = \frac{4b^2(a-c)^2}{4\sum a^2b^2 - 3abc\sum a}.$$

Therefore, it suffices to show that

$$\frac{\sum a^4+2\sum a^2b^2+8b^2(a-c)^2}{4\sum a^2b^2-3abc\sum a}\geq 3.$$

By Schur's inequality of degree four, we have

$$\sum a^4+abc\sum a\geq \sum ab(a^2+b^2)\geq 2\sum a^2b^2.$$

Thus,it is enough to prove that

$$\frac{4\sum a^2b^2-abc\sum a+8b^2(a-c)^2}{4\sum a^2b^2-3abc\sum a}\geq 3,$$

which is equivalent to

$$abc\sum a+b^2(a-c)^2\geq \sum a^2b^2,$$

$$ac(a-b)(b-c)\geq 0.$$

The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.77. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{2a^2-bc}{b^2-bc+c^2}+\frac{2b^2-ca}{c^2-ca+a^2}+\frac{2c^2-ab}{a^2-ab+b^2}\geq 3.$$

(Vasile Cîrtoaje, 2005)

Solution. Write the inequality such that the numerators of the fractions are non-negative and as small as possible:

$$\sum\left(\frac{2a^2-bc}{b^2-bc+c^2}+1\right)\geq 6,$$

$$\sum\frac{2a^2+(b-c)^2}{b^2-bc+c^2}\geq 6.$$

Applying the Cauchy-Schwarz inequality, we get

$$\sum\frac{2a^2+(b-c)^2}{b^2-bc+c^2}\geq\frac{4\left(2\sum a^2-\sum ab\right)^2}{\sum[2a^2+(b-c)^2](b^2-bc+c^2)}.$$

Thus, we still have to prove that

$$2\left(2\sum a^2-\sum ab\right)^2\geq 3\sum[2a^2+(b-c)^2](b^2-bc+c^2).$$

This inequality is equivalent to

$$2\sum a^4 + 2abc\sum a + \sum ab(a^2+b^2) \ge 6\sum a^2b^2.$$

We can obtain it by summing up Schur's inequality of degree four

$$\sum a^4 + abc\sum a \ge \sum ab(a^2+b^2)$$

and

$$\sum ab(a^2+b^2) \ge 2\sum a^2b^2,$$

multiplied by 2 and 3, respectively. The equality holds for $a = b = c$, and for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 1.78. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a^2}{2b^2 - bc + 2c^2} + \frac{b^2}{2c^2 - ca + 2a^2} + \frac{c^2}{2a^2 - ab + 2b^2} \ge 1.$$

(Vasile Cîrtoaje, 2005)

Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a^2}{2b^2 - bc + 2c^2} \ge \frac{\left(\sum a^2\right)^2}{\sum a^2(2b^2 - bc + 2c^2)}.$$

Therefore, it suffices to show that

$$\left(\sum a^2\right)^2 \ge \sum a^2(2b^2 - bc + 2c^2),$$

which is equivalent to

$$\sum a^4 + abc\sum a \ge 2\sum a^2b^2.$$

This inequality follows by adding Schur's inequality of degree four

$$\sum a^4 + abc\sum a \ge \sum ab(a^2+b^2)$$

and

$$\sum ab(a^2+b^2) \ge 2\sum a^2b^2.$$

The equality holds for $a = b = c$, and for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 1.79. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{1}{4b^2-bc+4c^2}+\frac{1}{4c^2-ca+4a^2}+\frac{1}{4a^2-ab+4b^2} \ge \frac{9}{7(a^2+b^2+c^2)}.$$

(Vasile Cîrtoaje, 2005)

Solution. Use the SOS method. Without loss of generality, assume that

$$a \ge b \ge c.$$

Write the inequality as

$$\sum \left[\frac{7(a^2+b^2+c^2)}{4b^2-bc+4c^2}-3\right] \ge 0,$$

$$\sum \frac{7a^2-5b^2-5c^2+3bc}{4b^2-bc+4c^2} \ge 0,$$

$$\sum \frac{5(2a^2-b^2-c^2)-3(a^2-bc)}{4b^2-bc+4c^2} \ge 0.$$

Since

$$2a^2-b^2-c^2=(a-b)(a+b)+(a-c)(a+c),$$

and

$$2(a^2-bc)=(a-b)(a+c)+(a-c)(a+b)$$

we have

$$\begin{aligned}&10(2a^2-b^2-c^2)-6(a^2-bc)=\\ &=(a-b)[10(a+b)-3(a+c)]+(a-c)[10(a+c)-3(a+b)]\\ &=(a-b)(7a+10b-3c)+(a-c)(7a+10c-3b).\end{aligned}$$

Thus, we can write the desired inequality as follows:

$$\sum \frac{(a-b)(7a+10b-3c)}{4b^2-bc+4c^2}+\sum \frac{(a-c)(7a+10c-3b)}{4b^2-bc+4c^2} \ge 0,$$

$$\sum \frac{(a-b)(7a+10b-3c)}{4b^2-bc+4c^2}+\sum \frac{(b-a)(7b+10a-3c)}{4c^2-ca+4a^2} \ge 0,$$

$$\sum \frac{(a-b)^2(28a^2+28b^2-9c^2+68ab-19bc-19ca)}{(4b^2-bc+4c^2)(4c^2-ca+4a^2)},$$

$$\sum \frac{(a-b)^2[(b-c)(28b+9c)+a(28a+68b-19c)]}{(4b^2-bc+4c^2)(4c^2-ca+4a^2)},$$

$$\sum (a-b)^2 R_c S_c \ge 0,$$

where

$$R_a = 4b^2 - bc + 4c^2, \quad R_b = 4c^2 - ca + 4a^2, \quad R_c = 4a^2 - ab + 4b^2,$$

$$S_a = (c-a)(28c+9a) + b(28b+68c-19a),$$

$$S_b = (a-b)(28a+9b) + c(28c+68a-19b),$$

$$S_c = (b-c)(28b+9c) + a(28a+68b-19c).$$

Since $S_b \ge 0$, $S_c > 0$ and $R_c \ge R_b \ge R_a > 0$, we have

$$\begin{aligned}\sum(b-c)^2R_aS_a &\ge (b-c)^2R_aS_a + (a-c)^2R_bS_b\\ &\ge (b-c)^2R_aS_a + (b-c)^2R_aS_b\\ &= (b-c)^2R_a(S_a+S_b).\end{aligned}$$

Thus, we only need to show that $S_a + S_b \ge 0$. Indeed,

$$S_a + S_b = 19(a-b)^2 + 49(a-b)c + 56c^2 \ge 0.$$

The equality holds for $a = b = c$, and for $a = b$ and $c = 0$ (or any cyclic permutation).

□

P 1.80. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{2a^2+bc}{b^2+c^2} + \frac{2b^2+ca}{c^2+a^2} + \frac{2c^2+ab}{a^2+b^2} \ge \frac{9}{2}.$$

(Vasile Cîrtoaje, 2005)

First Solution. We apply the SOS method. Since

$$\sum\left[\frac{2(2a^2+bc)}{b^2+c^2} - 3\right] = 2\sum\frac{2a^2-b^2-c^2}{b^2+c^2} - \sum\frac{(b-c)^2}{b^2+c^2}$$

and

$$\sum\frac{2a^2-b^2-c^2}{b^2+c^2} = \sum\frac{a^2-b^2}{b^2+c^2} + \sum\frac{a^2-c^2}{b^2+c^2} = \sum\frac{a^2-b^2}{b^2+c^2} + \sum\frac{b^2-a^2}{c^2+a^2}$$

$$= \sum(a^2-b^2)\left(\frac{1}{b^2+c^2} - \frac{1}{c^2+a^2}\right) = \sum\frac{(a^2-b^2)^2}{(b^2+c^2)(c^2+a^2)}$$

$$\ge \sum\frac{(a-b)^2(a^2+b^2)}{(b^2+c^2)(c^2+a^2)},$$

we can write the inequality as

$$2\sum \frac{(b-c)^2(b^2+c^2)}{(c^2+a^2)(a^2+b^2)} \ge \sum \frac{(b-c)^2}{b^2+c^2},$$

or

$$(b-c)^2 S_a + (c-a)^2 S_b + (a-b)^2 S_c \ge 0,$$

where

$$S_a = 2(b^2+c^2)^2 - (c^2+a^2)(a^2+b^2).$$

Without loss of generality, assume that $a \ge b \ge c$, which involves $S_a \le S_b \le S_c$. If

$$S_a + S_b \ge 0,$$

then

$$S_c \ge S_b \ge 0,$$

hence

$$\begin{aligned}(b-c)^2 S_a + (c-a)^2 S_b + (a-b)^2 S_c &\ge (b-c)^2 S_a + (a-c)^2 S_b \\ &\ge (b-c)^2(S_a+S_b) \ge 0.\end{aligned}$$

We have

$$S_a + S_b = (a^2-b^2)^2 + 2c^2(a^2+b^2+2c^2) \ge 0.$$

The equality holds for $a = b = c$, and for $a = b$ and $c = 0$ (or any cyclic permutation).

Second Solution. Since

$$bc \ge \frac{2b^2c^2}{b^2+c^2},$$

we have

$$\sum \frac{2a^2+bc}{b^2+c^2} \ge \sum \frac{2a^2 + \frac{2b^2c^2}{b^2+c^2}}{b^2+c^2} = 2(a^2b^2+b^2c^2+c^2a^2)\sum \frac{1}{(b^2+c^2)^2}.$$

Therefore, it suffices to show that

$$\sum \frac{1}{(b^2+c^2)^2} \ge \frac{9}{4(a^2b^2+b^2c^2+c^2a^2)},$$

which is just the known Iran-1996 inequality (see Remark from P 1.72).

Third Solution. We get the desired inequality by summing the inequality in P 1.60-(a), namely

$$\frac{2a^2-2bc}{b^2+c^2} + \frac{2b^2-2ca}{c^2+a^2} + \frac{2c^2-2ab}{a^2+b^2} + \frac{6(ab+bc+ca)}{a^2+b^2+c^2} \ge 6,$$

and the inequality

$$\frac{3bc}{b^2+c^2}+\frac{3ca}{c^2+a^2}+\frac{3ab}{a^2+b^2}+\frac{3}{2}\ge\frac{6(ab+bc+ca)}{a^2+b^2+c^2}.$$

This inequality is equivalent to

$$\sum\left(\frac{2bc}{b^2+c^2}+1\right)\ge\frac{4(ab+bc+ca)}{a^2+b^2+c^2}+2,$$

$$\sum\frac{(b+c)^2}{b^2+c^2}\ge\frac{2(a+b+c)^2}{a^2+b^2+c^2}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum\frac{(b+c)^2}{b^2+c^2}\ge\frac{\left[\sum(b+c)\right]^2}{\sum(b^2+c^2)}=\frac{2(a+b+c)^2}{a^2+b^2+c^2}.$$

□

P 1.81. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{2a^2+3bc}{b^2+bc+c^2}+\frac{2b^2+3ca}{c^2+ca+a^2}+\frac{2c^2+3ab}{a^2+ab+b^2}\ge 5.$$

(Vasile Cîrtoaje, 2005)

Solution. We apply the SOS method. Write the inequality as

$$\sum\left[\frac{3(2a^2+3bc)}{b^2+bc+c^2}-5\right]\ge 0,$$

or

$$\sum\frac{6a^2+4bc-5b^2-5c^2}{b^2+bc+c^2}\ge 0.$$

Since

$$2a^2-b^2-c^2=(a-b)(a+b)+(a-c)(a+c)$$

and

$$2(a^2-bc)=(a-b)(a+c)+(a-c)(a+b),$$

we have

$$6a^2+4bc-5b^2-5c^2=5(2a^2-b^2-c^2)-4(a^2-bc)$$
$$=(a-b)[5(a+b)-2(a+c)]+(a-c)[5(a+c)-2(a+b)]$$
$$=(a-b)(3a+5b-2c)+(a-c)(3a+5c-2b).$$

Thus, we can write the desired inequality as follows:

$$\sum\frac{(a-b)(3a+5b-2c)}{b^2+bc+c^2}+\sum\frac{(a-c)(3a+5c-2b)}{b^2+bc+c^2}\ge 0,$$

$$\sum \frac{(a-b)(3a+5b-2c)}{b^2+bc+c^2} + \sum \frac{(b-a)(3b+5a-2c)}{c^2+ca+a^2} \ge 0,$$

$$\sum \frac{(a-b)^2(3a^2+3b^2-4c^2+8ab+bc+ca)}{(b^2+bc+c^2)(c^2+ca+a^2)} \ge 0,$$

$$(b-c)^2 S_a + (c-a)^2 S_b + (a-b)^2 S_c \ge 0,$$

where

$$S_a = (b^2+bc+c^2)(-4a^2+3b^2+3c^2+ab+8bc+ca),$$

$$S_b = (c^2+ca+a^2)(-4b^2+3c^2+3a^2+bc+8ca+ab),$$

$$S_c = (a^2+ab+b^2)(-4c^2+3a^2+3b^2+ca+8ab+bc).$$

Assume that $a \ge b \ge c$. Since $S_c > 0$,

$$S_b = (c^2+ca+a^2)[(a-b)(3a+4b)+c(8a+b+3c)] \ge 0,$$

$$\begin{aligned} S_a + S_b &\ge (b^2+bc+c^2)(b-a)(3b+4a) + (c^2+ca+a^2)(a-b)(3a+4b) \\ &= (a-b)^2[3(a+b)(a+b+c)+ab-c^2] \ge 0, \end{aligned}$$

we have

$$\begin{aligned} (b-c)^2 S_a + (c-a)^2 S_b + (a-b)^2 S_c &\ge (b-c)^2 S_a + (a-c)^2 S_b \\ &\ge (b-c)^2 (S_a + S_b) \ge 0. \end{aligned}$$

The equality holds for $a = b = c$, and for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 1.82. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{2a^2+5bc}{(b+c)^2} + \frac{2b^2+5ca}{(c+a)^2} + \frac{2c^2+5ab}{(a+b)^2} \ge \frac{21}{4}.$$

(Vasile Cîrtoaje, 2005)

Solution. Use the SOS method.Write the inequality as follows:

$$\sum \left[\frac{2a^2+5bc}{(b+c)^2} - \frac{7}{4} \right] \ge 0,$$

$$\sum \frac{4(a^2-b^2)+4(a^2-c^2)-3(b-c)^2}{(b+c)^2} \ge 0,$$

$$4\sum \frac{b^2-c^2}{(c+a)^2} + 4\sum \frac{c^2-b^2}{(a+b)^2} - 3\sum \frac{(b-c)^2}{(b+c)^2} \ge 0,$$

$$4\sum \frac{(b-c)^2(b+c)(2a+b+c)}{(c+a)^2(a+b)^2} - 3\sum \frac{(b-c)^2}{(b+c)^2} \geq 0.$$

Substituting $b+c=x$, $c+a=y$ and $a+b=z$, we can rewrite the inequality in the form

$$(y-z)^2S_x + (z-x)^2S_y + (x-y)^2S_z \geq 0,$$

where

$$S_x = 4x^3(y+z) - 3y^2z^2, \quad S_y = 4y^3(z+x) - 3z^2x^2, \quad S_z = 4z^3(x+y) - 3x^2y^2.$$

Without loss of generality, assume that

$$0 < x \leq y \leq z, \qquad z \leq x+y,$$

which involves $S_x \leq S_y \leq S_z$. If

$$S_x + S_y \geq 0,$$

then

$$S_z \geq S_y \geq 0,$$

hence

$$\begin{aligned}(y-z)^2S_x + (z-x)^2S_y + (x-y)^2S_z &\geq (y-z)^2S_x + (z-x)^2S_y \\ &\geq (y-z)^2(S_x+S_y) \geq 0.\end{aligned}$$

We have

$$\begin{aligned}S_x + S_y &= 4xy(x^2+y^2) + 4(x^3+y^3)z - 3(x^2+y^2)z^2 \\ &\geq 4xy(x^2+y^2) + 4(x^3+y^3)z - 3(x^2+y^2)(x+y)z \\ &= 4xy(x^2+y^2) + (x^2-4xy+y^2)(x+y)z.\end{aligned}$$

For the nontrivial case $x^2-4xy+y^2 < 0$, we get

$$\begin{aligned}S_x + S_y &\geq 4xy(x^2+y^2) + (x^2-4xy+y^2)(x+y)^2 \\ &\geq 2xy(x+y)^2 + (x^2-4xy+y^2)(x+y)^2 \\ &= (x-y)^2(x+y)^2.\end{aligned}$$

The equality holds for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.83. *Let a,b,c be nonnegative real numbers, no two of which are zero. If $k > -2$, then*

$$\sum \frac{2a^2+(2k+1)bc}{b^2+kbc+c^2} \geq \frac{3(2k+3)}{k+2}.$$

(Vasile Cîrtoaje, 2005)

First Solution. There are two cases to consider.

Case 1: $-2<k\le -1/2$. Write the inequality as

$$\sum\left[\frac{2a^2+(2k+1)bc}{b^2+kbc+c^2}-\frac{2k+1}{k+2}\right]\ge\frac{6}{k+2},$$

$$\sum\frac{2(k+2)a^2-(2k+1)(b-c)^2}{b^2+kbc+c^2}\ge 6.$$

Since $2(k+2)a^2-(2k+1)(b-c)^2\ge 0$ for $-2<k\le -1/2$, we can apply the Cauchy-Schwarz inequality. Thus, it suffices to show that

$$\frac{\left[2(k+2)\sum a^2-(2k+1)\sum(b-c)^2\right]^2}{\sum[2(k+2)a^2-(2k+1)(b-c)^2](b^2+kbc+c^2)}\ge 6,$$

which is equivalent to each of the following inequalities

$$\frac{2[(1-k)\sum a^2+(2k+1)\sum ab]^2}{\sum[2(k+2)a^2-(2k+1)(b-c)^2](b^2+kbc+c^2)}\ge 3,$$

$$2(k+2)\sum a^4+2(k+2)abc\sum a-(2k+1)\sum ab(a^2+b^2)\ge 6\sum a^2b^2,$$

$$2(k+2)\left[\sum a^4+abc\sum a-\sum ab(a^2+b^2)\right]+3\sum ab(a-b)^2\ge 0.$$

The last inequality is true since, by Schur's inequality of degree four, we have

$$\sum a^4+abc\sum a-\sum ab(a^2+b^2)\ge 0.$$

Case 2: $k\ge -9/5$. Use the SOS method. Without loss of generality, assume that $a\ge b\ge c$. Write the inequality as

$$\sum\left[\frac{2a^2+(2k+1)bc}{b^2+kbc+c^2}-\frac{2k+3}{k+2}\right]\ge 0,$$

$$\sum\frac{2(k+2)a^2-(2k+3)(b^2+c^2)+2(k+1)bc}{b^2+kbc+c^2}\ge 0,$$

$$\sum\frac{(2k+3)(2a^2-b^2-c^2)-2(k+1)(a^2-bc)}{b^2+kbc+c^2}\ge 0.$$

Since

$$2a^2-b^2-c^2=(a-b)(a+b)+(a-c)(a+c)$$

and

$$2(a^2-bc)=(a-b)(a+c)+(a-c)(a+b),$$

we have

$$(2k+3)(2a^2-b^2-c^2)-2(k+1)(a^2-bc)=$$

$$=(a-b)[(2k+3)(a+b)-(k+1)(a+c)]+(a-c)[(2k+3)(a+c)-(k+1)(a+b)]$$

$= (a-b)[(k+2)a+(2k+3)b-(k+1)c]+(a-c)[(k+2)a+(2k+3)c-(k+1)b)].$

Thus, we can write the desired inequality as

$$\sum \frac{(a-b)[(k+2)a+(2k+3)b-(k+1)c]}{b^2+kbc+c^2}+$$

$$+\sum \frac{(a-c)[(k+2)a+(2k+3)c-(k+1)b]}{b^2+kbc+c^2} \geq 0,$$

or

$$\sum \frac{(a-b)[(k+2)a+(2k+3)b-(k+1)c]}{b^2+kbc+c^2}+$$

$$+\sum \frac{(b-a)[(k+2)b+(2k+3)a-(k+1)c]}{c^2+kca+a^2} \geq 0,$$

or

$$(b-c)^2R_aS_a+(c-a)^2R_bS_b+(a-b)^2R_cS_c \geq 0,$$

where

$$R_a = b^2+kbc+c^2, \quad R_b = c^2+kca+a^2, \quad R_c = a^2+kab+b^2,$$

$$S_a = (k+2)(b^2+c^2)-(k+1)^2a^2+(3k+5)bc+(k^2+k-1)a(b+c)$$

$$= -(a-b)[(k+1)^2a+(k+2)b]+c[(k^2+k-1)a+(3k+5)b+(k+2)c],$$

$$S_b = (k+2)(c^2+a^2)-(k+1)^2b^2+(3k+5)ca+(k^2+k-1)b(c+a)$$

$$= (a-b)[(k+2)a+(k+1)^2b]+c[(3k+5)a+(k^2+k-1)b+(k+2)c],$$

$$S_c = (k+2)(a^2+b^2)-(k+1)^2c^2+(3k+5)ab+(k^2+k-1)c(a+b)$$

$$= (k+2)(a^2+b^2)+(3k+5)ab+c[(k^2+k-1)(a+b)-(k+1)^2c]$$

$$\geq (5k+9)ab+c[(k^2+k-1)(a+b)-(k+1)^2c].$$

We have $S_b \geq 0$, since for the nontrivial case

$$(3k+5)a+(k^2+k-1)b+(k+2)c < 0,$$

we get

$$S_b \geq (a-b)[(k+2)a+(k+1)^2b]+b[(3k+5)a+(k^2+k-1)b+(k+2)c]$$

$$= (k+2)(a^2-b^2)+(k+2)^2ab+(k+2)bc > 0.$$

Also, we have $S_c \geq 0$ for $k \geq -9/5$, since

$$\begin{aligned}
(5k+9)ab+c[(k^2+k-1)(a+b)-(k+1)^2c] &\geq \\
&\geq (5k+9)ac+c[(k^2+k-1)(a+b)-(k+1)^2c] \\
&= (k+2)(k+4)ac+(k^2+k-1)bc-(k+1)^2c^2 \\
&\geq (2k^2+7k+7)bc-(k+1)^2c^2 \\
&\geq (k+2)(k+3)c^2 \geq 0.
\end{aligned}$$

Therefore, it suffices to show that $R_aS_a + R_bS_b \ge 0$. From

$$bR_b - aR_a = (a-b)(ab-c^2) \ge 0,$$

we get

$$R_aS_a + R_bS_b \ge R_a\left(S_a + \frac{a}{b}S_b\right).$$

Thus, it suffices to show that

$$S_a + \frac{a}{b}S_b \ge 0.$$

We have

$$\begin{aligned} bS_a + aS_b &= (k+2)(a+b)(a-b)^2 + cf(a,b,c) \\ &\ge 2(k+2)b(a-b)^2 + cf(a,b,c), \end{aligned}$$

hence

$$S_a + \frac{a}{b}S_b \ge 2(k+2)(a-b)^2 + \frac{c}{b}f(a,b,c),$$

where

$$f(a,b,c) = b[(k^2+k-1)a + (3k+5)b] + a[(3k+5)a + (k^2+k-1)b]$$

$$+(k+2)c(a+b) = (3k+5)(a^2+b^2) + 2(k^2+k-1)ab + (k+2)c(a+b).$$

For the nontrivial case $f(a,b,c) < 0$, we have

$$S_a + \frac{a}{b}S_b \ge 2(k+2)(a-b)^2 + f(a,b,c)$$

$$\ge 2(k+2)(a-b)^2 + (3k+5)(a^2+b^2) + 2(k^2+k-1)ab$$

$$= (5k+9)(a^2+b^2) + 2(k^2-k-5)ab \ge 2(k+2)^2ab \ge 0.$$

The proof is completed. The equality holds for $a=b=c$, and for $a=b$ and $c=0$ (or any cyclic permutation).

Second Solution. We use the *highest coefficient method* (see P 3.76 in Volume 1). Let

$$p = a+b+c, \quad q = ab+bc+ca.$$

Write the inequality as $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c) = (k+2)\sum[2a^2 + (2k+1)bc](a^2+kab+b^2)(a^2+kac+c^2)$$

$$-3(2k+3)\prod(b^2+kbc+c^2).$$

Since

$$(a^2+kab+b^2)(a^2+kac+c^2) = (p^2-2q+kab-c^2)(p^2-2q+kac-b^2),$$

$f_6(a,b,c)$ has the same highest coefficient A as

$$(k+2)P_2(a,b,c)-3(2k+3)P_4(a,b,c),$$

where

$$P_2(a,b,c)=\sum[2a^2+(2k+1)bc](kab-c^2)(kac-b^2),$$
$$P_4(a,b,c)=\prod(b^2+kbc+c^2).$$

According to Remark 2 from the proof of P 2.75 in Volume 1, we have

$$A=(k+2)P_2(1,1,1)-3(2k+3)(k-1)^3=9(2k+3)(k-1)^2.$$

On the other hand,

$$f_6(a,1,1)=2(k+2)a(a^2+ka+1)(a-1)^2(a+k+2)\geq 0,$$

$$\frac{f_6(0,b,c)}{(b-c)^2}=2(k+2)(b^2+c^2)^2+2(k+2)^2bc(b^2+c^2)+(4k^2+6k-1)b^2c^2.$$

For $-2<k\leq -3/2$, we have $A\leq 0$. According to P 3.76-(a) in Volume 1, it suffices to show that $f_6(a,1,1)\geq 0$ and $f_6(0,b,c)\geq 0$ for all $a,b,c\geq 0$. The first condition is clearly satisfied. The second condition is satisfied for all $k>-2$ since

$$\begin{aligned}2(k+2)(b^2+c^2)^2+(4k^2+6k-1)b^2c^2 &\geq [8(k+2)+4k^2+6k-1]b^2c^2\\ &=(4k^2+14k+15)b^2c^2\geq 0.\end{aligned}$$

For $k>-3/2$, when $A>0$, we will apply the *highest coefficient cancellation method*. Consider two cases: $p^2\leq 4q$ and $p^2>4q$.

Case 1: $p^2\leq 4q$. Since

$$f_6(1,1,1)=f_6(0,1,1)=0,$$

define the homogeneous function

$$P(a,b,c)=abc+B(a+b+c)^3+C(a+b+c)(ab+bc+ca)$$

such that $P(1,1,1)=P(0,1,1)=0$; that is,

$$P(a,b,c)=abc+\frac{1}{9}(a+b+c)^3-\frac{4}{9}(a+b+c)(ab+bc+ca).$$

We will prove the sharper inequality $g_6(a,b,c)\geq 0$, where

$$g_6(a,b,c)=f_6(a,b,c)-9(2k+3)(k-1)^2P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient $A=0$. Then, according to Remark 1 from the proof of P 3.76 in Volume 1, it suffices to prove that $g_6(a,1,1)\geq 0$ for $0\leq a\leq 4$. We have

$$P(a,1,1)=\frac{a(a-1)^2}{9},$$

hence

$$g_6(a,1,1) = f_6(a,1,1) - 9(2k+3)(k-1)^2P^2(a,1,1) = \frac{a(a-1)^2g(a)}{9},$$

where

$$g(a) = 18(k+2)(a^2+ka+1)(a+k+2) - (2k+3)(k-1)^2a(a-1)^2.$$

Since $a^2+ka+1 \geq (k+2)a$, it suffices to show that

$$18(k+2)^2(a+k+2) \geq (2k+3)(k-1)^2(a-1)^2.$$

Also, since $(a-1)^2 \leq 2a+1$, it is enough to prove that $h(a) \geq 0$, where

$$h(a) = 18(k+2)^2(a+k+2) - (2k+3)(k-1)^2(2a+1).$$

Since $h(a)$ is a linear function, the inequality $h(a) \geq 0$ is true if $h(0) \geq 0$ and $h(4) \geq 0$. Setting $x = 2k+3$, $x > 0$, we get

$$h(0) = 18(k+2)^3 - (2k+3)(k-1)^2 = \frac{1}{4}(8x^3+37x^2+2x+9) > 0.$$

Also,

$$\frac{1}{9}h(4) = 2(k+2)^2(k+6) - (2k+3)(k-1)^2 = 3(7k^2+20k+15) > 0.$$

Case 2: $p^2 > 4q$. We will prove the sharper inequality $g_6(a,b,c) \geq 0$, where

$$g_6(a,b,c) = f_6(a,b,c) - 9(2k+3)(k-1)^2a^2b^2c^2.$$

We see that $g_6(a,b,c)$ has the highest coefficient $A = 0$. According to Remark 1 from the proof of P 3.76 in Volume 1, it suffices to prove that $g_6(a,1,1) \geq 0$ for $a > 4$ and $g_6(0,b,c) \geq 0$ for all $b,c \geq 0$. We have

$$g_6(a,1,1) = f_6(a,1,1) - 9(2k+3)(k-1)^2a^2$$
$$= a[2(k+2)(a^2+ka+1)(a-1)^2(a+k+2) - 9(2k+3)(k-1)^2a].$$

Since

$$a^2+ka+1 > (k+2)a, \quad (a-1)^2 > 9,$$

it suffices to show that

$$2(k+2)^2(a+k+2) \geq (2k+3)(k-1)^2.$$

Indeed,

$$2(k+2)^2(a+k+2) - (2k+3)(k-1)^2 > 2(k+2)^2(k+6) - (2k+3)(k-1)^2$$
$$= 3(7k^2+20k+15) > 0.$$

Also,

$$g_6(0,b,c) = f_6(0,b,c) \geq 0.$$

□

P 1.84. *Let a,b,c be nonnegative real numbers, no two of which are zero. If $k>-2$, then*

$$\sum \frac{3bc-2a^2}{b^2+kbc+c^2} \le \frac{3}{k+2}.$$

(Vasile Cîrtoaje, 2011)

First Solution. Write the inequality as

$$\sum \left[\frac{2a^2-3bc}{b^2+kbc+c^2} + \frac{3}{k+2} \right] \ge \frac{6}{k+2},$$

$$\sum \frac{2(k+2)a^2+3(b-c)^2}{b^2+kbc+c^2} \ge 6.$$

Applying the Cauchy-Schwarz inequality, it suffices to show that

$$\frac{\left[2(k+2)\sum a^2+3\sum(b-c)^2\right]^2}{\sum[2(k+2)a^2+3(b-c)^2](b^2+kbc+c^2)} \ge 6,$$

which is equivalent to each of the following inequalities

$$\frac{2\left[(k+5)\sum a^2-3\sum ab\right]^2}{\sum[2(k+2)a^2+3(b-c)^2](b^2+kbc+c^2)} \ge 3,$$

$$2(k+8)\sum a^4+2(2k+19)\sum a^2b^2 \ge 6(k+2)abc\sum a+21\sum ab(a^2+b^2),$$

$$2(k+2)f(a,b,c)+3g(a,b,c) \ge 0,$$

where

$$f(a,b,c)=\sum a^4+2\sum a^2b^2-3abc\sum a,$$

$$g(a,b,c)=4\sum a^4+10\sum a^2b^2-7\sum ab(a^2+b^2).$$

We need to show that $f(a,b,c)\ge 0$ and $g(a,b,c)\ge 0$. Indeed,

$$f(a,b,c)=\left(\sum a^2\right)^2-3abc\sum a \ge \left(\sum ab\right)^2-3abc\sum a \ge 0$$

and

$$\begin{aligned} g(a,b,c) &= \sum[2(a^4+b^4)+10a^2b^2-7ab(a^2+b^2)] \\ &= \sum(a-b)^2(2a^2-3ab+2b^2) \ge 0. \end{aligned}$$

The equality occurs for $a=b=c$.

Second Solution. Write the inequality in P 1.83 as

$$\sum \left[2-\frac{2a^2+(2k+1)bc}{b^2+kbc+c^2} \right] \le \frac{3}{k+2},$$

$$\sum \frac{2(b^2+c^2)-bc-2a^2}{b^2+kbc+c^2} \le \frac{3}{k+2}.$$

Since $b^2+c^2 \ge 2bc$, we get

$$\sum \frac{3bc-2a^2}{b^2+kbc+c^2} \le \frac{3}{k+2},$$

which is just the desired inequality.

□

P 1.85. *If a,b,c are nonnegative real numbers, no two of which are zero, then*

$$\frac{a^2+16bc}{b^2+c^2}+\frac{b^2+16ca}{c^2+a^2}+\frac{c^2+16ab}{a^2+b^2} \ge 10.$$

(Vasile Cîrtoaje, 2005)

Solution. Assume that $a \le b \le c$ and denote

$$E(a,b,c)=\frac{a^2+16bc}{b^2+c^2}+\frac{b^2+16ca}{c^2+a^2}+\frac{c^2+16ab}{a^2+b^2}.$$

Consider two cases.

Case 1: $16b^3 \ge ac^2$. We will show that

$$E(a,b,c) \ge E(0,b,c) \ge 10.$$

We have

$$E(a,b,c)-E(0,b,c)=\frac{a^2}{b^2+c^2}+\frac{a(16c^3-ab^2)}{c^2(c^2+a^2)}+\frac{a(16b^3-ac^2)}{b^2(a^2+b^2)} \ge 0.$$

Also,

$$\begin{aligned} E(0,b,c)-10 &= \frac{16bc}{b^2+c^2}+\frac{b^2}{c^2}+\frac{c^2}{b^2}-10 \\ &= \frac{(b-c)^4(b^2+c^2+4bc)}{b^2c^2(b^2+c^2)} \ge 0. \end{aligned}$$

Case 2: $16b^3 \le ac^2$. It suffices to show that

$$\frac{c^2+16ab}{a^2+b^2} \ge 10.$$

Indeed,

$$\begin{aligned} \frac{c^2+16ab}{a^2+b^2}-10 &\ge \frac{\dfrac{16b^3}{a}+16ab}{a^2+b^2}-10 \\ &= \frac{16b}{a}-10 \ge 16-10 > 0. \end{aligned}$$

This completes the proof. The equality holds for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.86. *If a,b,c are nonnegative real numbers, no two of which are zero, then*

$$\frac{a^2+128bc}{b^2+c^2}+\frac{b^2+128ca}{c^2+a^2}+\frac{c^2+128ab}{a^2+b^2}\ge 46.$$

(Vasile Cîrtoaje, 2005)

Solution. Let

$$a\le b\le c,$$

$$E(a,b,c)=\frac{a^2+128bc}{b^2+c^2}+\frac{b^2+128ca}{c^2+a^2}+\frac{c^2+128ab}{a^2+b^2}.$$

Consider two cases.

Case 1: $128b^3\ge ac^2$. We will show that

$$E(a,b,c)\ge E(0,b,c)\ge 46.$$

We have

$$E(a,b,c)-E(0,b,c)=\frac{a^2}{b^2+c^2}+\frac{a(128c^3-ab^2)}{c^2(c^2+a^2)}+\frac{a(128b^3-ac^2)}{b^2(a^2+b^2)}\ge 0.$$

Also,

$$\begin{aligned}E(0,b,c)-46&=\frac{128bc}{b^2+c^2}+\frac{b^2}{c^2}+\frac{c^2}{b^2}-46\\&=\frac{(b^2+c^2-4bc)^2(b^2+c^2+8bc)}{b^2c^2(b^2+c^2)}\ge 0.\end{aligned}$$

Case 2: $128b^3\le ac^2$. It suffices to show that

$$\frac{c^2+128ab}{a^2+b^2}\ge 46.$$

Indeed,

$$\begin{aligned}\frac{c^2+128ab}{a^2+b^2}-46&\ge\frac{\dfrac{128b^3}{a}+128ab}{a^2+b^2}-46\\&=\frac{128b}{a}-46\ge 128-46>0.\end{aligned}$$

This completes the proof. The equality holds for $a=0$ and $\dfrac{b}{c}+\dfrac{c}{b}=4$ (or any cyclic permutation).

$\square$

P 1.87. *If a,b,c are nonnegative real numbers, no two of which are zero, then*

$$\frac{a^2+64bc}{(b+c)^2}+\frac{b^2+64ca}{(c+a)^2}+\frac{c^2+64ab}{(a+b)^2}\ge 18.$$

(Vasile Cîrtoaje, 2005)

Solution. Let

$$a\le b\le c,$$

$$E(a,b,c)=\frac{a^2+64bc}{(b+c)^2}+\frac{b^2+64ca}{(c+a)^2}+\frac{c^2+64ab}{(a+b)^2}.$$

Consider two cases.

Case 1: $64b^3\ge c^2(a+2b)$. We will show that

$$E(a,b,c)\ge E(0,b,c)\ge 18.$$

We have

$$E(a,b,c)-E(0,b,c)=\frac{a^2}{(b+c)^2}+\frac{a[64c^3-b^2(a+2c)]}{c^2(c+a)^2}+\frac{a[64b^3-c^2(a+2b)]}{b^2(a+b)^2}$$
$$\ge 0.$$

Also,

$$E(0,b,c)-18=\frac{64bc}{(b+c)^2}+\frac{b^2}{c^2}+\frac{c^2}{b^2}-18$$
$$=\frac{(b-c)^4(b^2+c^2+6bc)}{b^2c^2(b+c)^2}\ge 0.$$

Case 2: $64b^3\le c^2(a+2b)$. It suffices to show that

$$\frac{c^2+64ab}{(a+b)^2}\ge 18.$$

Indeed,

$$\frac{c^2+64ab}{(a+b)^2}-18\ge\frac{\dfrac{64b^3}{a+2b}+64ab}{(a+b)^2}-18$$
$$=\frac{64b}{a+2b}-18\ge\frac{64}{3}-18>0.$$

This completes the proof. The equality holds for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.88. *Let a, b, c be nonnegative real numbers, no two of which are zero. If $k \ge -1$, then*

$$\sum \frac{a^2(b+c)+kabc}{b^2+kbc+c^2} \ge a+b+c.$$

Solution. We apply the SOS method. Write the inequality as follows:

$$\sum \left[\frac{a^2(b+c)+kabc}{b^2+kbc+c^2} - a \right] \ge 0,$$

$$\sum \frac{a(ab+ac-b^2-c^2)}{b^2+kbc+c^2} \ge 0,$$

$$\sum \frac{ab(a-b)}{b^2+kbc+c^2} + \sum \frac{ac(a-c)}{b^2+kbc+c^2} \ge 0,$$

$$\sum \frac{ab(a-b)}{b^2+kbc+c^2} + \sum \frac{ba(b-a)}{c^2+kca+a^2} \ge 0,$$

$$\sum ab(a^2+kab+b^2)(a+b+kc)(a-b)^2 \ge 0.$$

Without loss of generality, assume that

$$a \ge b \ge c.$$

Since $a+b+kc \ge a+b-c > 0$, it suffices to show that

$$b(b^2+kbc+c^2)(b+c+ka)(b-c)^2 + a(c^2+kca+a^2)(c+a+kb)(c-a)^2 \ge 0.$$

Since

$$c+a+kb \ge c+a-b \ge 0, \qquad c^2+kca+a^2 \ge b^2+kbc+c^2,$$

it is enough to prove that

$$b(b+c+ka)(b-c)^2 + a(c+a+kb)(c-a)^2 \ge 0.$$

We have

$$\begin{aligned} b(b+c+ka)(b-c)^2 + a(c+a+kb)(c-a)^2 &\ge \\ \ge [b(b+c+ka)+a(c+a+kb)](b-c)^2 & \\ = [a^2+b^2+2kab+c(a+b)](b-c)^2 & \\ \ge [(a-b)^2+c(a+b)](b-c)^2 \ge 0. & \end{aligned}$$

The equality holds for $a=b=c$, and for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 1.89. *Let a,b,c be nonnegative real numbers, no two of which are zero. If $k \ge \frac{-3}{2}$, then*

$$\sum \frac{a^3+(k+1)abc}{b^2+kbc+c^2} \ge a+b+c.$$

(Vasile Cîrtoaje, 2009)

Solution. Use the SOS method. Write the inequality as follows:

$$\sum \left[\frac{a^3+(k+1)abc}{b^2+kbc+c^2} - a\right] \ge 0, \quad \sum \frac{a^3-a(b^2-bc+c^2)}{b^2+kbc+c^2} \ge 0,$$

$$\sum \frac{(b+c)a^3-a(b^3+c^3)}{(b+c)(b^2+kbc+c^2)} \ge 0, \quad \sum \frac{ab(a^2-b^2)+ac(a^2-c^2)}{(b+c)(b^2+kbc+c^2)} \ge 0,$$

$$\sum \frac{ab(a^2-b^2)}{(b+c)(b^2+kbc+c^2)} + \sum \frac{ba(b^2-a^2)}{(c+a)(c^2+kca+a^2)} \ge 0,$$

$$\sum (a^2-b^2)^2 ab(a^2+kab+b^2)[a^2+b^2+ab+(k+1)c(a+b+c)] \ge 0,$$

$$\sum (b^2-c^2)^2 bc(b^2+kbc+c^2)S_a \ge 0,$$

where

$$S_a = b^2+c^2+bc+(k+1)a(a+b+c).$$

Without loss of generality, assume that

$$a \ge b \ge c.$$

Since $S_c > 0$, it suffices to show that

$$(b^2-c^2)^2 b(b^2+kbc+c^2)S_a + (c^2-a^2)^2 a(c^2+kca+a^2)S_b \ge 0.$$

Since

$$(c^2-a^2)^2 \ge (b^2-c^2)^2, \qquad a \ge b,$$

$$c^2+kca+a^2-(b^2+kbc+c^2) = (a-b)(a+b+kc) \ge 0,$$

$$S_b = a^2+c^2+ac+(k+1)b(a+b+c) \ge a^2+c^2+ac-\frac{1}{2}b(a+b+c)$$
$$= \frac{(a-b)(2a+b)+c(2a+2c-b)}{2} \ge 0,$$

it is enough to show that $S_a+S_b \ge 0$. Indeed,

$$S_a+S_b = a^2+b^2+2c^2+c(a+b)+(k+1)(a+b)(a+b+c)$$
$$\ge a^2+b^2+2c^2+c(a+b)-\frac{1}{2}(a+b)(a+b+c)$$
$$= \frac{(a-b)^2+c(a+b+4c)}{2} \ge 0.$$

This completes the proof. The equality holds for $a=b=c$, and also for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 1.90. *Let a,b,c be nonnegative real numbers, no two of which are zero. If $k>0$, then*

$$\frac{2a^k-b^k-c^k}{b^2-bc+c^2}+\frac{2b^k-c^k-a^k}{c^2-ca+a^2}+\frac{2c^k-a^k-b^k}{a^2-ab+b^2}\ge 0.$$

(Vasile Cîrtoaje, 2004)

Solution. Let

$$X=b^k-c^k,\quad Y=c^k-a^k,\quad Z=a^k-b^k,$$

$$A=b^2-bc+c^2,\ \ B=c^2-ca+a^2,\ \ C=a^2-ab+b^2.$$

Without loss of generality, assume that $a\ge b\ge c$. This involves

$$A\le B,\quad A\le C,\qquad X\ge 0,\quad Z\ge 0.$$

Since

$$\begin{aligned}\sum\frac{2a^k-b^k-c^k}{b^2-bc+c^2}&=\frac{X+2Z}{A}+\frac{X-Z}{B}-\frac{2X+Z}{C}\\&=X\left(\frac{1}{A}+\frac{1}{B}-\frac{2}{C}\right)+Z\left(\frac{2}{A}-\frac{1}{B}-\frac{1}{C}\right),\end{aligned}$$

it suffices to prove that

$$\frac{1}{A}+\frac{1}{B}-\frac{2}{C}\ge 0.$$

Write this inequality as

$$\frac{1}{A}-\frac{1}{C}\ge\frac{1}{C}-\frac{1}{B},$$

that is,

$$(a-c)(a+c-b)(a^2-ac+c^2)\ge(b-c)(a-b-c)(b^2-bc+c^2).$$

For the nontrivial case $a>b+c$, this inequality follows from

$$a-c\ge b-c,$$

$$a+c-b\ge a-b-c,$$

$$a^2-ac+c^2>b^2-bc+c^2.$$

This completes the proof. The equality holds for $a=b=c$, and for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 1.91. *If a, b, c are the lengths of the sides of a triangle, then*

$$\text{(a)} \qquad \frac{b+c-a}{b^2-bc+c^2}+\frac{c+a-b}{c^2-ca+a^2}+\frac{a+b-c}{a^2-ab+b^2} \ge \frac{2(a+b+c)}{a^2+b^2+c^2};$$

$$\text{(b)} \qquad \frac{2bc-a^2}{b^2-bc+c^2}+\frac{2ca-b^2}{c^2-ca+a^2}+\frac{2ab-c^2}{a^2-ab+b^2} \ge 0.$$

(Vasile Cîrtoaje, 2009)

Solution. (a) By the Cauchy-Schwarz inequality, we get

$$\sum \frac{b+c-a}{b^2-bc+c^2} \ge \frac{\left[\sum(b+c-a)\right]^2}{\sum(b+c-a)(b^2-bc+c^2)}$$
$$= \frac{\left(\sum a\right)^2}{2\sum a^3-\sum a^2(b+c)+3abc}.$$

On the other hand, from

$$(b+c-a)(c+a-b)(a+b-c) \ge 0,$$

we get

$$2abc \le \sum a^2(b+c) - \sum a^3,$$

hence

$$2\sum a^3-\sum a^2(b+c)+3abc \le \frac{\sum a^3+\sum a^2(b+c)}{2} = \frac{\left(\sum a\right)\left(\sum a^2\right)}{2}.$$

Therefore,

$$\sum \frac{b+c-a}{b^2-bc+c^2} \ge \frac{2\sum a}{\sum a^2}.$$

The equality holds for a degenerate triangle with $a = b + c$ (or any cyclic permutation).

(b) Since

$$\frac{2bc-a^2}{b^2-bc+c^2} = \frac{(b-c)^2+(b+c)^2-a^2}{b^2-bc+c^2} - 2,$$

we can write the inequality as

$$\sum \frac{(b-c)^2}{b^2-bc+c^2} + (a+b+c)\sum \frac{b+c-a}{b^2-bc+c^2} \ge 6.$$

Using the inequality in (a), it suffices to prove that

$$\sum \frac{(b-c)^2}{b^2-bc+c^2} + \frac{2(a+b+c)^2}{a^2+b^2+c^2} \ge 6.$$

Write this inequality as

$$\sum \frac{(b-c)^2}{b^2-bc+c^2} \ge \sum \frac{2(b-c)^2}{a^2+b^2+c^2},$$

$$\sum \frac{(b-c)^2(a-b+c)(a+b-c)}{b^2-bc+c^2} \ge 0.$$

Clearly, the last inequality is true. The equality holds for degenerate triangles with either $a/2 = b = c$ (or any cyclic permutation), or $a = 0$ and $b = c$ (or any cyclic permutation).

Remark. The following generalization of the inequality in (b) holds (*Vasile Cîrtoaje*, 2009):

- *Let a, b, c be the lengths of the sides of a triangle. If $k \ge -1$, then*

$$\sum \frac{2(k+2)bc - a^2}{b^2+kbc+c^2} \ge 0.$$

with equality for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 1.92. *If a, b, c are nonnegative real numbers, then*

(a) $$\frac{a^2}{5a^2+(b+c)^2} + \frac{b^2}{5b^2+(c+a)^2} + \frac{c^2}{5c^2+(a+b)^2} \le \frac{1}{3};$$

(b) $$\frac{a^3}{13a^3+(b+c)^3} + \frac{b^3}{13b^3+(c+a)^3} + \frac{c^3}{13c^3+(a+b)^3} \le \frac{1}{7}.$$

(Vo Quoc Ba Can and Vasile Cîrtoaje, 2009)

Solution. (a) Apply the Cauchy-Schwarz inequality in the following manner

$$\frac{9}{5a^2+(b+c)^2} = \frac{(1+2)^2}{(a^2+b^2+c^2)+2(2a^2+bc)} \le \frac{1}{a^2+b^2+c^2} + \frac{2}{2a^2+bc}.$$

Then,

$$\sum \frac{9a^2}{5a^2+(b+c)^2} \le \sum \frac{a^2}{a^2+b^2+c^2} + \sum \frac{2a^2}{2a^2+bc} = 1 + 2\sum \frac{a^2}{2a^2+bc},$$

and it remains to show that

$$\sum \frac{a^2}{2a^2+bc} \le 1.$$

For the nontrivial case $a, b, c > 0$, this is equivalent to

$$\sum \frac{1}{2+bc/a^2} \le 1,$$

which follows immediately from P 1.2-(b). The equality holds for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

(b) By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\frac{49}{13a^3+(b+c)^3} &= \frac{(1+6)^2}{(a^3+b^3+c^3)+12a^3+3bc(b+c))}\\ &\le \frac{1}{a^3+b^3+c^3}+\frac{36}{12a^3+3bc(b+c)},\end{aligned}$$

hence

$$\begin{aligned}\sum\frac{49a^3}{13a^3+(b+c)^3} &\le \sum\frac{a^3}{a^3+b^3+c^3}+\sum\frac{36a^3}{12a^3+3bc(b+c)}\\ &= 1+\sum\frac{12a^3}{4a^3+bc(b+c)}.\end{aligned}$$

Thus, it suffices to show that

$$\sum\frac{2a^3}{4a^3+bc(b+c)}\le 1.$$

For the nontrivial case $a,b,c>0$, this is equivalent to

$$\sum\frac{1}{2+bc(b+c)/(2a^3)}\le 1.$$

Since

$$\prod bc(b+c)/(2a^3)\ge\prod bc\sqrt{bc}/a^3=1,$$

the inequality follows immediately from P 1.2-(b). The equality holds for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.93. *If a,b,c are nonnegative real numbers, then*

$$\frac{b^2+c^2-a^2}{2a^2+(b+c)^2}+\frac{c^2+a^2-b^2}{2b^2+(c+a)^2}+\frac{a^2+b^2-c^2}{2c^2+(a+b)^2}\ge\frac{1}{2}.$$

(Vasile Cîrtoaje, 2011)

Solution. We apply the SOS method. Write the inequality as follows:

$$\sum\left[\frac{b^2+c^2-a^2}{2a^2+(b+c)^2}-\frac{1}{6}\right]\ge 0,$$

$$\sum\frac{5(b^2+c^2-2a^2)+2(a^2-bc)}{2a^2+(b+c)^2}\ge 0,$$

$$\sum \frac{5(b^2-a^2)+5(c^2-a^2)+(a-b)(a+c)+(a-c)(a+b)}{2a^2+(b+c)^2} \ge 0,$$

$$\sum \frac{(b-a)[5(b+a)-(a+c)]}{2a^2+(b+c)^2} + \sum \frac{(c-a)[5(c+a)-(a+b)]}{2a^2+(b+c)^2} \ge 0,$$

$$\sum \frac{(b-a)[5(b+a)-(a+c)]}{2a^2+(b+c)^2} + \sum \frac{(a-b)[5(a+b)-(b+c)]}{2b^2+(c+a)^2} \ge 0,$$

$$\sum (a-b)^2[2c^2+(a+b)^2][2(a^2+b^2)+c^2+3ab-3c(a+b)] \ge 0,$$

$$\sum (b-c)^2 R_a S_a \ge 0,$$

where

$$R_a = 2a^2+(b+c)^2, \quad S_a = a^2+2(b^2+c^2)+3bc-3a(b+c).$$

Without loss of generality, assume that $a \ge b \ge c$. We have

$$S_b = b^2+2(c^2+a^2)+3ca-3b(c+a) = (a-b)(2a-b)+2c^2+3c(a-b) \ge 0,$$

$$S_c = c^2+2(a^2+b^2)+3ab-3c(a+b) \ge 7ab-3c(a+b) \ge 3a(b-c)+3b(a-c) \ge 0,$$

$$S_a + S_b = 3(a-b)^2+4c^2 \ge 0.$$

Since

$$\sum (b-c)^2 R_a S_a \ge (b-c)^2 R_a S_a + (c-a)^2 R_b S_b$$

$$= (b-c)^2 R_a (S_a+S_b) + [(c-a)^2 R_b - (b-c)^2 R_a] S_b,$$

it suffices to prove that

$$(a-c)^2 R_b \ge (b-c)^2 R_a.$$

We can get this by multiplying the inequalities

$$b^2(a-c)^2 \ge a^2(b-c)^2$$

and

$$a^2 R_b \ge b^2 R_a.$$

The equality holds for $a=b=c$, and for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 1.94. *Let a,b,c be positive real numbers. If $k>0$, then*

$$\frac{3a^2-2bc}{ka^2+(b-c)^2} + \frac{3b^2-2ca}{kb^2+(c-a)^2} + \frac{3c^2-2ab}{kc^2+(a-b)^2} \le \frac{3}{k}.$$

(Vasile Cîrtoaje, 2011)

Solution. Use the SOS method. Write the inequality as follows:

$$\sum\left[\frac{1}{k}-\frac{3a^2-2bc}{ka^2+(b-c)^2}\right]\ge 0,$$

$$\sum\frac{b^2+c^2-2a^2+2(k-1)(bc-a^2)}{ka^2+(b-c)^2}\ge 0;$$

$$\sum\frac{(b^2-a^2)+(c^2-a^2)+(k-1)[(a+b)(c-a)+(a+c)(b-a)]}{ka^2+(b-c)^2}\ge 0;$$

$$\sum\frac{(b-a)[b+a+(k-1)(a+c)]}{ka^2+(b-c)^2}+\sum\frac{(c-a)[c+a+(k-1)(a+b)]}{ka^2+(b-c)^2}\ge 0;$$

$$\sum\frac{(b-a)[b+a+(k-1)(a+c)]}{ka^2+(b-c)^2}+\sum\frac{(a-b)[a+b+(k-1)(b+c)]}{kb^2+(c-a)^2}\ge 0;$$

$$\sum(a-b)^2[kc^2+(a-b)^2][(k-1)c^2+2c(a+b)+(k^2-1)(ab+bc+ca)]\ge 0.$$

For $k\ge 1$, the inequality is clearly true. Consider further that $0<k<1$. Since

$$(k-1)c^2+2c(a+b)+(k^2-1)(ab+bc+ca)>$$

$$>-c^2+2c(a+b)-(ab+bc+ca)=(b-c)(c-a),$$

it suffices to prove that

$$(a-b)(b-c)(c-a)\sum(a-b)[kc^2+(a-b)^2]\ge 0.$$

Since

$$\sum(a-b)[kc^2+(a-b)^2]=k\sum(a-b)c^2+\sum(a-b)^3$$
$$=(3-k)(a-b)(b-c)(c-a),$$

we have

$$(a-b)(b-c)(c-a)\sum(a-b)[kc^2+(a-b)^2]=$$

$$=(3-k)(a-b)^2(b-c)^2(c-a)^2\ge 0.$$

This completes the proof. The equality holds for $a=b=c$.

□

P 1.95. *Let a,b,c be nonnegative real numbers, no two of which are zero. If $k\ge 3+\sqrt{7}$, then*

(a) $$\frac{a}{a^2+kbc}+\frac{b}{b^2+kca}+\frac{c}{c^2+kab}\ge\frac{9}{(1+k)(a+b+c)};$$

(b) $$\frac{1}{ka^2+bc}+\frac{1}{kb^2+ca}+\frac{1}{kc^2+ab}\ge\frac{9}{(k+1)(ab+bc+ca)}.$$

(Vasile Cîrtoaje, 2005)

Solution. (a) Assume that $a = \max\{a,b,c\}$. Setting

$$t = \frac{b+c}{2}, \qquad t \le a,$$

by the Cauchy-Schwarz inequality, we get

$$\frac{b}{b^2+kca} + \frac{c}{c^2+kab} \ge \frac{(b+c)^2}{b(b^2+kca)+c(c^2+kab)} = \frac{4t^2}{8t^3-6bct+2kabc}$$
$$= \frac{2t^2}{4t^3+(ka-3t)bc} \ge \frac{2t^2}{4t^3+(ka-3t)t^2} = \frac{2}{t+ka}.$$

On the other hand,

$$\frac{a}{a^2+kbc} \ge \frac{a}{a^2+kt^2}.$$

Therefore, it suffices to prove that

$$\frac{a}{a^2+kt^2} + \frac{2}{t+ka} \ge \frac{9}{(k+1)(a+2t)},$$

which is equivalent to

$$(a-t)^2[(k^2-6k+2)a+k(4k-5)t] \ge 0.$$

This inequality is true, since $k^2-6k+2 \ge 0$ and $4k-5>0$. The equality holds for $a=b=c$.

(b) For $a=0$, the inequality becomes

$$\frac{1}{b^2} + \frac{1}{c^2} \ge \frac{k(8-k)}{(k+1)bc}.$$

We have

$$\frac{1}{b^2} + \frac{1}{c^2} - \frac{k(8-k)}{(k+1)bc} \ge \frac{2}{bc} - \frac{k(8-k)}{(k+1)bc} = \frac{k^2-6k+2}{(k+1)bc} \ge 0.$$

For $a,b,c>0$, the desired inequality follows from the inequality in (a) by substituting a,b,c with $1/a, 1/b, 1/c$, respectively. The equality holds for $a=b=c$. In the case $k=3+\sqrt{7}$, the equality also holds for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.96. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{1}{2a^2+bc} + \frac{1}{2b^2+ca} + \frac{1}{2c^2+ab} \ge \frac{6}{a^2+b^2+c^2+ab+bc+ca}.$$

(Vasile Cîrtoaje, 2005)

Solution. Applying the Cauchy-Schwarz inequality, we have

$$\sum \frac{1}{2a^2+bc} \ge \frac{\left[\sum(b+c)\right]^2}{\sum(b+c)^2(2a^2+bc)} = \frac{4(a+b+c)^2}{\sum(b+c)^2(2a^2+bc)}.$$

Thus, it suffices to show that

$$2(a+b+c)^2(a^2+b^2+c^2+ab+bc+ca) \ge 3\sum(b+c)^2(2a^2+bc),$$

which is equivalent to

$$2\sum a^4+3\sum ab(a^2+b^2)+2abc\sum a \ge 10\sum a^2b^2.$$

This follows by adding Schur's inequality

$$2\sum a^4+2abc\sum a \ge 2\sum ab(a^2+b^2)$$

to the inequality

$$5\sum ab(a^2+b^2) \ge 10\sum a^2b^2.$$

The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.97. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{1}{22a^2+5bc}+\frac{1}{22b^2+5ca}+\frac{1}{22c^2+5ab} \ge \frac{1}{(a+b+c)^2}.$$

(Vasile Cîrtoaje, 2005)

Solution. Applying the Cauchy-Schwarz inequality, we have

$$\sum \frac{1}{22a^2+5bc} \ge \frac{\left[\sum(b+c)\right]^2}{\sum(b+c)^2(22a^2+5bc)} = \frac{4(a+b+c)^2}{\sum(b+c)^2(22a^2+5bc)}.$$

Thus, it suffices to show that

$$4(a+b+c)^4 \ge \sum(b+c)^2(22a^2+5bc),$$

which is equivalent to

$$4\sum a^4+11\sum ab(a^2+b^2)+4abc\sum a \ge 30\sum a^2b^2.$$

This follows by adding Schur's inequality

$$4\sum a^4+4abc\sum a \ge 4\sum ab(a^2+b^2)$$

to the inequality

$$15\sum ab(a^2+b^2) \ge 30\sum a^2b^2.$$

The equality holds for $a=b=c$.

□

P 1.98. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab}\geq\frac{8}{(a+b+c)^2}.$$

(Vasile Cîrtoaje, 2005)

First Solution. Applying the Cauchy-Schwarz inequality, we have

$$\sum\frac{1}{2a^2+bc}\geq\frac{\left[\sum(b+c)\right]^2}{\sum(b+c)^2(2a^2+bc)}=\frac{4(a+b+c)^2}{\sum(b+c)^2(2a^2+bc)}.$$

Thus, it suffices to show that

$$(a+b+c)^4\geq 2\sum(b+c)^2(2a^2+bc),$$

which is equivalent to

$$\sum a^4+2\sum ab(a^2+b^2)+4abc\sum a\geq 6\sum a^2b^2.$$

We will prove the sharper inequality

$$\sum a^4+2\sum ab(a^2+b^2)+abc\sum a\geq 6\sum a^2b^2.$$

This follows by adding Schur's inequality

$$\sum a^4+abc\sum a\geq\sum ab(a^2+b^2)$$

to the inequality

$$3\sum ab(a^2+b^2)\geq 6\sum a^2b^2.$$

The equality holds for $a=0$ and $b=c$ (or any cyclic permutation).

Second Solution. Without loss of generality, we may assume that $a\geq b\geq c$. Since the equality holds for $c=0$ and $a=b$, when

$$\frac{1}{2a^2+bc}=\frac{1}{2b^2+ca}=\frac{1}{4c^2+2ab},$$

write the inequality as

$$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{4c^2+2ab}+\frac{1}{4c^2+2ab}\geq\frac{8}{(a+b+c)^2},$$

then apply the Cauchy-Schwarz inequality. Thus, it suffices to prove that

$$\frac{16}{(2a^2+bc)+(2b^2+ca)+(4c^2+2ab)+(4c^2+2ab)}\geq\frac{8}{(a+b+c)^2},$$

which is equivalent to the obvious inequality

$$c(a+b-2c)\geq 0.$$

□

P 1.99. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{1}{a^2+bc}+\frac{1}{b^2+ca}+\frac{1}{c^2+ab} \ge \frac{12}{(a+b+c)^2}.$$

(Vasile Cîrtoaje, 2005)

Solution. Write the inequality such that the numerators of the fractions are non-negative and as small as possible:

$$\sum \left[\frac{1}{a^2+bc}-\frac{1}{(a+b+c)^2}\right] \ge \frac{9}{(a+b+c)^2},$$

$$\sum \frac{(a+b+c)^2-a^2-bc}{a^2+bc} \ge 9.$$

Assuming that $a+b+c=1$, the inequality becomes

$$\sum \frac{1-a^2-bc}{a^2+bc} \ge 9.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{1-a^2-bc}{a^2+bc} \ge \frac{\left[\sum(1-a^2-bc)\right]^2}{\sum(1-a^2-bc)(a^2+bc)}.$$

Then, it suffices to prove that

$$\left(3-\sum a^2-\sum bc\right)^2 \ge 9\sum(a^2+bc)-9\sum(a^2+bc)^2,$$

which is equivalent to

$$(1-4q)(4-7q)+36abc \ge 0, \quad q=ab+bc+ca.$$

For $q \le 1/4$, this inequality is clearly true. Consider further that $q > 1/4$. By Schur's inequality of degree three

$$(a+b+c)^3+9abc \ge 4(a+b+c)(ab+bc+ca),$$

we get $1+9abc \ge 4q$, and hence $36abc \ge 16q-4$. Thus,

$$(1-4q)(4-7q)+36abc \ge (1-4q)(4-7q)+16q-4 = 7q(4q-1) > 0.$$

The equality holds for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.100. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

(a) $$\frac{1}{a^2+2bc}+\frac{1}{b^2+2ca}+\frac{1}{c^2+2ab} \ge \frac{1}{a^2+b^2+c^2}+\frac{2}{ab+bc+ca};$$

(b) $$\frac{a(b+c)}{a^2+2bc}+\frac{b(c+a)}{b^2+2ca}+\frac{c(a+b)}{c^2+2ab} \ge 1+\frac{ab+bc+ca}{a^2+b^2+c^2}.$$

(Darij Grinberg and Vasile Cîrtoaje, 2005)

Solution. (a) Write the inequality as

$$\frac{\sum(b^2+2ca)(c^2+2ab)}{(a^2+2bc)(b^2+2ca)(c^2+2ab)} \ge \frac{ab+bc+ca+2a^2+2b^2+2c^2}{(a^2+b^2+c^2)(ab+bc+ca)}.$$

Since

$$\sum(b^2+2ca)(c^2+2ab) = (ab+bc+ca)(ab+bc+ca+2a^2+2b^2+2c^2),$$

it suffices to show that

$$(a^2+b^2+c^2)(ab+bc+ca)^2 \ge (a^2+2bc)(b^2+2ca)(c^2+2ab),$$

which is just the inequality (a) in P 2.16 in Volume 1. The equality holds for $a=b$, or $b=c$, or $c=a$.

(b) Write the inequality in (a) as

$$\sum \frac{ab+bc+ca}{a^2+2bc} \ge 2+\frac{ab+bc+ca}{a^2+b^2+c^2},$$

or

$$\sum \frac{a(b+c)}{a^2+2bc}+\sum \frac{bc}{a^2+2bc} \ge 2+\frac{ab+bc+ca}{a^2+b^2+c^2}.$$

The desired inequality follows by adding this inequality to

$$1 \ge \sum \frac{bc}{a^2+2bc}.$$

The last inequality is equivalent to

$$\sum \frac{a^2}{a^2+2bc} \ge 1,$$

which follows by applying the AM-GM inequality as follows:

$$\sum \frac{a^2}{a^2+2bc} \ge \sum \frac{a^2}{a^2+b^2+c^2} = 1.$$

The equality holds for $a=b=c$.

□

P 1.101. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\text{(a)} \qquad \frac{a}{a^2+2bc}+\frac{b}{b^2+2ca}+\frac{c}{c^2+2ab}\le\frac{a+b+c}{ab+bc+ca};$$

$$\text{(b)} \qquad \frac{a(b+c)}{a^2+2bc}+\frac{b(c+a)}{b^2+2ca}+\frac{c(a+b)}{c^2+2ab}\le 1+\frac{a^2+b^2+c^2}{ab+bc+ca}.$$

(Vasile Cîrtoaje, 2008)

Solution. (a) Use the SOS method. Write the inequality as

$$\sum a\left(1-\frac{ab+bc+ca}{a^2+2bc}\right)\ge 0,$$

$$\sum\frac{a(a-b)(a-c)}{a^2+2bc}\ge 0.$$

Assume that $a\ge b\ge c$. Since $(c-a)(c-b)\ge 0$, it suffices to show that

$$\frac{a(a-b)(a-c)}{a^2+2bc}+\frac{b(b-a)(b-c)}{b^2+2ca}\ge 0.$$

This inequality is equivalent to

$$c(a-b)^2[2a(a-c)+2b(b-c)+3ab]\ge 0,$$

which is clearly true. The equality holds for $a=b=c$, and for $a=b$ and $c=0$ (or any cyclic permutation).

(b) Since

$$\frac{a(b+c)}{a^2+2bc}=\frac{a(a+b+c)}{a^2+2bc}-\frac{a^2}{a^2+2bc},$$

we can write the inequality as

$$(a+b+c)\sum\frac{a}{a^2+2bc}\le 1+\frac{a^2+b^2+c^2}{ab+bc+ca}+\sum\frac{a^2}{a^2+2bc}.$$

According to the inequality in (a), it suffices to show that

$$\frac{(a+b+c)^2}{ab+bc+ca}\le 1+\frac{a^2+b^2+c^2}{ab+bc+ca}+\sum\frac{a^2}{a^2+2bc},$$

which is equivalent to

$$\sum\frac{a^2}{a^2+2bc}\ge 1.$$

Indeed,

$$\sum\frac{a^2}{a^2+2bc}\ge\sum\frac{a^2}{a^2+b^2+c^2}=1.$$

The equality holds for $a=b=c$.

□

P 1.102. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

(a) $$\frac{a}{2a^2+bc}+\frac{b}{2b^2+ca}+\frac{c}{2c^2+ab} \ge \frac{a+b+c}{a^2+b^2+c^2};$$

(b) $$\frac{b+c}{2a^2+bc}+\frac{c+a}{2b^2+ca}+\frac{a+b}{2c^2+ab} \ge \frac{6}{a+b+c}.$$

(Vasile Cîrtoaje, 2008)

Solution. Assume that

$$a \ge b \ge c.$$

(a) Multiplying by $a+b+c$, we can write the inequality as follows:

$$\sum \frac{a(a+b+c)}{2a^2+bc} \ge \frac{(a+b+c)^2}{a^2+b^2+c^2},$$

$$3-\frac{(a+b+c)^2}{a^2+b^2+c^2} \ge \sum\left[1-\frac{a(a+b+c)}{2a^2+bc}\right],$$

$$2\sum(a-b)(a-c) \ge (a^2+b^2+c^2)\sum\frac{(a-b)(a-c)}{2a^2+bc},$$

$$\sum\frac{3a^2-(b-c)^2}{2a^2+bc}(a-b)(a-c) \ge 0,$$

$$3f(a,b,c)+(a-b)(b-c)(c-a)g(a,b,c) \ge 0,$$

where

$$f(a,b,c)=\sum\frac{a^2(a-b)(a-c)}{2a^2+bc}, \quad g(a,b,c)=\sum\frac{b-c}{2a^2+bc}.$$

It suffices to show that $f(a,b,c) \ge 0$ and $g(a,b,c) \le 0$. We have

$$\begin{aligned}
f(a,b,c) &\ge \frac{a^2(a-b)(a-c)}{2a^2+bc}+\frac{b^2(b-a)(b-c)}{2b^2+ca}\\
&\ge \frac{a^2(a-b)(b-c)}{2a^2+bc}+\frac{b^2(b-a)(b-c)}{2b^2+ca}\\
&= \frac{a^2c(a-b)^2(b-c)(a^2+ab+b^2)}{(2a^2+bc)(2b^2+ca)} \ge 0.
\end{aligned}$$

Also,

$$\begin{aligned}
g(a,b,c) &= \frac{b-c}{2a^2+bc}-\frac{(a-b)+(b-c)}{2b^2+ca}+\frac{a-b}{2c^2+ab}\\
&= (a-b)\left(\frac{1}{2c^2+ab}-\frac{1}{2b^2+ca}\right)+(b-c)\left(\frac{1}{2a^2+bc}-\frac{1}{2b^2+ca}\right)\\
&= \frac{(a-b)(b-c)}{2b^2+ca}\left[\frac{2b+2c-a}{2c^2+ab}-\frac{2b+2a-c}{2a^2+bc}\right]=\\
&= \frac{2(a-b)(b-c)(c-a)(a^2+b^2+c^2-ab-bc-ca)}{(2a^2+bc)(2b^2+ca)(2c^2+ab)} \le 0.
\end{aligned}$$

The equality holds for $a=b=c$, and for $a=b$ and $c=0$ (or any cyclic permutation).

(b) We apply the SOS method. Write the inequality as follows:

$$\sum\left[\frac{(b+c)(a+b+c)}{2a^2+bc}-2\right]\geq 0,$$

$$\sum\frac{(b^2+ab-2a^2)+(c^2+ca-2a^2)}{2a^2+bc}\geq 0,$$

$$\sum\frac{(b-a)(b+2a)+(c-a)(c+2a)}{2a^2+bc}\geq 0,$$

$$\sum\frac{(b-a)(b+2a)}{2a^2+bc}+\sum\frac{(a-b)(a+2b)}{2b^2+ca}\geq 0,$$

$$\sum(a-b)\left(\frac{a+2b}{2b^2+ca}-\frac{b+2a}{2a^2+bc}\right)\geq 0,$$

$$\sum(a-b)^2(2c^2+ab)(a^2+b^2+3ab-ac-bc)\geq 0.$$

It suffices to show that

$$\sum(a-b)^2(2c^2+ab)(a^2+b^2+2ab-ac-bc)\geq 0,$$

which is equivalent to

$$\sum(a-b)^2(2c^2+ab)(a+b)(a+b-c)\geq 0.$$

This inequality is true if

$$(b-c)^2(2a^2+bc)(b+c)(b+c-a)+(c-a)^2(2b^2+ca)(c+a)(c+a-b)\geq 0;$$

that is,

$$(a-c)^2(2b^2+ca)(a+c)(a+c-b)\geq(b-c)^2(2a^2+bc)(b+c)(a-b-c).$$

Since

$$a+c\geq b+c,\qquad a+c-b\geq a-b-c,$$

it is enough to prove that

$$(a-c)^2(2b^2+ca)\geq(b-c)^2(2a^2+bc).$$

We can obtain this inequality by multiplying the inequalities

$$b^2(a-c)^2\geq a^2(b-c)^2$$

and

$$a^2(2b^2+ca)\geq b^2(2a^2+bc).$$

The equality holds for $a=b=c$, and for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 1.103. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a(b+c)}{a^2+bc}+\frac{b(c+a)}{b^2+ca}+\frac{c(a+b)}{c^2+ab}\ge\frac{(a+b+c)^2}{a^2+b^2+c^2}.$$

(Pham Huu Duc, 2006)

Solution. Assume that $a\ge b\ge c$ and write the inequality as follows:

$$3-\frac{(a+b+c)^2}{a^2+b^2+c^2}\ge\sum\left(1-\frac{ab+ac}{a^2+bc}\right),$$

$$2\sum(a-b)(a-c)\ge(a^2+b^2+c^2)\sum\frac{(a-b)(a-c)}{a^2+bc},$$

$$\sum\frac{(a-b)(a-c)(a+b-c)(a-b+c)}{a^2+bc}\ge 0.$$

It suffices to show that

$$\frac{(b-c)(b-a)(b+c-a)(b-c+a)}{b^2+ca}+\frac{(c-a)(c-b)(c+a-b)(c-a+b)}{c^2+ab}\ge 0,$$

which is equivalent to the obvious inequality

$$\frac{(b-c)^2(c-a+b)^2(a^2+bc)}{(b^2+ca)(c^2+ab)}\ge 0.$$

The equality holds for $a=b=c$, and for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 1.104. *Let a,b,c be nonnegative real numbers, no two of which are zero. If $k>0$, then*

$$\frac{b^2+c^2+\sqrt{3}bc}{a^2+kbc}+\frac{c^2+a^2+\sqrt{3}ca}{b^2+kca}+\frac{a^2+b^2+\sqrt{3}ab}{c^2+kab}\ge\frac{3(2+\sqrt{3})}{1+k}.$$

(Vasile Cîrtoaje, 2013)

Solution. We use the *highest coefficient method*. Write the inequality in the form $f_6(a,b,c)\ge 0$, where

$$f_6(a,b,c)=(1+k)\sum(b^2+c^2+\sqrt{3}bc)(b^2+kca)(c^2+kab)$$

$$-3(2+\sqrt{3}\,)(a^2+kbc)(b^2+kca)(c^2+kab).$$

Clearly, $f_6(a,b,c)$ has the same highest coefficient A as

$$(1+k)P_2(a,b,c)-3(2+\sqrt{3}\,)P_3(a,b,c),$$

where

$$P_2(a,b,c) = \sum(\sqrt{3}bc - a^2)(b^2 + kca)(c^2 + kab),$$

$$P_3(a,b,c) = (a^2 + kbc)(b^2 + kca)(c^2 + kab).$$

According to Remark 2 from the proof of P 2.75 in Volume 1, we have

$$\begin{aligned} A &= (1+k)P_2(1,1,1) - 3(2+\sqrt{3}\,)P_3(1,1,1) \\ &= 3(\sqrt{3}-1)(1+k)^3 - 3(2+\sqrt{3}\,)(1+k)^3 = -9(1+k)^3. \end{aligned}$$

Since $A \le 0$, according to P 3.76-(a) in Volume 1, it suffices to prove the original inequality for $b = c = 1$ and for $a = 0$.

In the first case ($b = c = 1$), the inequality is equivalent to

$$\frac{2+\sqrt{3}}{a^2+k} + \frac{2(a^2+\sqrt{3}a+1)}{ka+1} \ge \frac{3(2+\sqrt{3})}{1+k},$$

$$\frac{2(a^2+\sqrt{3}a+1)}{ka+1} \ge \frac{(2+\sqrt{3})(3a^2+2k-1)}{(k+1)(a^2+k)},$$

$$(a-1)^2\left[(k+1)a^2 - \left(1+\frac{\sqrt{3}}{2}\right)(k-2)a + \left(k - \frac{1+\sqrt{3}}{2}\right)^2\right] \ge 0.$$

For the nontrivial case $k > 2$, we have

$$(k+1)a^2 + \left(k - \frac{1+\sqrt{3}}{2}\right)^2 \ge 2\sqrt{k+1}\left(k - \frac{1+\sqrt{3}}{2}\right)a$$

$$\ge 2\sqrt{3}\left(k - \frac{1+\sqrt{3}}{2}\right)a \ge \left(1+\frac{\sqrt{3}}{2}\right)(k-2)a.$$

In the second case ($a = 0$), the original inequality can be written as

$$\frac{1}{k}\left(\frac{b}{c} + \frac{c}{b} + \sqrt{3}\right) + \left(\frac{b^2}{c^2} + \frac{c^2}{b^2}\right) \ge \frac{3(2+\sqrt{3})}{1+k}.$$

It suffices to show that

$$\frac{1}{k}(2+\sqrt{3}) + 2 \ge \frac{3(2+\sqrt{3})}{1+k},$$

which is equivalent to

$$\left(k - \frac{1+\sqrt{3}}{2}\right)^2 \ge 0.$$

The equality holds for $a = b = c$. If $k = \dfrac{1+\sqrt{3}}{2}$, then the equality holds also for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 1.105. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}+\frac{8}{a^2+b^2+c^2}\geq\frac{6}{ab+bc+ca}.$$

(Vasile Cîrtoaje, 2013)

Solution. Multiplying by $a^2+b^2+c^2$, the inequality becomes

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}+11\geq\frac{6(a^2+b^2+c^2)}{ab+bc+ca}.$$

Since

$$\left(\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}\right)(a^2b^2+b^2c^2+c^2a^2)=$$

$$=a^4+b^4+c^4+a^2b^2c^2\left(\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}\right)\geq a^4+b^4+c^4,$$

it suffices to show that

$$\frac{a^4+b^4+c^4}{a^2b^2+b^2c^2+c^2a^2}+11\geq\frac{6(a^2+b^2+c^2)}{ab+bc+ca},$$

which is equivalent to

$$\frac{(a^2+b^2+c^2)^2}{a^2b^2+b^2c^2+c^2a^2}+9\geq\frac{6(a^2+b^2+c^2)}{ab+bc+ca}.$$

Clearly, it is enough to prove that

$$\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^2+9\geq\frac{6(a^2+b^2+c^2)}{ab+bc+ca},$$

which is

$$\left(\frac{a^2+b^2+c^2}{ab+bc+ca}-3\right)^2\geq 0.$$

The equality holds for $a=0$ and $\dfrac{b}{c}+\dfrac{c}{b}=3$ (or any cyclic permutation).

□

P 1.106. *If a, b, c are the lengths of the sides of a triangle, then*

$$\frac{a(b+c)}{a^2+2bc}+\frac{b(c+a)}{b^2+2ca}+\frac{c(a+b)}{c^2+2ab}\leq 2.$$

(Vo Quoc Ba Can and Vasile Cîrtoaje, 2010)

Solution. Write the inequality as

$$\sum\left(1-\frac{ab+ac}{a^2+2bc}\right)\ge 1,$$

$$\sum\frac{a^2+2bc-ab-ac}{a^2+2bc}\ge 1.$$

Since

$$a^2+2bc-ab-ac=bc-(a-c)(b-a)\ge |a-c||b-a|-(a-c)(b-a)\ge 0,$$

by the Cauchy-Schwarz inequality, we have

$$\sum\frac{a^2+2bc-ab-ac}{a^2+2bc}\ge\frac{\left[\sum(a^2+2bc-ab-ac)\right]^2}{\sum(a^2+2bc)(a^2+2bc-ab-ac)}.$$

Thus, it suffices to prove that

$$(a^2+b^2+c^2)^2\ge\sum(a^2+2bc)(a^2+2bc-ab-ac),$$

which reduces to the obvious inequality

$$ab(a-b)^2+bc(b-c)^2+ca(c-a)^2\ge 0.$$

The equality holds for an equilateral triangle, and for a degenerate triangle with $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.107. *If a,b,c are real numbers, then*

$$\frac{a^2-bc}{2a^2+b^2+c^2}+\frac{b^2-ca}{2b^2+c^2+a^2}+\frac{c^2-ab}{2c^2+a^2+b^2}\ge 0.$$

(Nguyen Anh Tuan, 2005)

First Solution. Rewrite the inequality as

$$\sum\left(\frac{1}{2}-\frac{a^2-bc}{2a^2+b^2+c^2}\right)\le\frac{3}{2},$$

$$\sum\frac{(b+c)^2}{2a^2+b^2+c^2}\le 3.$$

If two of a,b,c are zero, then the inequality is trivial. Otherwise, applying the Cauchy-Schwarz inequality, we get

$$\sum\frac{(b+c)^2}{2a^2+b^2+c^2}=\sum\frac{(b+c)^2}{(a^2+b^2)+(a^2+c^2)}\le\sum\left(\frac{b^2}{a^2+b^2}+\frac{c^2}{a^2+c^2}\right)$$
$$=\sum\frac{b^2}{a^2+b^2}+\sum\frac{a^2}{b^2+a^2}=3.$$

The equality holds for $a = b = c$.

Second Solution. Use the SOS method. We have

$$
\begin{aligned}
2\sum \frac{a^2 - bc}{2a^2 + b^2 + c^2} &= \sum \frac{(a-b)(a+c) + (a-c)(a+b)}{2a^2 + b^2 + c^2} \\
&= \sum \frac{(a-b)(a+c)}{2a^2 + b^2 + c^2} + \sum \frac{(b-a)(b+c)}{2b^2 + c^2 + a^2} \\
&= \sum (a-b)\left(\frac{a+c}{2a^2 + b^2 + c^2} - \frac{b+c}{2b^2 + c^2 + a^2}\right) \\
&= (a^2 + b^2 + c^2 - ab - bc - ca)\sum \frac{(a-b)^2}{(2a^2 + b^2 + c^2)(2b^2 + c^2 + a^2)} \ge 0.
\end{aligned}
$$

□

P 1.108. *If a, b, c are nonnegative real numbers, then*

$$\frac{3a^2 - bc}{2a^2 + b^2 + c^2} + \frac{3b^2 - ca}{2b^2 + c^2 + a^2} + \frac{3c^2 - ab}{2c^2 + a^2 + b^2} \le \frac{3}{2}.$$

(Vasile Cîrtoaje, 2008)

First Solution. Write the inequality as

$$\sum \left(\frac{3}{2} - \frac{3a^2 - bc}{2a^2 + b^2 + c^2}\right) \ge 3,$$

$$\sum \frac{8bc + 3(b-c)^2}{2a^2 + b^2 + c^2} \ge 6.$$

By the Cauchy-Schwarz inequality, we have

$$8bc + 3(b-c)^2 \ge \frac{[4bc + (b-c)^2]^2}{2bc + \frac{1}{3}(b-c)^2} = \frac{2(b+c)^4}{b^2 + c^2 + 4bc}.$$

Therefore, it suffices to prove that

$$\sum \frac{(b+c)^4}{(2a^2 + b^2 + c^2)(b^2 + c^2 + 4bc)} \ge 2.$$

Using again the Cauchy-Schwarz inequality, we get

$$\sum \frac{(b+c)^4}{(2a^2 + b^2 + c^2)(b^2 + c^2 + 4bc)} \ge \frac{\left[\sum (b+c)^2\right]^2}{\sum (2a^2 + b^2 + c^2)(b^2 + c^2 + 4bc)} = 2.$$

The equality holds for $a = b = c$, for $a = 0$ and $b = c$ (or any cyclic permutation), and for $b = c = 0$ (or any cyclic permutation).

Second Solution. Use the SOS method. Write the inequality as

$$\sum\left(\frac{1}{2}-\frac{3a^2-bc}{2a^2+b^2+c^2}\right)\geq 0,$$

$$\sum\frac{(b+c+2a)(b+c-2a)}{2a^2+b^2+c^2}\geq 0,$$

$$\sum\frac{(b+c+2a)(b-a)+(b+c+2a)(c-a)}{2a^2+b^2+c^2}\geq 0,$$

$$\sum\frac{(b+c+2a)(b-a)}{2a^2+b^2+c^2}+\sum\frac{(c+a+2b)(a-b)}{2b^2+c^2+a^2}\geq 0,$$

$$\sum(a-b)\left(\frac{c+a+2b}{2b^2+c^2+a^2}-\frac{b+c+2a}{2a^2+b^2+c^2}\right)\geq 0,$$

$$\sum(3ab+bc+ca-c^2)(2c^2+a^2+b^2)(a-b)^2\geq 0.$$

Clearly, it suffices to show that

$$\sum c(a+b-c)(2c^2+a^2+b^2)(a-b)^2\geq 0.$$

Assume that $a\geq b\geq c$. It is enough to prove that

$$a(b+c-a)(2a^2+b^2+c^2)(b-c)^2+b(c+a-b)(2b^2+c^2+a^2)(c-a)^2\geq 0;$$

that is,

$$b(c+a-b)(2b^2+c^2+a^2)(a-c)^2\geq a(a-b-c)(2a^2+b^2+c^2)(b-c)^2.$$

Since $c+a-b\geq a-b-c$, it suffices to prove that

$$b(2b^2+c^2+a^2)(a-c)^2\geq a(2a^2+b^2+c^2)(b-c)^2.$$

We can obtain this inequality by multiplying the inequalities

$$b^2(a-c)^2\geq a^2(b-c)^2$$

and

$$a(2b^2+c^2+a^2)\geq b(2a^2+b^2+c^2).$$

The last inequality is equivalent to

$$(a-b)[(a-b)^2+ab+c^2]\geq 0.$$

□

P 1.109. *If a, b, c are nonnegative real numbers, then*

$$\frac{(b+c)^2}{4a^2+b^2+c^2}+\frac{(c+a)^2}{4b^2+c^2+a^2}+\frac{(a+b)^2}{4c^2+a^2+b^2}\ge 2.$$

(Vasile Cîrtoaje, 2005)

Solution. By the Cauchy-Schwarz inequality, we have

$$\sum\frac{(b+c)^2}{4a^2+b^2+c^2}\ge\frac{\left[\sum(b+c)^2\right]^2}{\sum(b+c)^2(4a^2+b^2+c^2)}$$

$$=2\text{ḑ}ot\frac{\sum a^4+3\sum a^2b^2+4abc\sum a+2\sum ab(a^2+b^2)}{\sum a^4+5\sum a^2b^2+4abc\sum a+\sum ab(a^2+b^2)}\ge 2$$

because

$$\sum ab(a^2+b^2)\ge 2\sum a^2b^2.$$

The equality holds for $a=b=c$, and for $b=c=0$ (or any cyclic permutation).

□

P 1.110. *If a, b, c are positive real numbers, then*

(a)
$$\sum\frac{1}{11a^2+2b^2+2c^2}\le\frac{3}{5(ab+bc+ca)};$$

(b)
$$\sum\frac{1}{4a^2+b^2+c^2}\le\frac{1}{2(a^2+b^2+c^2)}+\frac{1}{ab+bc+ca}.$$

(Vasile Cîrtoaje, 2008)

Solution. We will prove that

$$\sum\frac{k+2}{ka^2+b^2+c^2}\le\frac{11-2k}{a^2+b^2+c^2}+\frac{2(k-1)}{ab+bc+ca}$$

for any $k>1$. Due to homogeneity, we may assume that $a^2+b^2+c^2=3$. On this hypothesis, we need to show that

$$\sum\frac{k+2}{(k-1)a^2+3}\le\frac{11-2k}{3}+\frac{2(k-1)}{ab+bc+ca}.$$

Using the substitution $m=3/(k-1)$, $m>0$, the inequality can be written as

$$m(m+1)\sum\frac{1}{a^2+m}\le 3m-2+\frac{6}{ab+bc+ca}.$$

By the Cauchy-Schwarz inequality, we have

$$(a^2+m)[m+(m+1-a)^2] \ge [a\sqrt{m}+\sqrt{m}(m+1-a)]^2 = m(m+1)^2,$$

and hence

$$\frac{m(m+1)}{a^2+m} \le \frac{a^2-1}{m+1}+m+2-2a,$$

$$m(m+1)\sum\frac{1}{a^2+m} \le 3(m+2)-2\sum a.$$

Thus, it suffices to show that

$$3(m+2)-2\sum a \le 3m-2+\frac{6}{ab+bc+ca};$$

that is,

$$(4-a-b-c)(ab+bc+ca) \le 3.$$

Let $p=a+b+c$. Since

$$2(ab+bc+ca)=(a+b+c)^2-(a^2+b^2+c^2)=p^2-3,$$

we get

$$\begin{aligned}6-2(4-a-b-c)(ab+bc+ca) &= 6-(4-p)(p^2-3)\\ &= (p-3)^2(p+2) \ge 0.\end{aligned}$$

This completes the proof. The equality holds for $a=b=c$.

□

P 1.111. *If a, b, c are nonnegative real numbers such that $ab+bc+ca=3$, then*

$$\frac{\sqrt{a}}{b+c}+\frac{\sqrt{b}}{c+a}+\frac{\sqrt{c}}{a+b} \ge \frac{3}{2}.$$

(Vasile Cîrtoaje, 2006)

Solution. By the Cauchy-Schwarz inequality, we have

$$\sum\frac{\sqrt{a}}{b+c} \ge \frac{\left(\sum a^{3/4}\right)^2}{\sum a(b+c)} = \frac{1}{6}\left(\sum a^{3/4}\right)^2.$$

Thus, it suffices to show that

$$a^{3/4}+b^{3/4}+c^{3/4} \ge 3,$$

which follows immediately from Remark 1 from the proof of the inequality in P 3.33 in Volume 1. The equality occurs for $a=b=c=1$.

Remark. Analogously, according to Remark 2 from the proof of P 3.33 in Volume 1, we can prove that

$$\frac{a^k}{b+c}+\frac{b^k}{c+a}+\frac{c^k}{a+b}\ge\frac{3}{2}$$

for all $k\ge 3-\dfrac{4\ln 2}{\ln 3}\approx 0.476$. For $k=3-\dfrac{4\ln 2}{\ln 3}$, the equality occurs for $a=b=c=1$, and also for $a=0$ and $b=c=\sqrt{3}$ (or any cyclic permutation).

□

P 1.112. *If a,b,c are nonnegative real numbers such that $ab+bc+ca\ge 3$, then*

$$\frac{1}{2+a}+\frac{1}{2+b}+\frac{1}{2+c}\ge\frac{1}{1+b+c}+\frac{1}{1+c+a}+\frac{1}{1+a+b}.$$

(Vasile Cîrtoaje, 2014)

Solution. Consider $c=\min\{a,b,c\}$, and denote

$$E(a,b,c)=\frac{1}{2+a}+\frac{1}{2+b}+\frac{1}{2+c}-\frac{1}{1+b+c}-\frac{1}{1+c+a}-\frac{1}{1+a+b}.$$

If $c\ge 1$, the desired inequality $E(a,b,c)\ge 0$ follows by summing the obvious inequalities

$$\frac{1}{2+a}\ge\frac{1}{1+c+a},$$
$$\frac{1}{2+b}\ge\frac{1}{1+a+b},$$
$$\frac{1}{2+c}\ge\frac{1}{1+b+c}.$$

Consider further that $c<1$. From

$$E(a,b,c)=-\frac{1-c}{(2+a)(1+c+a)}-\frac{1}{1+a+b}+\frac{1}{2+b}+\frac{1}{2+c}-\frac{1}{1+b+c}$$

and

$$E(a,b,c)=-\frac{1-c}{(2+b)(1+b+c)}-\frac{1}{1+a+b}+\frac{1}{2+a}+\frac{1}{2+c}-\frac{1}{1+c+a},$$

it follows that $E(a,b,c)$ is increasing in a and b. Based on this result, it suffices to prove the desired inequality only for

$$ab+bc+ca=3.$$

Applying the AM-GM inequality, we get

$$3=ab+bc+ca\ge 3(abc)^{2/3},\quad abc\le 1,$$

$$a+b+c \ge 3\sqrt[3]{abc} \ge 3.$$

We will show that

$$\frac{1}{2+a}+\frac{1}{2+b}+\frac{1}{2+c} \ge 1 \ge \frac{1}{1+b+c}+\frac{1}{1+c+a}+\frac{1}{1+a+b}.$$

By direct calculation, we can show that the left inequality is equivalent to $abc \le 1$, while the right inequality is equivalent to $a+b+c \ge 2+abc$. Clearly, these are true and the proof is completed. The equality occurs for $a=b=c=1$.

□

P 1.113. *If a, b, c are the lengths of the sides of a triangle, then*

(a) $$\frac{a^2-bc}{3a^2+b^2+c^2}+\frac{b^2-ca}{3b^2+c^2+a^2}+\frac{c^2-ab}{3c^2+a^2+b^2} \le 0;$$

(b) $$\frac{a^4-b^2c^2}{3a^4+b^4+c^4}+\frac{b^4-c^2a^2}{3b^4+c^4+a^4}+\frac{c^4-a^2b^2}{3c^4+a^4+b^4} \le 0.$$

(Nguyen Anh Tuan and Vasile Cîrtoaje, 2006)

Solution. (a) Apply the SOS method. We have

$$2\sum \frac{a^2-bc}{3a^2+b^2+c^2} = \sum \frac{(a-b)(a+c)+(a-c)(a+b)}{3a^2+b^2+c^2}$$

$$= \sum \frac{(a-b)(a+c)}{3a^2+b^2+c^2}+\sum \frac{(b-a)(b+c)}{3b^2+c^2+a^2}$$

$$= \sum (a-b)\left(\frac{a+c}{3a^2+b^2+c^2}-\frac{b+c}{3b^2+c^2+a^2}\right)$$

$$= (a^2+b^2+c^2-2ab-2bc-2ca)\sum \frac{(a-b)^2}{(3a^2+b^2+c^2)(3b^2+c^2+a^2)}.$$

Since

$$a^2+b^2+c^2-2ab-2bc-2ca = a(a-b-c)+b(b-c-a)+c(c-a-b) \le 0,$$

the conclusion follows. The equality holds for an equilateral triangle, and for a degenerate triangle with $a=0$ and $b=c$ (or any cyclic permutation).

(b) Using the same way as above, we get

$$2\sum \frac{a^4-b^2c^2}{3a^4+b^4+c^4} = A\sum \frac{(a^2-b^2)^2}{(3a^4+b^4+c^4)(3b^4+c^4+a^4)},$$

where

$$\begin{aligned}A &= a^4+b^4+c^4-2a^2b^2-2b^2c^2-2c^2a^2\\ &= -(a+b+c)(a+b-c)(b+c-a)(c+a-b)\le 0.\end{aligned}$$

The equality holds for an equilateral triangle, and for a degenerate triangle with $a=b+c$ (or any cyclic permutation).

□

P 1.114. *If a,b,c are the lengths of the sides of a triangle, then*

$$\frac{bc}{4a^2+b^2+c^2}+\frac{ca}{4b^2+c^2+a^2}+\frac{ab}{4c^2+a^2+b^2}\ge\frac{1}{2}.$$

(Vasile Cîrtoaje and Vo Quoc Ba Can, 2010)

Solution. We apply the SOS method. Write the inequality as

$$\sum\left(\frac{2bc}{4a^2+b^2+c^2}-\sum\frac{b^2c^2}{a^2b^2+b^2c^2+c^2a^2}\right)\ge 0,$$

$$\sum\frac{bc(2a^2-bc)(b-c)^2}{4a^2+b^2+c^2}\ge 0.$$

Without loss of generality, assume that $a\ge b\ge c$. Then, it suffices to prove that

$$\frac{c(2b^2-ca)(c-a)^2}{4b^2+c^2+a^2}+\frac{b(2c^2-ab)(a-b)^2}{4c^2+a^2+b^2}\ge 0.$$

Since

$$2b^2-ca\ge c(b+c)-ca=c(b+c-a)\ge 0$$

and

$$\begin{aligned}(2b^2-ca)+(2c^2-ab) &= 2(b^2+c^2)-a(b+c)\ge (b+c)^2-a(b+c)\\ &= (b+c)(b+c-a)\ge 0,\end{aligned}$$

it is enough to show that

$$\frac{c(a-c)^2}{4b^2+c^2+a^2}\ge\frac{b(a-b)^2}{4c^2+a^2+b^2}.$$

This follows by multiplying the inequalities

$$c^2(a-c)^2\ge b^2(a-b)^2$$

and

$$\frac{b}{4b^2+c^2+a^2}\ge\frac{c}{4c^2+a^2+b^2}.$$

These inequalities are true, since

$$c(a-c)-b(a-b)=(b-c)(b+c-a)\geq 0,$$

$$b(4c^2+a^2+b^2)-c(4b^2+c^2+a^2)=(b-c)[(b-c)^2+a^2-bc]\geq 0.$$

The equality occurs for an equilateral triangle, and for a degenerate triangle with $a=b$ and $c=0$ (or any cyclic permutation).

□

P 1.115. *If a,b,c are the lengths of the sides of a triangle, then*

$$\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}+\frac{1}{a^2+b^2}\leq\frac{9}{2(ab+bc+ca)}.$$

(Vo Quoc Ba Can, 2008)

Solution. Apply the SOS method. Write the inequality as

$$\sum\left[\frac{3}{2}-\frac{ab+bc+ca}{b^2+c^2}\right]\geq 0,$$

$$\sum\frac{3(b^2+c^2)-2(ab+bc+ca)}{b^2+c^2}\geq 0,$$

$$\sum\frac{3b(b-a)+3c(c-a)+c(a-b)+b(a-c)}{b^2+c^2}\geq 0,$$

$$\sum\frac{(a-b)(c-3b)+(a-c)(b-3c)}{b^2+c^2}\geq 0,$$

$$\sum\frac{(a-b)(c-3b)}{b^2+c^2}+\sum\frac{(b-a)(c-3a)}{c^2+a^2}\geq 0,$$

$$\sum(a^2+b^2)(a-b)^2(ca+cb+3c^2-3ab)\geq 0.$$

Without loss of generality, assume that $a\geq b\geq c$. Since

$$ab+ac+3a^2-3bc>0,$$

it suffices to prove that

$$(a^2+b^2)(a-b)^2(ca+cb+3c^2-3ab)+(a^2+c^2)(a-c)^2(ab+bc+3b^2-3ac)\geq 0,$$

or, equivalently,

$$(a^2+c^2)(a-c)^2(ab+bc+3b^2-3ac)\geq(a^2+b^2)(a-b)^2(3ab-3c^2-ca-cb).$$

Since

$$\begin{aligned} ab + bc + 3b^2 - 3ac &= a\left(\frac{bc + 3b^2}{a} + b - 3c\right) \\ &\geq a\left(\frac{bc + 3b^2}{b + c} + b - 3c\right) \\ &= \frac{a(b - c)(4b + 3c)}{b + c} \geq 0 \end{aligned}$$

and

$$\begin{aligned} (ab + bc + 3b^2 - 3ac) - (3ab - 3c^2 - ca - cb) &= 3(b^2 + c^2) + 2bc - 2a(b + c) \\ &\geq 3(b^2 + c^2) + 2bc - 2(b + c)^2 \\ &= (b - c)^2 \geq 0, \end{aligned}$$

it suffices to show that

$$(a^2 + c^2)(a - c)^2 \geq (a^2 + b^2)(a - b)^2.$$

This is equivalent to $(b - c)A \geq 0$, where

$$\begin{aligned} A &= 2a^3 - 2a^2(b + c) + 2a(b^2 + bc + c^2) - (b + c)(b^2 + c^2) \\ &= 2a\left(a - \frac{b + c}{2}\right)^2 + \frac{a(3b^2 + 2bc + 3c^2)}{2} - (b + c)(b^2 + c^2) \\ &\geq \frac{b(3b^2 + 2bc + 3c^2)}{2} - (b + c)(b^2 + c^2) \\ &= \frac{(b - c)(b^2 + bc + 2c^2)}{2} \geq 0. \end{aligned}$$

The equality occurs for an equilateral triangle, and for a degenerate triangle with $a/2 = b = c$ (or any cyclic permutation).

□

P 1.116. *If a, b, c are the lengths of the sides of a triangle, then*

(a)
$$\left|\frac{a + b}{a - b} + \frac{b + c}{b - c} + \frac{c + a}{c - a}\right| > 5;$$

(b)
$$\left|\frac{a^2 + b^2}{a^2 - b^2} + \frac{b^2 + c^2}{b^2 - c^2} + \frac{c^2 + a^2}{c^2 - a^2}\right| \geq 3.$$

(Vasile Cîrtoaje, 2003)

Solution. Since the inequalities are symmetric, we consider

$$a > b > c.$$

(a) Let $x = a - c$ and $y = b - c$. From $a > b > c$ and $a \le b + c$, it follows

$$x > y > 0, \qquad c \ge x - y.$$

We have

$$\begin{aligned}
\frac{a+b}{a-b} + \frac{b+c}{b-c} + \frac{c+a}{c-a} &= \frac{2c+x+y}{x-y} + \frac{2c+y}{y} - \frac{2c+x}{x} \\
&= 2c\left(\frac{1}{x-y} + \frac{1}{y} - \frac{1}{x}\right) + \frac{x+y}{x-y} \\
&> \frac{2c}{y} + \frac{x+y}{x-y} \ge \frac{2(x-y)}{y} + \frac{x+y}{x-y} \\
&= 2\left(\frac{x-y}{y} + \frac{y}{x-y}\right) + 1 \ge 5.
\end{aligned}$$

(b) We will show that

$$\frac{a^2+b^2}{a^2-b^2} + \frac{b^2+c^2}{b^2-c^2} + \frac{c^2+a^2}{c^2-a^2} \ge 3;$$

that is,

$$\frac{b^2}{a^2-b^2} + \frac{c^2}{b^2-c^2} \ge \frac{a^2}{a^2-c^2}.$$

Since

$$\frac{a^2}{a^2-c^2} \le \frac{(b+c)^2}{a^2-c^2},$$

it suffices to prove that

$$\frac{b^2}{a^2-b^2} + \frac{c^2}{b^2-c^2} \ge \frac{(b+c)^2}{a^2-c^2}.$$

This is equivalent to each of the following inequalities:

$$b^2\left(\frac{1}{a^2-b^2} - \frac{1}{a^2-c^2}\right) + c^2\left(\frac{1}{b^2-c^2} - \frac{1}{a^2-c^2}\right) \ge \frac{2bc}{a^2-c^2},$$

$$\frac{b^2(b^2-c^2)}{a^2-b^2} + \frac{c^2(a^2-b^2)}{b^2-c^2} \ge 2bc,$$

$$[b(b^2-c^2) - c(a^2-b^2)]^2 \ge 0.$$

This completes the proof. If $a > b > c$, then the equality holds for a degenerate triangle with $a = b + c$ and $b/c = x_1$, where $x_1 \approx 1.5321$ is the positive root of the equation $x^3 - 3x - 1 = 0$.

□

P 1.117. *If a, b, c are the lengths of the sides of a triangle, then*

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}+3 \geq 6\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right).$$

Solution. We apply the SOS method. Write the inequality as

$$\sum \frac{b+c}{a}-6 \geq 3\left(\sum \frac{2a}{b+c}-3\right).$$

Since

$$\sum \frac{b+c}{a}-6=\sum\left(\frac{b}{c}+\frac{c}{b}\right)-6=\sum \frac{(b-c)^2}{bc}$$

and

$$\begin{aligned}\sum \frac{2a}{b+c}-3 &= \sum \frac{2a-b-c}{b+c}=\sum \frac{a-b}{b+c}+\sum \frac{a-c}{b+c} \\ &= \sum \frac{a-b}{b+c}+\sum \frac{b-a}{c+a}=\sum \frac{(a-b)^2}{(b+c)(c+a)} \\ &= \sum \frac{(b-c)^2}{(c+a)(a+b)},\end{aligned}$$

we can rewrite the inequality as

$$\sum a(b+c)(b-c)^2 S_a \geq 0,$$

where

$$S_a = a(a+b+c)-2bc.$$

Without loss of generality, assume that $a \geq b \geq c$. Since $S_a > 0$,

$$S_b = b(a+b+c)-2ca=(b-c)(a+b+c)+c(b+c-a) \geq 0$$

and

$$\begin{aligned}\sum a(b+c)(b-c)^2 S_a &\geq b(c+a)(c-a)^2 S_b + c(a+b)(a-b)^2 S_c \\ &\geq (a-b)^2[b(c+a)S_b + c(a+b)S_c],\end{aligned}$$

it suffices to prove that

$$b(c+a)S_b + c(a+b)S_c \geq 0.$$

This is equivalent to each of the following inequalities

$$(a+b+c)[a(b^2+c^2)+bc(b+c)] \geq 2abc(2a+b+c),$$

$$a(a+b+c)(b-c)^2+(a+b+c)[2abc+bc(b+c)] \geq 2abc(2a+b+c),$$

$$a(a+b+c)(b-c)^2+bc(2a+b+c)(b+c-a) \geq 0.$$

Since the last inequality is true, the proof is completed. The equality occurs for an equilateral triangle, and for a degenerate triangle with $a/2 = b = c$ (or any cyclic permutation).

$\square$

P 1.118. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\sum \frac{3a(b+c)-2bc}{(b+c)(2a+b+c)} \ge \frac{3}{2}.$$

(Vasile Cîrtoaje, 2009)

Solution. Use the SOS method. Write the inequality as follows:

$$\sum \left[\frac{3a(b+c)-2bc}{(b+c)(2a+b+c)} - \frac{1}{2}\right] \ge 0,$$

$$\sum \frac{4a(b+c)-6bc-b^2-c^2}{(b+c)(2a+b+c)} \ge 0,$$

$$\sum \frac{b(a-b)+c(a-c)+3b(a-c)+3c(a-b)}{(b+c)(2a+b+c)} \ge 0,$$

$$\sum \frac{(a-b)(b+3c)+(a-c)(c+3b)}{(b+c)(2a+b+c)} \ge 0,$$

$$\sum \frac{(a-b)(b+3c)}{(b+c)(2a+b+c)} + \sum \frac{(b-a)(a+3c)}{(c+a)(2b+c+a)} \ge 0,$$

$$\sum (a-b)\left[\frac{b+3c}{(b+c)(2a+b+c)} - \frac{a+3c}{(c+a)(2b+c+a)}\right] \ge 0,$$

$$(a-b)(b-c)(c-a)\sum (a^2-b^2)(a+b+2c) \ge 0.$$

Since

$$\sum (a^2-b^2)(a+b+2c) = (a-b)(b-c)(c-a),$$

the conclusion follows. The equality holds for $a=b$, or $b=c$, or $c=a$.

$\square$

P 1.119. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\sum \frac{a(b+c)-2bc}{(b+c)(3a+b+c)} \ge 0.$$

(Vasile Cîrtoaje, 2009)

Solution. We apply the SOS method. Since

$$\sum \frac{a(b+c)-2bc}{(b+c)(3a+b+c)} = \sum \frac{b(a-c)+c(a-b)}{(b+c)(3a+b+c)}$$

$$= \sum \frac{c(b-a)}{(c+a)(3b+c+a)} + \sum \frac{c(a-b)}{(b+c)(3a+b+c)}$$

$$= \sum \frac{c(a+b-c)(a-b)^2}{(b+c)(c+a)(3a+b+c)(3b+c+a)},$$

the inequality is equivalent to

$$\sum c(a+b)(3c+a+b)(a+b-c)(a-b)^2 \geq 0.$$

Without loss of generality, assume that $a \geq b \geq c$. Since $a+b-c \geq 0$, it suffices to show that

$$b(c+a)(3b+c+a)(c+a-b)(a-c)^2 \geq a(b+c)(3a+b+c)(a-b-c)(b-c)^2.$$

This is true since

$$c+a-b \geq a-b-c,$$

$$b^2(a-c)^2 \geq a^2(b-c)^2,$$

$$c+a \geq b+c,$$

$$a(3b+c+a) \geq b(3a+b+c).$$

The equality holds for $a=b=c$, and for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 1.120. *Let a,b,c be positive real numbers such that $a^2+b^2+c^2 \geq 3$. Prove that*

$$\frac{a^5-a^2}{a^5+b^2+c^2} + \frac{b^5-b^2}{b^5+c^2+a^2} + \frac{c^5-c^2}{c^5+a^2+b^2} \geq 0.$$

(Vasile Cîrtoaje, 2005)

Solution. The inequality is equivalent to

$$\frac{1}{a^5+b^2+c^2} + \frac{1}{b^5+c^2+a^2} + \frac{1}{c^5+a^2+b^2} \leq \frac{3}{a^2+b^2+c^2}.$$

Setting $a=tx$, $b=ty$ and $c=tz$, where

$$x,y,z>0, \qquad x^2+y^2+z^2=3,$$

the condition $a^2+b^2+c^2 \geq 3$ implies $t \geq 1$, and the inequality becomes

$$\frac{1}{t^3x^5+y^2+z^2} + \frac{1}{t^3y^5+z^2+x^2} + \frac{1}{t^3z^5+x^2+y^2} \leq 1.$$

We see that it suffices to prove this inequality for $t=1$, when it becomes

$$\frac{1}{x^5-x^2+3} + \frac{1}{y^5-y^2+3} + \frac{1}{z^5-z^2+3} \leq 1.$$

Without loss of generality, assume that $x \ge y \ge z$. There are two cases to consider.

Case 1: $z \le y \le x \le \sqrt{2}$. The desired inequality follows by adding the inequalities

$$\frac{1}{x^5-x^2+3} \le \frac{3-x^2}{6}, \quad \frac{1}{y^5-y^2+3} \le \frac{3-y^2}{6}, \quad \frac{1}{z^5-z^2+3} \le \frac{3-z^2}{6}.$$

We have

$$\frac{1}{x^5-x^2+3} - \frac{3-x^2}{6} = \frac{(x-1)^2(x^5+2x^4-3x^2-6x-3)}{6(x^5-x^2+3)} \le 0$$

since

$$\begin{aligned} x^5+2x^4-3x^2-6x-3 &= x^2\left(x^3+2x^2-3-\frac{6}{x}-\frac{3}{x^2}\right) \\ &\le x^2\left(2\sqrt{2}+4-3-3\sqrt{2}-\frac{3}{2}\right) \\ &= -x^2(\sqrt{2}+\frac{1}{2}) < 0. \end{aligned}$$

Case 2: $x > \sqrt{2}$. From $x^2+y^2+z^2=3$, it follows that $y^2+z^2<1$. Since

$$\frac{1}{x^5-x^2+3} < \frac{1}{(2\sqrt{2}-1)x^2+3} < \frac{1}{2(2\sqrt{2}-1)+3} < \frac{1}{6}$$

and

$$\frac{1}{y^5-y^2+3} + \frac{1}{z^5-z^2+3} < \frac{1}{3-y^2} + \frac{1}{3-z^2},$$

it suffices to prove that

$$\frac{1}{3-y^2} + \frac{1}{3-z^2} \le \frac{5}{6}.$$

Indeed, we have

$$\frac{1}{3-y^2} + \frac{1}{3-z^2} - \frac{5}{6} = \frac{9(y^2+z^2-1)-5y^2z^2}{6(3-y^2)(3-z^2)} < 0,$$

which completes the proof. The equality occurs for $a=b=c=1$.

Remark. Since $abc \ge 1$ involves $a^2+b^2+c^2 \ge 3\sqrt[3]{a^2b^2c^2} \ge 3$, the inequality is also true under the condition $abc \ge 1$. A proof of this inequality (which is a problem from IMO-2005 - proposed by *Hojoo Lee*) is the following:

$$\sum \frac{a^5-a^2}{a^5+b^2+c^2} \ge \sum \frac{a^5-a^2}{a^5+a^3(b^2+c^2)} = \frac{1}{a^2+b^2+c^2}\sum\left(a^2-\frac{1}{a}\right),$$

$$\sum\left(a^2-\frac{1}{a}\right) \ge \sum(a^2-bc) = \frac{1}{2}\sum(a-b)^2 \ge 0.$$

□

P 1.121. *Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = a^3 + b^3 + c^3$. Prove that*

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \ge \frac{3}{2}.$$

(Pham Huu Duc, 2008)

First Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a^2}{b+c} \ge \frac{\left(\sum a^3\right)^2}{\sum a^4(b+c)} = \frac{\left(\sum a^3\right)\left(\sum a^2\right)}{\left(\sum a^3\right)\left(\sum ab\right) - abc\sum a^2}.$$

Therefore, it is enough to show that

$$2\left(\sum a^3\right)\left(\sum a^2\right) + 3abc\sum a^2 \ge 3\left(\sum a^3\right)\left(\sum ab\right).$$

Write this inequality as follows:

$$3\left(\sum a^3\right)\left(\sum a^2 - \sum ab\right) - \left(\sum a^3 - 3abc\right)\left(\sum a^2\right) \ge 0,$$

$$3\left(\sum a^3\right)\left(\sum a^2 - \sum ab\right) - \left(\sum a\right)\left(\sum a^2 - \sum ab\right)\left(\sum a^2\right) \ge 0,$$

$$\left(\sum a^2 - \sum ab\right)\left[3\sum a^3 - \left(\sum a\right)\left(\sum a^2\right)\right] \ge 0.$$

The last inequality is true since

$$2\left(\sum a^2 - \sum ab\right) = \sum (a-b)^2 \ge 0$$

and

$$\begin{aligned} 3\sum a^3 - \left(\sum a\right)\left(\sum a^2\right) &= \sum (a^3 + b^3) - \sum ab(a+b) \\ &= \sum (a+b)(a-b)^2 \ge 0. \end{aligned}$$

The equality occurs for $a = b = c = 1$.

Second Solution. Write the inequality in the homogeneous form $A \ge B$, where

$$A = 2\sum \frac{a^2}{b+c} - \sum a, \quad B = \frac{3(a^3 + b^3 + c^3)}{a^2 + b^2 + c^2} - \sum a.$$

Since

$$\begin{aligned} A = \sum \frac{a(a-b) + a(a-c)}{b+c} &= \sum \frac{a(a-b)}{b+c} + \sum \frac{b(b-a)}{c+a} \\ &= (a+b+c)\sum \frac{(a-b)^2}{(b+c)(c+a)} \end{aligned}$$

and
$$B=\frac{\sum(a^3+b^3)-\sum ab(a+b)}{a^2+b^2+c^2}=\frac{\sum(a+b)(a-b)^2}{a^2+b^2+c^2},$$
we can write the inequality as
$$\sum\left[\frac{a+b+c}{(b+c)(c+a)}-\frac{a+b}{a^2+b^2+c^2}\right](a-b)^2\ge 0,$$
$$(a^3+b^3+c^3-2abc)\sum\frac{(a-b)^2}{(b+c)(c+a)}\ge 0.$$
Since $a^3+b^3+c^3\ge 3abc$, the conclusion follows.

□

P 1.122. *If* $a,b,c\in[0,1]$, *then*
$$\frac{a}{bc+2}+\frac{b}{ca+2}+\frac{c}{ab+2}\le 1.$$
(Vasile Cîrtoaje, 2010)

Solution. (a) ***First Solution***. It suffices to show that
$$\frac{a}{abc+2}+\frac{b}{abc+2}+\frac{c}{abc+2}\le 1,$$
which is equivalent to
$$abc+2\ge a+b+c.$$
We have
$$abc+2-a-b-c=(1-b)(1-c)+(1-a)(1-bc)\ge 0.$$
The equality holds for $a=b=c=1$, and for $a=0$ and $b=c=1$ (or any cyclic permutation).

Second Solution. Assume that $a=\max\{a,b,c\}$. It suffices to show that
$$\frac{a}{bc+2}+\frac{b}{bc+2}+\frac{c}{bc+2}\le 1.$$
that is,
$$a+b+c)\le 2+bc.$$
We have
$$2+bc-a-b-c)=1-a+(1-b)(1-c)\ge 0.$$

□

P 1.123. *Let a, b, c be positive real numbers such that $a+b+c=2$. Prove that*

$$5(1-ab-bc-ca)\left(\frac{1}{1-ab}+\frac{1}{1-bc}+\frac{1}{1-ca}\right)+9 \ge 0.$$

(Vasile Cîrtoaje, 2011)

Solution. Write the inequality as

$$24-\frac{5a(b+c)}{1-bc}-\frac{5b(c+a)}{1-ca}-\frac{5c(a+b)}{1-ab} \ge 0.$$

Since

$$4(1-bc) \ge 4-(b+c)^2=(a+b+c)^2-(b+c)^2=a(a+2b+2c),$$

it suffices to show that

$$6-5\left(\frac{b+c}{a+2b+2c}-\frac{c+a}{b+2c+2a}-\frac{a+b}{c+2a+2b}\right) \ge 0,$$

which is equivalent to

$$\sum 5\left(1-\frac{b+c}{a+2b+2c}\right) \ge 9,$$

$$5(a+b+c)\sum\frac{1}{a+2b+2c} \ge 9,$$

$$\left[\sum(a+2b+2c)\right]\left(\sum\frac{1}{a+2b+2c}\right) \ge 9.$$

The last inequality follows immediately from the AM-HM inequality. The equality holds for $a=b=c=2/3$.

□

P 1.124. *Let a, b, c be nonnegative real numbers such that $a+b+c=2$. Prove that*

$$\frac{2-a^2}{2-bc}+\frac{2-b^2}{2-ca}+\frac{2-c^2}{2-ab} \le 3.$$

(Vasile Cîrtoaje, 2011)

First Solution. Write the inequality as follows:

$$\sum\left(1-\frac{2-a^2}{2-bc}\right) \ge 0,$$

$$\sum\frac{a^2-bc}{2-bc} \ge 0,$$

$$\sum(a^2-bc)(2-ca)(2-ab) \ge 0,$$

$$\sum(a^2-bc)[4-2a(b+c)+a^2bc]\ge 0,$$

$$4\sum(a^2-bc)-2\sum a(b+c)(a^2-bc)+abc\sum a(a^2-bc)\ge 0.$$

By virtue of the AM-GM inequality,

$$\sum a(a^2-bc)=a^3+b^3+c^3-3abc\ge 0.$$

Then, it suffices to prove that

$$2\sum(a^2-bc)\ge\sum a(b+c)(a^2-bc).$$

Indeed, we have

$$\sum a(b+c)(a^2-bc)=\sum a^3(b+c)-abc\sum(b+c)$$

$$=\sum a(b^3+c^3)-abc\sum(b+c)=\sum a(b+c)(b-c)^2$$

$$\le\sum\left[\frac{a+(b+c)}{2}\right]^2(b-c)^2=\sum(b-c)^2=2\sum(a^2-bc).$$

The equality holds for $a=b=c=2/3$, and for $a=0$ and $b=c=1$ (or any cyclic permutation).

Second Solution. We apply the SOS method. Write the inequality as follows:

$$\sum\frac{a^2-bc}{2-bc}\ge 0,$$

$$\sum\frac{(a-b)(a+c)+(a-c)(a+b)}{2-bc}\ge 0,$$

$$\sum\frac{(a-b)(a+c)}{2-bc}+\sum\frac{(b-a)(b+c)}{2-ca}\ge 0,$$

$$\sum\frac{(a-b)^2[2-c(a+b)-c^2]}{(2-bc)(2-ca)}\ge 0,$$

$$\sum(a-b)^2(2-ab)(1-c)\ge 0.$$

Assuming that $a\ge b\ge c$, it suffices to prove that

$$(b-c)^2(2-bc)(1-a)+(c-a)^2(2-ca)(1-b)\ge 0.$$

Since

$$2(1-b)=a-b+c\ge 0,\qquad (c-a)^2\ge(b-c)^2,$$

it suffices to show that

$$(2-bc)(1-a)+(2-ca)(1-b)\ge 0.$$

We have

$$(2-bc)(1-a)+(2-ca)(1-b)=4-2(a+b)-c(a+b)+2abc$$

$$\geq 4-(a+b)(2+c)\geq 4-\left[\frac{(a+b)+(2+c)}{2}\right]^2=0.$$

□

P 1.125. *Let a,b,c be nonnegative real numbers such that $a+b+c=3$. Prove that*

$$\frac{3+5a^2}{3-bc}+\frac{3+5b^2}{3-ca}+\frac{3+5c^2}{3-ab}\geq 12.$$

(Vasile Cîrtoaje, 2010)

Solution. Use the SOS method. Write the inequality as follows:

$$\sum\left(\frac{3+5a^2}{3-bc}-4\right)\geq 0,$$

$$\sum\frac{5a^2+4bc-9}{3-bc}\geq 0,$$

$$\sum\frac{5a^2+4bc-(a+b+c)^2}{3-bc}\geq 0,$$

$$\sum\frac{4a^2-b^2-c^2-2ab+2bc-2ca}{3-bc}\geq 0,$$

$$\sum\frac{2a^2-b^2-c^2+2(a-b)(a-c)}{3-bc}\geq 0,$$

$$\sum\frac{(a-b)(a+b)+(a-c)(a+c)+2(a-b)(a-c)}{3-bc}\geq 0,$$

$$\sum\frac{[(a-b)(a+b)+(a-b)(a-c)]+[(a-c)(a+c)+(a-c)(a-b)]}{3-bc}\geq 0,$$

$$\sum\frac{(a-b)(2a+b-c)+(a-c)(2a+c-b)}{3-bc}\geq 0,$$

$$\sum\frac{(a-b)(2a+b-c)}{3-bc}+\sum\frac{(b-a)(2b+a-c)}{3-ca}\geq 0,$$

$$\sum\frac{(a-b)^2[3-2c(a+b)+c^2]}{(3-bc)(3-ca)}\geq 0,$$

$$\sum\frac{(a-b)^2(c-1)^2}{(3-bc)(3-ca)}\geq 0.$$

The equality holds for $a=b=c=1$.

□

P 1.126. *Let a,b,c be nonnegative real numbers such that $a+b+c=2$. If*

$$\frac{-1}{7} \le m \le \frac{7}{8},$$

then

$$\frac{a^2+m}{3-2bc}+\frac{b^2+m}{3-2ca}+\frac{c^2+m}{3-2ab} \ge \frac{3(4+9m)}{19}.$$

(Vasile Cîrtoaje, 2010)

Solution. We apply the SOS method. Write the inequality as

$$\sum \left(\frac{a^2+m}{3-2bc} - \frac{4+9m}{19} \right) \ge 0,$$

$$\sum \frac{19a^2+2(4+9m)bc-12-8m}{3-2bc} \ge 0.$$

Since

$$19a^2+2(4+9m)bc-12-8m =$$

$$= 19a^2+2(4+9m)bc-(3+2m)(a+b+c)^2$$

$$=(16-2m)a^2-(3+2m)(b^2+c^2+2ab+2ac)+2(1+7m)bc$$

$$=(3+2m)(2a^2-b^2-c^2)+2(5-3m)(a^2+bc-ab-ac)+(4-10m)(ab+ac-2bc)$$

$$=(3+2m)(a^2-b^2)+(5-3m)(a-b)(a-c)+(4-10m)c(a-b)$$

$$+(3+2m)(a^2-c^2)+(5-3m)(a-c)(a-b)+(4-10m)b(a-c)$$

$$=(a-b)B+(a-c)C,$$

where

$$B=(8-m)a+(3+2m)b-(1+7m)c,$$

$$C=(8-m)a+(3+2m)c-(1+7m)b,$$

the inequality can be written as

$$B_1+C_1 \ge 0,$$

where

$$B_1 = \sum \frac{(a-b)[(8-m)a+(3+2m)b-(1+7m)c]}{3-2bc},$$

$$C_1 = \sum \frac{(b-a)[(8-m)b+(3+2m)a-(1+7m)c]}{3-2ca}.$$

We have

$$B_1+C_1 = \sum \frac{(a-b)^2 S_c}{(3-2bc)(3-2ca)},$$

where

$$\begin{aligned}S_c =&3(5-3m)-2(8-m)c(a+b)+2(1+7m)c^2\\=&6(2m+3)c^2-4(8-m)c+3(5-3m)\\=&6(2m+3)\left[c-\frac{8-m}{3(2m+3)}\right]^2+\frac{(1+7m)(7-8m)}{3(2m+3)}.\end{aligned}$$

Since $S_c \ge 0$ for $-1/7 \le m \le 7/8$, the proof is completed. The equality holds for $a=b=c=2/3$. If $m=-1/7$, then the equality holds also for $a=0$ and $b=c=1$ (or any cyclic permutation). If $m=7/8$, then the equality holds also for $a=1$ and $b=c=1/2$ (or any cyclic permutation).

Remark. The following more general statement holds:

- *Let a,b,c be nonnegative real numbers such that $a+b+c=3$. If*

$$0<k\le 3, \qquad m_1 \le m \le m_2,$$

where

$$m_1=\begin{cases}-\infty, & 0<k\le \dfrac{3}{2}\\ \dfrac{(3-k)(4-k)}{2(3-2k)}, & \dfrac{3}{2}<k\le 3\end{cases},$$

$$m_2=\frac{36-4k-k^2+4(9-k)\sqrt{3(3-k)}}{72+k},$$

then

$$\frac{a^2+mbc}{9-kbc}+\frac{b^2+mca}{9-kca}+\frac{c^2+mab}{9-kab}\ge\frac{3(1+m)}{9-k},$$

with equality for $a=b=c=1$. If $3/2<k\le 3$ and $m=m_1$, then the equality holds also for

$$a=0, \quad b=c=\frac{3}{2}.$$

If $m=m_2$, then the equality holds also for

$$a=\frac{3k-6+2\sqrt{3(3-k)}}{k}, \quad b=c=\frac{3-\sqrt{3(3-k)}}{k}.$$

The inequalities in P 1.124, P 1.125 and P 1.126 are particular cases of this result (for $k=2$ and $m=m_1=-1$, for $k=3$ and $m=m_2=1/5$, and for $k=8/3$, respectively).

□

P 1.127. *Let a,b,c be nonnegative real numbers such that $a+b+c=3$. Prove that*

$$\frac{47-7a^2}{1+bc}+\frac{47-7b^2}{1+ca}+\frac{47-7c^2}{1+ab}\ge 60.$$

(Vasile Cîrtoaje, 2011)

Solution. We apply the SOS method. Write the inequality as follows:

$$\sum\left(\frac{47-7a^2}{1+bc}-20\right)\ge 0,$$

$$\sum\frac{27-7a^2-20bc}{1+bc}\ge 0,$$

$$\sum\frac{3(a+b+c)^2-7a^2-20bc}{1+bc}\ge 0,$$

$$\sum\frac{-3(2a^2-b^2-c^2)+2(a-b)(a-c)+8(ab-2bc+ca)}{1+bc}\ge 0,$$

$$\sum\frac{-3(a-b)(a+b)+(a-b)(a-c)+8c(a-b)}{1+bc}+$$

$$+\sum\frac{-3(a-c)(a+c)+(a-c)(a-b)+8b(a-c)}{1+bc}\ge 0,$$

$$\sum\frac{(a-b)(-2a-3b+7c)}{1+bc}+\sum\frac{(a-c)(-2a-3c+7b)}{1+bc}\ge 0,$$

$$\sum\frac{(a-b)(-2a-3b+7c)}{1+bc}+\sum\frac{(b-a)(-2b-3a+7c)}{1+ca}\ge 0,$$

$$\sum\frac{(a-b)^2[1-2c(a+b)+7c^2]}{(1+bc)(1+ca)}\ge 0,$$

$$\sum\frac{(a-b)^2(3c-1)^2}{(1+bc)(1+ca)}\ge 0,$$

The equality holds for $a=b=c=1$, and for $a=7/3$ and $b=c=1/3$ (or any cyclic permutation).

Remark. The following more general statement holds:

- *Let a,b,c be nonnegative real numbers such that $a+b+c=3$. If*

$$k>0,\qquad m\ge m_1,$$

where

$$m_1=\begin{cases}\dfrac{36+4k-k^2+4(9+k)\sqrt{3(3+k)}}{72-k}, & k\ne 72\\[2mm] \dfrac{238}{5}, & k=72\end{cases},$$

then

$$\frac{a^2+mbc}{9+kbc}+\frac{b^2+mca}{9+kca}+\frac{c^2+mab}{9+kab}\le\frac{3(1+m)}{9+k},$$

with equality for $a=b=c=1$. *If* $m=m_1$, *then the equality holds also for*

$$a=\frac{3k+6-2\sqrt{3(3+k)}}{k},\quad b=c=\frac{\sqrt{3(3+k)}-3}{k}.$$

The inequality in P 1.127 is a particular case of this result (for $k=9$ and $m=m_1=47/7$).

□

P 1.128. *Let* a,b,c *be nonnegative real numbers such that* $a+b+c=3$. *Prove that*

$$\frac{26-7a^2}{1+bc}+\frac{26-7b^2}{1+ca}+\frac{26-7c^2}{1+ab}\le\frac{57}{2}.$$

(Vasile Cîrtoaje, 2011)

Solution. Use the SOS method. Write the inequality as follows:

$$\sum\left(\frac{19}{2}-\frac{26-7a^2}{1+bc}\right)\ge 0,$$

$$\sum\frac{14a^2+19bc-33}{1+bc}\ge 0,$$

$$\sum\frac{42a^2+57bc-11(a+b+c)^2}{1+bc}\ge 0,$$

$$\sum\frac{11(2a^2-b^2-c^2)+9(a-b)(a-c)-13(ab-2bc+ca)}{1+bc}\ge 0,$$

$$\sum\frac{22(a-b)(a+b)+9(a-b)(a-c)-26c(a-b)}{1+bc}+$$

$$+\sum\frac{22(a-c)(a+c)+9(a-c)(a-b)-26b(a-c)}{1+bc}\ge 0,$$

$$\sum\frac{(a-b)(31a+22b-35c)}{1+bc}+\sum\frac{(a-c)(31a+22c-35b)}{1+bc}\ge 0,$$

$$\sum\frac{(a-b)(31a+22b-35c)}{1+bc}+\sum\frac{(b-a)(31b+22a-35c)}{1+ca}\ge 0,$$

$$\sum\frac{(a-b)^2[9+31c(a+b)-35c^2]}{(1+bc)(1+ca)}\ge 0,$$

$$\sum(a-b)^2(1+ab)(1+11c)(3-2c)\ge 0.$$

Assume that $a \ge b \ge c$. Since $3-2c > 0$, it suffices to show that

$$(b-c)^2(1+bc)(1+11a)(3-2a)+(c-a)^2(1+ab)(1+11b)(3-2b) \ge 0;$$

that is,

$$(a-c)^2(1+ab)(1+11b)(3-2b) \ge (b-c)^2(1+bc)(1+11a)(2a-3).$$

Since $3-2b = a-b+c \ge 0$, we get this inequality by multiplying the inequalities

$$3-2b \ge 2a-3,$$

$$a(1+ab) \ge b(1+bc),$$

$$a(1+11b) \ge b(1+11a),$$

$$b^2(a-c)^2 \ge a^2(b-c)^2.$$

The equality holds for $a=b=c=1$, and for $a=b=3/2$ and $c=0$ (or any cyclic permutation).

Remark. The following more general statement holds:

- *Let a,b,c be nonnegative real numbers such that $a+b+c=3$. If*

$$k>0, \qquad m \le m_2, \quad m_2 = \frac{(3+k)(4+k)}{2(3+2k)},$$

then

$$\frac{a^2+mbc}{9+kbc}+\frac{b^2+mca}{9+kca}+\frac{c^2+mab}{9+kab} \ge \frac{3(1+m)}{9+k},$$

with equality for $a=b=c=1$. When $m=m_2$, the equality holds also for $a=0$ and $b=c=3/2$ (or any cyclic permutation).

The inequalities in P 1.128 is a particular cases of this result (for $k=9$ and $m=m_2=26/7$).

□

P 1.129. *If a,b,c are nonnegative real numbers, then*

$$\sum \frac{5a(b+c)-6bc}{a^2+b^2+c^2+bc} \le 3.$$

(Vasile Cîrtoaje, 2010)

First Solution. Apply the SOS method. If two of a, b, c are zero, then the inequality is trivial. Consider further that

$$a^2+b^2+c^2=1, \qquad a \ge b \ge c, \quad b>0,$$

and write the inequality as follows:

$$\sum \left[1-\frac{5a(b+c)-6bc}{a^2+b^2+c^2+bc}\right] \ge 0,$$

$$\sum \frac{a^2+b^2+c^2-5a(b+c)+7bc}{a^2+b^2+c^2+bc} \ge 0,$$

$$\sum \frac{(7b+2c-a)(c-a)-(7c+2b-a)(a-b)}{1+bc} \ge 0,$$

$$\sum \frac{(7c+2a-b)(a-b)}{1+ca} - \sum \frac{(7c+2b-a)(a-b)}{1+bc} \ge 0,$$

$$\sum (a-b)^2(1+ab)(3+ac+bc-7c^2) \ge 0.$$

Since

$$3+ac+bc-7c^2 = 3a^2+3b^2+ac+bc-4c^2 > 0,$$

it suffices to prove that

$$(1+bc)(3+ab+ac-7a^2)(b-c)^2+(1+ac)(3+ab+bc-7b^2)(a-c)^2 \ge 0.$$

Since

$$3+ab+ac-7b^2 = 3(a^2-b^2)+3c^2+b(a-b)+bc \ge 0$$

and $1+ac \ge 1+bc$, it is enough to show that

$$(3+ab+ac-7a^2)(b-c)^2+(3+ab+bc-7b^2)(a-c)^2 \ge 0.$$

From $b(a-c) \ge a(b-c) \ge 0$, we get $b^2(a-c)^2 \ge a^2(b-c)^2$, hence

$$b(a-c)^2 \ge a(b-c)^2.$$

Thus, it suffices to show that

$$b(3+ab+ac-7a^2)+a(3+ab+bc-7b^2) \ge 0.$$

This is true if

$$b(3+ab-7a^2)+a(3+ab-7b^2) \ge 0.$$

Indeed,

$$b(3+ab-7a^2)+a(3+ab-7b^2) = 3(a+b)(1-2ab) \ge 0,$$

since

$$1-2ab = (a-b)^2+c^2 \ge 0.$$

The equality holds for $a=b=c$, and for $a=b$ and $c=0$ (or any cyclic permutation).

Second Solution. Without loss of generality, assume that $a^2+b^2+c^2=1$ and $a\le b\le c$. Setting

$$p=a+b+c,\quad q=ab+bc+ca,\quad r=abc,$$

the inequality becomes

$$\sum\frac{5q-11bc}{1+bc}\le 3,$$

$$3\prod(1+bc)+\sum(11bc-5q)(1+ca)(1+ab)\ge 0,$$

$$3(1+q+pr+r^2)+11(q+2pr+3r^2)-5q(3+2q+pr)\ge 0,$$

$$36r^2+5(5-q)pr+3-q-10q^2\ge 0.$$

According to P 3.57-(a) in Volume 1, for fixed p and q, the product $r=abc$ is minimum when $b=c$ or $a=0$. Therefore, since $5-q\ge 4>0$, it suffices to prove the original homogeneous inequality for $a=0$, and for $b=c=1$. For $a=0$, the original inequality becomes

$$\frac{-6bc}{b^2+c^2+bc}+\frac{10bc}{b^2+c^2}\le 3,$$

$$(b-c)^2(3b^2+5bc+3b^2)\ge 0,$$

while for $b=c=1$, the original inequality becomes

$$\frac{10a-6}{a^2+3}+2\frac{5-a}{a^2+a+2}\le 3,$$

which is equivalent to

$$a(3a+1)(a-1)^2\ge 0.$$

Remark. Similarly, we can prove the following generalization:

- *Let a,b,c be nonnegative real numbers. If $k>0$, then*

$$\sum\frac{(2k+3)a(b+c)+(k+2)(k-3)bc}{a^2+b^2+c^2+kbc}\le 3k,$$

with equality for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.130. *Let a, b, c be nonnegative real numbers, no two of which are zero, and let*

$$x = \frac{a^2+b^2+c^2}{ab+bc+ca}.$$

Prove that

(a) $$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{1}{2} \geq x+\frac{1}{x};$$

(b) $$6\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right) \geq 5x+\frac{4}{x};$$

(c) $$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2} \geq \frac{1}{3}\left(x-\frac{1}{x}\right).$$

(Vasile Cîrtoaje, 2011)

Solution. We will prove the more general inequality

$$\frac{2a}{b+c}+\frac{2b}{c+a}+\frac{2c}{a+b}+1-3k \geq (2-k)x+\frac{2(1-k)}{x},$$

where

$$0 \leq k \leq k_0, \qquad k_0 = \frac{21+6\sqrt{6}}{25} \approx 1.428.$$

For $k=0$, $k=1/3$ and $k=4/3$, we get the inequalities in (a), (b) and (c), respectively. Let $p=a+b+c$ and $q=ab+bc+ca$. Since $x=(p^2-2q)/q$, we can write the inequality as follows:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq f(p,q),$$

$$\sum\left(\frac{a}{b+c}+1\right) \geq 3+f(p,q),$$

$$\frac{p(p^2+q)}{pq-abc} \geq 3+f(p,q).$$

According to P 3.57-(a) in Volume 1, for fixed p and q, the product abc is minimum when $b=c$ or $a=0$. Therefore, it suffices to prove the inequality for $a=0$, and for $b=c=1$. For $a=0$, using the substitution $y=b/c+c/b$, the desired inequality becomes

$$2y+1-3k \geq (2-k)y+\frac{2(1-k)}{y},$$

$$\frac{(y-2)[k(y-1)+1]}{y} \geq 0.$$

Since $y \geq 2$, this inequality is clearly true. For $b = c = 1$, the desired inequality becomes

$$a + \frac{4}{a+1} + 1 - 3k \geq \frac{(2-k)(a^2+2)}{2a+1} + \frac{2(1-k)(2a+1)}{a^2+2},$$

which is equivalent to

$$a(a-1)^2[ka^2 + 3(1-k)a + 6 - 4k] \geq 0.$$

For $0 \leq k \leq 1$, this is obvious, and for $1 < k \leq (21+6\sqrt{6})/25$, we have

$$ka^2 + 3(1-k)a + 6 - 4k \geq [2\sqrt{k(6-4k)} + 3(1-k)]a \geq 0.$$

The equality holds for $a = b = c$, and for $a = 0$ and $b = c$ (or any cyclic permutation). If $k = k_0$, then the equality holds also for $(2+\sqrt{6})a = 2b = 2c$ (or any cyclic permutation).

□

P 1.131. *If a, b, c are real numbers, then*

$$\frac{1}{a^2+7(b^2+c^2)} + \frac{1}{b^2+7(c^2+a^2)} + \frac{1}{c^2+7(a^2+b^2)} \leq \frac{9}{5(a+b+c)^2}.$$

(Vasile Cîrtoaje, 2008)

Solution. We use the *highest coefficient method*. Let

$$p = a+b+c, \qquad q = ab+bc+ca.$$

Write the inequality as $f_6(a,b,c) \geq 0$, where

$$f_6(a,b,c) = 9\prod(a^2+7b^2+7c^2) - 5p^2\sum(b^2+7c^2+7a^2)(c^2+7a^2+7b^2).$$

Since

$$\prod(a^2+7b^2+7c^2) = \prod[7(p^2-2q) - 6a^2],$$

$f_6(a,b,c)$ has the highest coefficient

$$A = 9(-6)^3 < 0.$$

According to P 2.75 in Volume 1, it suffices to prove the original inequality for $b = c = 1$, when the inequality reduces to

$$\frac{1}{a^2+14} + \frac{2}{7a^2+8} \leq \frac{9}{5(a+2)^2},$$

$$(a-1)^2(a-4)^2 \geq 0.$$

Thus, the proof is completed. The equality holds for $a = b = c$, and for $a/4 = b = c$ (or any cyclic permutation).

□

P 1.132. *If a, b, c are real numbers, then*

$$\frac{bc}{3a^2+b^2+c^2}+\frac{ca}{3b^2+c^2+a^2}+\frac{ab}{3c^2+a^2+b^2}\le\frac{3}{5}.$$

(Vasile Cîrtoaje and Pham Kim Hung, 2005)

Solution. Use the *highest coefficient method*. Write the inequality as $f_6(a,b,c)\ge 0$, where

$$f_6(a,b,c)=3\prod(3a^2+b^2+c^2)-5\sum bc(3b^2+c^2+a^2)(3c^2+a^2+b^2).$$

Let

$$p=a+b+c,\qquad q=ab+bc+ca.$$

From

$$f_6(a,b,c)=3\prod(2a^2+p^2-2q)-5\sum bc(2b^2+p^2-2q)(2c^2+p^2-2q),$$

it follows that $f_6(a,b,c)$ has the same highest coefficient A as

$$24a^2b^2c^2-20\sum b^3c^3;$$

that is,

$$A=24-60<0.$$

According to P 2.75 in Volume 1, it suffices to prove the original inequality for $b=c=1$, when the inequality is equivalent to

$$\frac{1}{3a^2+2}+\frac{2a}{a^2+4}\le\frac{3}{5},$$

$$(a-1)^2(3a-2)^2\ge 0.$$

Thus, the proof is completed. The equality holds for $a=b=c$, and for $3a/2=b=c$ (or any cyclic permutation).

Remark. The inequality in P 1.132 is a particular case ($k=3$) of the following more general result (*Vasile Cîrtoaje*, 2008):

- *Let a, b, c be real numbers. If $k>1$, then*

$$\sum\frac{k(k-3)a^2+2(k-1)bc}{ka^2+b^2+c^2}\le\frac{3(k+1)(k-2)}{k+2},$$

with equality for $a=b=c$, and for $ka/2=b=c$ (or any cyclic permutation).

□

P 1.133. *If a, b, c are real numbers such that $a+b+c=3$, then*

$$\frac{1}{8+5(b^2+c^2)}+\frac{1}{8+5(c^2+a^2)}+\frac{1}{8+5(a^2+b^2)}\le\frac{1}{6}.$$

(Vasile Cîrtoaje, 2006)

Solution. Use the *highest coefficient method*. Denote

$$p=a+b+c,\qquad q=ab+bc+ca,$$

and write the inequality in the homogeneous form

$$\frac{1}{8p^2+45(b^2+c^2)}+\frac{1}{8p^2+45(c^2+a^2)}+\frac{1}{8p^2+45(a^2+b^2)}\le\frac{1}{6p^2},$$

which is equivalent to $f_6(a,b,c)\ge 0$, where

$$\begin{aligned} f_6(a,b,c)=&\prod(53p^2-90q-45a^2)\\ &-6p^2\sum(53p^2-90q-45b^2)(53p^2-90q-45c^2). \end{aligned}$$

Clearly, $f_6(a,b,c)$ has the highest coefficient

$$A=(-45)^3<0.$$

According to P 2.75 in Volume 1, it suffices to prove the homogeneous inequality for $b=c=1$; that is,

$$\frac{1}{8(a+2)^2+90}+\frac{2}{8(a+2)^2+45(1+a^2)}\le\frac{1}{6(a+2)^2}.$$

Using the substitution

$$a+2=3x,$$

the inequality becomes as follows:

$$\frac{1}{72x^2+90}+\frac{2}{72x^2+45+45(3x-2)^2)}\le\frac{1}{54x^2},$$

$$\frac{1}{8x^2+10}+\frac{2}{53x^2-60x+25}\le\frac{1}{6x^2},$$

$$x^4-12x^3+46x^2-60x+25\ge 0,$$

$$(x-1)^2(x-5)^2\ge 0,$$

$$(a-1)^2(a-13)^2\ge 0.$$

The equality holds for $a=b=c=1$, and for $a=13/5$ and $b=c=1/5$ (or any cyclic permutation).

□

P 1.134. *If a, b, c are real numbers, then*

$$\frac{(a+b)(a+c)}{a^2+4(b^2+c^2)}+\frac{(b+c)(b+a)}{b^2+4(c^2+a^2)}+\frac{(c+a)(c+b)}{c^2+4(a^2+b^2)}\le\frac{4}{3}.$$

(Vasile Cîrtoaje, 2008)

Solution. Use the *highest coefficient method*. Let

$$p=a+b+c,\qquad q=ab+bc+ca.$$

Write the inequality as $f_6(a,b,c)\ge 0$, where

$$f_6(a,b,c)=4\prod(a^2+4b^2+4c^2)$$

$$-3\sum(a+b)(a+c)(b^2+4c^2+4a^2)(c^2+4a^2+4b^2)$$

$$=4\prod(4p^2-8q-3a^2)-3\sum(a^2+q)(4p^2-8q-3b^2)(4p^2-8q-3c^2).$$

Thus, $f_6(a,b,c)$ has the highest coefficient

$$A=4(-3)^3-3^4<0.$$

By P 2.75 in Volume 1, it suffices to prove the original inequality for $b=c=1$, when the inequality is equivalent to

$$\frac{(a+1)^2}{a^2+8}+\frac{4(a+1)}{4a^2+5}\le\frac{4}{3},$$

$$(a-1)^2(2a-7)^2\ge 0.$$

The equality holds for $a=b=c$, and for $2a/7=b=c$ (or any cyclic permutation).

□

P 1.135. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\sum\frac{1}{(b+c)(7a+b+c)}\le\frac{1}{2(ab+bc+ca)}.$$

(Vasile Cîrtoaje, 2009)

First Solution. Write the inequality as

$$\sum\left[1-\frac{4(ab+bc+ca)}{(b+c)(7a+b+c)}\right]\ge 1,$$

$$\sum\frac{(b-c)^2+3a(b+c)}{(b+c)(7a+b+c)}\ge 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{(b-c)^2+3a(b+c)}{(b+c)(7a+b+c)} \ge \frac{4(a+b+c)^4}{\sum[(b-c)^2+3a(b+c)](b+c)(7a+b+c)}.$$

Therefore, it suffices to show that

$$4(a+b+c)^4 \ge \sum(b^2+c^2-2bc+3ca+3ab)(b+c)(7a+b+c).$$

Write this inequality as

$$\sum a^4 + abc\sum a + 3\sum ab(a^2+b^2) - 8\sum a^2b^2 \ge 0,$$

$$\sum a^4 + abc\sum a - \sum ab(a^2+b^2) + 4\sum ab(a-b)^2 \ge 0.$$

Since

$$\sum a^4 + abc\sum a - \sum ab(a^2+b^2) \ge 0$$

(Schur's inequality of degree four), the conclusion follows. The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

Second Solution. Use the *highest coefficient method*. We need to prove that $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c) = \prod(b+c)(7a+b+c)$$

$$-2(ab+bc+ca)\sum(a+b)(a+c)(7b+c+a)(7c+a+b).$$

Let $p=a+b+c$. Clearly, $f_6(a,b,c)$ has the same highest coefficient A as $f(a,b,c)$, where

$$f(a,b,c) = \prod(b+c)(7a+b+c) = \prod(p-a)(p+6a);$$

that is,

$$A = (-6)^3 < 0.$$

Thus, by P 3.76-(a) in Volume 1, it suffices to prove the original inequality for $b=c=1$, and for $a=0$.

For $b=c=1$, the inequality reduces to

$$\frac{1}{2(7a+2)} + \frac{2}{(a+1)(a+8)} \le \frac{1}{2(2a+1)},$$

$$a(a-1)^2 \ge 0.$$

For $a=0$, the inequality can be written as

$$\frac{1}{(b+c)^2} + \frac{1}{c(7b+c)} + \frac{1}{b(7c+b)} \le \frac{1}{2bc},$$

$$\frac{1}{(b+c)^2} + \frac{b^2+c^2+14bc}{bc[7(b^2+c^2)+50bc]} \le \frac{1}{2bc},$$

$$\frac{1}{x+2}+\frac{x+14}{7x+50}\leq\frac{1}{2},$$

where

$$x=\frac{b}{c}+\frac{c}{b},\qquad x\geq 2.$$

This reduces to the obvious inequality

$$(x-2)(5x+28)\geq 0.$$

□

P 1.136. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\sum\frac{1}{b^2+c^2+4a(b+c)}\leq\frac{9}{10(ab+bc+ca)}.$$

(Vasile Cîrtoaje, 2009)

Solution. Use the *highest coefficient method*. Let

$$p=a+b+c,\qquad q=ab+bc+ca.$$

We need to prove that $f_6(a,b,c)\geq 0$, where

$$f_6(a,b,c)=9\prod[b^2+c^2+4a(b+c)]$$

$$-10(ab+bc+ca)\sum[a^2+b^2+4c(a+b)][a^2+c^2+4b(a+c)]$$

$$=9\prod(p^2+2q-a^2-4bc)-10q\sum(p^2+2q-c^2-4ab)(p^2+2q-b^2-4ca).$$

Clearly, $f_6(a,b,c)$ has the same highest coefficient A as $P_3(a,b,c)$, where

$$P_3(a,b,c)=-9\prod(a^2+4bc).$$

According to Remark 2 from the proof of P 2.75 in Volume 1,

$$A=P_3(1,1,1)=-9\cdot 125<0.$$

Thus, by P 3.76-(a) in Volume 1, it suffices to prove the original inequality for $b=c=1$, and for $a=0$.

For $b=c=1$, the inequality reduces to

$$\frac{1}{2(4a+1)}+\frac{2}{a^2+4a+5}\leq\frac{9}{10(2a+1)},$$

$$a(a-1)^2\geq 0.$$

For $a=0$, the inequality becomes

$$\frac{1}{b^2+c^2}+\frac{1}{b^2+4bc}+\frac{1}{c^2+4bc}\le\frac{9}{10bc},$$

$$\frac{1}{b^2+c^2}+\frac{b^2+c^2+8bc}{4bc(b^2+c^2)+17b^2c^2}\le\frac{9}{10bc}.$$

$$\frac{1}{x}+\frac{x+8}{4x+17}\le\frac{9}{10},$$

$$(x-2)(26x+85)\ge 0,$$

where

$$x=\frac{b}{c}+\frac{c}{b},\qquad x\ge 2.$$

The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.137. *Let a,b,c be nonnegative real numbers, no two of which are zero. If $a+b+c=3$, then*

$$\frac{1}{3-ab}+\frac{1}{3-bc}+\frac{1}{3-ca}\le\frac{9}{2(ab+bc+ca)}.$$

(Vasile Cîrtoaje, 2011)

First Solution. We apply the SOS method. Write the inequality as

$$\sum\left(\frac{3}{2}-\frac{ab+bc+ca}{3-bc}\right)\ge 0.$$

$$\sum\frac{9-2a(b+c)-5bc}{3-bc}\ge 0,$$

$$\sum\frac{a^2+b^2+c^2-3bc}{3-bc}\ge 0.$$

Since

$$2(a^2+b^2+c^2-3bc)=2(a^2-bc)+2(b^2+c^2-ab-ac)+2(ab+ac-2bc)$$

$$=(a-b)(a+c)+(a-c)(a+b)-2b(a-b)-2c(a-c)+2c(a-b)+2b(a-c)$$

$$=(a-b)(a-2b+3c)+(a-c)(a-2c+3b),$$

the required inequality is equivalent to

$$\sum\frac{(a-b)(a-2b+3c)+(a-c)(a-2c+3b)}{3-bc}\ge 0,$$

$$\sum \frac{(a-b)(a-2b+3c)}{3-bc} + \sum \frac{(b-a)(b-2a+3c)}{3-ca} \ge 0,$$

$$\sum \frac{(a-b)^2[9-c(a+b+3c)]}{(3-bc)(3-ca)} \ge 0,$$

$$\sum (a-b)^2(3-ab)(3+c)(3-2c) \ge 0.$$

Without loss of generality, assume that $a \ge b \ge c$. It suffices to prove that

$$(b-c)^2(3-bc)(3+a)(3-2a) + (c-a)^2(3-ca)(3+b)(3-2b) \ge 0,$$

which is equivalent to

$$(a-c)^2(3-ac)(3+b)(3-2b) \ge (b-c)^2(3-bc)(a+3)(2a-3).$$

Since $3-2b = a-b+c \ge 0$, we can obtain this inequality by multiplying the inequalities

$$b^2(a-c)^2 \ge a^2(b-c)^2,$$

$$a(3-ac) \ge b(3-bc),$$

$$a(3+b)(3-2b) \ge b(a+3)(2a-3) \ge 0.$$

We have

$$\begin{aligned} a(3-ac)-b(3-bc) &= (a-b)[3-c(a+b)] = (a-b)(3-3c+c^2) \\ &\ge (a-b)(3-3c) \ge 0. \end{aligned}$$

Also, since $a+b \le a+b+c = 3$, we have

$$a(3+b)(3-2b) - b(a+3)(2a-3) = 9(a+b) - 6ab - 2ab(a+b)$$

$$\ge 9(a+b) - 12ab \ge 3(a+b)^2 - 12ab = 3(a-b)^2 \ge 0.$$

The equality holds for $a=b=c=1$, and for $a=0$ and $b=c=3/2$ (or any cyclic permutation).

Second Solution. Write the inequality in the homogeneous form

$$\frac{1}{p^2-3ab} + \frac{1}{p^2-3bc} + \frac{1}{p^2-3ca} \le \frac{3}{2q},$$

where

$$p = a+b+c, \quad q = ab+bc+ca.$$

We need to prove that $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c) = 3\prod(p^2-3bc) - 2q\sum(p^2-3ca)(p^2-3ab).$$

Clearly, $f_6(a,b,c)$ has the highest coefficient

$$A = 3(-3)^3 < 0.$$

Thus, by P 3.76-(a) in Volume 1, it suffices to prove the homogeneous inequality for $b = c = 1$, and for $a = 0$.

For $b = c = 1$, the homogeneous inequality reduces to

$$\frac{2}{(a+2)^2-3a}+\frac{1}{(a+2)^2-3}\le\frac{3}{2(2a+1)},$$

$$\frac{a^2+3a+2}{(a^2+a+4)(a^2+4a+1)}\le\frac{3}{2(2a+1)},$$

$$a(a+3)(a-1)^2\ge 0.$$

For $a = 0$, the homogeneous inequality can be written as

$$\frac{2}{(b+c)^2}+\frac{1}{(b+c)^2-3bc}\le\frac{3}{2bc},$$

$$\frac{(b-c)^2(b^2+c^2+bc)}{2bc(b+c)^2(b^2+c^2-bc)}\ge 0.$$

□

P 1.138. *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

$$\frac{bc}{a^2+a+6}+\frac{ca}{b^2+b+6}+\frac{ab}{c^2+c+6}\le\frac{3}{8}.$$

(Vasile Cîrtoaje, 2009)

Solution. Write the inequality in the homogeneous form

$$\frac{bc}{3a^2+ap+2p^2}+\frac{ca}{3b^2+bp+2p^2}+\frac{ab}{3c^2+cp+2p^2}\le\frac{1}{8},\qquad p=a+b+c.$$

We need to prove that $f_6(a,b,c)\ge 0$, where

$$f_6(a,b,c)=\prod(3a^2+ap+2p^2)-8\sum bc(3b^2+bp+2p^2)(3c^2+cp+2p^2).$$

Clearly, $f_6(a,b,c)$ has the same highest coefficient as

$$27a^2b^2c^2-72\sum b^3c^3;$$

that is,

$$A=27-216<0.$$

By P 3.76-(a) in Volume 1, it suffices to prove the homogeneous inequality for $b=c=1$, and for $a=0$.

For $b = c = 1$, the homogeneous inequality reduces to

$$\frac{1}{2(3a^2+5a+4)} + \frac{2a}{2a^2+9a+13} \le \frac{1}{8},$$

$$6a^4 - 11a^3 + 4a^2 + a \ge 0,$$

$$a(6a+1)(a-1)^2 \ge 0.$$

For $a = 0$, the homogeneous inequality can be written as

$$\frac{bc}{2(b+c)^2} \le \frac{1}{8},$$

$$(b-c)^2 \ge 0.$$

The equality holds for $a = b = c = 1$, and for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

$\square$

P 1.139. *If a, b, c are nonnegative real numbers such that $ab + bc + ca = 3$, then*

$$\frac{1}{8a^2-2bc+21} + \frac{1}{8b^2-2ca+21} + \frac{1}{8c^2-2ab+21} \ge \frac{1}{9}.$$

(Michael Rozenberg, 2013)

Solution. Write the inequality in the homogeneous form

$$\frac{1}{8a^2-2bc+7q} + \frac{1}{8b^2-2ca+7q} + \frac{1}{8c^2-2ab+7q} \ge \frac{1}{3q}, \qquad q = ab+bc+ca.$$

We need to prove that $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c) = 3q\sum(8b^2-2ca+7q)(8c^2-2ab+7q) - \prod(8a^2-2bc+7q).$$

Clearly, $f_6(a,b,c)$ has the same highest coefficient as $P_2(a,b,c)$, where

$$P_2(a,b,c) = -\prod(8a^2-2bc).$$

According to Remark 2 from the proof of P 2.75 in Volume 1, we have

$$A = P_2(1,1,1) = -6^3 < 0.$$

By P 3.76-(a) in Volume 1, it suffices to prove the homogeneous inequality for $b = c = 1$, and for $a = 0$.

For $b = c = 1$, the homogeneous inequality reduces to

$$\frac{1}{8a^2+14a+5} + \frac{2}{12a+15} \ge \frac{1}{3(2a+1)},$$

$$\frac{1}{(4a+5)(2a+1)}+\frac{2}{3(4a+5)} \geq \frac{1}{3(2a+1)},$$

which is an identity.

For $a=0$, the homogeneous inequality can be written as

$$\frac{1}{b(8b+7c)}+\frac{1}{c(8c+7b)} \geq \frac{2}{15bc},$$

$$\frac{c}{8b+7c}+\frac{b}{8c+7b} \geq \frac{2}{15},$$

$$(b-c)^2 \geq 0.$$

The equality holds when two of a,b,c are equal.

Remark. The following identity holds for $ab+bc+ca=3$:

$$\sum \frac{9}{8a^2-2bc+21}-1=\frac{8\prod(a-b)^2}{\prod(8a^2-2bc+21)}.$$

□

P 1.140. *Let a,b,c be real numbers, no two of which are zero. Prove that*

(a) $$\frac{a^2+bc}{b^2+c^2}+\frac{b^2+ca}{c^2+a^2}+\frac{c^2+ab}{a^2+b^2} \geq \frac{(a+b+c)^2}{a^2+b^2+c^2};$$

(b) $$\frac{a^2+3bc}{b^2+c^2}+\frac{b^2+3ca}{c^2+a^2}+\frac{c^2+3ab}{a^2+b^2} \geq \frac{6(ab+bc+ca)}{a^2+b^2+c^2}.$$

(Vasile Cîrtoaje, 2014)

Solution. (a) Using the known inequality

$$\sum \frac{a^2}{b^2+c^2} \geq \frac{3}{2}$$

and the Cauchy-Schwarz inequality yields

$$\sum \frac{a^2+bc}{b^2+c^2}=\sum \frac{a^2}{b^2+c^2}+\sum \frac{bc}{b^2+c^2} \geq \sum\left(\frac{1}{2}+\frac{bc}{b^2+c^2}\right)$$

$$=\sum \frac{(b+c)^2}{2(b^2+c^2)} \geq \frac{\left[\sum(b+c)\right]^2}{\sum 2(b^2+c^2)}=\frac{(a+b+c)^2}{a^2+b^2+c^2}.$$

The equality holds for $a=b=c$.

(b) We have

$$\sum \frac{a^2+3bc}{b^2+c^2} = \sum \frac{a^2}{b^2+c^2} + \sum \frac{3bc}{b^2+c^2} \ge \frac{3}{2} + \sum \frac{3bc}{b^2+c^2}$$
$$= -3 + 3\sum\left(\frac{1}{2} + \frac{bc}{b^2+c^2}\right) = -3 + 3\sum \frac{(b+c)^2}{2(b^2+c^2)}$$
$$\ge -3 + \frac{3\left[\sum(b+c)\right]^2}{\sum 2(b^2+c^2)} = -3 + \frac{3\left(\sum a\right)^2}{\sum a^2} = \frac{6(ab+bc+ca)}{a^2+b^2+c^2}.$$

The equality holds for $a = b = c$.

□

P 1.141. *Let a, b, c be real numbers such that $ab + bc + ca \ge 0$ and no two of which are zero. Prove that*

$$\frac{a(b+c)}{b^2+c^2} + \frac{b(c+a)}{c^2+a^2} + \frac{c(a+b)}{a^2+b^2} \ge \frac{3}{10}.$$

(Vasile Cîrtoaje, 2014)

Solution. Since the problem remains unchanged by replacing a, b, c with $-a, -b, -c$, it suffices to consider the cases $a, b, c \ge 0$ and $a < 0,\ b \ge 0,\ c \ge 0$.

Case 1: $a, b, c \ge 0$. We have

$$\sum \frac{a(b+c)}{b^2+c^2} \ge \sum \frac{a(b+c)}{(b+c)^2}$$
$$= \sum \frac{a}{b+c} \ge \frac{3}{2} > \frac{3}{10}.$$

Case 2: $a < 0,\ b \ge 0,\ c \ge 0$. Replacing a by $-a$, we need to show that

$$\frac{b(c-a)}{a^2+c^2} + \frac{c(b-a)}{a^2+b^2} - \frac{a(b+c)}{b^2+c^2} \ge \frac{3}{10},$$

where

$$a, b, c \ge 0, \qquad a \le \frac{bc}{b+c}.$$

We show first that

$$\frac{b(c-a)}{a^2+c^2} \ge \frac{b(c-x)}{x^2+c^2},$$

where $x = \dfrac{bc}{b+c}$, $x \ge a$. This is equivalent to

$$b(x-a)[(c-a)x + ac + c^2] \ge 0,$$

which is true because

$$(c-a)x+ac+c^2=\frac{c^2(a+2b+c)}{b+c}\geq 0.$$

Similarly, we can show that

$$\frac{c(b-a)}{a^2+b^2}\geq\frac{c(b-x)}{x^2+b^2}.$$

In addition, since

$$\frac{a(b+c)}{b^2+c^2}\leq\frac{x(b+c)}{b^2+c^2}.$$

it suffices to prove that

$$\frac{b(c-x)}{x^2+c^2}+\frac{c(b-x)}{x^2+b^2}-\frac{x(b+c)}{b^2+c^2}\geq\frac{3}{10}.$$

Denote

$$p=\frac{b}{b+c},\quad q=\frac{c}{b+c},\quad p+q=1.$$

Since

$$\frac{b(c-x)}{x^2+c^2}=\frac{p}{1+p^2},\quad \frac{c(b-x)}{x^2+b^2}=\frac{q}{1+q^2},$$

$$\frac{x(b+c)}{b^2+c^2}=\frac{bc}{b^2+c^2}=\frac{pq}{1-2pq},$$

we need to show that

$$\frac{p}{1+p^2}+\frac{q}{1+q^2}-\frac{pq}{1-2pq}\geq\frac{3}{10}.$$

This inequality is equivalent to

$$\frac{1+pq}{2-2pq+p^2q^2}-\frac{pq}{1-2pq}\geq\frac{3}{10},$$

$$(pq+2)^2(1-4pq)\geq 0.$$

Since

$$1-4pq=(p+q)^2-4pq=(p-q)^2\geq 0,$$

the proof is completed. The equality holds for $-2a=b=c$ (or any cyclic permutation).

□

P 1.142. *If a,b,c are positive real numbers such that $abc>1$, then*

$$\frac{1}{a+b+c-3}+\frac{1}{abc-1}\geq\frac{4}{ab+bc+ca-3}.$$

(Vasile Cîrtoaje, 2011)

Solution *(by Vo Quoc Ba Can)*. By the AM-GM inequality, we have

$$a+b+c \ge 3\sqrt[3]{abc} > 3,$$

$$ab+bc+ca \ge \sqrt[3]{a^2b^2c^2} > 3.$$

Without loss of generality, assume that $a = \min\{a,b,c\}$. By the Cauchy-Schwarz inequality, we have

$$\left(\frac{1}{a+b+c-3}+\frac{1}{abc-1}\right)\left[a(a+b+c-3)+\frac{abc-1}{a}\right] \ge \left(\sqrt{a}+\frac{1}{\sqrt{a}}\right)^2.$$

Therefore, it suffices to prove that

$$\frac{(a+1)^2}{4a} \ge \frac{a(a+b+c-3)+\dfrac{abc-1}{a}}{ab+bc+ca-3}.$$

Since

$$a(a+b+c-3)+\frac{abc-1}{a} = ab+bc+ca-3+\frac{(a-1)^3}{a},$$

this inequality can be written as follows:

$$\frac{(a+1)^2}{4a}-1 \ge \frac{(a-1)^3}{a(ab+bc+ca-3)},$$

$$\frac{(a-1)^2}{4a} \ge \frac{(a-1)^3}{a(ab+bc+ca-3)},$$

$$(a-1)^2(ab+bc+ca+1-4a) \ge 0.$$

This is true since

$$bc \ge \sqrt[3]{(abc)^2} > 1,$$

hence

$$ab+bc+ca+1-4a > a^2+1+a^2+1-4a = 2(a-1)^2 \ge 0.$$

The equality holds for $a=b=1$ and $c>1$ (or any cyclic permutation).

Remark. Using this inequality, we can prove P 3.84 in Volume 1, which states that

$$(a+b+c-3)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-3\right)+abc+\frac{1}{abc} \ge 2$$

for any positive real numbers a,b,c. This inequality is clearly true for $abc=1$. In addition, it remains unchanged by substituting a, b, c with $1/a$, $1/b$, $1/c$, respectively. Therefore, it suffices to consider the case $abc>1$. Since $a+b+c \ge 3\sqrt[3]{abc} > 3$, we can write the required inequality as $E \ge 0$, where

$$E = ab+bc+ca-3abc+\frac{(abc-1)^2}{a+b+c-3}.$$

According to the inequality in P 1.142, we have

$$E \ge ab+bc+ca-3abc+(abc-1)^2\left(\frac{4}{ab+bc+ca-3}-\frac{1}{abc-1}\right)$$

$$=(ab+bc+ca-3)+\frac{4(abc-1)^2}{ab+bc+ca-3}-4(abc-1)$$

$$\ge 2\sqrt{(ab+bc+ca-3)\cdot\frac{4(abc-1)^2}{ab+bc+ca-3}}-4(abc-1)=0.$$

□

P 1.143. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\sum\frac{(4b^2-ac)(4c^2-ab)}{b+c}\le\frac{27}{2}abc.$$

(Vasile Cîrtoaje, 2011)

Solution. Use the SOS method. Since

$$\begin{aligned}\sum\frac{(4b^2-ac)(4c^2-ab)}{b+c}&=\sum\frac{bc(16bc+a^2)}{b+c}-4\sum\frac{a(b^3+c^3)}{b+c}\\&=\sum\frac{bc(16bc+a^2)}{b+c}-4\sum a(b^2+c^2)+12abc\\&=\sum bc\left[\frac{a^2}{b+c}+\frac{16bc}{b+c}-4(b+c)\right]+12abc\\&=\sum bc\left[\frac{a^2}{b+c}-4\frac{(b-c)^2}{b+c}\right]+12abc\end{aligned}$$

we can write the inequality as follows:

$$\sum bc\left[\frac{a}{2}-\frac{a^2}{b+c}+\frac{4(b-c)^2}{b+c}\right]\ge 0,$$

$$8\sum\frac{bc(b-c)^2}{b+c}\ge abc\sum\frac{2a-b-c}{b+c}.$$

In addition, since

$$\begin{aligned}\sum\frac{2a-b-c}{b+c}&=\sum\frac{(a-b)+(a-c)}{b+c}=\sum\frac{a-b}{b+c}+\sum\frac{b-a}{c+a}\\&=\sum\frac{(a-b)^2}{(b+c)(c+a)}=\sum\frac{(b-c)^2}{(c+a)(a+b)},\end{aligned}$$

the inequality can be restated as

$$8\sum\frac{bc(b-c)^2}{b+c}\ge abc\sum\frac{(b-c)^2}{(c+a)(a+b)},$$

$$\sum \frac{bc(b-c)^2(8a^2+8bc+7ab+7ac)}{(a+b)(b+c)(c+a)} \geq 0.$$

Since the last form is obvious, the proof is completed. The equality holds for $a = b = c$, and also for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 1.144. *Let a, b, c be nonnegative real numbers, no two of which are zero, such that*

$$a+b+c=3.$$

Prove that

$$\frac{a}{3a+bc}+\frac{b}{3b+ca}+\frac{c}{3c+ab} \geq \frac{2}{3}.$$

Solution. Since

$$3a+bc = a(a+b+c)+bc = (a+b)(a+c),$$

we can write the inequality as follows:

$$a(b+c)+b(c+a)+c(a+b) \geq \frac{2}{3}(a+b)(b+c)(c+a),$$

$$6(ab+bc+ca) \geq 2[(a+b+c)(ab+bc+ca)-abc],$$

$$2abc \geq 0.$$

The equality holds for $a = 0$, or $b = 0$, or $c = 0$.

□

P 1.145. *Let a, b, c be positive real numbers such that*

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = 10.$$

Prove that

$$\frac{19}{12} \leq \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \leq \frac{5}{3}.$$

(Vasile Cîrtoaje, 2012)

First Solution. Write the hypothesis

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=10$$

as

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}=7$$

and

$$(a+b)(b+c)(c+a)=9abc.$$

Using the substitution

$$x=\frac{b+c}{a}, \qquad y=\frac{c+a}{b}, \qquad z=\frac{a+b}{c},$$

we need to show that $x+y+z=7$ and $xyz=9$ involve

$$\frac{19}{12}\le\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\le\frac{5}{3},$$

or, equivalently,

$$\frac{19}{12}\le\frac{1}{x}+\frac{x(7-x)}{9}\le\frac{5}{3}.$$

Clearly, $x,y,z\in(0,7)$. The left inequality is equivalent to

$$(x-4)(2x-3)^2\le 0,$$

while the right inequality is equivalent to

$$(x-1)(x-3)^2\ge 0.$$

These inequalities are true if $1\le x\le 4$. To show that $1\le x\le 4$, from $(y+z)^2\ge 4yz$, we get

$$(7-x)^2\ge\frac{36}{x},$$

$$(x-1)(x-4)(x-9)\ge 0,$$

$$1\le x\le 4.$$

Thus, the proof is completed. The left inequality is an equality for $2a=b=c$ (or any cyclic permutation), and the right inequality is an equality for $a/2=b=c$ (or any cyclic permutation).

Second Solution. Due to homogeneity, assume that $b+c=2$; this involves $bc\le 1$. From the hypothesis

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=10,$$

we get

$$bc = \frac{2a(a+2)}{9a-2}.$$

Since

$$bc - 1 = \frac{(a-2)(2a-1)}{9a-2},$$

from the condition $bc \le 1$, we get

$$\frac{1}{2} \le a \le 2.$$

We have

$$\begin{aligned}\frac{b}{c+a} + \frac{c}{a+b} &= \frac{a(b+c)+b^2+c^2}{a^2+(b+c)a+bc} = \frac{2a+4-2bc}{a^2+2a+bc}\\ &= \frac{2(7a^2+12a-4)}{9a^2(a+2)} = \frac{2(7a-2)}{9a^2},\end{aligned}$$

hence

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{a}{2} + \frac{2(7a-2)}{9a^2} = \frac{9a^3+28a-8}{18a^2}.$$

Thus, we need to show that

$$\frac{19}{12} \le \frac{9a^3+28a-8}{18a^2} \le \frac{5}{3}.$$

These inequalities are true, since the left inequality is equivalent to

$$(2a-1)(3a-4)^2 \ge 0,$$

and the right inequality is equivalent to

$$(a-2)(3a-2)^2 \le 0.$$

Remark. Similarly, we can prove the following generalization.

- *Let a, b, c be positive real numbers such that*

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = 9 + \frac{8k^2}{1-k^2},$$

where $k \in (0,1)$. Then,

$$\frac{k^2}{1+k} \le \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - \frac{3}{2} \le \frac{k^2}{1-k}.$$

□

P 1.146. *Let a,b,c be nonnegative real numbers, no two of which are zero, such that $a+b+c=3$. Prove that*

$$\frac{9}{10}<\frac{a}{2a+bc}+\frac{b}{2b+ca}+\frac{c}{2c+ab}\le 1.$$

(Vasile Cîrtoaje, 2012)

Solution. (a) Since

$$\frac{a}{2a+bc}-\frac{1}{2}=\frac{-bc}{2(2a+bc)},$$

we can write the right inequality as

$$\sum\frac{bc}{2a+bc}\ge 1.$$

According to the Cauchy-Schwarz inequality, we have

$$\sum\frac{bc}{2a+bc}\ge\frac{\left(\sum bc\right)^2}{\sum bc(2a+bc)}=\frac{\sum b^2c^2+2abc\sum a}{6abc+\sum b^2c^2}=1.$$

The equality holds for $a=b=c=1$, and also for $a=0$, or $b=0$, or $c=0$.

(b) **First Solution.** For the nontrivial case $a,b,c>0$, we can write the left inequality as

$$\sum\frac{1}{2+\dfrac{bc}{a}}>\frac{9}{10}.$$

Using the substitution

$$x=\sqrt{\frac{bc}{a}},\quad y=\sqrt{\frac{ca}{b}},\quad z=\sqrt{\frac{ab}{c}},$$

we need to show that

$$\sum\frac{1}{2+x^2}>\frac{9}{10}$$

for all positive real numbers x,y,z satisfying $xy+yz+zx=3$. By expanding, the inequality becomes

$$4\sum x^2+48>9x^2y^2z^2+8\sum x^2y^2.$$

Since

$$\sum x^2y^2=\left(\sum xy\right)^2-2xyz\sum x=9-2xyz\sum x,$$

we can write the desired inequality as

$$4\sum x^2+16xyz\sum x>9x^2y^2z^2+24,$$

which is equivalent to

$$4(p^2-12)+16xyzp > 9x^2y^2z^2,$$

where $p = x+y+z$. Using Schur's inequality

$$p^3+9xyz \ge 4p(xy+yz+zx),$$

which is equivalent to

$$p(p^2-12) \ge -9xyz,$$

it suffices to prove that

$$-\frac{36xyz}{p}+16xyzp > 9x^2y^2z^2.$$

This is true if

$$-\frac{36}{p}+16p > 9xyz.$$

Since

$$x+y+z \ge \sqrt{3(xy+yz+zx)} = 3$$

and

$$1 = \frac{xy+yz+zx}{3} \ge \sqrt[3]{x^2y^2z^2},$$

we have

$$-\frac{36}{p}+16p-9xyz \ge -\frac{36}{3}+48-9 > 0.$$

Second Solution. As it is shown at the first solution, it suffices to show that

$$\sum \frac{1}{2+x^2} > \frac{9}{10}$$

for all positive real numbers x, y, z satisfying $xy+yz+zx = 3$. Rewrite this inequality as

$$\sum \frac{x^2}{2+x^2} < \frac{6}{5}.$$

Let p and q be two positive real numbers such that

$$p+q = \sqrt{3}.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{x^2}{2+x^2} = \frac{3x^2}{2(xy+yz+zx)+3x^2} = \frac{(px+qx)^2}{2x(x+y+z)+(x^2+2yz)}$$

$$\le \frac{p^2x}{2(x+y+z)}+\frac{q^2x^2}{x^2+2yz}.$$

Therefore,

$$\sum \frac{x^2}{2+x^2} \le \sum \frac{p^2x}{2(x+y+z)} + \sum \frac{q^2x^2}{x^2+2yz} = \frac{p^2}{2} + q^2 \sum \frac{x^2}{x^2+2yz}.$$

Thus, it suffices to prove that

$$\frac{p^2}{2} + q^2 \sum \frac{x^2}{x^2+2yz} < \frac{6}{5}.$$

We claim that

$$\sum \frac{x^2}{x^2+2yz} < 2.$$

Under this assumption, we only need to show that

$$\frac{p^2}{2} + 2q^2 \le \frac{6}{5}.$$

Indeed, choosing $p = \dfrac{4\sqrt{3}}{5}$ and $q = \dfrac{\sqrt{3}}{5}$, we have $p+q = \sqrt{3}$ and $\dfrac{p^2}{2} + 2q^2 = \dfrac{6}{5}$. To complete the proof, we need to prove the homogeneous inequality $\sum \dfrac{x^2}{x^2+2yz} < 2$, which is equivalent to

$$\sum \frac{yz}{x^2+2yz} > \frac{1}{2}.$$

By the Cauchy-Schwarz inequality, we get

$$\sum \frac{yz}{x^2+2yz} \ge \frac{\left(\sum yz\right)^2}{\sum yz(x^2+2yz)} = \frac{\sum y^2z^2 + 2xyz\sum x}{xyz\sum x + 2\sum y^2z^2} > \frac{1}{2}.$$

□

P 1.147. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a^3}{2a^2+bc} + \frac{b^3}{2b^2+ca} + \frac{c^3}{2c^2+ab} \le \frac{a^3+b^3+c^3}{a^2+b^2+c^2}.$$

(Vasile Cîrtoaje, 2011)

Solution. Use the SOS method. Write the inequality as follows:

$$\sum \left[\frac{a^3}{a^2+b^2+c^2} - \frac{a^3}{2a^2+bc} \right] \ge 0,$$

$$\sum \frac{a^3(a^2+bc-b^2-c^2)}{2a^2+bc} \ge 0,$$

$$\sum \frac{a^3[a^2(b+c)-b^3-c^3]}{(b+c)(2a^2+bc)} \ge 0,$$

$$\sum \frac{a^3b(a^2-b^2)+a^3c(a^2-c^2)}{(b+c)(2a^2+bc)} \ge 0,$$

$$\sum \frac{a^3b(a^2-b^2)}{(b+c)(2a^2+bc)} + \sum \frac{b^3a(b^2-a^2)}{(c+a)(2b^2+ca)} \ge 0,$$

$$\sum \frac{ab(a+b)(a-b)^2[2a^2b^2+c(a^3+a^2b+ab^2+b^3)+c^2(a^2+ab+b^2)]}{(b+c)(c+a)(2a^2+bc)(2b^2+ca)} \ge 0.$$

The equality holds for $a = b = c$, and also for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 1.148. *If a, b, c are positive real numbers, then*

$$\frac{a^3}{4a^2+bc} + \frac{b^3}{4b^2+ca} + \frac{c^3}{4c^2+ab} \ge \frac{a+b+c}{5}.$$

(Vasile Cîrtoaje, 2011)

Solution. Use the SOS method. Write the inequality as follows:

$$\sum \left(\frac{a^3}{4a^2+bc} - \frac{a}{5} \right) \ge 0,$$

$$\sum \frac{a(a^2-bc)}{4a^2+bc} \ge 0,$$

$$\sum \frac{a[(a-b)(a+c)+(a-c)(a+b)]}{4a^2+bc} \ge 0,$$

$$\sum \frac{a(a-b)(a+c)}{4a^2+bc} + \sum \frac{b(b-a)(b+c)}{4b^2+ca} \ge 0,$$

$$\sum \frac{c(a-b)^2[(a-b)^2+bc+ca-ab]}{(4a^2+bc)(4b^2+ca)} \ge 0.$$

Clearly, it suffices to show that

$$\sum \frac{c(a-b)^2(bc+ca-ab)}{(4a^2+bc)(4b^2+ca)} \ge 0,$$

which can be written as

$$\sum (a-b)^2(bc+ca-ab)(4c^3+abc) \ge 0.$$

Assume that $a \ge b \ge c$. Since $ca+ab-bc > 0$, it is enough to prove that

$$(c-a)^2(ab+bc-ca)(4b^3+abc)+(a-b)^2(bc+ca-ab)(4c^3+abc) \ge 0,$$

which is equivalent to

$$(a-c)^2(ab+bc-ca)(4b^3+abc) \ge (a-b)^2(ab-bc-ca)(4c^3+abc).$$

This inequality is true since $ab+bc-ca>0$ and

$$(a-c)^2 \ge (a-b)^2, \quad 4b^3+abc \ge 4c^3+abc, \quad ab+bc-ca \ge ab-bc-ca.$$

The equality holds for $a=b=c$.

□

P 1.149. *If a,b,c are positive real numbers, then*

$$\frac{1}{(2+a)^2}+\frac{1}{(2+b)^2}+\frac{1}{(2+c)^2} \ge \frac{3}{6+ab+bc+ca}.$$

(Vasile Cîrtoaje, 2013)

Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{1}{(2+a)^2} \ge \frac{4(a+b+c)^2}{\sum(2+a)^2(b+c)^2}.$$

Thus, it suffices to show that

$$4(a+b+c)^2(6+ab+bc+ca) \ge 3\sum(2+a)^2(b+c)^2.$$

This inequality is equivalent to

$$2p^2q-3q^2+3pr+12q \ge 6(pq+3r),$$

where

$$p=a+b+c, \quad q=ab+bc+ca, \quad r=abc.$$

According to AM-GM inequality, we have

$$(2p^2q-3q^2+3pr)+12q \ge 2\sqrt{12q(2p^2q-3q^2+3pr)}.$$

Therefore, it is enough to prove the homogeneous inequality

$$4q(2p^2q-3q^2+3pr) \ge 3(pq+3r)^2,$$

which can be written as

$$5p^2q^2 \ge 12q^3+6pqr+27r^2.$$

Since $pq \ge 9r$, we have

$$\begin{aligned} 3(5p^2q^2-12q^3-6pqr-27r^2) &\ge 15p^2q^2-36q^3-2p^2q^2-p^2q^2 \\ &= 12q^2(p^2-3q) \ge 0. \end{aligned}$$

The equality holds for $a=b=c=1$.

□

P 1.150. *If a, b, c are positive real numbers, then*

$$\frac{1}{1+3a}+\frac{1}{1+3b}+\frac{1}{1+3c}\geq\frac{3}{3+abc}.$$

(Vasile Cîrtoaje, 2013)

Solution. Set

$$p=a+b+c,\quad q=ab+bc+ca,\quad r=\sqrt[3]{abc},$$

and write the inequality as follows:

$$(3+r^3)\sum(1+3b)(1+3c)\geq 3(1+3a)(1+3b)(1+3c),$$

$$(3+r^3)(3+6p+9q)\geq 3(1+3p+9q+27r^3),$$

$$r^3(2p+3q)+2+3p\geq 26r^3.$$

By virtue of the AM-GM inequality, we have

$$p\geq 3r,\quad q\geq 3r^2.$$

Therefore, it suffices to show that

$$r^3(6r+9r^2)+2+9r\geq 26r^3,$$

which is equivalent to the obvious inequality

$$(r-1)^2(9r^3+24r^2+13r+2)\geq 0.$$

The equality holds for $a=b=c=1$.

□

P 1.151. *Let a, b, c be real numbers, no two of which are zero. If $1<k\leq 3$, then*

$$\left(k+\frac{2ab}{a^2+b^2}\right)\left(k+\frac{2bc}{b^2+c^2}\right)\left(k+\frac{2ca}{c^2+a^2}\right)\geq(k-1)(k^2-1).$$

(Vasile Cîrtoaje and Vo Quoc Ba Can, 2011)

Solution. If a, b, c have the same sign, then

$$\left(k+\frac{2ab}{a^2+b^2}\right)\left(k+\frac{2bc}{b^2+c^2}\right)\left(k+\frac{2ca}{c^2+a^2}\right)>k^3>(k-1)(k^2-1).$$

Since the inequality remains unchanged by replacing a, b, c with $-a, -b, -c$, it suffices to consider further that $a\leq 0$ and $b, c\geq 0$. Setting $-a$ for a, we need to show that

$$\left(k-\frac{2ab}{a^2+b^2}\right)\left(k+\frac{2bc}{b^2+c^2}\right)\left(k-\frac{2ca}{c^2+a^2}\right)\geq(k-1)(k^2-1)$$

for $a,b,c \ge 0$. Since

$$\left(k-\frac{2ab}{a^2+b^2}\right)\left(k-\frac{2ca}{c^2+a^2}\right)=\left[k-1+\frac{(a-b)^2}{a^2+b^2}\right]\left[k-1+\frac{(a-c)}{c^2+a^2}\right]$$

$$\ge (k-1)^2+(k-1)\left[\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{c^2+a^2}\right],$$

it suffices to prove that

$$\left[k-1+\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{c^2+a^2}\right]\left(k+\frac{2bc}{b^2+c^2}\right)\ge k^2-1.$$

According to Lemma below, we have

$$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{c^2+a^2}\ge\frac{(b-c)^2}{(b+c)^2}.$$

Thus, it suffices to show that

$$\left[k-1+\frac{(b-c)^2}{(b+c)^2}\right]\left(k+\frac{2bc}{b^2+c^2}\right)\ge k^2-1,$$

which is equivalent to the obvious inequality

$$(b-c)^4+2(3-k)bc(b-c)^2\ge 0.$$

The equality holds for $a=b=c$.

Lemma. *If $a,b,c \ge 0$, no two of which are zero, then*

$$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{a^2+c^2}\ge\frac{(b-c)^2}{(b+c)^2}.$$

Proof. Consider two cases: $a^2\le bc$ and $a^2\ge bc$.

Case 1: $a^2\le bc$. By the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2)}{a^2+c^2}\ge\frac{[(b-a)+(a-c)]^2}{(a^2+b^2)+(a^2+c^2)}=\frac{(b-c)^2}{2a^2+b^2+c^2}.$$

Thus, it suffices to show that

$$\frac{1}{2a^2+b^2+c^2}\ge\frac{1}{(b+c)^2},$$

which is equivalent to $a^2\le bc$.

Case 2: $a^2\ge bc$. By the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{a^2+c^2}\ge\frac{[c(b-a)+b(a-c)]^2}{c^2(a^2+b^2)+b^2(a^2+c^2)}=\frac{a^2(b-c)^2}{a^2(b^2+c^2)+2b^2c^2}.$$

Therefore, it suffices to prove that

$$\frac{a^2}{a^2(b^2+c^2)+2b^2c^2} \geq \frac{1}{(b+c)^2},$$

which reduces to $bc(a^2-bc) \geq 0$.

□

P 1.152. *If a,b,c are non-zero and distinct real numbers, then*

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+3\left[\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}\right] \geq 4\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right).$$

Solution. Write the inequality as

$$\left(\sum\frac{1}{a^2}-\sum\frac{1}{bc}\right)+3\sum\frac{1}{(b-c)^2} \geq 3\sum\frac{1}{bc}.$$

In virtue of the AM-GM inequality, it suffices to prove that

$$2\sqrt{3\left(\sum\frac{1}{a^2}-\sum\frac{1}{bc}\right)\left[\sum\frac{1}{(b-c)^2}\right]} \geq 3\sum\frac{1}{bc},$$

which is true if

$$4\left(\sum\frac{1}{a^2}-\sum\frac{1}{bc}\right)\left[\sum\frac{1}{(b-c)^2}\right] \geq 3\left(\sum\frac{1}{bc}\right)^2.$$

Since

$$\sum\frac{1}{(b-c)^2}=\left(\sum\frac{1}{b-c}\right)^2=\frac{\left(\sum a^2-\sum ab\right)^2}{(a-b)^2(b-c)^2(c-a)^2},$$

we can rewrite this inequality as

$$4\left(\sum a^2b^2-abc\sum a\right)\left(\sum a^2-\sum ab\right)^2 \geq 3(a+b+c)^2(a-b)^2(b-c)^2(c-a)^2.$$

Using the notations

$$p=a+b+c,\quad q=ab+bc+ca,\quad r=abc,$$

and the identity

$$(a-b)^2(b-c)^2(c-a)^2=-27r^2-2(2p^2-9q)pr+p^2q^2-4q^3,$$

the inequality can be written as

$$4(q^2-3pr)(p^2-3q)^2 \geq 3p^2[-27r^2-2(2p^2-9q)pr+p^2q^2-4q^3],$$

which is equivalent to

$$(9pr+p^2q-6q^2)^2 \geq 0.$$

□

P 1.153. *Let a, b, c be positive real numbers, and let*

$$A = \frac{a}{b} + \frac{b}{a} + k, \quad B = \frac{b}{c} + \frac{c}{b} + k, \quad C = \frac{c}{a} + \frac{a}{b} + k,$$

where $-2 < k \le 4$. Prove that

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C} \le \frac{1}{k+2} + \frac{4}{A+B+C-k-2}.$$

(Vasile Cîrtoaje, 2009)

Solution. Let us denote

$$x = \frac{a}{b}, \quad y = \frac{b}{c}, \quad z = \frac{c}{a}.$$

We need to show that

$$\sum \frac{x}{x^2+kx+1} \le \frac{1}{k+2} + \frac{4}{\sum x + \sum xy + 2k - 2}$$

for all positive real numbers x, y, z satisfying $xyz = 1$. Write this inequality as follows:

$$\sum \left(\frac{1}{k+2} - \frac{x}{x^2+kx+1} \right) \ge \frac{2}{k+2} - \frac{4}{\sum x + \sum xy + 2k - 2},$$

$$\sum \frac{(x-1)^2}{x^2+kx+1} \ge \frac{2\sum yz(x-1)^2}{\sum x + \sum xy + 2k - 2},$$

$$\sum \frac{(x-1)^2[-x+y+z+x(y+z)-yz-2]}{x^2+kx+1} \ge 0.$$

Since

$$-x+y+z+x(y+z)-yz-2 = (x+1)(y+z)-(x+yz+2)$$

$$= (x+1)(y+z)-(x+1)(yz+1) = -(x+1)(y-1)(z-1),$$

the inequality is equivalent to

$$-(x-1)(y-1)(z-1)\sum \frac{x^2-1}{x^2+kx+1} \ge 0;$$

that is, $E \ge 0$, where

$$E = -(x-1)(y-1)(z-1)\sum (x^2-1)(y^2+ky+1)(z^2+kz+1).$$

We have

$$\sum (x^2-1)(y^2+ky+1)(z^2+kz+1) =$$

$$= k(k-2)\left(\sum x - \sum xy\right) + \left(\sum x^2y^2 - \sum x^2\right)$$

$$= k(k-2)(x-1)(y-1)(z-1) - (x^2-1)(y^2-1)(z^2-1)$$
$$= -(x-1)(y-1)(z-1)[(x+1)(y+1)(z+1) - k(k-2)],$$

hence

$$E = (x-1)^2(y-1)^2(z-1)^2[(x+1)(y+1)(z+1) - k(k-2)].$$

Since

$$(x+1)(y+1)(z+1) - k(k-2) \geq (2\sqrt{x})(2\sqrt{y})(2\sqrt{z}) - k(k-2)$$
$$= (2+k)(4-k) \geq 0,$$

it follows that $E \geq 0$. The equality holds for $a = b$, or $b = c$, or $c = a$.

□

P 1.154. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

$$\frac{1}{b^2+bc+c^2} + \frac{1}{c^2+ca+a^2} + \frac{1}{a^2+ab+b^2} \geq \frac{1}{2a^2+bc} + \frac{1}{2b^2+ca} + \frac{1}{2c^2+ab}.$$

(Vasile Cîrtoaje, 2014)

Solution. Write the inequality as follows:

$$\sum \left(\frac{1}{b^2+bc+c^2} - \frac{1}{2a^2+bc} \right) \geq 0,$$

$$\sum \frac{(a^2-b^2)+(a^2-c^2)}{(b^2+bc+c^2)(2a^2+bc)} \geq 0,$$

$$\sum \frac{a^2-b^2}{(b^2+bc+c^2)(2a^2+bc)} + \sum \frac{b^2-a^2}{(c^2+ca+a^2)(2b^2+ca)} \geq 0,$$

$$(a^2+b^2+c^2-ab-bc-ca) \sum \frac{c(a^2-b^2)(a-b)}{(b^2+bc+c^2)(c^2+ca+a^2)(2a^2+bc)(2b^2+ca)} \geq 0.$$

Clearly, the last inequality is obvious. The equality holds for $a = b = c$.

□

P 1.155. *If a, b, c are nonnegative real numbers such that $a+b+c \leq 3$, then*

(a) $$\frac{1}{2a+1} + \frac{1}{2b+1} + \frac{1}{2c+1} \geq \frac{1}{a+2} + \frac{1}{b+2} + \frac{1}{c+2};$$

(b) $$\frac{1}{2ab+1} + \frac{1}{2bc+1} + \frac{1}{2ca+1} \geq \frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2}.$$

(Vasile Cîrtoaje, 2014)

Solution. Denote

$$p = a + b + c, \qquad \sqrt{3q} \le p \le 3,$$
$$q = ab + bc + ca, \qquad 0 \le q \le 3.$$

(a) Use the SOS method. Write the inequality as follows

$$\sum \left(\frac{1}{2a+1} - \frac{1}{a+2} \right) \ge 0,$$

$$\sum \frac{1-a}{(2a+1)(a+2)} \ge 0,$$

$$\sum \frac{(a+b+c)-3a}{(2a+1)(a+2)} \ge 0,$$

$$\sum \frac{(b-a)+(c-a)}{(2a+1)(a+2)} \ge 0,$$

$$\sum \frac{b-a}{(2a+1)(a+2)} + \sum \frac{a-b}{(2b+1)(b+2)} \ge 0,$$

$$\sum (a-b) \left[\frac{1}{(2b+1)(b+2)} - \frac{1}{(2a+1)(a+2)} \right],$$

$$\sum (a-b)^2(2a+2b+5)(2c+1)(c+2) \ge 0.$$

The equality holds for $a = b = c = 1$.

(b) Write the inequality as

$$\sum \frac{1}{2ab+1} \ge \sum \left(\frac{1}{a^2+2} - \frac{1}{2} \right) + \frac{3}{2},$$

$$\sum \frac{2}{2ab+1} + \sum \frac{a^2}{a^2+2} \ge 3.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{1}{2ab+1} \ge \frac{9}{\sum (2ab+1)} = \frac{9}{2q+3}$$

and

$$\sum \frac{a^2}{a^2+2} \ge \frac{\left(\sum a\right)^2}{\sum (a^2+2)} = \frac{p^2}{p^2-2q+6}$$
$$= 1 - \frac{2(3-q)}{p^2-2q+6} \ge 1 - \frac{2(3-q)}{q+6} = \frac{3q}{q+6}.$$

Therefore, it suffices to show that

$$\frac{18}{2q+3} + \frac{3q}{q+6} \ge 3,$$

which is equivalent to the obvious inequality $q \le 3$. The equality holds for $a = b = c = 1$.

$\square$

P 1.156. *If a,b,c are nonnegative real numbers such that $a+b+c=4$, then*

$$\frac{1}{ab+2}+\frac{1}{bc+2}+\frac{1}{ca+2}\ge\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2}.$$

(Vasile Cîrtoaje, 2014)

First Solution *(by Nguyen Van Quy)*. Use the SOS method. Rewrite the inequality as follows:

$$\sum\left(\frac{2}{ab+2}-\frac{1}{a^2+2}-\frac{1}{b^2+2}\right)\ge 0,$$

$$\sum\left[\frac{a(a-b)}{(ab+2)(a^2+2)}+\frac{b(b-a)}{(ab+2)(b^2+2)}\right]\ge 0,$$

$$\sum\frac{(2-ab)(a-b)^2(c^2+2)}{ab+2}\ge 0.$$

Without loss of generality, assume that $a\ge b\ge c\ge 0$. Then,

$$bc\le ac\le\frac{a(b+c)}{2}\le\frac{(a+b+c)^2}{8}=2$$

and

$$\begin{aligned}\sum\frac{(2-ab)(a-b)^2(c^2+2)}{ab+2}&\ge\frac{(2-ab)(a-b)^2(c^2+2)}{ab+2}+\frac{(2-ac)(a-c)^2(b^2+2)}{ac+2}\\&\ge\frac{(2-ab)(a-b)^2(c^2+2)}{ab+2}+\frac{(2-ac)(a-b)^2(c^2+2)}{ab+2}\\&=\frac{(4-ab-ac)(a-b)^2(c^2+2)}{ab+2}\\&=\frac{(a-b-c)^2(a-b)^2(c^2+2)}{4(ab+2)}\end{aligned}$$

The equality holds for $a=b=c=4/3$, and also for $a=2$ and $b=c=1$ (or any cyclic permutation).

Second Solution. Write the inequality as

$$\sum\frac{1}{bc+2}\ge\sum\left(\frac{1}{a^2+2}-\frac{1}{2}\right)+\frac{3}{2},$$

$$\sum\frac{1}{bc+2}+\sum\frac{a^2}{2(a^2+2)}\ge\frac{3}{2}.$$

Assume that $a\ge b\ge c$, and denote

$$s=\frac{b+c}{2},\quad p=bc,\qquad 0\le s\le\frac{4}{3},\ \ 0\le p\le s^2.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{b^2}{2(b^2+2)}+\frac{c^2}{2(c^2+2)} \geq \frac{(b+c)^2}{2(b^2+2)+2(c^2+2)+4}=\frac{s^2}{2s^2-p+2}.$$

In addition,

$$\frac{1}{ca+2}+\frac{1}{ab+2}=\frac{a(b+c)+4}{(ab+2)(ac+2)}=\frac{2as+4}{a^2p+4as+4}.$$

Therefore, it suffices to show that $E(a,b,c) \geq 0$, where

$$E(a,b,c)=\frac{1}{p+2}+\frac{s^2}{2s^2-p+2}+\frac{2(as+2)}{a^2p+4as+4}+\frac{a^2}{2(a^2+2)}-\frac{3}{2}.$$

Use the mixing variables method. We will prove that

$$E(a,b,c) \geq E(a,s,s) \geq 0.$$

We have

$$\begin{aligned}
E(a,b,c)-E(a,s,s)=&\left(\frac{1}{p+2}-\frac{1}{s^2+2}\right)+s^2\left(\frac{1}{2s^2-p+2}-\frac{1}{s^2+2}\right)\\
&+2(as+2)\left(\frac{1}{a^2p+4as+4}-\frac{1}{a^2s^2+4as+4}\right)\\
=&\frac{s^2-p}{(p+2)(s^2+2)}-\frac{s^2(s^2-p)}{(s^2+2)(2s^2-p+2)}\\
&+\frac{2a^2(s^2-p)}{(a^2p+4as+4)(as+2)}.
\end{aligned}$$

Since $s^2-p \geq 0$, we need to show that

$$\frac{1}{(p+2)(s^2+2)}+\frac{2a^2}{(a^2p+4as+4)(as+2)} \geq \frac{s^2}{(s^2+2)(2s^2-p+2)},$$

which is equivalent to

$$\frac{2a^2}{(a^2p+4as+4)(as+2)} \geq \frac{p(s^2+1)-2}{(p+2)(s^2+2)(2s^2-p+2)}.$$

Since

$$a^2p+4as+4 \leq a^2s^2+4as+4=(as+2)^2$$

and

$$2s^2-p+2 \geq s^2+2,$$

it is enough to prove that

$$\frac{2a^2}{(as+2)^3} \geq \frac{p(s^2+1)-2}{(p+2)(s^2+2)^2}.$$

In addition, since

$$as+2=(4-2s)s+2\le 4$$

and

$$\frac{p(s^2+1)-2}{p+2}=s^2+1-\frac{2(s^2+2)}{p+2}\le s^2+1-\frac{2(s^2+2)}{s^2+2}=s^2-1,$$

it suffices to show that

$$\frac{a^2}{32}\ge\frac{s^2-1}{(s^2+2)^2},$$

which is equivalent to

$$(2-s)^2(2+s^2)^2\ge 8(s^2-1).$$

Indeed, for the nontrivial case $1<s\le\frac{4}{3}$, we have

$$\begin{aligned}(2-s)^2(2+s^2)^2-8(s^2-1)&\ge\left(2-\frac{4}{3}\right)^2(2+s^2)^2-8(s^2-1)\\&=\frac{4}{9}(s^4-14s^2+22)=\frac{4}{9}\left[(7-s^2)^2-27\right]\\&\ge\frac{4}{9}\left[\left(7-\frac{16}{9}\right)^2-27\right]=\frac{88}{729}>0.\end{aligned}$$

To end the proof, we need to show that $E(a,s,s)\ge 0$. We have

$$\begin{aligned}E(a,s,s)&=\frac{1}{s^2+2}+\frac{s^2}{s^2+2}+\frac{2}{as+2}+\frac{a^2}{2(a^2+2)}-\frac{3}{2}\\&=\frac{(s-1)^2(3s-4)^2}{2(s^2+2)(1+2s-s^2)(2s^2-8s+9)}\ge 0.\end{aligned}$$

□

P 1.157. *If a,b,c are nonnegative real numbers, no two of which are zero, then*

(a) $$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(a-b)^2(b-c)^2(c-a)^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}\le 1;$$

(b) $$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(a-b)^2(b-c)^2(c-a)^2}{(a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2)}\le 1.$$

(Vasile Cîrtoaje, 2014)

Solution. (a) ***First Solution.*** Consider the nontrivial case where a,b,c are distinct and write the inequality as follows:

$$\frac{(a-b)^2(b-c)^2(c-a)^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}\le\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2(a^2+b^2+c^2)},$$

$$\frac{(a^2+b^2)+(b^2+c^2)+(c^2+a^2)}{(a^2+b^2)(b^2+c^2)(c^2+a^2)} \le \frac{(a-b)^2+(b-c)^2+(c-a)^2}{(a-b)^2(b-c)^2(c-a)^2},$$

$$\sum \frac{1}{(b^2+c^2)(c^2+a^2)} \le \sum \frac{1}{(b-c)^2(c-a)^2}.$$

Since

$$a^2+b^2 \ge (a-b)^2, \quad b^2+c^2 \ge (b-c)^2, \quad c^2+a^2 \ge (c-a)^2,$$

the conclusion follows. The equality holds for $a=b=c$.

Second Solution. Assume that $a \ge b \ge c$. We have

$$\begin{aligned}\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(a-b)^2(b-c)^2(c-a)^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)} &\le \frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(a-b)^2(a-c)^2}{(a^2+b^2)(a^2+c^2)}\\ &\le \frac{2ab+c^2}{a^2+b^2+c^2}+\frac{(a-b)^2a^2}{a^2(a^2+b^2+c^2)}\\ &= \frac{2ab+c^2+(a-b)^2}{a^2+b^2+c^2}=1.\end{aligned}$$

(b) Consider the nontrivial case where a,b,c are distinct and write the inequality as follows:

$$\frac{(a-b)^2(b-c)^2(c-a)^2}{(a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2)} \le \frac{(a-b)^2+(b-c)^2+(c-a)^2}{2(a^2+b^2+c^2)},$$

$$\frac{2(a^2+b^2+c^2)}{(a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2)} \le \frac{(a-b)^2+(b-c)^2+(c-a)^2}{(a-b)^2(b-c)^2(c-a)^2},$$

$$\sum \frac{1}{(a-b)^2(a-c)^2} \ge \frac{2(a^2+b^2+c^2)}{(a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2)}.$$

Assume that $a=\min\{a,b,c\}$, and use the substitution

$$b=a+x, \quad c=a+y, \quad x,y \ge 0.$$

The inequality can be written as

$$\frac{1}{x^2y^2}+\frac{1}{x^2(x-y)^2}+\frac{1}{y^2(x-y)^2} \ge 2f(a),$$

where

$$f(a)=\frac{3a^2+2(x+y)a+x^2+y^2}{(a^2+xa+x^2)(a^2+ya+y^2)[a^2+(x+y)a+x^2-xy+y^2]}.$$

We will show that

$$\frac{1}{x^2y^2}+\frac{1}{x^2(x-y)^2}+\frac{1}{y^2(x-y)^2} \ge 2f(0) \ge 2f(a).$$

The left inequality is equivalent to

$$\frac{x^2+y^2-xy}{x^2y^2(x-y)^2} \geq \frac{x^2+y^2}{x^2y^2(x^2-xy+y^2)}.$$

Indeed,

$$\frac{x^2+y^2-xy}{x^2y^2(x-y)^2} - \frac{x^2+y^2}{x^2y^2(x^2-xy+y^2)} = \frac{1}{(x-y)^2(x^2-xy+y^2)} \geq 0.$$

Also, since

$$(a^2+xa+x^2)(a^2+ya+y^2) \geq (x^2+y^2)a^2+xy(x+y)a+x^2y^2$$

and

$$a^2+(x+y)a+x^2-xy+y^2 \geq x^2-xy+y^2,$$

we get $f(a) \leq g(a)$, where

$$g(a) = \frac{3a^2+2(x+y)a+x^2+y^2}{[(x^2+y^2)a^2+xy(x+y)a+x^2y^2](x^2-xy+y^2)}.$$

Therefore,

$$\begin{aligned} f(0)-f(a) &\geq \frac{x^2+y^2}{x^2y^2(x^2-xy+y^2)} - g(a) \\ &= \frac{(x^4-x^2y^2+y^4)a^2+xy(x+y)(x-y)^2a}{x^2y^2(x^2-xy+y^2)[(x^2+y^2)a^2+xy(x+y)a+x^2y^2]} \geq 0. \end{aligned}$$

Thus, the proof is completed. The equality holds for $a=b=c$.

□

P 1.158. *If a,b,c are nonnegative real numbers, no two of which are zero, then*

$$\frac{a^2+b^2+c^2}{ab+bc+ca} \geq 1 + \frac{9(a-b)^2(b-c)^2(c-a)^2}{(a+b)^2(b+c)^2(c+a)^2}.$$

(Vasile Cîrtoaje, 2014)

Solution. Consider the nontrivial case where

$$0 \leq a < b < c,$$

and write the inequality as follows:

$$\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2(ab+bc+ca)} \geq \frac{9(a-b)^2(b-c)^2(c-a)^2}{(a+b)^2(b+c)^2(c+a)^2},$$

$$\frac{(a-b)^2+(b-c)^2+(c-a)^2}{(a-b)^2(b-c)^2(c-a)^2} \ge \frac{18(ab+bc+ca)}{(a+b)^2(b+c)^2(c+a)^2},$$

$$\sum \frac{1}{(b-a)^2(c-a)^2} \ge \frac{18(ab+bc+ca)}{(a+b)^2(a+c)^2(b+c)^2}.$$

Since

$$\sum \frac{1}{(b-a)^2(c-a)^2} \ge \frac{1}{b^2c^2} + \frac{1}{b^2(b-c)^2} + \frac{1}{c^2(b-c)^2} = \frac{2(b^2+c^2-bc)}{b^2c^2(b-c)^2}$$

and

$$\frac{ab+bc+ca}{(a+b)^2(a+c)^2(b+c)^2} \le \frac{ab+bc+ca}{(ab+bc+ca)^2(b+c)^2} \le \frac{1}{bc(b+c)^2},$$

it suffices to show that

$$\frac{b^2+c^2-bc}{b^2c^2(b-c)^2} \ge \frac{9}{bc(b+c)^2}.$$

Write this inequality as follows:

$$\frac{(b+c)^2-3bc}{bc} \ge \frac{9(b+c)^2-36bc}{(b+c)^2},$$

$$\frac{(b+c)^2}{bc} - 12 + \frac{36bc}{(b+c)^2} \ge 0,$$

$$(b+c)^4 - 12bc(b+c)^2 + 36b^2c^2 \ge 0,$$

$$[(b+c)^2-6bc]^2 \ge 0.$$

Thus, the proof is completed. The equality holds for $a=b=c$, and also for $a=0$ and $b/c+c/b=4$ (or any cyclic permutation).

□

P 1.159. *If a,b,c are nonnegative real numbers, no two of which are zero, then*

$$\frac{a^2+b^2+c^2}{ab+bc+ca} \ge 1+(1+\sqrt{2})^2\frac{(a-b)^2(b-c)^2(c-a)^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}.$$

(Vasile Cîrtoaje, 2014)

Solution. Consider the nontrivial case where a,b,c are distinct and denote $k = 1+\sqrt{2}$. Write the inequality as follows:

$$\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2(ab+bc+ca)} \ge \frac{k^2(a-b)^2(b-c)^2(c-a)^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)},$$

$$\frac{(a-b)^2+(b-c)^2+(c-a)^2}{(a-b)^2(b-c)^2(c-a)^2} \ge \frac{2k^2(ab+bc+ca)}{(a^2+b^2)(b^2+c^2)(c^2+a^2)},$$

$$\sum \frac{1}{(b-a)^2(c-a)^2} \geq \frac{2k^2(ab+bc+ca)}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}.$$

Assume that $a = \min\{a,b,c\}$, and use the substitution

$$b = a+x, \quad c = a+y, \quad x,y \geq 0.$$

The inequality becomes

$$\frac{1}{x^2y^2} + \frac{1}{x^2(x-y)^2} + \frac{1}{y^2(x-y)^2} \geq 2k^2 f(a),$$

where

$$f(a) = \frac{3a^2+2(x+y)a+xy}{(2a^2+2xa+x^2)(2a^2+2ya+y^2)[2a^2+2(x+y)a+x^2+y^2]}.$$

We will show that

$$\frac{1}{x^2y^2} + \frac{1}{x^2(x-y)^2} + \frac{1}{y^2(x-y)^2} \geq 2k^2 f(0) \geq 2k^2 f(a).$$

We have

$$\begin{aligned} \frac{1}{x^2y^2} + \frac{1}{x^2(x-y)^2} + \frac{1}{y^2(x-y)^2} - 2k^2 f(0) &= \frac{2(x^2+y^2-xy)}{x^2y^2(x-y)^2} - \frac{2k^2xy}{x^2y^2(x^2+y^2)} \\ &= \frac{2[x^2+y^2-(2+\sqrt{2}\,)xy]^2}{x^2y^2(x-y)^2(x^2-xy+y^2)} \geq 0. \end{aligned}$$

Also, since

$$(2a^2+2xa+x^2)(2a^2+2ya+y^2) \geq 2(x^2+y^2)a^2 + 2xy(x+y)a + x^2y^2$$

and

$$2a^2+2(x+y)a+x^2+y^2 \geq x^2+y^2,$$

we get $f(a) \leq g(a)$, where

$$g(a) = \frac{3a^2+2(x+y)a+xy}{[2(x^2+y^2)a^2+2xy(x+y)a+x^2y^2](x^2+y^2)}.$$

Therefore,

$$\begin{aligned} f(0) - f(a) &\geq \frac{1}{xy(x^2+y^2)} - g(a) \\ &= \frac{(2x^2+2y^2-3xy)a^2}{xy(x^2+y^2)[2(x^2+y^2)a^2+2xy(x+y)a+x^2y^2]} \geq 0. \end{aligned}$$

Thus, the proof is completed. The equality holds for $a = b = c$, and also for $a = 0$ and $b/c + c/b = 2 + \sqrt{2}$ (or any cyclic permutation).

□

P 1.160. *If a,b,c are nonnegative real numbers, no two of which are zero, then*

$$\frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a}\ge\frac{5}{3a+b+c}+\frac{5}{3b+c+a}+\frac{5}{3c+a+b}.$$

Solution. Use the SOS method. Write the inequality as follows:

$$\sum\left(\frac{2}{b+c}-\frac{5}{3a+b+c}\right)\ge 0,$$

$$\sum\frac{2a-b-c}{(b+c)(3a+b+c)}\ge 0,$$

$$\sum\frac{a-b}{(b+c)(3a+b+c)}+\sum\frac{a-c}{(b+c)(3a+b+c)}\ge 0,$$

$$\sum\frac{a-b}{(b+c)(3a+b+c)}+\sum\frac{b-a}{(c+a)(3b+c+a)}\ge 0,$$

$$\sum\frac{(a-b)^2(a+b-c)}{(b+c)(c+a)(3a+b+c)(3b+c+a)},$$

$$\sum(b-c)^2S_a\ge 0,$$

where

$$S_a=(b+c-a)(b+c)(3a+b+c).$$

Assume that $a\ge b\ge c$. Since $S_c>0$, it suffices to show that

$$(b-c)^2S_a+(a-c)^2S_b\ge 0.$$

Since $S_b\ge 0$, we have

$$(b-c)^2S_a+(a-c)^2S_b\ge(b-c)^2S_a+(b-c)^2S_b=(b-c)^2(S_a+S_b).$$

Thus, it is enough to prove that $S_a+S_b\ge 0$, which is equivalent to

$$(c+a-b)(c+a)(3b+c+a)\ge(b+c-a)(b+c)(3a+b+c).$$

For the nontrivial case $b+c-a>0$, since $c+a-b\ge b+c-a$, we only need to show that

$$(c+a)(3b+c+a)\ge(b+c)(3a+b+c).$$

Indeed,

$$(c+a)(3b+c+a)-(b+c)(3a+b+c)=(a-b)(a+b-c)\ge 0.$$

The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.161. *If a, b, c are real numbers, no two of which are zero, then*

$$(a) \qquad \frac{8a^2+3bc}{b^2+bc+c^2}+\frac{8b^2+3ca}{c^2+ca+a^2}+\frac{8c^2+3ab}{a^2+ab+b^2} \ge 11;$$

$$(b) \qquad \frac{8a^2-5bc}{b^2-bc+c^2}+\frac{8b^2-5ca}{c^2-ca+a^2}+\frac{8c^2-5ab}{a^2-ab+b^2} \ge 9.$$

(Vasile Cîrtoaje, 2011)

Solution. Consider the more general inequality

$$\frac{a^2+mbc}{b^2+kbc+c^2}+\frac{b^2+mca}{c^2+kca+a^2}+\frac{c^2+mab}{a^2+kab+b^2} \ge \frac{3(m+1)}{k+2},$$

which can be written as $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c) = (k+2)\sum(a^2+mbc)(a^2+kab+b^2)(a^2+kac+c^2)$$

$$-3(m+1)\prod(b^2+kbc+c^2).$$

Let

$$p = a+b+c, \qquad q = ab+bc+ca.$$

From

$$f_6(a,b,c) = (k+2)\sum(a^2+mbc)(kab-c^2+p^2-2q)(kac-b^2+p^2-2q)$$

$$-3(m+1)\prod(kbc-a^2+p^2-2q).$$

it follows that $f_6(a,b,c)$ has the same highest coefficient A as

$$(k+2)P_2(a,b,c)-3(m+1)P_3(a,b,c),$$

where

$$P_2(a,b,c) = \sum(a^2+mbc)(kab-c^2)(kac-b^2),$$

$$P_3(a,b,c) = \prod(kbc-a^2).$$

According to Remark 2 from the proof of P 2.75 in Volume 1,

$$\begin{aligned} A &= (k+2)P_2(1,1,1)-3(m+1)P_3(1,1,1) \\ &= 3(k+2)(m+1)(k-1)^2-3(m+1)(k-1)^3 = 9(m+1)(k-1)^2. \end{aligned}$$

Also, we have

$$f_6(a,1,1) = (k+2)(a^2+ka+1)(a-1)^2[a^2+(k+2)a+1+2k-2m].$$

(a) For our particular case $m = 3/8$ and $k = 1$, we have $A = 0$. Therefore, according to P 2.75 in Volume 1, it suffices to prove that $f_6(a,1,1) \ge 0$ for all real a. Indeed,

$$f_6(a,1,1) = 3(a^2+a+1)(a-1)^2\left(a+\frac{3}{2}\right)^2 \ge 0.$$

Thus, the proof is completed. The equality holds for $a = b = c$, and also for $-2a/3 = b = c$ (or any cyclic permutation).

(b) For $m = -5/8$ and $k = -1$, we have

$$A = \frac{27}{2}$$

and

$$f_6(a,1,1) = \frac{1}{4}(a^2-a+1)(a-1)^2(2a+1)^2.$$

Since $A > 0$, we will use the *highest coefficient cancellation method*. Consider the homogeneous polynomial

$$P(a,b,c) = abc + Bp^3 + Cpq,$$

where B and C are real constants. Since the desired inequality becomes an equality for $a = b = c = 1$, and also for $a = -1/2$ and $b = c = 1$, we will determine B and C such that $P(1,1,1) = P(-1/2,1,1) = 0$; that is,

$$B = \frac{4}{27}, \quad C = \frac{-5}{9},$$

when

$$P(a,b,c) = abc + \frac{4p^3}{27} - \frac{5pq}{9},$$

$$P(a,1,1) = \frac{2}{27}(a-1)^2(2a+1).$$

We will show that

$$f_6(a,b,c) \ge \frac{27}{2}P^2(a,b,c).$$

Let us denote

$$g_6(a,b,c) = f_6(a,b,c) - \frac{27}{2}P^2(a,b,c).$$

Since $g_6(a,b,c)$ has the highest coefficient equal to zero, it suffices to prove that $g_6(a,1,1) \ge 0$ for all real a (see P 2.75 in Volume 1). Indeed,

$$g_6(a,1,1) = f_6(a,1,1) - \frac{27}{2}P^2(a,1,1) = \frac{1}{108}(a-1)^2(2a+1)^2(19a^2-11a+19) \ge 0.$$

The equality holds for $a = b = c$, and also for $-2a = b = c$ (or any cyclic permutation).

□

P 1.162. *If a,b,c are real numbers, no two of which are zero, then*

$$\frac{4a^2+bc}{4b^2+7bc+4c^2}+\frac{4b^2+ca}{4c^2+7ca+4a^2}+\frac{4c^2+ab}{4a^2+7ab+4b^2}\geq 1.$$

(Vasile Cîrtoaje, 2011)

Solution. Write the inequality as $f_6(a,b,c)\geq 0$, where

$$f_6(a,b,c)=\sum(4a^2+bc)(4a^2+7ab+4b^2)(4a^2+7ac+4c^2)-\prod(4b^2+7bc+4c^2).$$

Let

$$p=a+b+c,\quad q=ab+bc+ca.$$

From

$$f_6(a,b,c)=\sum(4a^2+bc)(7ab-4c^2+4p^2-8q)(7ac-4b^2+4p^2-8q)$$

$$-\prod(7bc-4a^2+4p^2-8q),$$

it follows that $f_6(a,b,c)$ has the same highest coefficient A as

$$P_2(a,b,c)-P_3(a,b,c),$$

where

$$P_2(a,b,c)=\sum(4a^2+bc)(7ab-4c^2)(7ac-4b^2),$$

$$P_3(a,b,c)=\prod(7bc-4a^2).$$

According to Remark 2 from the proof of P 2.75 in Volume 1,

$$A=P_2(1,1,1)-P_3(1,1,1)=135-27=108.$$

Since $A>0$, we will apply the *highest coefficient cancellation method*. Consider the homogeneous polynomial

$$P(a,b,c)=abc+Bp^3+Cpq,$$

where B and C are real constants. We will show that there are two real numbers B and C such that the following sharper inequality holds

$$f_6(a,b,c)\geq 108P^2(a,b,c).$$

Let us denote

$$g_6(a,b,c)=f_6(a,b,c)-108P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient equal to zero. Then, by P 2.75 in Volume 1, it suffices to prove that there exist B and C such that $g_6(a,1,1)\geq 0$ for all real a.

We have

$$g_6(a,1,1)=f_6(a,1,1)-108P^2(a,1,1),$$

where

$$f_6(a,1,1)=4(4a^2+7a+4)(a-1)^2(4a^2+15a+16),$$

$$P(a,1,1)=a+B(a+2)^3+C(a+2)(2a+1).$$

Let us denote $g(a)=f_6(a,1,1)$. Since

$$g(-2)=0,$$

the condition

$$g'(-2)=0,$$

which involves $C=-5/9$, is necessary to have $g(a)\ge 0$ in the vicinity of $a=-2$. On the other hand, from $g(1)=0$, we get $B=4/27$. For these values of B and C, we get

$$P(a,1,1)=\frac{2(a-1)^2(2a+1)}{27},$$

$$g_6(a,1,1)=\frac{4}{27}(a-1)^2(a+2)^2(416a^2+728a+431)\ge 0.$$

The proof is completed. The equality holds for $a=b=c$, and for $a=0$ and $b+c=0$ (or any cyclic permutation).

□

P 1.163. *If a,b,c are real numbers, no two of which are equal, then*

$$\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}\ge\frac{27}{4(a^2+b^2+c^2-ab-bc-ca)}.$$

First Solution. Write the inequality as follows:

$$\left[(a-b)^2+(b-c)^2+(a-c)^2\right]\left[\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(a-c)^2}\right]\ge\frac{27}{2},$$

$$\left[\frac{(a-b)^2}{(a-c)^2}+\frac{(b-c)^2}{(a-c)^2}+1\right]\left[\frac{(a-c)^2}{(a-b)^2}+\frac{(a-c)^2}{(b-c)^2}+1\right]\ge\frac{27}{2},$$

$$(x^2+y^2+1)\left(\frac{1}{x^2}+\frac{1}{y^2}+1\right)\ge\frac{27}{2},$$

where

$$x=\frac{a-b}{a-c},\quad y=\frac{b-c}{a-c},\quad x+y=1.$$

We have

$$(x^2+y^2+1)\left(\frac{1}{x^2}+\frac{1}{y^2}+1\right)-\frac{27}{2}=\frac{(x+1)^2(x-2)^2(2x-1)^2}{2x^2(1-x)^2}\ge 0.$$

The proof is completed. The equality holds for $2a=b+c$ (or any cyclic permutation).

Second Solution. Assume that $a>b>c$. We have

$$\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}\ge\frac{2}{(a-b)(b-c)}\ge\frac{8}{[(a-b)+(b-c)]^2}=\frac{8}{(a-c)^2}.$$

Therefore, it suffices to show that

$$\frac{9}{(a-c)^2}\ge\frac{27}{4(a^2+b^2+c^2-ab-bc-ca)},$$

which is equivalent to

$$(a-2b+c)^2\ge 0.$$

Third Solution. Write the inequality as $f_6(a,b,c)\ge 0$, where

$$f_6(a,b,c)=4(a^2+b^2+c^2-ab-bc-ca)\sum(a-b)^2(a-c)^2-27(a-b)^2(b-c)^2(c-a)^2.$$

Clearly, $f_6(a,b,c)$ has the same highest coefficient A as

$$-27(a-b)^2(b-c)^2(c-a)^2;$$

that is,

$$A=-27(-27)=729.$$

Since $A>0$, we will use the *highest coefficient cancellation method*. Define the homogeneous polynomial

$$P(a,b,c)=abc+B(a+b+c)^3-\left(3B+\frac{1}{9}\right)(a+b+c)(ab+bc+ca)$$

which satisfies $P(1,1,1)=0$. We will show that there is a real value of B such that the following sharper inequality holds

$$f_6(a,b,c)\ge 729P^2(a,b,c).$$

Let us denote

$$g_6(a,b,c)=f_6(a,b,c)-729P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient equal to zero. Then, by P 2.75 in Volume 1, it suffices to prove that $g_6(a,1,1)\ge 0$ for all real a. We have

$$f_6(a,1,1)=4(a-1)^6$$

and

$$P(a,1,1)=\frac{1}{9}(a-1)^2[9B(a+2)+2],$$

hence

$$\begin{aligned}g_6(a,1,1)&=f_6(a,1,1)-729P^2(a,1,1)\\&=(27B+2)(a+2)(a-1)^4[(2-27B)a-54B-8].\end{aligned}$$

Choosing $B=-2/27$, we get $g_6(a,1,1)=0$ for all real a.

Remark. The inequality is equivalent to

$$(a-2b+c)^2(b-2c+a)^2(c-2a+b)^2\geq 0.$$

□

P 1.164. *If a,b,c are real numbers, no two of which are zero, then*

$$\frac{1}{a^2-ab+b^2}+\frac{1}{b^2-bc+c^2}+\frac{1}{c^2-ca+a^2}\geq\frac{14}{3(a^2+b^2+c^2)}.$$

(Vasile Cîrtoaje and BJSL, 2014)

Solution. Write the inequality as $f_6(a,b,c)\geq 0$, where

$$\begin{aligned}f_6(a,b,c)=&3(a^2+b^2+c^2)\sum(a^2-ab+b^2)(a^2-ac+c^2)\\&-14(a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2).\end{aligned}$$

Clearly, $f_6(a,b,c)$ has the same highest coefficient A as

$$-14(a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2);$$

that is, according to Remark 2 from the proof of P 2.75 in Volume 1,

$$A=-14(-1-1)^3=112.$$

Since $A>0$, we apply the *highest coefficient cancellation method*. Consider the homogeneous polynomial

$$P(a,b,c)=abc+B(a+b+c)^3+C(a+b+c)(ab+bc+ca).$$

We will show that there are two real numbers B and C such that the following sharper inequality holds

$$f_6(a,b,c)\geq 112P^2(a,b,c).$$

Let us denote

$$g_6(a,b,c)=f_6(a,b,c)-112P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient equal to zero. By P 2.75 in Volume 1, it suffices to prove that $g_6(a,1,1) \ge 0$ for all real a. We have

$$g_6(a,1,1) = f_6(a,1,1) - 112P^2(a,1,1),$$

where

$$f_6(a,1,1) = (a^2 - a + 1)(3a^4 - 3a^3 + a^2 + 8a + 4),$$

$$P(a,1,1) = a + B(a+2)^3 + C(a+2)(2a+1).$$

Let us denote $g(a) = g_6(a,1,1)$. Since

$$g(-2) = 0,$$

the condition

$$g'(-2) = 0,$$

which involves $C = -4/7$, is necessary to have $g(a) \ge 0$ in the vicinity of $a = -2$. In addition, setting $B = 9/56$, we get

$$P(a,1,1) = \frac{1}{56}(9a^3 - 10a^2 + 4a + 8),$$

$$\begin{aligned} g_6(a,1,1) &= \frac{3}{28}(a^6 + 4a^5 + 8a^4 + 16a^3 + 20a^2 + 16a + 16) \\ &= \frac{3(a+2)^2(a^2+2)^2}{28} \ge 0. \end{aligned}$$

The proof is completed. The equality holds for $a = 0$ and $b + c = 0$ (or any cyclic permutation).

□

P 1.165. *If a,b,c are real numbers, then*

$$\frac{a^2+bc}{2a^2+b^2+c^2} + \frac{b^2+ca}{a^2+2b^2+c^2} + \frac{c^2+ab}{a^2+b^2+2c^2} \ge \frac{1}{6}.$$

(Vasile Cîrtoaje, 2010)

Solution. Write the inequality as $f_6(a,b,c) \ge 0$, where

$$\begin{aligned} f_6(a,b,c) =& 6\sum(a^2+bc)(a^2+2b^2+c^2)(a^2+b^2+2c^2) \\ & -(2a^2+b^2+c^2)(a^2+2b^2+c^2)(a^2+b^2+2c^2). \end{aligned}$$

Clearly, $f_6(a,b,c)$ has the same highest coefficient A as $f(a,b,c)$, where

$$f(a,b,c) = 6\sum(a^2+bc)b^2c^2 - a^2b^2c^2 = 17a^2b^2c^2 + 6(a^3b^3 + b^3c^3 + c^3a^3);$$

that is,
$$A = 17 + 6 \cdot 3 = 35.$$

Since $A > 0$, we apply the *highest coefficient cancellation method*. Consider the homogeneous polynomial

$$P(a,b,c) = abc + B(a+b+c)^3 + C(a+b+c)(ab+bc+ca)$$

and show that there are two real numbers B and C such that the following sharper inequality holds
$$f_6(a,b,c) \ge 35P^2(a,b,c).$$

Let us denote
$$g_6(a,b,c) = f_6(a,b,c) - 35P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient equal to zero. By P 2.75 in Volume 1, it suffices to prove that $g_6(a,1,1) \ge 0$ for all real a. We have

$$g_6(a,1,1) = f_6(a,1,1) - 35P^2(a,1,1),$$

where
$$f_6(a,1,1) = 4(a^2+1)(a^2+3)(a+3)^2,$$
$$P(a,1,1) = a + B(a+2)^3 + C(a+2)(2a+1).$$

Let
$$g(a) = g_6(a,1,1).$$

Since $g(-2) = 0$, we can have $g(a) \ge 0$ in the vicinity of $a = -2$ only if $g'(-2) = 0$, which involves $C = 19/35$. Since $f_6(-3,1,1) = 0$, we enforce $P(-3,1,1) = 0$, which provides $B = -2/7$. Thus,

$$P(a,1,1) = a - \frac{2}{7}(a+1)^3 + \frac{19}{35}(a+2)(2a+1) = \frac{-2(a+3)(5a^2-4a+7)}{35}$$

and

$$\begin{aligned} g_6(a,1,1) &= 4(a^2+1)(a^2+3)(a+3)^2 - \frac{4}{35}(a+3)^2(5a^2-4a+7)^2 \\ &= \frac{8}{35}(a+3)^2(a+2)^2(5a^2+7) \ge 0. \end{aligned}$$

The proof is completed. The equality holds for $a = 0$ and $b + c = 0$ (or any cyclic permutation), and also for $-a/3 = b = c$ (or any cyclic permutation).

□

P 1.166. *If a,b,c are real numbers, then*

$$\frac{2b^2+2c^2+3bc}{(a+3b+3c)^2} + \frac{2c^2+2a^2+3ca}{(b+3c+3a)^2} + \frac{2a^2+2b^2+3ab}{(c+3a+3b)^2} \ge \frac{3}{7}.$$

(Vasile Cîrtoaje, 2010)

Solution. Write the inequality as $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c) = 7\sum(2b^2+2c^2+3bc)(b+3c+3a)^2(c+3a+3b)^2-3\prod(a+3b+3c)^2.$$

We have

$$f_6(a,1,1) = (a-1)^2(a-8)^2(3a+4)^2.$$

Let

$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc.$$

From

$$f_6(a,b,c) = 7\sum(2p^2-4q+3bc-2a^2)(3p-2b)^2(3p-2c)^2-3\prod(3p-2a)^2,$$

it follows that $f(a,b,c)$ has the same highest coefficient A as $g(a,b,c)$, where

$$g(a,b,c) = 7\sum(3bc-2a^2)(-2b)^2(-2c)^2-3\prod(-2a)^2 = 48\left(7\sum b^3c^3-18a^2b^2c^2\right);$$

that is,

$$A = 48(21-18) = 144.$$

Since the highest coefficient A is positive, we will use the *highest coefficient cancellation method*. There are two cases to consider: $p^2+q \ge 0$ and $p^2+q < 0$.

Case 1: $p^2+q \ge 0$. Since

$$f_6(1,1,1) = f_6(8,1,1) = 0,$$

define the homogeneous function

$$P(a,b,c) = r + Bp^3 + Cpq$$

such that $P(1,1,1) = P(8,1,1) = 0$; that is,

$$P(a,b,c) = r + \frac{1}{45}p^3 - \frac{8}{45}pq,$$

which leads to

$$P(a,1,1) = \frac{45a+(a+2)^3-8(a+2)(2a+1)}{45} = \frac{(a-1)^2(a-8)}{45}.$$

We will show that the following sharper inequality holds for $p^2+q \ge 0$:

$$f_6(a,b,c) \ge 144P^2(a,b,c).$$

Let us denote

$$g_6(a,b,c) = f_6(a,b,c) - 144P^2(a,b,c).$$

Since the highest coefficient of $g_6(a,b,c)$ is zero, it suffices to prove that $g_6(a,1,1) \ge 0$ for all real a such that $(a+2)^2 + 2a + 1 \ge 0$, that is

$$a \in (-\infty, -5] \cup [-1, \infty)$$

(see Remark 3 from the proof of P 2.75 in Volume 1). We have

$$\begin{aligned} g_6(a,1,1) &= f_6(a,1,1) - 144P^2(a,1,1) \\ &= \frac{1}{225}(a-1)^2(a-8)^2[225(3a+4)^2 - 16(a-1)^2] \\ &= \frac{7}{225}(a-1)^2(a-8)^2(41a+64)(7a+8) \ge 0. \end{aligned}$$

Case 2: $p^2 + q < 0$. Since

$$f_6(1,1,1) = f_6(-4/3,1,1) = 0,$$

define the homogeneous function

$$P(a,b,c) = r + Bp^3 + Cpq$$

such that $P(1,1,1) = P(-4/3,1,1) = 0$; that is,

$$P(a,b,c) = r + \frac{1}{3}p^3 - \frac{10}{9}pq,$$

which leads to

$$P(a,1,1) = \frac{9a + 3(a+2)^3 - 10(a+2)(2a+1)}{9} = \frac{(a-1)^2(3a+4)}{9}.$$

We will show that the following sharper inequality holds for $p^2 + q < 0$:

$$f_6(a,b,c) \ge 144P^2(a,b,c).$$

Let us denote

$$g_6(a,b,c) = f_6(a,b,c) - 144P^2(a,b,c).$$

Since the highest coefficient of $g_6(a,b,c)$ is zero, it suffices to prove that $g_6(a,1,1) \ge 0$ for all real a such that $(a+2)^2 + 2a + 1 < 0$, that is

$$a \in (-5, -1)$$

(see Remark 3 from the proof of P 2.75 in Volume 1). We have

$$\begin{aligned} g_6(a,1,1) &= f_6(a,1,1) - 144P^2(a,1,1) \\ &= \frac{1}{9}(a-1)^2(3a+4)^2[9(a-8)^2 - 16(a-1)^2] \\ &= \frac{7}{9}(a-1)^2(3a+4)^2(20+a)(4-a) \ge 0. \end{aligned}$$

The proof is completed. The equality holds for $a = b = c$, for $a/8 = b = c$ (or any cyclic permutation), and also for $-3a/4 = b = c$ (or any cyclic permutation).

□

P 1.167. *If a, b, c are nonnegative real numbers, then*

$$\frac{6b^2+6c^2+13bc}{(a+2b+2c)^2}+\frac{6c^2+6a^2+13ca}{(b+2c+2a)^2}+\frac{6a^2+6b^2+13ab}{(c+2a+2b)^2}\le 3.$$

(Vasile Cîrtoaje, 2010)

Solution. Write the inequality as $f_6(a,b,c)\ge 0$, where

$$f_6(a,b,c)=3\prod(a+2b+2c)^2-\sum(6b^2+6c^2+13bc)(b+2c+2a)^2(c+2a+2b)^2.$$

Let

$$p=a+b+c,\quad q=ab+bc+ca.$$

From

$$f_6(a,b,c)=3\prod(2p-a)^2-\sum(6p^2-12q+13bc-6a^2)(2p-b)^2(2p-c)^2,$$

it follows that $f(a,b,c)$ has the same highest coefficient A as $g(a,b,c)$, where

$$g(a,b,c)=3\prod(-a)^2-\sum(13bc-6a^2)(-b)^2(-c)^2=21a^2b^2c^2-13\sum b^3c^3;$$

that is,

$$A=21-39=-18.$$

Since the highest coefficient A is negative, it suffices to prove the desired inequality for $b=c=1$, and for $a=0$ (see P 3.76-(a) in Volume 1).

For $b=c=1$, the inequality becomes

$$\frac{25}{(a+4)^2}+\frac{2(6a^2+13a+6)}{(2a+3)^2}\le 3,$$

$$\frac{2(6a^2+13a+6)}{(2a+3)^2}\le\frac{3a^2+24a+23}{(a+4)^2},$$

$$\frac{5(2a+3)(a-1)^2}{(2a+3)^2(a+4)^2}\ge 0.$$

For $a=0$, the inequality turns into

$$\frac{6b^2+6c^2+13bc}{4(b+c)^2}+\frac{6c^2}{(b+2c)^2}+\frac{6b^2}{(2b+c)^2}\le 3,$$

$$\frac{6b^2+6c^2+13bc}{4(b+c)^2}+\frac{6[(b^2+c^2)^2+4bc(b^2+c^2)+6b^2c^2]}{(2b^2+2c^2+5bc)^2}\le 3.$$

If $bc=0$, then the inequality is an identity. For $bc\ne 0$, we may consider $bc=1$ (due to homogeneity). Denoting

$$x=b^2+c^2,\qquad x\ge 2,$$

the inequality becomes

$$\frac{6x+13}{4(x+2)}+\frac{6(x^2+4x+6)}{(2x+5)^2}\le 3,$$

which reduces to the obvious inequality

$$20x^2+34x-13\ge 0.$$

The equality holds for $a=b=c$, and also for $a=b=0$ (or any cyclic permutation).

□

P 1.168. *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

$$\frac{3a^2+8bc}{9+b^2+c^2}+\frac{3b^2+8ca}{9+c^2+a^2}+\frac{3c^2+8ab}{9+a^2+b^2}\le 3.$$

(Vasile Cîrtoaje, 2010)

Solution. Let

$$p=a+b+c,\quad q=ab+bc+ca.$$

Write the inequality in the homogeneous form

$$\frac{3a^2+8bc}{p^2+b^2+c^2}+\frac{3b^2+8ca}{p^2+c^2+a^2}+\frac{3c^2+8ab}{p^2+a^2+b^2}\le 3,$$

which is equivalent to $f_6(a,b,c)\ge 0$, where

$$f_6(a,b,c)=3\prod(p^2+b^2+c^2)-\sum(3a^2+8bc)(p^2+c^2+a^2)(p^2+a^2+b^2).$$

From

$$f_6(a,b,c)=3\prod(2p^2-2q-a^2)-\sum(3a^2+8bc)(2p^2-2q-b^2)(2p^2-2q-c^2),$$

it follows that $f(a,b,c)$ has the same highest coefficient A as $g(a,b,c)$, where

$$g(a,b,c)=3\prod(-a)^2-\sum(3a^2+8bc)(-b^2)(-c^2)=-12a^2b^2c^2-8\sum b^3c^3;$$

that is,

$$A=-12-24=-36.$$

Since the highest coefficient A is negative, it suffices to prove the homogeneous inequality for $b=c=1$ and for $a=0$ (see P 3.76-(a) in Volume 1).

For $b=c=1$, we need to show that

$$\frac{3a^2+8}{(a+2)^2+2}+\frac{2(3+8a)}{(a+2)^2+a^2+1}\le 3,$$

which is equivalent to

$$\frac{3a^2+8}{a^2+4a+6}+\frac{2(8a+3)}{2a^2+4a+5}\le 3,$$

$$\frac{8a+3}{2a^2+4a+5}\le\frac{6a+5}{a^2+4a+6},$$

$$4a^3-a^2-10a+7\ge 0,$$

$$(a-1)^2(4a+7)\ge 0.$$

For $a=0$, we need to show that

$$\frac{8bc}{(b+c)^2+b^2+c^2}+\frac{3b^2}{(b+c)^2+c^2}+\frac{3c^2}{(b+c)^2+b^2}\le 3.$$

Clearly, it suffices to show that

$$\frac{8bc}{(b+c)^2+b^2+c^2}+\frac{3(b^2+c^2)}{(b+c)^2}\le 3,$$

which is equivalent to

$$\frac{4bc}{b^2+c^2+bc}\le\frac{6bc}{(b+c)^2},$$

$$bc(b-c)^2\ge 0.$$

The equality holds for $a=b=c=1$, and also for $a=b=0$ and $c=3$ (or any cyclic permutation).

□

P 1.169. *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

$$\frac{5a^2+6bc}{9+b^2+c^2}+\frac{5b^2+6ca}{9+c^2+a^2}+\frac{5c^2+6ab}{9+a^2+b^2}\ge 3.$$

(Vasile Cîrtoaje, 2010)

Solution. We use the *highest coefficient method*. Let

$$p=a+b+c,\qquad q=ab+bc+ca.$$

Write the inequality in the homogeneous form $f_6(a,b,c)\ge 0$, where

$$f_6(a,b,c)=\sum(5a^2+6bc)(p^2+c^2+a^2)(p^2+a^2+b^2)-3\prod(p^2+b^2+c^2).$$

From

$$f_6(a,b,c)=\sum(5a^2+6bc)(2p^2-2q-b^2)(2p^2-2q-c^2)-3\prod(2p^2-2q-a^2),$$

it follows that $f_6(a,b,c)$ has the same highest coefficient A as

$$f(a,b,c)=\sum(5a^2+6bc)(-b^2)(-c^2)-3(-a^2)(-b^2)(-c^2)=18a^2b^2c^2+6\sum b^3c^3;$$

therefore,

$$A=18+18=36.$$

On the other hand,

$$f_6(a,1,1)=4a(2a^2+4a+5)(a+1)(a-1)^2\geq 0$$

and

$$\begin{aligned}f_6(0,b,c)&=6bcBC+5b^2AB+5c^2AC-3ABC\\&=-3(A-2bc)BC+5A(b^2B+c^2C),\end{aligned}$$

where

$$A=(b+c)^2+b^2+c^2,\quad B=(b+c)^2+b^2,\quad C=(b+c)^2+c^2.$$

Substituting

$$(b+c)^2=4x,\quad bc=y,\quad x\geq y,$$

we have

$$A=2(4x-y),\quad B=4x+b^2,\quad C=4x+c^2,$$

$$A-2bc=4(2x-y),$$

$$BC=16x^2+4x(b^2+c^2)+b^2c^2=16x^2+4x(4x-2y)+y^2=32x^2-8xy+y^2,$$

$$b^2B+c^2C=4x(b^2+c^2)+b^4+c^4=2(16x^2-12xy+y^2),$$

therefore

$$\begin{aligned}f_6(0,b,c)&=-12(2x-y)(32x^2-8xy+y^2)+20(4x-y)(16x^2-12xy+y^2)\\&=8(64x^3-88x^2y+25xy^2-y^3)=8(x-y)(64x^2-24xy+y^2).\end{aligned}$$

Since

$$f_6(1,1,1)=f_6(0,1,1)=0,$$

define the homogeneous function

$$P(a,b,c)=abc+B(a+b+c)^3+C(a+b+c)(ab+bc+ca)$$

such that $P(1,1,1)=P(0,1,1)=0$; that is,

$$P(a,b,c)=abc+\frac{1}{9}(a+b+c)^3-\frac{4}{9}(a+b+c)(ab+bc+ca).$$

We have

$$P(a,1,1)=\frac{a(a-1)^2}{9},\quad P^2(a,1,1)=\frac{a^2(a-1)^4}{81},$$

$$P(0,b,c)=\frac{(b+c)(b-c)^2}{9},\qquad P^2(0,b,c)=\frac{64x(x-y)^2}{81}.$$

We will prove the sharper inequality $g_6(a,b,c)\ge 0$, where

$$g_6(a,b,c)=f_6(a,b,c)-36P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient $A=0$. Then, according to P 3.76-(a) in Volume 1, it suffices to prove that $g_6(a,1,1)\ge 0$ and $g_6(0,b,c)\ge 0$ for $a,b,c\ge 0$.

We have

$$g_6(a,1,1)=f_6(a,1,1)-36P^2(a,1,1)=\frac{4a(a-1)^2h(a)}{9},$$

where

$$\begin{aligned}h(a)&=9(2a^2+4a+5)(a+1)-a(a-1)^2\\&>(a-1)^2(a+1)-a(a-1)^2=(a-1)^2\ge 0.\end{aligned}$$

Also, we have

$$g_6(0,b,c)=f_6(0,b,c)-36P^2(0,b,c)=\frac{8(x-y)g(x,y)}{9},$$

where

$$\begin{aligned}g(x,y)&=9(64x^2-24xy+y^2)-32x(x-y)\\&>(64x^2-24xy+y^2)-32x(x-y)=32x^2+8xy+y^2>0.\end{aligned}$$

The equality holds for $a=b=c=1$, and also for $a=0$ and $b=c=3/2$ (or any cyclic permutation).

□

P 1.170. *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

$$\frac{1}{a^2+bc+12}+\frac{1}{b^2+ca+12}+\frac{1}{c^2+ab+12}\le\frac{3}{14}.$$

(Vasile Cîrtoaje, 2010)

Solution. Write the inequality in the homogeneous form

$$\frac{1}{3(a^2+bc)+4p^2}+\frac{1}{3(b^2+ca)+4p^2}+\frac{1}{3(c^2+ab)+4p^2}\le\frac{9}{14p^2},$$

where

$$p=a+b+c.$$

The inequality is equivalent to $f_6(a,b,c)\ge 0$, where

$$f_6(a,b,c)=9\prod(3a^2+3bc+4p^2)-14p^2\sum(3b^2+3ca+4p^2)(3c^2+3ab+4p^2).$$

Clearly, $f_6(a,b,c)$ has the same highest coefficient A as

$$g(a,b,c)=243\prod(a^2+bc).$$

According to Remark 2 from the proof of P 2.75 in Volume 1, we have

$$A=g(1,1,1)=243\cdot 8=1944.$$

Since the highest coefficient A is positive, we will apply the *highest coefficient cancellation method*. We have

$$\begin{aligned}f_6(a,1,1)=&9[3a^2+3+4(a+2)^2][3a+3+4(a+2)^2]^2\\&-14(a+2)^2[3a+3+4(a+2)^2]^2\\&-28(a+2)^2[3a+3+4(a+2)^2][3a^2+3+4(a+2)^2]\\=&9(7a^2+16a+19)(4a^2+19a+19)^2-14(a+2)^2(4a^2+19a+19)^2\\&-28(a+2)^2(4a^2+19a+19)(7a^2+16a+19)\\=&3(4a^2+19a+19)f(a),\end{aligned}$$

where

$$\begin{aligned}f(a)&=3(7a^2+16a+19)(4a^2+19a+19)-14(a+2)^2(6a^2+17a+19)\\&=17a^3-15a^2-21a+19=(a-1)^2(17a+19);\end{aligned}$$

therefore,

$$f_6(a,1,1)=3(4a^2+19a+19)(a-1)^2(17a+19).$$

Since

$$f_6(1,1,1)=f_6(1,0,0)=0,$$

define the homogeneous function

$$P(a,b,c)=abc+B(a+b+c)^3+C(a+b+c)(ab+bc+ca)$$

such that $P(1,1,1)=P(1,0,0)=0$; that is,

$$P(a,b,c)=abc-\frac{1}{9}(a+b+c)(ab+bc+ca).$$

We will prove the sharper inequality $g_6(a,b,c)\ge 0$, where

$$g_6(a,b,c)=f_6(a,b,c)-1944P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient $A=0$. Then, it suffices to prove that $g_6(a,1,1)\ge 0$ and $g_6(0,b,c)\ge 0$ for $a,b,c\ge 0$ (see P 3.76-(a) in Volume 1).

To show that $g_6(a,1,1) \ge 0$, which can be written as

$$f_6(a,1,1) - 1944P^2(a,1,1) \ge 0,$$

we see that

$$P(a,1,1) = a - \frac{(a+2)(2a+1)}{9} = \frac{-2(a-1)^2}{9},$$

$$P^2(a,1,1) = \frac{4(a-1)^4}{81},$$

hence

$$\begin{aligned} g_6(a,1,1) &= 3(4a^2+19a+19)(a-1)^2(17a+19) - 96(a-1)^4 \\ &= 3(a-1)^2 h(a), \end{aligned}$$

where

$$h(a) = (4a^2+19a+19)(17a+19) - 32(a-1)^2.$$

We need to show that $h(a) \ge 0$ for $a \ge 0$. Indeed, since

$$(4a^2+19a+19)(17a+19) > (19a+19)(17a+17) > 32(a+1)^2,$$

we get

$$h(a) > 32[(a+1)^2 - (a-1)^2] = 128a \ge 0.$$

To show that $g_6(0,b,c) \ge 0$, denote

$$x = (b+c)^2, \qquad y = bc.$$

We have

$$f_6(0,b,c) = 9ABC - 14x[BC + A(B+C)] = (9A - 14x)BC - 14xA(B+C),$$

where

$$A = 4x + 3y, \quad B = 4x + 3b^2, \quad C = 4x + 3c^2.$$

Since

$$9A - 14x = 22x + 27y, \quad B + C = 8x + 3(x - 2y) = 11x - 6y,$$

$$BC = 16x^2 + 12x(x-2y) + 9y^2 = 28x^2 - 24xy + 9y^2,$$

we get

$$\begin{aligned} f_6(0,b,c) &= (22x+27y)(28x^2-24xy+9y^2) - 14x(4x+3y)(11x-6y) \\ &= 3y(34x^2 - 66xy + 81y^2). \end{aligned}$$

Also,

$$P(0,b,c) = \frac{-bc(b+c)}{9}, \qquad P^2(0,b,c) = \frac{xy^2}{81}.$$

Hence

$$\begin{aligned} g_6(0,b,c) &= f_6(0,b,c) - 1944P^2(0,b,c) = 3y(34x^2 - 74xy + 81y^2) \\ &\ge 3y(25x^2 - 90xy + 81y^2) = 3y(5x-9y)^2 \ge 0. \end{aligned}$$

The equality holds for $a = b = c$, and also for $a = b = 0$ (or any cyclic permutation).

□

P 1.171. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

$$\frac{1}{a^2+b^2} + \frac{1}{b^2+c^2} + \frac{1}{c^2+a^2} \ge \frac{45}{8(a^2+b^2+c^2)+2(ab+bc+ca)}.$$

(Vasile Cîrtoaje, 2014)

First Solution *(by Nguyen Van Quy).* Multiplying by $a^2+b^2+c^2$, the inequality becomes

$$\sum \frac{a^2}{b^2+c^2} + 3 \ge \frac{45(a^2+b^2+c^2)}{8(a^2+b^2+c^2)+2(ab+bc+ca)}.$$

Applying the Cauchy-Schwarz inequality, we have

$$\sum \frac{a^2}{b^2+c^2} \ge \frac{\left(\sum a^2\right)^2}{\sum a^2(b^2+c^2)} = \frac{(a^2+b^2+c^2)^2}{2(a^2b^2+b^2c^2+c^2a^2)}.$$

Therefore, it suffices to show that

$$\frac{(a^2+b^2+c^2)^2}{2(a^2b^2+b^2c^2+c^2a^2)} + 3 \ge \frac{45(a^2+b^2+c^2)}{8(a^2+b^2+c^2)+2(ab+bc+ca)},$$

which is equivalent to

$$\frac{(a^2+b^2+c^2)^2}{a^2b^2+b^2c^2+c^2a^2} - 3 \ge \frac{45(a^2+b^2+c^2)}{4(a^2+b^2+c^2)+ab+bc+ca} - 9,$$

$$\frac{a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2}{a^2b^2+b^2c^2+c^2a^2} \ge \frac{9(a^2+b^2+c^2-ab-bc-ca)}{4(a^2+b^2+c^2)+ab+bc+ca}.$$

By Schur's inequality of degree four, we have

$$a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2 \ge (a^2+b^2+c^2-ab-bc-ca)(ab+bc+ca).$$

Therefore, it suffices to show that

$$[4(a^2+b^2+c^2)+ab+bc+ca](ab+bc+ca) \ge 9(a^2b^2+b^2c^2+c^2a^2).$$

Since

$$(ab+bc+ca)^2 \ge a^2b^2+b^2c^2+c^2a^2,$$

this inequality is true if

$$4(a^2+b^2+c^2)(ab+bc+ca) \geq 8(a^2b^2+b^2c^2+c^2a^2),$$

which is equivalent to the obvious inequality

$$ab(a-b)^2+bc(b-c)^2+ca(c-a)^2+abc(a+b+c) \geq 0.$$

The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

Second Solution. Write the inequality as $f_6(a,b,c) \geq 0$, where

$$f_6(a,b,c) = \left[8(a^2+b^2+c^2)+2(ab+bc+ca)\right]\sum(a^2+b^2)(a^2+c^2) - 45\prod(b^2+c^2).$$

Clearly, $f_6(a,b,c)$ has the same highest coefficient A as

$$f(a,b,c) = -45\prod(b^2+c^2) = -45\prod(p^2-2q-a^2),$$

where $p=a+b+c$ and $q=ab+bc+ca$; that is,

$$A=45.$$

Since $A>0$, we will apply the *highest coefficient cancellation method*. We have

$$f_6(a,1,1) = 4a(2a+5)(a^2+1)(a-1)^2,$$

$$f_6(0,b,c) = (b-c)^2[8(b^4+c^4)+18bc(b^2+c^2)+15b^2c^2].$$

Since

$$f_6(1,1,1) = f_6(0,1,1) = 0,$$

define the homogeneous function

$$P(a,b,c) = abc + B(a+b+c)^3 + C(a+b+c)(ab+bc+ca)$$

such that $P(1,1,1) = P(0,1,1) = 0$; that is,

$$P(a,b,c) = abc + \frac{1}{9}(a+b+c)^3 - \frac{4}{9}(a+b+c)(ab+bc+ca).$$

We will show that the following sharper inequality holds

$$f_6(a,b,c) \geq 45P^2(a,b,c).$$

Let us denote

$$g_6(a,b,c) = f_6(a,b,c) - 45P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient equal to zero. By P 3.76-(a) in Volume 1, it suffices to prove that $g_6(a,1,1) \geq 0$ and $g_6(0,b,c) \geq 0$ for all $a,b,c \geq 0$. We have

$$P(a,1,1) = \frac{a(a-1)^2}{9},$$

hence

$$g_6(a,1,1)=f_6(a,1,1)-45P^2(a,1,1)=\frac{a(a-1)^2(67a^3+190a^2+67a+180)}{9}\ge 0.$$

Also, we have

$$P(0,b,c)=\frac{(b+c)(b-c)^2}{9},$$

hence

$$\begin{aligned}g_6(0,b,c)&=f_6(0,b,c)-45P^2(0,b,c)\\&=\frac{(b-c)^2[67(b^4+c^4)+162bc(b^2+c^2)+145b^2c^2]}{9}\ge 0.\end{aligned}$$

□

P 1.172. *If a,b,c are real numbers, no two of which are zero, then*

$$\frac{a^2-7bc}{b^2+c^2}+\frac{b^2-7ca}{a^2+b^2}+\frac{c^2-7ab}{a^2+b^2}+\frac{9(ab+bc+ca)}{a^2+b^2+c^2}\ge 0.$$

(Vasile Cîrtoaje, 2014)

Solution. Let

$$p=a+b+c,\quad q=ab+bc+ca,\quad r=abc.$$

Write the inequality as $f_8(a,b,c)\ge 0$, where

$$\begin{aligned}f_8(a,b,c)=&(a^2+b^2+c^2)\sum(a^2-7bc)(a^2+b^2)(a^2+c^2)\\&+9(ab+bc+ca)\prod(b^2+c^2)\end{aligned}$$

is a symmetric homogeneous polynomial of degree eight. Notice that any symmetric homogeneous polynomial of degree eight $f_8(a,b,c)$ can be written in the form

$$f_8(a,b,c)=A(p,q)r^2+B(p,q)r+C(p,q),$$

where the *highest polynomial* $A(p,q)$ has the form

$$A(p,q)=\alpha p^2+\beta q.$$

Since

$$\begin{aligned}f_8(a,b,c)=&(p^2-2q)\sum(a^2-7bc)(p^2-2q-c^2)(p^2-2q-b^2)\\&+9q\prod(p^2-2q-a^2),\end{aligned}$$

$f_8(a,b,c)$ has the same highest polynomial as

$$\begin{aligned} g_8(a,b,c) =& (p^2-2q)\sum(a^2-7bc)(-c^2)(-b^2)+9q(-a^2)(-b^2)(-c^2) \\ =& (p^2-2q)\left(3r^2-7\sum b^3c^3\right)-9qr^2; \end{aligned}$$

that is,

$$A(p,q)=(p^2-2q)(3-21)-9q=-9(p^2-3q).$$

Since $A(p,q)\le 0$ for all real a,b,c, it suffices to prove the original inequality for $b=c=1$ (see Lemma below). We need to show that

$$\frac{a^2-7}{2}-\frac{2(7a-1)}{a^2+1}+\frac{9(2a+1)}{a^2+2}\ge 0,$$

which is equivalent to

$$(a-1)^2(a+2)^2(a^2-2a+3)\ge 0.$$

The equality holds for $a=b=c$, and also for $-a/2=b=c$ (or any cyclic permutation).

Lemma. *Let*

$$p=a+b+c, \quad q=ab+bc+ca, \quad r=abc,$$

and let $f_8(a,b,c)$ be a symmetric homogeneous polynomial of degree eight written in the form

$$f_8(a,b,c)=A(p,q)r^2+B(p,q)r+C(p,q),$$

where $A(p,q)\le 0$ for all real a,b,c. The inequality $f_8(a,b,c)\ge 0$ holds for all real numbers a,b,c if and only if $f_8(a,1,1)\ge 0$ for all real a.

Proof. For fixed p and q,

$$h_8(r)=A(p,q)r^2+B(p,q)r+C(p,q)$$

is a concave quadratic function of r which is minimum when r is minimum or maximum; that is, according to P 2.53 in Volume 1, when two of a,b,c are equal. Thus, the inequality $f_8(a,b,c)\ge 0$ holds for all real numbers a,b,c if and only if $f_8(a,1,1)\ge 0$ and $f_8(a,0,0)\ge 0$ for all real a. The last condition is not necessary because it follows from the first condition as follows:

$$f_8(a,0,0)=\lim_{t\to 0}f_8(a,t,t)=\lim_{t\to 0}t^8 f_8(a/t,1,1)\ge 0.$$

Notice that $A(p,q)$ is called the *highest polynomial* of $f_8(a,b,c)$.

Remark. This Lemma can be extended for the case where the highest polynomial $A(p,q$ is not nonnegative for all real a,b,c.

- *The inequality $f_8(a,b,c)\ge 0$ in the preceding Lemma holds for all real numbers a,b,c satisfying*

$$A(p,q)\le 0$$

if and only if $f_8(a,1,1)\ge 0$ for all real a satisfying $A(a+2,2a+1)\le 0$.

□

P 1.173. *If a,b,c are nonnegative real numbers, no two of which are zero, then*

$$\frac{a^2-4bc}{b^2+c^2}+\frac{b^2-4ca}{c^2+a^2}+\frac{c^2-4ab}{a^2+b^2}+\frac{9(ab+bc+ca)}{a^2+b^2+c^2}\ge\frac{9}{2}.$$

(Vasile Cîrtoaje, 2014)

Solution. Let

$$p=a+b+c,\quad q=ab+bc+ca,\quad r=abc.$$

Write the inequality as $f_8(a,b,c)\ge 0$, where

$$\begin{aligned}f_8(a,b,c)=&2(a^2+b^2+c^2)\sum(a^2-4bc)(a^2+b^2)(a^2+c^2)\\&+9(2ab+2bc+2ca-a^2-b^2-c^2)\prod(b^2+c^2)\end{aligned}$$

is a symmetric homogeneous polynomial of degree eight. Any symmetric homogeneous polynomial of degree eight can be written in the form

$$f_8(a,b,c)=A(p,q)r^2+B(p,q)r+C(p,q),$$

where $A(p,q)=\alpha p^2+\beta q$ is called the *highest polynomial* of $f_8(a,b,c)$. From

$$\begin{aligned}f_8(a,b,c)=&2(p^2-2q)\sum(a^2-4bc)(p^2-2q-c^2)(p^2-2q-b^2)\\&+9(4q-p^2\prod(p^2-2q-a^2),\end{aligned}$$

it follows that $f_8(a,b,c)$ has the same highest polynomial as

$$\begin{aligned}g_8(a,b,c)=&2(p^2-2q)\sum(a^2-4bc)b^2c^2+9(4q-p^2)(-a^2b^2c^2)\\=&2(p^2-2q)\left(3r^2-4\sum b^3c^3\right)-9(4q-p^2)r^2;\end{aligned}$$

that is,

$$A(p,q)=2(p^2-2q)(3-12)-9(4q-p^2)=-9p^2.$$

Since $A(p,q)\le 0$ for all $a,b,c\ge 0$, it suffices to prove the original inequality for $b=c=1$, and for $a=0$ (see Lemma below).

For $b=c=1$, the original inequality becomes

$$\frac{a^2-4}{2}-\frac{2(4a-1)}{a^2+1}+\frac{9(2a+1)}{a^2+2}\ge\frac{9}{2},$$

which is equivalent to

$$a(a+4)(a-1)^4\ge 0.$$

For $a=0$, the original inequality turns into

$$\frac{b^2}{c^2}+\frac{c^2}{b^2}+\frac{5bc}{b^2+c^2}\ge\frac{9}{2}.$$

Substituting
$$x=\frac{b}{c}+\frac{c}{b},\qquad x\ge 2,$$
the inequality becomes
$$(x^2-2)+\frac{5}{x}\ge\frac{9}{2},$$
$$(x-2)(2x^2+4x-5)\ge 0.$$
Thus, the proof is completed. The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

Lemma. *Let*
$$p=a+b+c,\quad q=ab+bc+ca,\quad r=abc,$$
and let $f_8(a,b,c)$ be a symmetric homogeneous polynomial of degree eight written in the form
$$f_8(a,b,c)=A(p,q)r^2+B(p,q)r+C(p,q),$$
where $A(p,q)\le 0$ for all $a,b,c\ge 0$. The inequality $f_8(a,b,c)\ge 0$ holds for all $a,b,c\ge 0$ if and only if the inequalities $f_8(a,1,1)\ge 0$ and $f_8(0,b,c)\ge 0$ hold for all $a,b,c\ge 0$.

Proof. For fixed p and q,
$$h_8(r)=A(p,q)r^2+B(p,q)r+C(p,q)$$
is a concave quadratic function of r. Therefore, $h_8(r)$ is minimum when r is minimum or maximum; that is, according to P 3.57 in Volume 1, when $b=c$ or $a=0$. Thus, the conclusion follows. Notice that $A(p,q)$ is called the *highest polynomial* of $f_8(a,b,c)$.

Remark. This Lemma can be extended for the case where the highest polynomial $A(p,q$ is not nonnegative for all $a,b,c\ge 0$.

- *The inequality $f_8(a,b,c)\ge 0$ in the preceding Lemma holds for all $a,b,c\ge 0$ satisfying $A(p,q)\le 0$ if and only if the inequalities $f_8(a,1,1)\ge 0$ and $f_8(0,b,c)\ge 0$ hold for all $a,b,c\ge 0$ satisfying $A(a+2,2a+1)\le 0$ and $A(b+c,bc)\le 0$.*

□

P 1.174. *If a,b,c are real numbers such that $abc\ne 0$, then*
$$\frac{(b+c)^2}{a^2}+\frac{(c+a)^2}{b^2}+\frac{(a+b)^2}{c^2}\ge 2+\frac{10(a+b+c)^2}{3(a^2+b^2+c^2)}.$$

(Vasile Cîrtoaje and Michael Rozenberg, 2014)

Solution. Let

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{(b+c)^2}{a^2} \ge \frac{\left[\sum (b+c)^2\right]^2}{\sum a^2(b+c)^2} = \frac{2\left(\sum a^2 + \sum ab\right)^2}{\sum a^2b^2 + abc\sum a} = \frac{2(p^2-q)^2}{q^2-pr}.$$

Therefore, it suffices to show that

$$\frac{2(p^2-q)^2}{q^2-pr} \ge 2 + \frac{10p^2}{3(p^2-2q)},$$

which is equivalent to

$$\frac{3(p^2-q)^2}{q^2-pr} \ge \frac{8p^2-6q}{p^2-2q}.$$

Using Schur's inequality

$$p^3 + 9r \ge 4pq,$$

we get

$$q^2 - pr \le q^2 - p \cdot \frac{4pq - p^3}{9} = \frac{p^4 - 4p^2q + 9q^2}{9}.$$

Thus, it suffices to prove that

$$\frac{27(p^2-q)^2}{p^4-4p^2q+9q^2} \ge \frac{8p^2-6q}{p^2-2q},$$

which is equivalent to the obvious inequality

$$p^2(p^2-3q)(19p^2-13q) \ge 0.$$

The equality holds for $a = b = c$.

□

P 1.175. *Let a, b, c be real numbers such that $ab + bc + ca \ge 0$ and no two of which are zero. Prove that*

(a)
$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2};$$

(b) $if\ ab \le 0$, then

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge 2.$$

(Vasile Cîrtoaje, 2014)

Solution. Let as show first that $b+c \neq 0$, $c+a \neq 0$ and $a+b \neq 0$. Indeed, if $b+c=0$, then $ab+bc+ca \geq 0$ yields $bc \geq 0$, hence $b=c=0$, which is not possible (because, by hypothesis, at most one of a,b,c can be zero).

(a) Use the SOS method. Write the inequality as follows:

$$\sum\left(\frac{a}{b+c}+1\right) \geq \frac{9}{2},$$

$$\left[\sum(b+c)\right]\left(\sum\frac{1}{b+c}\right) \geq 9,$$

$$\sum\left(\frac{a+b}{a+c}+\frac{a+c}{a+b}-2\right) \geq 0,$$

$$\sum\frac{(b-c)^2}{(a+b)(a+c)} \geq 0,$$

$$\sum\frac{(b-c)^2}{a^2+(ab+bc+ca)} \geq 0.$$

Clearly, the last inequality is true. The equality holds for $a=b=c \neq 0$.

(b) From $ab+bc+ca \geq 0$, it follows that if one of a,b,c is zero, then the others are the same sign. In this case, the desired inequality is trivial. Consider further that $abc \neq 0$. Since the problem remains unchanged by replacing a,b,c with $-a,-b,-c$, it suffices to consider

$$a<0<b \leq c.$$

First Solution. We will show that

$$F(a,b,c) > F(0,b,c) \geq 2,$$

where

$$F(a,b,c)=\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}.$$

The right inequality is true because

$$F(0,b,c)=\frac{b}{c}+\frac{c}{b} \geq 2.$$

Since

$$F(a,b,c)-F(0,b,c)=a\left[\frac{1}{b+c}-\frac{b}{c(c+a)}-\frac{c}{b(a+b)}\right],$$

the left inequality is true if

$$\frac{b}{c(c+a)}+\frac{c}{b(a+b)} > \frac{1}{b+c}.$$

From $ab+bc+ca \ge 0$, we get

$$c+a \ge \frac{-ca}{b} > 0, \quad a+b \ge \frac{-ab}{c} > 0,$$

hence

$$\frac{b}{c(c+a)} > \frac{b}{c^2}, \quad \frac{c}{b(a+b)} > \frac{c}{b^2}.$$

Therefore, it suffices to prove that

$$\frac{b}{c^2} + \frac{c}{b^2} \ge \frac{1}{b+c}.$$

Indeed, by virtue of the AM-GM inequality, we have

$$\frac{b}{c^2} + \frac{c}{b^2} - \frac{1}{b+c} \ge \frac{2}{\sqrt{bc}} - \frac{1}{2\sqrt{bc}} > 0.$$

This completes the proof. The equality holds for $a=0$ and $b=c$, or $b=0$ and $a=c$.

Second Solution. From $b+c>0$ and

$$(b+c)(a+b) = b^2 + (ab+bc+ca) > 0,$$

$$(b+c)(c+a) = c^2 + (ab+bc+ca) > 0,$$

it follows that

$$a+b>0, \quad c+a>0.$$

By virtue of the Cauchy-Schwarz inequality and AM-GM inequality, we have

$$\begin{aligned}
\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} &\ge \frac{a}{b+c} + \frac{(b+c)^2}{b(c+a)+c(a+b)} \\
&= \frac{a}{b+c} + \frac{(b+c)^2}{2bc+a(b+c)} \\
&> \frac{a}{2a+b+c} + \frac{(b+c)^2}{\dfrac{(b+c)^2}{2} + a(b+c)} \\
&> \frac{4a}{2a+b+c} + \frac{2(b+c)}{2a+b+c} = 2.
\end{aligned}$$

□

P 1.176. *If a,b,c are nonnegative real numbers, then*

$$\frac{a}{7a+b+c} + \frac{b}{7b+c+a} + \frac{c}{7c+a+b} \ge \frac{ab+bc+ca}{(a+b+c)^2}.$$

(Vasile Cîrtoaje, 2014)

First Solution. Use the SOS method. Write the inequality as follows:

$$\sum\left[\frac{2a}{7a+b+c}-\frac{a(b+c)}{(a+b+c)^2}\right]\geq 0,$$

$$\sum\frac{a[(a-b)+(a-c)](a-b-c)}{7a+b+c}\geq 0,$$

$$\sum\frac{a(a-b)(a-b-c)}{7a+b+c}+\sum\frac{a(a-c)(a-b-c)}{7a+b+c}\geq 0,$$

$$\sum\frac{a(a-b)(a-b-c)}{7a+b+c}+\sum\frac{b(b-a)(b-c-a)}{7b+c+a}\geq 0,$$

$$\sum(a-b)\left[\frac{a(a-b-c)}{7a+b+c}-\frac{b(b-c-a)}{7b+c+a}\right]\geq 0,$$

$$\sum(a-b)^2(a^2+b^2-c^2+14ab)(a+b+7c)\geq 0.$$

Since

$$a^2+b^2-c^2+14ab\geq(a+b)^2-c^2=(a+b+c)(a+b-c),$$

it suffices to show that

$$\sum(a-b)^2(a+b-c)(a+b+7c)\geq 0.$$

Assume that $a\geq b\geq c$. It is enough to show that

$$(a-c)^2(a-b+c)(a+7b+c)+(b-c)^2(-a+b+c)(7a+b+c)\geq 0.$$

For the nontrivial case $b>0$, we have

$$(a-c)^2\geq\frac{a^2}{b^2}(b-c)^2\geq\frac{a}{b}(b-c)^2.$$

Thus, it suffices to prove that

$$a(a-b+c)(a+7b+c)+b(-a+b+c)(7a+b+c)\geq 0.$$

Since

$$a(a+7b+c)\geq b(7a+b+c),$$

we have

$$a(a-b+c)(a+7b+c)+b(-a+b+c)(7a+b+c)\geq$$

$$\geq b(a-b+c)(7a+b+c)+b(-a+b+c)(7a+b+c)$$

$$=2bc(7a+b+c)\geq 0.$$

This completes the proof. The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

Second Solution. Assume that

$$a \le b \le c, \qquad a+b+c=3,$$

and use the substitution

$$x=\frac{2a+1}{3}, \quad y=\frac{2b+1}{3}, \quad z=\frac{2c+1}{3}.$$

We have $b+c \ge 2$ and

$$\frac{1}{3} \le x \le y \le z, \qquad x+y+z=3, \quad x \le 1, \quad y+z \ge 2.$$

The inequality can be written as follows:

$$\frac{a}{2a+1}+\frac{b}{2b+1}+\frac{c}{2c+1} \ge \frac{9-a^2-b^2-c^2}{6},$$

$$\frac{a^2+b^2+c^2}{3} \ge \frac{1}{2a+1}+\frac{1}{2b+1}+\frac{1}{2c+1},$$

$$\frac{(2a+1)^2+(2b+1)^2+(2c+1)^2-15}{12} \ge \frac{1}{2a+1}+\frac{1}{2b+1}+\frac{1}{2c+1},$$

$$9(x^2+y^2+z^2) \ge 4\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+15.$$

We will use the mixing variables method. More precisely, we will show that

$$E(x,y,z) \ge E(x,t,t) \ge 0,$$

where

$$t=(y+z)/2=(3-x)/2,$$

$$E(x,y,z)=9(x^2+y^2+z^2)-4\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)-15.$$

We have

$$\begin{aligned} E(x,y,z)-E(x,t,t) &= 9(y^2+z^2-2t^2)-4\left(\frac{1}{y}+\frac{1}{z}-\frac{2}{t}\right) \\ &= \frac{(y-z)^2[9yz(y+z)-8]}{2yz(y+z)} \ge 0 \end{aligned}$$

since

$$9yz=(2b+1)(2c+1) \ge 2(b+c)+1 \ge 5, \qquad y+z \ge 2.$$

Also,

$$E(x,t,t)=9x^2+2t^2-15-\frac{4}{x}-\frac{8}{t}=\frac{(x-1)^2(3x-1)(8-3x)}{2x(3-x)} \ge 0.$$

Third Solution. Write the inequality as $f_5(a,b,c) \geq 0$, where $f_5(a,b,c)$ is a symmetric homogeneous inequality of degree five. According to P 3.68-(a) in Volume 1, it suffices to prove the inequality for $a = 0$ and for $b = c = 1$, when the inequality is equivalent to

$$(b-c)^2(b^2+c^2+11bc) \geq 0$$

and

$$a(a-1)^2(a+14) \geq 0,$$

respectively.

□

P 1.177. *If a,b,c are positive real numbers such that $abc = 1$, then*

$$\frac{a+b+c}{30} + \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} \geq \frac{8}{5}.$$

(Vasile Cîrtoaje, 2018)

Solution. Assume that $a \geq b \geq c$, which involves $ab \geq 1$. Since $a+b \geq 2\sqrt{ab}$ and

$$\frac{1}{a+1} + \frac{1}{b+1} - \frac{2}{\sqrt{ab}+1} = \frac{(\sqrt{a}-\sqrt{b})^2(\sqrt{ab}-1)}{(a+1)(b+1)(\sqrt{ab}+1)} \geq 0,$$

it suffices to show that

$$\frac{2\sqrt{ab}+c}{30} + \frac{2}{\sqrt{ab}+1} + \frac{1}{c+1} \geq \frac{8}{5}.$$

Substituting $\sqrt{ab} = 1/t$, which implies $c = t^2$, the inequality becomes

$$\frac{t^3+2}{30t} + \frac{2t}{t+1} + \frac{1}{t^2+1} \geq \frac{8}{5},$$

$$t^6 + t^5 + 13t^4 - 45t^3 + 44t^2 - 16t + 2 \geq 0,$$

$$(t-1)^2[t^4 + 3t^3 + 2(3t-1)^2] \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 1.178. *Let f be a real function defined on an interval $\mathbb{I}$, and let $x,y,s \in \mathbb{I}$ such that $x+my=(1+m)s$, where $m>0$. Prove that the inequality*

$$f(x)+mf(y)\ge(1+m)f(s)$$

holds if and only if

$$h(x,y)\ge 0,$$

where

$$h(x,y)=\frac{g(x)-g(y)}{x-y},\qquad g(u)=\frac{f(u)-f(s)}{u-s}.$$

(Vasile Cîrtoaje, 2006)

Solution. From

$$\begin{aligned}f(x)+mf(y)-(1+m)f(s)&=[f(x)-f(s)]+m[f(y)-f(s)]\\&=(x-s)g(x)+m(y-s)g(y)\\&=\frac{m}{1+m}(x-y)[g(x)-g(y)]\\&=\frac{m}{1+m}(x-y)^2h(x,y),\end{aligned}$$

the conclusion follows.

Remark. From the proof above, it follows that P 1.178 is also valid for the case where f is defined on $\mathbb{I}\setminus\{u_0\}$ and $x,y,s\neq u_0$.

□

P 1.179. *Let $a,b,c\le 8$ be real numbers such that $a+b+c=3$. Prove that*

$$\frac{13a-1}{a^2+23}+\frac{13b-1}{b^2+23}+\frac{13c-1}{c^2+23}\le\frac{3}{2}.$$

(Vasile Cîrtoaje, 2008)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)\ge\frac{-3}{2},$$

where

$$f(u)=\frac{1-13u}{u^2+23}.$$

Assume that $a\le b\le c$, and denote

$$s=\frac{b+c}{2}.$$

We have

$$s = \frac{3-a}{2}, \quad 1 \le s \le 8.$$

We claim that

$$f(b) + f(c) \ge 2f(s).$$

To show this, according to P 1.178, it suffices to show that

$$h(b,c) \ge 0,$$

where

$$h(b,c) = \frac{g(b) - g(c)}{b - c}, \qquad g(u) = \frac{f(u) - f(s)}{u - s}.$$

We have

$$g(u) = \frac{(13s-1)u - s - 299}{(s^2+23)(u^2+23)},$$

$$h(b,c) = \frac{(1-13s)bc + (s+299)(b+c) + 23(13s-1)}{(s^2+23)(b^2+23)(c^2+23)}.$$

Since $1 - 13s < 0$ and $bc \le s^2$, we get

$$\begin{aligned}
h(b,c) &\ge \frac{(1-13s)s^2 + (s+299)(2s) + 23(13s-1)}{(s^2+23)(b^2+23)(c^2+23)} \\
&= \frac{-13s^3 + 3s^2 + 897s - 23}{(s^2+23)(b^2+23)(c^2+23)} \\
&> \frac{-13s^3 + 3s^2 + 897s - 712}{(s^2+23)(b^2+23)(c^2+23)} \\
&= \frac{(8-s)(13s^2 + 101s - 89)}{(s^2+23)(b^2+23)(c^2+23)} \ge 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
f(a) + f(b) + f(c) + \frac{3}{2} \ge f(a) + 2f(s) + \frac{3}{2} &= f(a) + 2f\left(\frac{3-a}{2}\right) + \frac{3}{2} \\
&= \frac{1-13a}{a^2+23} + \frac{4(13a-37)}{a^2-6a+101} + \frac{3}{2} \\
&= \frac{3(a-1)^2(a+11)^2}{2(a^2+23)(a^2-6a+101)} \ge 0.
\end{aligned}$$

The equality holds for $a = b = c = 1$, and also for $a = -11$ and $b = c = 7$ (or any cyclic permutation).

$\square$

P 1.180. *Let $a,b,c \neq \frac{3}{4}$ be nonnegative real numbers such that $a+b+c=3$. Prove that*

$$\frac{1-a}{(4a-3)^2}+\frac{1-b}{(4b-3)^2}+\frac{1-c}{(4c-3)^2} \ge 0.$$

(Vasile Cîrtoaje, 2006)

Solution. Write the inequality as

$$f(a)+f(b)+f(c) \ge 0,$$

where

$$f(u)=\frac{1-u}{(4u-3)^2}.$$

Assume that $a \le b \le c$, and denote

$$s=\frac{b+c}{2}.$$

We have

$$s=\frac{3-a}{2}, \quad 1 \le s \le \frac{3}{2}.$$

We claim that

$$f(b)+f(c) \ge 2f(s).$$

According to Remark from P 1.178, it suffices to show that

$$h(b,c) \ge 0,$$

where

$$h(b,c)=\frac{g(b)-g(c)}{b-c}, \qquad g(u)=\frac{f(u)-f(s)}{u-s}.$$

We have

$$g(u)=\frac{16(s-1)u-16s+15}{(4s-3)^2(4u-3)^2},$$

$$\frac{1}{8}h(b,c)=\frac{-32(s-1)bc+64s^2-90s+27}{(4s-3)^2(4b-3)^2(4c-3)^2}.$$

Since $s-1 \ge 0$ and $bc \le s^2$, we get

$$\begin{aligned}\frac{1}{8}h(b,c) &\ge \frac{-32(s-1)s^2+64s^2-90s+27}{(4s-3)^2(4b-3)^2(4c-3)^2}\\ &=\frac{-32s^3+96s^2-90s+27}{(4s-3)^2(4b-3)^2(4c-3)^2}\\ &=\frac{(3-2s)(3-4s)^2}{(4s-3)^2(4b-3)^2(4c-3)^2} \ge 0.\end{aligned}$$

Therefore,

$$f(a)+f(b)+f(c) \geq f(a)+2f(s) = f(a)+2f\left(\frac{3-a}{2}\right)$$
$$= \frac{1-a}{(4a-3)^2} + \frac{a-1}{(3-2a)^2} = \frac{12a(a-1)^2}{(4a-3)^2(3-2a)^2} \geq 0.$$

The equality holds for $a=b=c=1$, and also for $a=0$ and $b=c=3/2$ (or any cyclic permutation).

□

P 1.181. *If a,b,c are the lengths of the sides of a triangle, then*

$$\frac{a^2}{4a^2+5bc} + \frac{b^2}{4b^2+5ca} + \frac{c^2}{4c^2+5ab} \geq \frac{1}{3}.$$

(Vasile Cîrtoaje, 2009)

Solution. Use the highest coefficient method. Write the inequality as $f_6(a,b,c) \geq 0$, where

$$f_6(a,b,c) = 3\sum a^2(4b^2+5ca)(4c^2+5ab) - \prod(4a^2+5bc)$$
$$= -45a^2b^2c^2 - 25abc\sum a^3 + 40\sum a^3b^3.$$

Since $f_6(a,b,c)$ has the highest coefficient

$$A = -45-75+120 = 0,$$

according to P 3.76-(b) in Volume 1, it suffices to prove the original inequality for $b=c=1$ and $0 \leq a \leq 2$, and for $a=b+c$.

Case 1: $b=c=1$, $0 \leq a \leq 2$. The original inequality becomes

$$\frac{a^2}{4a^2+5} + \frac{2}{5a+4} \geq \frac{1}{3},$$

$$(2-a)(a-1)^2 \geq 0.$$

Case 2: $a=b+c$. Using the Cauchy-Schwarz inequality

$$\frac{b^2}{4b^2+5ca} + \frac{c^2}{4c^2+5ab} \geq \frac{(b+c)^2}{4(b^2+c^2)+5a(b+c)},$$

it suffices to show that

$$\frac{a^2}{4a^2+5bc} + \frac{(b+c)^2}{4(b^2+c^2)+5a(b+c)} \geq \frac{1}{3},$$

which is equivalent to

$$\frac{1}{4(b^2+c^2)+13bc}+\frac{1}{9(b^2+c^2)+10bc}\geq\frac{1}{3(b^2+c^2+2bc)}.$$

Using the substitution

$$x=\frac{b}{c}+\frac{c}{b},\qquad x\geq 2,$$

the inequality becomes

$$\frac{1}{4x+13}+\frac{1}{9x+10}\geq\frac{1}{3(x+2)},$$

$$(x-2)(3x-4)\geq 0.$$

The equality holds for an equilateral triangle, and for a degenerate triangle with $a/2=b=c$ (or any cyclic permutation).

□

P 1.182. *If a,b,c are the lengths of the sides of a triangle, then*

$$\frac{1}{7a^2+b^2+c^2}+\frac{1}{7b^2+c^2+a^2}+\frac{1}{7c^2+a^2+b^2}\geq\frac{3}{(a+b+c)^2}.$$

(Vo Quoc Ba Can, 2010)

Solution. Use the highest coefficient method. Denote

$$p=a+b+c,\qquad q=ab+bc+ca,$$

and write the inequality as $f_6(a,b,c)\geq 0$, where

$$\begin{aligned}f_6(a,b,c)&=p^2\sum(7b^2+c^2+a^2)(7c^2+a^2+b^2)-3\prod(7a^2+b^2+c^2)\\&=p^2\sum(6b^2+p^2-2q)(6c^2+p^2-2q)-3\prod(6a^2+p^2-2q).\end{aligned}$$

Since $f_6(a,b,c)$ has the highest coefficient

$$A=-3\cdot 6^3<0,$$

according to P 3.76-(b) in Volume 1, it suffices to prove the original inequality for $b=c=1$ and $0\leq a\leq 2$, and for $a=b+c$.

Case 1: $b=c=1$, $0\leq a\leq 2$. The original inequality reduces to

$$\frac{1}{7a^2+2}+\frac{2}{a^2+8}\geq\frac{3}{(a+2)^2},$$

$$a(8-a)(a-1)^2\geq 0.$$

Case 2: $a = b + c$. Write the inequality as

$$\frac{1}{4(b^2+c^2)+7bc} + \frac{1}{4b^2+c^2+bc} + \frac{1}{4c^2+b^2+bc} \geq \frac{3}{2(b+c)^2}.$$

Since

$$\frac{3}{2(b+c)^2} - \frac{1}{4(b^2+c^2)+7bc} \leq \frac{3}{2(b+c)^2} - \frac{1}{4(b^2+c^2)+8bc} = \frac{5}{4(b+c)^2},$$

it suffices to show that

$$\frac{1}{4b^2+c^2+bc} + \frac{1}{4c^2+b^2+bc} \geq \frac{5}{4(b+c)^2},$$

which is equivalent to

$$4[5(b^2+c^2)+2bc][(b^2+c^2)+2bc] \geq 5(4b^2+c^2+bc)(4c^2+b^2+bc),$$

$$4[5(b^2+c^2)^2+12bc(b^2+c^2)+4b^2c^2] \geq 5[4(b^2+c^2)^2+5bc(b^2+c^2)+10b^2c^2],$$

$$bc[23(b-c)^2+12bc] \geq 0.$$

The equality holds for an equilateral triangle, and for a degenerate triangle with $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 1.183. *Let a, b, c be the lengths of the sides of a triangle. If $k > -2$, then*

$$\sum \frac{a(b+c)+(k+1)bc}{b^2+kbc+c^2} \leq \frac{3(k+3)}{k+2}.$$

(Vasile Cîrtoaje, 2009)

Solution. Use the highest coefficient method. Let

$$p = a+b+c, \qquad q = ab+bc+ca.$$

Write the inequality as $f_6(a,b,c) \geq 0$, where

$$f_6(a,b,c) = 3(k+3)\prod(b^2+kbc+c^2)$$

$$-(k+2)\sum[a(b+c)+(k+1)bc](c^2+kca+a^2)(a^2+kab+b^2).$$

From

$$f_6(a,b,c) = 3(k+3)\prod(p^2-2q+kbc-a^2)$$

$$-(k+2)\sum(q+kbc)(p^2-2q+kca-b^2)(p^2-2q+kab-c^2),$$

it follows that $f_6(a,b,c)$ has the same highest coefficient A as $f(a,b,c)$, where

$$f(a,b,c) = 3(k+3)P_3(a,b,c) - k(k+2)P_2(a,b,c),$$

$$P_3(a,b,c) = \prod(kbc - a^2), \qquad P_2(a,b,c) = \sum bc(kca - b^2)(kab - c^2).$$

According to Remark 2 from the proof of P 2.75 in Volume 1, we have

$$\begin{aligned} A &= 3(k+3)P_3(1,1,1) - k(k+2)P_2(1,1,1) \\ &= 3(k+3)(k-1)^3 - 3k(k+2)(k-1)^2 = -9(k-1)^2 \le 0. \end{aligned}$$

Taking into account P 3.76-(b) in Volume 1, it suffices to prove the original inequality for $b = c = 1$ and $0 \le a \le 2$, and for $a = b + c$.

Case 1: $b = c = 1$, $0 \le a \le 2$. The original inequality reduces to

$$\frac{2a+k+1}{k+2} + \frac{2(k+2)a+2}{a^2+ka+1} \le \frac{3(k+3)}{k+2},$$

$$\frac{a-k-4}{k+2} + \frac{(k+2)a+1}{a^2+ka+1} \le 0,$$

$$(2-a)(a-1)^2 \ge 0.$$

Case 2: $a = b + c$. Write the inequality as follows:

$$\sum\left[\frac{a(b+c)+(k+1)bc}{b^2+kbc+c^2} - 1\right] \le \frac{3}{k+2},$$

$$\sum \frac{ab+bc+ca-b^2-c^2}{b^2+kbc+c^2} \le \frac{3}{k+2},$$

$$\frac{3bc}{b^2+kbc+c^2} + \frac{bc-c^2}{b^2+(k+2)(bc+c^2)} + \frac{bc-b^2}{c^2+(k+2)(bc+b^2)} \le \frac{3}{k+2}.$$

Since

$$\frac{3bc}{b^2+kbc+c^2} \le \frac{3}{k+2},$$

it suffices to prove that

$$\frac{bc-c^2}{b^2+(k+2)(bc+c^2)} + \frac{bc-b^2}{c^2+(k+2)(bc+b^2)} \le 0.$$

This reduces to the obvious inequality

$$(b-c)^2(b^2+bc+c^2) \ge 0.$$

The equality holds for an equilateral triangle, and for a degenerate triangle with $a/2 = b = c$ (or any cyclic permutation).

□

P 1.184. *Let a,b,c be the lengths of the sides of a triangle. If $k>-2$, then*

$$\sum \frac{2a^2+(4k+9)bc}{b^2+kbc+c^2} \le \frac{3(4k+11)}{k+2}.$$

(Vasile Cîrtoaje, 2009)

Solution. Use the highest coefficient method. Let

$$p=a+b+c, \qquad q=ab+bc+ca.$$

Write the inequality as $f_6(a,b,c)\ge 0$, where

$$f_6(a,b,c)=3(4k+11)\prod(b^2+kbc+c^2)$$

$$-(k+2)\sum[2a^2+(4k+9)bc](c^2+kca+a^2)(a^2+kab+b^2).$$

From

$$f_6(a,b,c)=3(4k+11)\prod(p^2-2q+kbc-a^2)$$

$$-(k+2)\sum[2a^2+(4k+9)bc](p^2-2q+kca-b^2)(p^2-2q+kab-c^2),$$

it follows that $f_6(a,b,c)$ has the same highest coefficient A as $f(a,b,c)$, where

$$f(a,b,c)=3(4k+11)P_3(a,b,c)-(k+2)P_2(a,b,c),$$

$$P_3(a,b,c)=\prod(kbc-a^2),$$

$$P_2(a,b,c)=\sum[2a^2+(4k+9)bc](kca-b^2)(kab-c^2).$$

According to Remark 2 from the proof of P 2.75 in Volume 1, we have

$$\begin{aligned} A &= 3(4k+11)P_3(1,1,1)-(k+2)P_2(1,1,1)\\ &= 3(4k+11)(k-1)^3-3(k+2)(4k+11)(k-1)^2\\ &= -9(4k+11)(k-1)^2\le 0. \end{aligned}$$

Taking into account P 3.76-(b) in Volume 1, it suffices to prove the original inequality for $b=c=1$ and $0\le a\le 2$, and for $a=b+c$.

Case 1: $b=c=1$, $0\le a\le 2$. The original inequality reduces to

$$\frac{2a^2+4k+9}{k+2}+\frac{2(4k+9)a+4}{a^2+ka+1}\le \frac{3(4k+11)}{k+2},$$

$$\frac{a^2-4k-12}{k+2}+\frac{(4k+9)a+2}{a^2+ka+1}\le 0,$$

$$(2-a)(a-1)^2\ge 0,$$

Case 2: $a = b + c$. Write the inequality as follows:

$$\sum \left[\frac{2a^2 + (4k+9)bc}{b^2 + kbc + c^2} - 4 \right] \le \frac{9}{k+2},$$

$$\sum \frac{2a^2 - 4b^2 - 4c^2 + 9bc}{b^2 + kbc + c^2} \le \frac{9}{k+2},$$

$$\frac{13bc - 2b^2 - 2c^2}{b^2 + kbc + c^2} + \frac{bc - 2b^2 + c^2}{b^2 + (k+2)(bc + c^2)} + \frac{bc - 2c^2 + b^2}{c^2 + (k+2)(bc + b^2)} \le \frac{9}{k+2}.$$

Since

$$\frac{9}{k+2} - \frac{13bc - 2b^2 - 2c^2}{b^2 + kbc + c^2} = \frac{(2k+13)(b-c)^2}{(k+2)(b^2 + kbc + c^2)}$$

and

$$\frac{bc - 2b^2 + c^2}{b^2 + (k+2)(bc + c^2)} + \frac{bc - 2c^2 + b^2}{c^2 + (k+2)(bc + b^2)} =$$

$$= \frac{(b-c)^2(b^2 + c^2 + 3bc) - 2(k+2)(b^2 - c^2)^2}{[b^2 + (k+2)(bc + c^2)][c^2 + (k+2)(bc + b^2]},$$

we only need to show that

$$\frac{2k+13}{(k+2)(b^2 + kbc + c^2)} + \frac{2(k+2)(b+c)^2 - b^2 - c^2 - 3bc}{[b^2 + (k+2)(bc + c^2)][c^2 + (k+2)(bc + b^2]} \ge 0.$$

Using the substitution

$$x = \frac{b}{c} + \frac{c}{b}, \qquad x \ge 2,$$

the inequality can be written as

$$\frac{2k+13}{(k+2)(x+k)} + \frac{(2k+3)x + 4k + 5}{(k+2)x^2 + (k+2)(k+3)x + 2k^2 + 6k + 5} \ge 0,$$

which is equivalent to

$$4(k+2)(k+4)x^2 + 2(k+2)Bx + C \ge 0,$$

where

$$B = 2k^2 + 13k + 22, \quad C = 8k^3 + 51k^2 + 98k + 65.$$

Since

$$B = 2(k+2)^2 + 5(k+2) + 4 > 0,$$

$$C = 8(k+2)^3 + 2k^2 + (k+1)^2 > 0,$$

the conclusion follows. The equality holds for an equilateral triangle, and for a degenerate triangle with $a/2 = b = c$ (or any cyclic permutation).

□

P 1.185. *If a,b,c are nnonnegative numbers such that $abc=1$, then*

$$\frac{1}{(a+1)^2}+\frac{1}{(b+1)^2}+\frac{1}{(c+1)^2}+\frac{1}{2(a+b+c-1)}\ge 1.$$

(Vasile Cîrtoaje, 2018)

Solution. Let

$$p=a+b+c, \qquad q=ab+bc+ca.$$

By the Cauchy-Schwarz inequality, we have

$$\sum\frac{1}{(a+1)^2}=\sum\frac{b^2c^2}{(1+bc)^2}\ge\frac{(\sum bc)^2}{\sum(1+bc)^2}$$

$$=\frac{q^2}{q^2+2q-2p+3}.$$

Thus we only need to show that

$$\frac{q^2}{q^2+2q-2p+3}+\frac{1}{2(p-1)}\ge 1,$$

which is equivalent to

$$(q-2p+3)^2\ge 0.$$

The equality occurs for $a=b=c=1$.

□

P 1.186. *If a,b,c are positive real numbers such that*

$$a\le b\le c, \qquad a^2bc\ge 1,$$

then

$$\frac{1}{1+a^3}+\frac{1}{1+b^3}+\frac{1}{1+c^3}\ge\frac{3}{1+abc}.$$

(Vasile Cîrtoaje, 2008)

Solution. Since

$$\frac{1}{1+x^2}+\frac{1}{1+y^2}-\frac{2}{1+xy}=\frac{(x-y)^2(xy-1)}{(1+x^2)(1+y^2)(1+xy)},$$

we have

$$\frac{1}{1+b^3}+\frac{1}{1+c^3}\ge\frac{2}{1+t^3},$$

where

$$t=\sqrt{bc}, \quad at\ge 1, \quad t\ge 1, \quad t\ge a.$$

So, we only need to show that

$$\frac{1}{1+a^3}+\frac{2}{1+t^3} \ge \frac{3}{1+at^2}\ ,$$

which is equivalent to

$$\frac{a(t^2-a^2)}{1+a^3} \ge \frac{2t^2(t-a)}{1+t^3}\ ,$$

$$(t-a)^2[at^2(2a+t)-a-2t] \ge 0.$$

This is true since

$$at^2(2a+t)-a-2t \ge t(2a+t)-a-2t = (t-1)^2+(at-1)+a(t-1) \ge 0.$$

The equality occurs for $a=b=c \ge 1$.

Remark 1. The inequality is true for the weaker condition

$$a^{8/5}bc \ge 1,$$

that is $a^4t^5 \ge 1$. Since $bc \ge 1$, it suffices to show that $at^2(2a+t)-a-2t \ge 0$. This is true if the following homogeneous inequality is true:

$$\frac{at^2}{(a^4t^5)^{1/3}}(2a+t) \ge a+2t,$$

that is

$$t^{1/3}(2a+t) \ge a^{1/3}(a+2t).$$

Setting $a=1$ and $t=z^3 \ge 1$, the inequality becomes as follows:

$$z(2+z^3) \ge 1+2z^3,$$

$$z^4-1 \ge 2z(z^2-1),$$

$$(z^2-1)(z-1)^2 \ge 0.$$

Remark 2. The inequality is also true for the condition

$$a^4b^5 \ge 1.$$

Indeed, if $a^4b^5 \ge 1$, then $b \ge 1$, $bc \ge b^2 \ge 1$ and

$$a^4(bc)^{5/2} \ge 1,$$

which is equivalent to to the condition $a^{8/5}bc \ge 1$ from Remark 1.

Remark 3. From P 1.186, the following statement follows (*V. Cirtoaje* and *V. Vornicu*):

- *If a, b, c, d are positive real numbers such that*

$$a \ge b \ge c \ge d, \qquad abcd \ge 1,$$

then

$$\frac{1}{1+a^3}+\frac{1}{1+b^3}+\frac{1}{1+c^3} \ge \frac{3}{1+abc}.$$

This is valid because $c \le b \le a$ and $c^2ba \ge 1$.

□

P 1.187. *If a, b, c are positive real numbers such that*

$$a \le b \le c, \qquad a^2c \ge 1,$$

then

$$\frac{1}{1+a^3}+\frac{1}{1+b^3}+\frac{1}{1+c^3} \ge \frac{3}{1+abc}.$$

(Vasile Cîrtoaje, 2021)

Solution. Denote

$$d=\sqrt{ac}, \quad d \ge 1.$$

If $d=1$, then $ac=1$ and $a^2c \ge 1$ yield $a=b=c=1$, and the required inequality is an equality. Consider next that $d>1$. For fixed a and c, write the inequality as $f(b) \ge 0$, where

$$f(b)=\frac{1}{1+a^3}+\frac{1}{1+b^3}+\frac{1}{1+c^3} - \frac{3}{1+abc}, \quad b \in [a,c],$$

and calculate the derivative

$$\frac{1}{3}f'(b)=\frac{d^2}{(1+d^2b)^2}-\frac{b^2}{(1+b^3)^2}$$

$$=\frac{(db^2-1)(b-d)[d(1+b^3)+b(d^2b+1)]}{(1+d^2b)^2(1+b^3)^2}.$$

If $a \le \dfrac{1}{\sqrt{d}}$, then $f'(b) \le 0$ for $b \in [1/\sqrt{d}, d]$ and $f'(b) \ge 0$ for $b \in [a, 1/\sqrt{d}] \cup [d,c]$, hence $f(b)$ is decreasing on $[1/\sqrt{d}, d]$ and increasing on $[a, 1/\sqrt{d}] \cup [d,c]$. Thus, it suffices to show that $f(a) \ge 0$ and $f(d) \ge 0$. If $a \ge \dfrac{1}{\sqrt{d}}$, then $f'(b) \le 0$ for $b \in [a,d]$ and $f'(b) \ge 0$ for $b \in [d,c]$, $f(b)$ is decreasing on $[a,d]$ and increasing on $[d,c]$, hence it suffices to show that $f(d) \ge 0$. In conclusion, we only need to show that $f(a) \ge 0$ and $f(d) \ge 0$. Write the inequality $f(a) \ge 0$ as follows:

$$\frac{2}{1+a^3}+\frac{1}{1+c^3} \ge \frac{3}{1+a^2c},$$

$$\frac{2a^2(c-a)}{1+a^3} \ge \frac{c(c^2-a^2)}{1+c^3},$$
$$(c-a)^2[a^2c(a+2c)-2a-c] \ge 0.$$

This is true because

$$a^2c(a+2c)-2a-c \ge (a+2c)-2a-c = c-a \ge 0.$$

Write now the inequality $f(d) \ge 0$ as

$$\frac{1}{1+a^3}+\frac{1}{1+c^3} \ \ge \ \frac{2}{1+(ac)^{3/2}}.$$

Since

$$\frac{1}{1+x^2}+\frac{1}{1+y^2}-\frac{2}{1+xy} = \frac{(x-y)^2(xy-1)}{(1+x^2)(1+y^2)(1+xy)},$$

the inequality is equivalent to

$$\left(a^{3/2}-c^{3/2}\right)^2\left[(ac)^{3/2}-1\right] \ge 0.$$

This is true because

$$(ac)^3 \ge (a^2c)^2 \ge 1.$$

The equality occurs for $a=b=c \ge 1$.

$\square$

P 1.188. *If a,b,c are positive real numbers such that*

$$a \le b \le c, \qquad 2a+c \ge 3,$$

then

$$\frac{1}{3+a^2}+\frac{1}{3+b^2}+\frac{1}{3+c^2} \ge \frac{3}{3+\left(\frac{a+b+c}{3}\right)^2}.$$

(Vasile Cîrtoaje, 2021)

Solution. Denote

$$s = \frac{a+b+c}{3}, \quad s \ge 1.$$

For fixed a and c, write the inequality as $f(b) \ge 0$, where

$$f(b) = \frac{1}{3+a^2}+\frac{1}{3+b^2}+\frac{1}{3+c^2} - \frac{3}{3+s^2}, \quad b \in [a,c],$$

and calculate the derivative

$$\frac{1}{2}f'(b) = \frac{s}{(3+s^2)^2}-\frac{b}{(3+b^2)^2} = \frac{(b-s)g(b)}{(3+s^2)^2(3+b^2)^2},$$

where

$$g(b) = bs(b^2 + bs + s^2 + 6) - 9.$$

Denote

$$d = \frac{a+c}{2}, \quad d \geq 1.$$

If $d = 1$, then $a + c = 2$ and $2a + c \geq 3$ yield $a = b = c = 1$, and the required inequality is an equality. Consider next that $d > 1$. Since

$$s = \frac{b+2d}{3},$$

we have

$$b - s = \frac{2(b-d)}{3},$$

$$g(b) = \frac{b(b+2d)}{3}\left[b^2 + \frac{b(b+2d)}{3} + \frac{(b+2d)^2}{9} + 6\right] - 9.$$

Since $g(b)$ is strictly increasing, $g(0) = -9$ and

$$g(d) = 3(d^4 + 2d^2 - 3) > 0,$$

there is an unique $d_1 \in (0, d)$ such that $g(d_1) = 0$, $g(b) \leq 0$ for $b \leq d_1$ and $g(b) \geq 0$ for $b \geq d_1$. If $a \leq d_1$, then $f'(b) \leq 0$ for $b \in [d_1, d]$ and $f'(b) \geq 0$ for $b \in [a, d_1] \cup [d, c]$, hence $f(b)$ is decreasing on $[d_1, d]$ and increasing on $[a, d_1] \cup [d, c]$. Thus, it suffices to show that $f(a) \geq 0$ and $f(d) \geq 0$. If $a \geq d_1$, then $f'(b) \leq 0$ for $b \in [a, d]$ and $f'(b) \geq 0$ for $b \in [d, c]$, $f(b)$ is decreasing on $[a, d]$ and increasing on $[d, c]$, hence it suffices to show that $f(d) \geq 0$. In conclusion, we only need to show that $f(a) \geq 0$ and $f(d) \geq 0$. Denoting

$$p = \frac{2a+c}{3},$$

we may write the inequality $f(a) \geq 0$ as follows:

$$\frac{2}{3+a^2} + \frac{1}{3+c^2} \geq \frac{3}{3+p^2},$$

$$\frac{2(p^2-a^2)}{3+a^2} \geq \frac{c^2-p^2}{3+c^2},$$

$$(a-c)^2[(a+c)p + ac - 3] \geq 0,$$

$$(a-c)^2(2a^2 + 6ac + c^2 - 9) \geq 0.$$

This is true because

$$2a^2 + 6ac + c^2 - 9 = (2a+c)^2 - 9 + 2a(c-a) \geq 0.$$

Write now the inequality $f(d) \geq 0$ as follows:

$$\frac{1}{3+a^2} + \frac{1}{3+c^2} \geq \frac{2}{3+d^2},$$

$$\frac{d^2-a^2}{3+a^2} \geq \frac{c^2-d^2}{3+c^2},$$

$$(a-c)^2[(a+c)d+ac)-3] \geq 0,$$

$$(a-c)^2(a^2+4ac+c^2-6) \geq 0.$$

This is true because

$$3(a^2+4ac+c^2)-18 \geq 3(a^2+4ac+c^2)-2(2a+c)^2 = (c-a)(c+5a) \geq 0.$$

The equality occurs for $a = b = c \geq 1$, and also for $a = b = 0$ and $c = 3$.

□

P 1.189. *If a, b, c are positive real numbers such that*

$$a \leq b \leq c, \qquad 9a+8b \geq 17,$$

then

$$\frac{1}{3+a^2} + \frac{1}{3+b^2} + \frac{1}{3+c^2} \geq \frac{3}{3+\left(\frac{a+b+c}{3}\right)^2}.$$

(Vasile Cîrtoaje, 2021)

Solution. From $a \leq b \leq c$ and $9a+8b \geq 17$, it follows that

$$1 \leq b \leq c, \quad a+b+c \geq 3.$$

As in the preceding P 1.188, denote

$$s = \frac{a+b+c}{3}, \qquad 1 \leq s \leq c,$$

and, for fixed a and b, write the inequality as $f(c) \geq 0$, where

$$f(c) = \frac{1}{3+a^2} + \frac{1}{3+b^2} + \frac{1}{3+c^2} - \frac{3}{3+s^2}, \qquad c \geq b.$$

We show that

$$f(c) \geq f(b) \geq 0.$$

Since

$$\frac{1}{2}f'(c) = \frac{s}{(3+s^2)^2} - \frac{c}{(3+c^2)^2} = \frac{(c-s)[cs(c^2+cs+s^2+6)-9]}{(3+s^2)^2(3+c^2)^2} \geq 0,$$

$f(c)$ is increasing, therefore $f(c) \geq f(b)$. Denote

$$p = \frac{a+2b}{3},$$

Write now the inequality $f(b) \geq 0$ as follows:

$$\frac{1}{3+a^2} + \frac{2}{3+b^2} \geq \frac{3}{3+p^2} ,$$

$$\frac{p^2-a^2}{3+a^2} \geq \frac{2(b^2-p^2)}{3+b^2} ,$$

$$(a-b)^2[(a+b)p+ab-3] \geq 0 ,$$

$$(a-b)^2(a^2+6ab+2b^2-9) \geq 0 .$$

This is true if

$$16(a^2+6ab+2b^2) \geq (7a+5b)^2,$$

which is equivalent to

$$(b-a)(b+220a) \geq 0.$$

The equality occurs for $a = b = c \geq 1$.

Remark. Actually, the inequality is valid for the weaker condition

$$ka + b \geq k+1, \qquad k = \frac{3}{\sqrt{2}} - 1,$$

when the inequality

$$(k+1)^2(a^2+6ab+2b^2) \geq 9(ka+b)^2,$$

reduces to the form

$$a(b-a) \geq 0.$$

The equality occurs for $a = b = c \geq 1$, and also for $a = 0$ and $b = c = \dfrac{3}{\sqrt{2}}$.

□

P 1.190. *Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that*

$$\sum \frac{1}{1+ab+bc+ca} \leq 1.$$

Solution. From

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}+\frac{1}{\sqrt{ab}} = \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{abc}},$$

we get

$$ab+bc+ca \geq \sqrt{abc}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right) = \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{d}}.$$

Therefore,

$$\sum \frac{1}{1+ab+bc+ca} \leq \sum \frac{\sqrt{d}}{\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}} = 1,$$

which is just the required inequality. The equality occurs for $a=b=c=d=1$.

□

P 1.191. *Let a,b,c,d be positive real numbers such that $abcd=1$. Prove that*

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}+\frac{1}{(1+c)^2}+\frac{1}{(1+d)^2} \geq 1.$$

(Vasile Cîrtoaje, 1995)

First Solution. The inequality follows by summing the following inequalities (see P 1.1):

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2} \geq \frac{1}{1+ab},$$

$$\frac{1}{(1+c)^2}+\frac{1}{(1+d)^2} \geq \frac{1}{1+cd} = \frac{ab}{1+ab}.$$

The equality occurs for $a=b=c=d=1$.

Second Solution. Using the substitution

$$a=\frac{1}{x^4}, \quad b=\frac{1}{y^4}, \quad c=\frac{1}{z^4}, \quad d=\frac{1}{t^4},$$

where x,y,z,t are positive real numbers such that $xyzt=1$, the inequality becomes

$$\frac{x^6}{\left(x^3+\frac{1}{x}\right)^2}+\frac{y^6}{\left(y^3+\frac{1}{y}\right)^2}+\frac{z^6}{\left(z^3+\frac{1}{z}\right)^2}+\frac{t^6}{\left(t^3+\frac{1}{t}\right)^2} \geq 1.$$

By the Cauchy-Schwarz inequality, we get

$$\sum \frac{x^6}{\left(x^3+\frac{1}{x}\right)^2} \geq \frac{\left(\sum x^3\right)^2}{\sum\left(x^3+\frac{1}{x}\right)^2} = \frac{\left(\sum x^3\right)^2}{\sum x^6+2\sum x^2+\sum x^2y^2z^2}.$$

Thus, it suffices to prove the homogeneous inequality

$$2(x^3y^3+x^3z^3+x^3t^3+y^3z^3+y^3t^3+z^3t^3)\geq 2xyzt\sum x^2+\sum x^2y^2z^2.$$

We can get it by summing the inequalities

$$4(x^3y^3+x^3z^3+x^3t^3+y^3z^3+y^3t^3+z^3t^3)\geq 6xyzt\sum x^2$$

and

$$2(x^3y^3+x^3z^3+x^3t^3+y^3z^3+y^3t^3+z^3t^3)\geq 3\sum x^2y^2z^2,$$

Write these inequalities as

$$\sum x^3(y^3+z^3+t^3-3yzt)\geq 0$$

and

$$\sum(x^3y^3+y^3z^3+z^3x^3-3x^2y^2z^2)\geq 0,$$

respectively. By the AM-GM inequality, we have

$$y^3+z^3+t^3\geq 3yzt,\qquad x^3y^3+y^3z^3+z^3x^3\geq 3x^2y^2z^2.$$

Thus the conclusion follows.

Third Solution. Using the substitution

$$a=\frac{yz}{x^2},\quad b=\frac{zt}{y^2},\quad c=\frac{tx}{z^2},\quad d=\frac{xy}{t^2},$$

where x,y,z,t are positive real numbers, the inequality becomes

$$\frac{x^4}{(x^2+yz)^2}+\frac{y^4}{(y^2+zt)^2}+\frac{z^4}{(z^2+tx)^2}+\frac{t^4}{(t^2+xy)^2}\geq 1.$$

Using the Cauchy-Schwarz inequality two times, we deduce

$$\frac{x^4}{(x^2+yz)^2}+\frac{z^4}{(z^2+tx)^2}\geq\frac{x^4}{(x^2+y^2)(x^2+z^2)}+\frac{z^4}{(z^2+t^2)(z^2+x^2)}$$

$$=\frac{1}{x^2+z^2}\left(\frac{x^4}{x^2+y^2}+\frac{z^4}{z^2+t^2}\right)\geq\frac{x^2+z^2}{x^2+y^2+z^2+t^2},$$

hence

$$\frac{x^4}{(x^2+yz)^2}+\frac{z^4}{(z^2+tx)^2}\geq\frac{x^2+z^2}{x^2+y^2+z^2+t^2}.$$

Adding this to the similar inequality

$$\frac{y^4}{(y^2+zt)^2}+\frac{t^4}{(t^2+xy)^2}\geq\frac{y^2+t^2}{x^2+y^2+z^2+t^2},$$

we get the required inequality.

Fourth Solution. Using the substitution

$$a=\frac{x}{y},\quad b=\frac{y}{z},\quad c=\frac{z}{t},\quad d=\frac{t}{x},$$

where x,y,z,t are positive real numbers, the inequality can be written as

$$\frac{y^2}{(x+y)^2}+\frac{z^2}{(y+z)^2}+\frac{t^2}{(z+t)^2}+\frac{x^2}{(t+x)^2}\ge 1.$$

By the Cauchy-Schwarz inequality and the AM-GM inequality, we get

$$\sum\frac{y^2}{(x+y)^2}\ge\frac{[\sum y(y+z)]^2}{\sum(x+y)^2(y+z)^2}$$

$$=\frac{[(x+y)^2+(y+z)^2+(z+t)^2+(t+x)^2]^2}{4[(x+y)^2+(z+t)^2][(y+z)^2+(t+x)^2]}\ge 1.$$

Remark. The following generalization holds true (*Vasile Cîrtoaje*, 2005):

- *Let $a_1,a_2,\ldots,a_n$ be positive real numbers such that $a_1a_2\cdots a_n=1$. If $k\ge\sqrt{n}-1$, then*

$$\frac{1}{(1+ka_1)^2}+\frac{1}{(1+ka_2)^2}+\cdots+\frac{1}{(1+ka_n)^2}\ge\frac{n}{(1+k)^2}.$$

□

P 1.192. *Let $a,b,c,d\neq\frac{1}{3}$ be positive real numbers such that $abcd=1$. Prove that*

$$\frac{1}{(3a-1)^2}+\frac{1}{(3b-1)^2}+\frac{1}{(3c-1)^2}+\frac{1}{(3d-1)^2}\ge 1.$$

(Vasile Cîrtoaje, 2006)

First Solution. It suffices to show that

$$\frac{1}{(3a-1)^2}\ge\frac{a^{-3}}{a^{-3}+b^{-3}+c^{-3}+d^{-3}}.$$

This inequality is equivalent to

$$6a^{-2}+b^{-3}+c^{-3}+d^{-3}\ge 9a^{-1},$$

which follows by the AM-GM inequality, as follows:

$$6a^{-2}+b^{-3}+c^{-3}+d^{-3}\ge 9\sqrt[9]{a^{-12}b^{-3}c^{-3}d^{-3}}=9a^{-1}.$$

The equality occurs for $a = b = c = d = 1$.

Second Solution. Let $a \le b \le c \le d$. If $a \le 2/3$, then

$$\frac{1}{(3a-1)^2} \ge 1,$$

and the desired inequality is clearly true. Otherwise, if $2/3 < a \le b \le c \le d$, we have

$$4a^3 - (3a-1)^2 = (a-1)^2(4a-1) \ge 0.$$

Using this result and the AM-GM inequality, we get

$$\sum \frac{1}{(3a-1)^2} \ge \frac{1}{4}\sum \frac{1}{a^3} \ge \sqrt[4]{\frac{1}{a^3b^3c^3d^3}} = 1.$$

Third Solution. We have

$$\frac{1}{(3a-1)^2} - \frac{1}{(a^3+1)^2} = \frac{a(a-1)^2(a+2)(a^2+3)}{(3a-1)^2(a^3+1)^2} \ge 0;$$

therefore,

$$\sum \frac{1}{(3a-1)^2} \ge \sum \frac{1}{(a^3+1)^2}.$$

Thus, it suffices to prove that

$$\sum \frac{1}{(a^3+1)^2} \ge 1,$$

which is an immediate consequence of the inequality in P 1.191.

□

P 1.193. *Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that*

$$\frac{1}{1+a+a^2+a^3} + \frac{1}{1+b+b^2+b^3} + \frac{1}{1+c+c^2+c^3} + \frac{1}{1+d+d^2+d^3} \ge 1.$$

(Vasile Cîrtoaje, 1999)

First Solution. We get the desired inequality by summing the inequalities

$$\frac{1}{1+a+a^2+a^3} + \frac{1}{1+b+b^2+b^3} \ge \frac{1}{1+(ab)^{3/2}},$$

$$\frac{1}{1+c+c^2+c^3} + \frac{1}{1+d+d^2+d^3} \ge \frac{1}{1+(cd)^{3/2}}.$$

Thus, it suffices to show that

$$\frac{1}{1+x^2+x^4+x^6} + \frac{1}{1+y^2+y^4+y^6} \ge \frac{1}{1+x^3y^3},$$

where x and y are positive real numbers. Putting $p = xy$ and $s = x^2 + xy + y^2$, this inequality becomes

$$p^3(x^6+y^6)+p^2(p-1)(x^4+y^4)-p^2(p^2-p+1)(x^2+y^2)-p^6-p^4+2p^3-p^2+1 \geq 0,$$

$$p^3(x^3-y^3)^2+p^2(p-1)(x^2-y^2)^2-p^2(p^2-p+1)(x-y)^2+p^6-p^4-p^2+1 \geq 0,$$

$$p^3s^2(x-y)^2+p^2(p-1)(s+p)^2(x-y)^2-p^2(p^2-p+1)(x-y)^2+p^6-p^4-p^2+1 \geq 0,$$

$$p^2(s+1)(ps-1)(x-y)^2+(p^2-1)(p^4-1) \geq 0.$$

If $ps-1 \geq 0$, then the inequality is clearly true. Consider further that $ps < 1$. From $ps < 1$ and $s \geq 3p$, we get $p^2 < 1/3$. Write the desired inequality in the form

$$p^2(1+s)(1-ps)(x-y)^2 \leq (1-p^2)(1-p^4).$$

Since

$$p(x-y)^2 = p(s-3p) < 1-3p^2 < 1-p^2,$$

it suffices to show that

$$p(1+s)(1-ps) \leq 1-p^4.$$

Indeed,

$$4p(1+s)(1-ps) \leq [p(1+s)+(1-ps)]^2 = (1+p)^2 < 2(1+p^2) < 4(1-p^4).$$

The equality occurs for $a = b = c = d = 1$.

Second Solution. Assume that $a \geq b \geq c \geq d$, and write the inequality as

$$\sum \frac{1}{(1+a)(1+a^2)} \geq 1.$$

Since

$$\frac{1}{1+a} \leq \frac{1}{1+b} \leq \frac{1}{1+c}, \quad \frac{1}{1+a^2} \leq \frac{1}{1+b^2} \leq \frac{1}{1+c^2},$$

by Chebyshev's inequality, it suffices to prove that

$$\frac{1}{3}\left(\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\right)\left(\frac{1}{1+a^2}+\frac{1}{1+b^2}+\frac{1}{1+c^2}\right)+\frac{1}{(1+d)(1+d^2)} \geq 1.$$

On the other hand, from Remark 3 of P 1.186, we have

$$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c} \geq \frac{3}{1+\sqrt[3]{abc}} = \frac{3\sqrt[3]{d}}{\sqrt[3]{d}+1}$$

and

$$\frac{1}{1+a^2}+\frac{1}{1+b^2}+\frac{1}{1+c^2} \geq \frac{3}{1+\sqrt[3]{a^2b^2c^2}} = \frac{3\sqrt[3]{d^2}}{\sqrt[3]{d^2}+1}.$$

Thus, it suffices to prove that

$$\frac{3d}{(1+\sqrt[3]{d})(1+\sqrt[3]{d^2}\,)}+\frac{1}{(1+d)(1+d^2)}\ge 1.$$

Putting $x=\sqrt[3]{d}$, this inequality becomes as follows:

$$\frac{3x^3}{(1+x)(1+x^2)}+\frac{1}{(1+x^3)(1+x^6)}\ge 1,$$

$$3x^3(1-x+x^2)(1-x^2+x^4)+1\ge(1+x^3)(1+x^6),$$

$$x^3(2-3x+2x^3-3x^5+2x^6)\ge 0,$$

$$x^3(1-x)^2(2+x+x^3+2x^4)\ge 0.$$

Remark. The following generalization holds true (*Vasile Cîrtoaje*, 2004):

- *If $a_1,a_2,\ldots,a_n$ are positive real numbers such that $a_1a_2\cdots a_n=1$, then*

$$\frac{1}{1+a_1+\cdots+a_1^{n-1}}+\frac{1}{1+a_2+\cdots+a_2^{n-1}}+\cdots+\frac{1}{1+a_n+\cdots+a_n^{n-1}}\ge 1.$$

□

P 1.194. *Let a,b,c,d be positive real numbers such that $abcd=1$. Prove that*

$$\frac{1}{1+a+2a^2}+\frac{1}{1+b+2b^2}+\frac{1}{1+c+2c^2}+\frac{1}{1+d+2d^2}\ge 1.$$

(Vasile Cîrtoaje, 2006)

Solution. We will show that

$$\frac{1}{1+a+2a^2}\ge\frac{1}{1+a^k+a^{2k}+a^{3k}},$$

where $k=5/6$. Then, it suffices to show that

$$\sum\frac{1}{1+a^k+a^{2k}+a^{3k}}\ge 1,$$

which immediately follows from the inequality in P 1.193. Setting $a=x^6$, $x>0$, the claimed inequality can be written as

$$\frac{1}{1+x^6+2x^{12}}\ge\frac{1}{1+x^5+x^{10}+x^{15}},$$

which is equivalent to

$$x^{10}+x^5+1\ge 2x^7+x.$$

We can prove it by summing the AM-GM inequalities

$$x^5 + 4 \geq 5x$$

and

$$5x^{10} + 4x^5 + 1 \geq 10x^7.$$

This completes the proof. The equality occurs for $a = b = c = d = 1$.

Remark. The inequalities in P 1.191, P 1.193 and P 1.194 are particular cases of the following more general inequality (*Vasile Cîrtoaje*, 2009):

- *Let $a_1, a_2, \ldots, a_n$ ($n \geq 4$) be positive real numbers such that $a_1 a_2 \cdots a_n = 1$. If p, q, r are nonnegative real numbers satisfying $p + q + r = n - 1$, then*

$$\sum_{i=1}^{i=n} \frac{1}{1 + pa_i + qa_i^2 + ra_i^3} \geq 1.$$

□

P 1.195. *Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that*

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{9}{a + b + c + d} \geq \frac{25}{4}.$$

Solution *(by Vo Quoc Ba Can).* Replacing a, b, c, d by a^4, b^4, c^4, d^4, respectively, the inequality becomes as follows:

$$\frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4} + \frac{1}{d^4} + \frac{9}{a^4 + b^4 + c^4 + d^4} \geq \frac{25}{4abcd},$$

$$\frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4} + \frac{1}{d^4} - \frac{4}{abcd} \geq \frac{9}{4abcd} - \frac{9}{a^4 + b^4 + c^4 + d^4},$$

$$\frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4} + \frac{1}{d^4} - \frac{4}{abcd} \geq \frac{9(a^4 + b^4 + c^4 + d^4 - 4abcd)}{4abcd(a^4 + b^4 + c^4 + d^4)}.$$

Using the identities

$$a^4 + b^4 + c^4 + d^4 - 4abcd = (a^2 - b^2)^2 + (c^2 - d^2)^2 + 2(ab - cd)^2,$$

$$\frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4} + \frac{1}{d^4} - \frac{4}{abcd} = \frac{(a^2 - b^2)^2}{a^4 b^4} + \frac{(c^2 - d^2)^2}{c^4 d^4} + \frac{2(ab - cd)^2}{a^2 b^2 c^2 d^2},$$

the inequality can be written as

$$\frac{(a^2 - b^2)^2}{a^4 b^4} + \frac{(c^2 - d^2)^2}{c^4 d^4} + \frac{2(ab - cd)^2}{a^2 b^2 c^2 d^2} \geq \frac{9[(a^2 - b^2)^2 + (c^2 - d^2)^2 + 2(ab - cd)^2]}{4abcd(a^4 + b^4 + c^4 + d^4)},$$

$$(a^2-b^2)^2\left[\frac{4cd(a^4+b^4+c^4+d^4)}{a^3b^3}-9\right]+(c^2-d^2)^2\left[\frac{4ab(a^4+b^4+c^4+d^4)}{c^3d^3}-9\right]$$
$$+2(ab-cd)^2\left[\frac{4(a^4+b^4+c^4+d^4)}{abcd}-9\right]\geq 0.$$

By the AM-GM inequality, we have

$$a^4+b^4+c^4+d^4\geq 4abcd.$$

Therefore, it suffices to show that

$$(a^2-b^2)^2\left[\frac{4cd(a^4+b^4+c^4+d^4)}{a^3b^3}-9\right]+(c^2-d^2)^2\left[\frac{4ab(a^4+b^4+c^4+d^4)}{c^3d^3}-9\right]\geq 0.$$

Without loss of generality, assume that $a\geq c\geq d\geq b$. Since

$$(a^2-b^2)^2\geq(c^2-d^2)^2$$

and

$$\frac{4cd(a^4+b^4+c^4+d^4)}{a^3b^3}\geq\frac{4(a^4+b^4+c^4+d^4)}{a^3b}\geq\frac{4(a^4+3b^4)}{a^3b}>9,$$

it is enough to prove that

$$\left[\frac{4cd(a^4+b^4+c^4+d^4)}{a^3b^3}-9\right]+\left[\frac{4ab(a^4+b^4+c^4+d^4)}{c^3d^3}-9\right]\geq 0,$$

which is equivalent to

$$2(a^4+b^4+c^4+d^4)\left(\frac{cd}{a^3b^3}+\frac{ab}{c^3d^3}\right)\geq 9.$$

Indeed, by the AM-GM inequality,

$$2(a^4+b^4+c^4+d^4)\left(\frac{cd}{a^3b^3}+\frac{ab}{c^3d^3}\right)\geq 8abcd\left(\frac{2}{abcd}\right)=16>9.$$

The equality occurs for $a=b=c=d=1$.

□

P 1.196. *If a,b,c,d are real numbers such that $a+b+c+d=0$, then*

$$\frac{(a-1)^2}{3a^2+1}+\frac{(b-1)^2}{3b^2+1}+\frac{(c-1)^2}{3c^2+1}+\frac{(d-1)^2}{3d^2+1}\leq 4.$$

Solution. Since

$$4 - \frac{3(a-1)^2}{3a^2+1} = \frac{(3a+1)^2}{3a^2+1},$$

we can write the inequality as

$$\sum \frac{(3a+1)^2}{3a^2+1} \ge 4.$$

On the other hand, since

$$4a^2 = 3a^2 + (b+c+d)^2 \le 3a^2 + 3(b^2+c^2+d^2) = 3(a^2+b^2+c^2+d^2),$$

$$3a^2 + 1 \le \frac{9}{4}(a^2+b^2+c^2+d^2) + 1 = \frac{9(a^2+b^2+c^2+d^2)+4}{4},$$

we have

$$\sum \frac{(3a+1)^2}{3a^2+1} \ge \frac{4\sum(3a+1)^2}{9(a^2+b^2+c^2+d^2)+4} = 4.$$

The equality holds for $a = b = c = d = 0$, and also for $a = 1$ and $b = c = d = -1/3$ (or any cyclic permutation).

Remark. The following generalization is also true.

- *If $a_1, a_2, \ldots, a_n$ are real numbers such that $a_1 + a_2 + \cdots + a_n = 0$, then*

$$\frac{(a_1-1)^2}{(n-1)a_1^2+1} + \frac{(a_2-1)^2}{(n-1)a_2^2+1} + \cdots + \frac{(a_n-1)^2}{(n-1)a_n^2+1} \le n,$$

with equality for $a_1 = a_2 = \cdots = a_n = 0$, and also for $a_1 = 1$ and $a_2 = a_3 = \cdots = a_n = -1/(n-1)$ (or any cyclic permutation).

□

P 1.197. *If $a, b, c, d \ge -5$ such that $a + b + c + d = 4$, then*

$$\frac{1-a}{(1+a)^2} + \frac{1-b}{(1+b)^2} + \frac{1-c}{(1+c)^2} + \frac{1-d}{(1+d)^2} \ge 0.$$

Solution. Assume that $a \le b \le c \le d$. We show first that $x \in \mathbb{R} \setminus \{-1\}$ involves

$$\frac{1-x}{(1+x)^2} \ge \frac{-1}{8},$$

and $x \in [-5, 1/3] \setminus \{-1\}$ involves

$$\frac{1-x}{(1+x)^2} \ge \frac{3}{8}.$$

Indeed, we have

$$\frac{1-x}{(1+x)^2}+\frac{1}{8}=\frac{(x-3)^2}{8(1+x)^2}\geq 0$$

and

$$\frac{1-x}{(1+x)^2}-\frac{3}{8}=\frac{(5+x)(1-3x)}{8(1+x)^2}\geq 0.$$

Therefore, if $a\leq 1/3$, then

$$\frac{1-a}{(1+a)^2}+\frac{1-b}{(1+b)^2}+\frac{1-c}{(1+c)^2}+\frac{1-d}{(1+d)^2}\geq\frac{3}{8}-\frac{1}{8}-\frac{1}{8}-\frac{1}{8}=0.$$

Assume now that $1/3\leq a\leq b\leq c\leq d$. Since

$$1-a\geq 1-b\geq 1-c\geq 1-d$$

and

$$\frac{1}{(1+a)^2}\geq\frac{1}{(1+b)^2}\geq\frac{1}{(1+c)^2}\geq\frac{1}{(1+d)^2},$$

by Chebyshev's inequality, we have

$$\frac{1-a}{(1+a)^2}+\frac{1-b}{(1+b)^2}+\frac{1-c}{(1+c)^2}+\frac{1-d}{(1+d)^2}\geq$$

$$\geq\frac{1}{4}\left[\sum(1-a)\right]\left[\sum\frac{1}{(1+a)^2}\right]=0.$$

The equality holds for $a=b=c=d=1$, and also for $a=-5$ and $b=c=d=3$ (or any cyclic permutation).

□

P 1.198. *Let $a_1,a_2,\ldots,a_n$ be positive real numbers such that $a_1+a_2+\cdots+a_n=n$. Prove that*

$$\sum\frac{1}{(n+1)a_1^2+a_2^2+\cdots+a_n^2}\leq\frac{1}{2}.$$

(Vasile Cîrtoaje, 2008)

First Solution. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\sum\frac{n^2}{(n+1)a_1^2+a_2^2+\cdots+a_n^2}&=\sum\frac{(a_1+a_2+\cdots+a_n)^2}{2a_1^2+(a_1^2+a_2^2)+\cdots+(a_1^2+a_n^2)}\\&\leq\sum\left(\frac{1}{2}+\frac{a_2^2}{a_1^2+a_2^2}+\cdots+\frac{a_n^2}{a_1^2+a_n^2}\right)\\&=\frac{n}{2}+\frac{n(n-1)}{2}=\frac{n^2}{2},\end{aligned}$$

from which the conclusion follows. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Second Solution. Write the inequality as

$$\sum \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{(n+1)a_1^2 + a_2^2 + \cdots + a_n^2} \le \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{2}.$$

Since

$$\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{(n+1)a_1^2 + a_2^2 + \cdots + a_n^2} = 1 - \frac{na_1^2}{(n+1)a_1^2 + a_2^2 + \cdots + a_n^2},$$

we need to prove that

$$\sum \frac{a_1^2}{(n+1)a_1^2 + a_2^2 + \cdots + a_n^2} + \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{2n} \ge 1.$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\sum \frac{a_1^2}{(n+1)a_1^2 + a_2^2 + \cdots + a_n^2} &\ge \frac{(a_1 + a_2 + \cdots + a_n)^2}{\sum[(n+1)a_1^2 + a_2^2 + \cdots + a_n^2]} \\ &= \frac{n}{2(a_1^2 + a_2^2 + \cdots + a_n^2)}.\end{aligned}$$

Then, it suffices to prove that

$$\frac{n}{a_1^2 + a_2^2 + \cdots + a_n^2} + \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n} \ge 2,$$

which follows immediately from the AM-GM inequality.

□

P 1.199. *Let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 + a_2 + \cdots + a_n = 0$. Prove that*

$$\frac{(a_1+1)^2}{a_1^2+n-1} + \frac{(a_2+1)^2}{a_2^2+n-1} + \cdots + \frac{(a_n+1)^2}{a_n^2+n-1} \ge \frac{n}{n-1}.$$

(Vasile Cîrtoaje, 2010)

Solution. Without loss of generality, assume that $a_n^2 = \max\{a_1^2, a_2^2, \cdots, a_n^2\}$. Since

$$\frac{(a_n+1)^2}{a_n^2+n-1} = \frac{n}{n-1} - \frac{(n-1-a_n)^2}{(n-1)(a_n^2+n-1)},$$

we can write the inequality as

$$\sum_{i=1}^{n-1} \frac{(a_i+1)^2}{a_i^2+n-1} \ge \frac{(n-1-a_n)^2}{(n-1)(a_n^2+n-1)}.$$

From the Cauchy-Schwarz inequality

$$\left[\sum_{i=1}^{n-1}(a_i^2+n-1)\right]\left[\sum_{i=1}^{n-1}\frac{(a_i+1)^2}{a_i^2+n-1}\right]\geq\left[\sum_{i=1}^{n-1}(a_i+1)\right]^2,$$

we get

$$\sum_{i=1}^{n-1}\frac{(a_i+1)^2}{a_i^2+n-1}\geq\frac{(n-1-a_n)^2}{\sum_{i=1}^{n-1}a_i^2+(n-1)^2}.$$

Thus, it suffices to show that

$$\sum_{i=1}^{n-1}a_i^2+(n-1)^2\leq(n-1)(a_n^2+n-1),$$

which is clearly true. The proof is completed. The equality holds for $\dfrac{-a_1}{n-1}=a_2=a_3=\cdots=a_n$ (or any cyclic permutation).

□

P 1.200. *Let $a_1,a_2,\ldots,a_n$ be positive real numbers such that $a_1a_2\cdots a_n=1$. Prove that*

$$(a)\qquad \frac{1}{1+(n-1)a_1}+\frac{1}{1+(n-1)a_2}+\cdots+\frac{1}{1+(n-1)a_n}\geq 1;$$

$$(b)\qquad \frac{1}{a_1+n-1}+\frac{1}{a_2+n-1}+\cdots+\frac{1}{a_n+n-1}\leq 1.$$

(Vasile Cîrtoaje, 1991)

Solution. (a) ***First Solution.*** Let $k=(n-1)/n$. We can get the required inequality by summing the inequalities

$$\frac{1}{1+(n-1)a_i}\geq\frac{a_i^{-k}}{a_1^{-k}+a_2^{-k}+\cdots+a_n^{-k}}$$

for $i=1,2,\cdots,n$. The inequality is equivalent to

$$a_1^{-k}+\cdots+a_{i-1}^{-k}+a_{i+1}^{-k}+\cdots+a_n^{-k}\geq(n-1)a_i^{1-k},$$

which follows from the AM-GM inequality. The equality holds for $a_1=a_2=\cdots=a_n=1$.

Second Solution. Replacing all a_i by $1/a_i$, the inequality becomes

$$\frac{a_1}{a_1+n-1}+\frac{a_2}{a_2+n-1}+\cdots+\frac{a_n}{a_n+n-1}\geq 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a_i}{a_i+n-1} \ge \frac{\left(\sum \sqrt{a_1}\right)^2}{\sum(a_1+n-1)}.$$

Thus, we still have to prove that

$$\left(\sum \sqrt{a_1}\right)^2 \ge \sum a_1 + n(n-1),$$

which is equivalent to

$$\sum_{1\le i<j\le n} 2\sqrt{a_ia_j} \ge n(n-1).$$

Since $a_1a_2\cdots a_n = 1$, this inequality follows from the AM-GM inequality.

Third Solution. Use the contradiction method. Assume that

$$\frac{1}{1+(n-1)a_1} + \frac{1}{1+(n-1)a_2} + \cdots + \frac{1}{1+(n-1)a_n} < 1$$

and show that $a_1a_2\cdots a_n > 1$ (which contradicts the hypothesis $a_1a_2\cdots a_n = 1$). Let

$$x_i = \frac{1}{1+(n-1)a_i}, \quad 0 < x_i < 1, \qquad i = 1,2,\cdots,n.$$

Since

$$a_i = \frac{1-x_i}{(n-1)x_i}, \qquad i = 1,2,\cdots,n,$$

we need to show that

$$x_1 + x_2 + \cdots + x_n < 1$$

implies

$$(1-x_1)(1-x_2)\cdots(1-x_n) > (n-1)^n x_1x_2\cdots x_n.$$

Using the AM-GM inequality, we have

$$1 - x_i > \sum_{k\ne i} x_k \ge (n-1)\left(\prod_{k\ne i} x_k\right)^{1/(n-1)}.$$

Multiplying the inequalities

$$1 - x_i > (n-1)\left(\prod_{k\ne i} x_k\right)^{1/(n-1)}, \quad i = 1,2,\cdots,n,$$

the conclusion follows.

(b) This inequality follows from the inequality in (a) by replacing all a_i with $1/a_i$. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark. The inequalities in P 1.200 are particular cases of the following more general results (*Vasile Cîrtoaje*, 2005):

- *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \cdots a_n = 1$. If*

$$0 < k \le n-1, \qquad p \ge n^{1/k} - 1,$$

then

$$\frac{1}{(1+pa_1)^k} + \frac{1}{(1+pa_2)^k} + \cdots + \frac{1}{(1+pa_n)^k} \ge \frac{n}{(1+p)^k}.$$

- *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \cdots a_n = 1$. If*

$$k \ge \frac{1}{n-1}, \qquad 0 < p \le \left(\frac{n}{n-1}\right)^{1/k} - 1,$$

then

$$\frac{1}{(1+pa_1)^k} + \frac{1}{(1+pa_2)^k} + \cdots + \frac{1}{(1+pa_n)^k} \le \frac{n}{(1+p)^k}.$$

□

P 1.201. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \cdots a_n = 1$. Prove that*

$$\frac{1}{1-a_1+na_1^2} + \frac{1}{1-a_2+na_2^2} + \cdots + \frac{1}{1-a_n+na_n^2} \ge 1.$$

(Vasile Cîrtoaje, 2009)

Solution. First, we show that

$$\frac{1}{1-x+nx^2} \ge \frac{1}{1+(n-1)x^k},$$

where $x > 0$ and $k = 2 + \frac{1}{n-1}$. Write the inequality as

$$(n-1)x^k + x \ge nx^2.$$

We can get this inequality using the AM-GM inequality as follows:

$$(n-1)x^k + x \ge n\sqrt[n]{x^{(n-1)k}x} = nx^2.$$

Thus, it suffices to show that

$$\frac{1}{1+(n-1)a_1^k} + \frac{1}{1+(n-1)a_2^k} + \cdots + \frac{1}{1+(n-1)a_n^k} \ge 1,$$

which follows immediately from the inequality (a) in the preceding P 1.200. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark 1. Similarly, we can prove the following more general statement.

• *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2\cdots a_n = 1$. If p and q are real numbers such that $p + q = n - 1$ and $n - 1 \le q \le (\sqrt{n} + 1)^2$, then*

$$\frac{1}{1+pa_1+qa_1^2}+\frac{1}{1+pa_2+qa_2^2}+\cdots+\frac{1}{1+pa_n+qa_n^2}\ge 1.$$

Remark 2. We can extend the inequality in Remark 1 as follows (*Vasile Cîrtoaje*, 2009).

• *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2\cdots a_n = 1$. If p and q are real numbers such that $p + q = n - 1$ and $0 \le q \le (\sqrt{n} + 1)^2$, then*

$$\frac{1}{1+pa_1+qa_1^2}+\frac{1}{1+pa_2+qa_2^2}+\cdots+\frac{1}{1+pa_n+qa_n^2}\ge 1.$$

□

P 1.202. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that*

$$a_1, a_2, \ldots, a_n \ge \frac{k(n-k-1)}{kn-k-1}, \quad k > 1$$

and

$$a_1a_2\cdots a_n = 1.$$

Prove that

$$\frac{1}{a_1+k}+\frac{1}{a_2+k}+\cdots+\frac{1}{a_n+k}\le\frac{n}{1+k}.$$

(Vasile Cîrtoaje, 2005)

Solution. We use the induction method. Let

$$E_n(a_1, a_2, \ldots, a_n) = \frac{1}{a_1+k}+\frac{1}{a_2+k}+\cdots+\frac{1}{a_n+k}-\frac{n}{1+k}.$$

For $n = 2$, we have

$$E_2(a_1, a_2) = \frac{(1-k)(\sqrt{a_1}-\sqrt{a_2})^2}{(1+k)(a_1+k)(a_2+k)}\le 0.$$

Assume that the inequality is true for $n - 1$ numbers ($n \ge 3$), and prove that $E_n(a_1, a_2, \ldots, a_n) \ge 0$ for $a_1a_2\cdots a_n = 1$ and $a_1, a_2, \ldots, a_n \ge p_n$, where

$$p_n = \frac{k(n-k-1)}{kn-k-1}.$$

Due to symmetry, we may assume that $a_1 \ge 1$ and $a_2 \le 1$. There are two cases to consider.

Case 1: $a_1a_2 \le k^2$. From $a_1a_2 \ge a_2$, $p_{n-1} < p_n$ and $a_1, a_2, \ldots, a_n \ge p_n$, it follows that

$$a_1a_2,\ a_3, \cdots, a_n > p_{n-1}.$$

Then, by the induction hypothesis, we have $E_{n-1}(a_1a_2, a_2, \ldots, a_n) \le 0$; thus, it suffices to show that

$$E_n(a_1, a_2, \ldots, a_n) \le E_{n-1}(a_1a_2, a_2, \ldots, a_n).$$

This is equivalent to

$$\frac{1}{a_1+k} + \frac{1}{a_2+k} - \frac{1}{a_1a_2+k} - \frac{1}{1+k} \le 0,$$

which reduces to the obvious inequality

$$(a_1-1)(1-a_2)(a_1a_2-k^2) \le 0.$$

Case 2: $a_1a_2 \ge k^2$. Since

$$\frac{1}{a_1+k} + \frac{1}{a_2+k} = \frac{a_1+a_2+2k}{a_1a_2+k(a_1+a_2)+k^2} \le \frac{a_1+a_2+2k}{k^2+k(a_1+a_2)+k^2} = \frac{1}{k}$$

and

$$\frac{1}{a_3+k} + \cdots + \frac{1}{a_n+k} \le \frac{n-2}{p_n+k} = \frac{kn-k-1}{k(k+1)},$$

we have

$$E_n(a_1, a_2, \ldots, a_n) \le \frac{1}{k} + \frac{kn-k-1}{k(k+1)} - \frac{n}{1+k} = 0.$$

Thus, the proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark. For $k = n-1$, we get the inequality (b) in P 1.200.

□

P 1.203. *If* $a_1, a_2, \ldots, a_n \ge 0$*, then*

$$\frac{1}{1+na_1} + \frac{1}{1+na_2} + \cdots + \frac{1}{1+na_n} \ge \frac{n}{n+a_1a_2\cdots a_n}.$$

(Vasile Cîrtoaje, 2013)

Solution. If one of $a_1, a_2, \ldots, a_n$ is zero, the inequality is obvious. Consider further that $a_1, a_2, \ldots, a_n > 0$ and let

$$r = \sqrt[n]{a_1a_2\cdots a_n}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{1}{1+na_1} \geq \frac{\left(\sum \sqrt{a_2a_3\cdots a_n}\right)^2}{\sum(1+na_1)a_2a_3\cdots a_n} = \frac{\left(\sum \sqrt{a_2a_3\cdots a_n}\right)^2}{\sum a_2a_3\cdots a_n + n^2r^n}.$$

Therefore, it suffices to show that

$$(n+r^n)\left(\sum \sqrt{a_2a_3\cdots a_n}\right)^2 \geq n\sum a_2a_3\cdots a_n + n^3r^n.$$

By the AM-GM inequality, we have

$$\left(\sum \sqrt{a_2a_3\cdots a_n}\right)^2 \geq \sum a_2a_3\cdots a_n + n(n-1)r^{n-1}.$$

Thus, it is enough to prove that

$$(n+r^n)\left[\sum a_2a_3\cdots a_n + n(n-1)r^{n-1}\right] \geq n\sum a_2a_3\cdots a_n + n^3r^n,$$

which is equivalent to

$$r^n\sum a_2a_3\cdots a_n + n(n-1)r^{2n-1} + n^2(n-1)r^{n-1} \geq n^3r^n.$$

Also, by the AM-GM inequality,

$$\sum a_2a_3\cdots a_n \geq nr^{n-1},$$

and it suffices to show the inequality

$$nr^{2n-1} + n(n-1)r^{2n-1} + n^2(n-1)r^{n-1} \geq n^3r^n,$$

which can be rewritten as

$$n^2r^{n-1}(r^n - nr + n - 1) \geq 0.$$

Indeed, by the AM-GM inequality, we get

$$r^n + n - 1 = r^n + 1 + \cdots + 1 \geq n\sqrt[n]{r^n\cdot 1\cdots 1} = nr.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Chapter 2

Symmetric Nonrational Inequalities

2.1 Applications

2.1. If a, b are nonnegative real numbers such that $a^2 + b^2 \le 1 + \frac{2}{\sqrt{3}}$, then

$$\frac{a}{2a^2+1} + \frac{b}{2b^2+1} \le \frac{\sqrt{2(a^2+b^2)}}{a^2+b^2+1}.$$

2.2. If a, b, c are real numbers, then

$$\sum \sqrt{a^2 - ab + b^2} \le \sqrt{6(a^2+b^2+c^2) - 3(ab+bc+ca)}.$$

2.3. If a, b, c are positive real numbers, then

$$a\sqrt{b+c} + b\sqrt{c+a} + c\sqrt{a+b} \ge \frac{2bc}{\sqrt{b+c}} + \frac{2ca}{\sqrt{c+a}} + \frac{2ab}{\sqrt{a+b}}.$$

2.4. If a, b, c are nonnegative real numbers, then

$$\sqrt{a^2-ab+b^2} + \sqrt{b^2-bc+c^2} + \sqrt{c^2-ca+a^2} \le 3\sqrt{\frac{a^2+b^2+c^2}{2}}.$$

2.5. If a, b, c are nonnegative real numbers, then

$$\sqrt{a^2+b^2-\frac{2}{3}ab} + \sqrt{b^2+c^2-\frac{2}{3}bc} + \sqrt{c^2+a^2-\frac{2}{3}ca} \ge 2\sqrt{a^2+b^2+c^2}.$$

2.6. If a,b,c are nonnegative real numbers, then

$$\sum \sqrt{a^2+ab+b^2} \ge \sqrt{4(a^2+b^2+c^2)+5(ab+bc+ca)}.$$

2.7. If a,b,c are positive real numbers, then

$$\sum \sqrt{a^2+ab+b^2} \le \sqrt{5(a^2+b^2+c^2)+4(ab+bc+ca)}.$$

2.8. If a,b,c are nonnegative real numbers, then

$$\sum \sqrt{a^2+ab+b^2} \le 2\sqrt{a^2+b^2+c^2}+\sqrt{ab+bc+ca}.$$

2.9. If a,b,c are nonnegative real numbers, then

$$\sqrt{a^2+2bc}+\sqrt{b^2+2ca}+\sqrt{c^2+2ab} \le \sqrt{a^2+b^2+c^2}+2\sqrt{ab+bc+ca}.$$

2.10. If a,b,c are nonnegative real numbers, then

$$\frac{1}{\sqrt{a^2+2bc}}+\frac{1}{\sqrt{b^2+2ca}}+\frac{1}{\sqrt{c^2+2ab}} \ge \frac{1}{\sqrt{a^2+b^2+c^2}}+\frac{2}{\sqrt{ab+bc+ca}}.$$

2.11. If a,b,c are positive real numbers, then

$$\sqrt{2a^2+bc}+\sqrt{2b^2+ca}+\sqrt{2c^2+ab} \le 2\sqrt{a^2+b^2+c^2}+\sqrt{ab+bc+ca}.$$

2.12. Let a,b,c be nonnegative real numbers such that $a+b+c=3$. If $k=\sqrt{3}-1$, then

$$\sum \sqrt{a(a+kb)(a+kc)} \le 3\sqrt{3}.$$

2.13. If a,b,c are nonnegative real numbers such that $a+b+c=3$, then

$$\sum \sqrt{a(2a+b)(2a+c)} \ge 9.$$

2.14. Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$\sqrt{b^2+c^2+a(b+c)}+\sqrt{c^2+a^2+b(c+a)}+\sqrt{a^2+b^2+c(a+b)} \geq 6.$$

2.15. Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that

(a) $$\sqrt{a(3a^2+abc)}+\sqrt{b(3b^2+abc)}+\sqrt{c(3c^2+abc)} \geq 6;$$

(b) $$\sqrt{3a^2+abc}+\sqrt{3b^2+abc}+\sqrt{3c^2+abc} \geq 3\sqrt{3+abc}.$$

2.16. Let a, b, c be positive real numbers such that $ab+bc+ca=3$. Prove that

$$a\sqrt{(a+2b)(a+2c)}+b\sqrt{(b+2c)(b+2a)}+c\sqrt{(c+2a)(c+2b)} \geq 9.$$

2.17. Let a, b, c be nonnegative real numbers such that $a+b+c=1$. Prove that

$$\sqrt{a+(b-c)^2}+\sqrt{b+(c-a)^2}+\sqrt{c+(a-b)^2} \geq \sqrt{3}.$$

2.18. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\sqrt{\frac{a(b+c)}{a^2+bc}}+\sqrt{\frac{b(c+a)}{b^2+ca}}+\sqrt{\frac{c(a+b)}{c^2+ab}} \geq 2.$$

2.19. Let a, b, c be positive real numbers such that $abc=1$. Prove that

$$\frac{1}{\sqrt[3]{a^2+25a+1}}+\frac{1}{\sqrt[3]{b^2+25b+1}}+\frac{1}{\sqrt[3]{c^2+25c+1}} \geq 1.$$

2.20. If a, b, c are nonnegative real numbers, then

$$\sqrt{a^2+bc}+\sqrt{b^2+ca}+\sqrt{c^2+ab} \leq \frac{3}{2}(a+b+c).$$

2.21. If a, b, c are nonnegative real numbers, then

$$\sqrt{a^2+9bc}+\sqrt{b^2+9ca}+\sqrt{c^2+9ab} \geq 5\sqrt{ab+bc+ca}.$$

2.22. If a, b, c are nonnegative real numbers, then

$$\sum \sqrt{(a^2+4bc)(b^2+4ca)} \geq 5(ab+ac+bc).$$

2.23. If a, b, c are nonnegative real numbers, then

$$\sum \sqrt{(a^2+9bc)(b^2+9ca)} \geq 7(ab+ac+bc).$$

2.24. If a, b, c are nonnegative real numbers, then

$$\sum \sqrt{(a^2+b^2)(b^2+c^2)} \leq (a+b+c)^2.$$

2.25. If a, b, c are nonnegative real numbers, then

$$\sum \sqrt{(a^2+ab+b^2)(b^2+bc+c^2)} \geq (a+b+c)^2.$$

2.26. If a, b, c are nonnegative real numbers, then

$$\sum \sqrt{(a^2+7ab+b^2)(b^2+7bc+c^2)} \geq 7(ab+ac+bc).$$

2.27. If a, b, c are nonnegative real numbers, then

$$\sum \sqrt{\left(a^2+\frac{7}{9}ab+b^2\right)\left(b^2+\frac{7}{9}bc+c^2\right)} \leq \frac{13}{12}(a+b+c)^2.$$

2.28. If a, b, c are nonnegative real numbers, then

$$\sum \sqrt{\left(a^2+\frac{1}{3}ab+b^2\right)\left(b^2+\frac{1}{3}bc+c^2\right)} \leq \frac{61}{60}(a+b+c)^2.$$

2.29. If a, b, c are nonnegative real numbers, then

$$\frac{a}{\sqrt{4b^2+bc+4c^2}}+\frac{b}{\sqrt{4c^2+ca+4a^2}}+\frac{c}{\sqrt{4a^2+ab+4b^2}} \geq 1.$$

2.30. If a, b, c are nonnegative real numbers, then

$$\frac{a}{\sqrt{b^2+bc+c^2}}+\frac{b}{\sqrt{c^2+ca+a^2}}+\frac{c}{\sqrt{a^2+ab+b^2}}\ge\frac{a+b+c}{\sqrt{ab+bc+ca}}.$$

2.31. If a, b, c are nonnegative real numbers, then

$$\frac{a}{\sqrt{a^2+2bc}}+\frac{b}{\sqrt{b^2+2ca}}+\frac{c}{\sqrt{c^2+2ab}}\le\frac{a+b+c}{\sqrt{ab+bc+ca}}.$$

2.32. If a, b, c are nonnegative real numbers, then

$$a^3+b^3+c^3+3abc\ge a^2\sqrt{a^2+3bc}+b^2\sqrt{b^2+3ca}+c^2\sqrt{c^2+3ab}.$$

2.33. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a}{\sqrt{4a^2+5bc}}+\frac{b}{\sqrt{4b^2+5ca}}+\frac{c}{\sqrt{4c^2+5ab}}\le 1.$$

2.34. Let a, b, c be nonnegative real numbers. Prove that

$$a\sqrt{4a^2+5bc}+b\sqrt{4b^2+5ca}+c\sqrt{4c^2+5ab}\ge(a+b+c)^2.$$

2.35. Let a, b, c be nonnegative real numbers. Prove that

$$a\sqrt{a^2+3bc}+b\sqrt{b^2+3ca}+c\sqrt{c^2+3ab}\ge 2(ab+bc+ca).$$

2.36. Let a, b, c be nonnegative real numbers. Prove that

$$a\sqrt{a^2+8bc}+b\sqrt{b^2+8ca}+c\sqrt{c^2+8ab}\le(a+b+c)^2.$$

2.37. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a^2+2bc}{\sqrt{b^2+bc+c^2}}+\frac{b^2+2ca}{\sqrt{c^2+ca+a^2}}+\frac{c^2+2ab}{\sqrt{a^2+ab+b^2}}\ge 3\sqrt{ab+bc+ca}.$$

2.38. Let a,b,c be nonnegative real numbers, no two of which are zero. If $k\ge 1$, then

$$\frac{a^{k+1}}{2a^2+bc}+\frac{b^{k+1}}{2b^2+ca}+\frac{c^{k+1}}{2c^2+ab}\le\frac{a^k+b^k+c^k}{a+b+c}.$$

2.39. If a,b,c are positive real numbers, then

(a) $$\frac{a^2-bc}{\sqrt{3a^2+2bc}}+\frac{b^2-ca}{\sqrt{3b^2+2ca}}+\frac{c^2-ab}{\sqrt{3c^2+2ab}}\ge 0;$$

(b) $$\frac{a^2-bc}{\sqrt{8a^2+(b+c)^2}}+\frac{b^2-ca}{\sqrt{8b^2+(c+a)^2}}+\frac{c^2-ab}{\sqrt{8c^2+(a+b)^2}}\ge 0.$$

2.40. Let a,b,c be positive real numbers. If $0\le k\le 1+2\sqrt{2}$, then

$$\frac{a^2-bc}{\sqrt{ka^2+b^2+c^2}}+\frac{b^2-ca}{\sqrt{kb^2+c^2+a^2}}+\frac{c^2-ab}{\sqrt{kc^2+a^2+b^2}}\ge 0.$$

2.41. If a,b,c are nonnegative real numbers, then

$$(a^2-bc)\sqrt{b+c}+(b^2-ca)\sqrt{c+a}+(c^2-ab)\sqrt{a+b}\ge 0.$$

2.42. If a,b,c are nonnegative real numbers, then

$$(a^2-bc)\sqrt{a^2+4bc}+(b^2-ca)\sqrt{b^2+4ca}+(c^2-ab)\sqrt{c^2+4ab}\ge 0.$$

2.43. If a,b,c are nonnegative real numbers, then

$$\sqrt{\frac{a^3}{a^3+(b+c)^3}}+\sqrt{\frac{b^3}{b^3+(c+a)^3}}+\sqrt{\frac{c^3}{c^3+(a+b)^3}}\ge 1.$$

2.44. If $a,\ b,\ c$ are positive real numbers, then

$$\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}\ge 1+\sqrt{1+\sqrt{(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}}.$$

2.45. If a, b, c are positive real numbers, then

$$5+\sqrt{2(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)-2} \ge (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

2.46. If a, b, c are real numbers, then

$$2(1+abc)+\sqrt{2(1+a^2)(1+b^2)(1+c^2)} \ge (1+a)(1+b)(1+c).$$

2.47. Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that

$$\sqrt{\frac{a^2+bc}{b^2+c^2}}+\sqrt{\frac{b^2+ca}{c^2+a^2}}+\sqrt{\frac{c^2+ab}{a^2+b^2}} \ge 2+\frac{1}{\sqrt{2}}.$$

2.48. If a,b,c are nonnegative real numbers, then

$$\sqrt{a(2a+b+c)}+\sqrt{b(2b+c+a)}+\sqrt{c(2c+a+b)} \ge \sqrt{12(ab+bc+ca)}.$$

2.49. Let a,b,c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$a\sqrt{(4a+5b)(4a+5c)}+b\sqrt{(4b+5c)(4b+5a)}+c\sqrt{(4c+5a)(4c+5b)} \ge 27.$$

2.50. Let a,b,c be nonnegative real numbers such that $ab+bc+ca=3$. Prove that

$$a\sqrt{(a+3b)(a+3c)}+b\sqrt{(b+3c)(b+3a)}+c\sqrt{(c+3a)(c+3b)} \ge 12.$$

2.51. Let a,b,c be nonnegative real numbers such that $a^2+b^2+c^2=3$. Prove that

$$\sqrt{2+7ab}+\sqrt{2+7bc}+\sqrt{2+7ca} \ge 3\sqrt{3(ab+bc+ca)}.$$

2.52. Let a,b,c be nonnegative real numbers such that $a^2+b^2+c^2=3$. Prove that

$$\frac{a}{2a^2+1}+\frac{b}{2b^2+1}+\frac{c}{2c^2+1} \le 1.$$

2.53. Let a,b,c be nonnegative real numbers such that $ab+bc+ca=3$. Prove that

(a) $$\sum\sqrt{a(b+c)(a^2+bc)}\ge 6;$$

(b) $$\sum a(b+c)\sqrt{a^2+2bc}\ge 6\sqrt{3};$$

(c) $$\sum a(b+c)\sqrt{(a+2b)(a+2c)}\ge 18.$$

2.54. Let a,b,c be nonnegative real numbers such that $ab+bc+ca=3$. Prove that

$$a\sqrt{bc+3}+b\sqrt{ca+3}+c\sqrt{ab+3}\ge 6.$$

2.55. Let a,b,c be nonnegative real numbers such that $a+b+c=3$. Prove that

(a) $$\sum(b+c)\sqrt{b^2+c^2+7bc}\ge 18;$$

(b) $$\sum(b+c)\sqrt{b^2+c^2+10bc}\le 12\sqrt{3}.$$

2.56. Let a,b,c be nonnegative real numbers such then $a+b+c=2$. Prove that

$$\sqrt{a+4bc}+\sqrt{b+4ca}+\sqrt{c+4ab}\ge 4\sqrt{ab+bc+ca}.$$

2.57. If a,b,c are nonnegative real numbers, then

$$\sqrt{a^2+b^2+7ab}+\sqrt{b^2+c^2+7bc}+\sqrt{c^2+a^2+7ca}\ge 5\sqrt{ab+bc+ca}.$$

2.58. If a,b,c are nonnegative real numbers, then

$$\sqrt{a^2+b^2+5ab}+\sqrt{b^2+c^2+5bc}+\sqrt{c^2+a^2+5ca}\ge\sqrt{21(ab+bc+ca)}.$$

2.59. Let a,b,c be nonnegative real numbers such that $ab+bc+ca=3$. Prove that

$$a\sqrt{a^2+5}+b\sqrt{b^2+5}+c\sqrt{c^2+5}\ge\sqrt{\frac{2}{3}(a+b+c)^2}.$$

2.60. Let a, b, c be nonnegative real numbers such that $a^2+b^2+c^2=1$. Prove that

$$a\sqrt{2+3bc}+b\sqrt{2+3ca}+c\sqrt{2+3ab}\geq (a+b+c)^2.$$

2.61. Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that

(a) $$a\sqrt{\frac{2a+bc}{3}}+b\sqrt{\frac{2b+ca}{3}}+c\sqrt{\frac{2c+ab}{3}}\geq 3;$$

(b) $$a\sqrt{\frac{a(1+b+c)}{3}}+b\sqrt{\frac{b(1+c+a)}{3}}+c\sqrt{\frac{c(1+a+b)}{3}}\geq 3.$$

2.62. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

$$\sqrt{8(a^2+bc)+9}+\sqrt{8(b^2+ca)+9}+\sqrt{8(c^2+ab)+9}\geq 15.$$

2.63. Let a, b, c be nonnegative real numbers such that $a+b+c=3$. If $k\geq \frac{9}{8}$, then

$$\sqrt{a^2+bc+k}+\sqrt{b^2+ca+k}+\sqrt{c^2+ab+k}\geq 3\sqrt{2+k}.$$

2.64. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

$$\sqrt{a^3+2bc}+\sqrt{b^3+2ca}+\sqrt{c^3+2ab}\geq 3\sqrt{3}.$$

2.65. If a, b, c are positive real numbers, then

$$\frac{\sqrt{a^2+bc}}{b+c}+\frac{\sqrt{b^2+ca}}{c+a}+\frac{\sqrt{c^2+ab}}{a+b}\geq \frac{3\sqrt{2}}{2}.$$

2.66. If a, b, c are nonnegative real numbers, no two of which are zero,then

$$\frac{\sqrt{bc+4a(b+c)}}{b+c}+\frac{\sqrt{ca+4b(c+a)}}{c+a}+\frac{\sqrt{ab+4c(a+b)}}{a+b}\geq \frac{9}{2}.$$

2.67. If a,b,c are nonnegative real numbers, no two of which are zero,then

$$\frac{a\sqrt{a^2+3bc}}{b+c}+\frac{b\sqrt{b^2+3ca}}{c+a}+\frac{c\sqrt{c^2+3ab}}{a+b}\ge a+b+c.$$

2.68. If a,b,c are nonnegative real numbers, no two of which are zero,then

$$\sqrt{\frac{2a(b+c)}{(2b+c)(b+2c)}}+\sqrt{\frac{2b(c+a)}{(2c+a)(c+2a)}}+\sqrt{\frac{2c(a+b)}{(2a+b)(a+2b)}}\ge 2.$$

2.69. If a,b,c are nonnegative real numbers such that $ab+bc+ca=3$, then

$$\sqrt{\frac{bc}{3a^2+6}}+\sqrt{\frac{ca}{3b^2+6}}+\sqrt{\frac{ab}{3c^2+6}}\le 1\le\sqrt{\frac{bc}{6a^2+3}}+\sqrt{\frac{ca}{6b^2+3}}+\sqrt{\frac{ab}{6c^2+3}}.$$

2.70. Let a,b,c be nonnegative real numbers such that $ab+bc+ca=3$. If $k>1$, than

$$a^k(b+c)+b^k(c+a)+c^k(a+b)\ge 6.$$

2.71. Let a,b,c be nonnegative real numbers such that $a+b+c=2$. If

$$2\le k\le 3,$$

than

$$a^k(b+c)+b^k(c+a)+c^k(a+b)\le 2.$$

2.72. Let a,b,c be nonnegative real numbers, no two of which are zero. If $m>n\ge 0$, than

$$\frac{b^m+c^m}{b^n+c^n}(b+c-2a)+\frac{c^m+a^m}{c^n+a^n}(c+a-2b)+\frac{a^m+b^m}{a^n+b^n}(a+b-2c)\ge 0.$$

2.73. Let a,b,c be positive real numbers such that $abc=1$. Prove that

$$\sqrt{a^2-a+1}+\sqrt{a^2-a+1}+\sqrt{a^2-a+1}\ge a+b+c.$$

2.74. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\sqrt{16a^2+9}+\sqrt{16b^2+9}+\sqrt{16b^2+9} \geq 4(a+b+c)+3.$$

2.75. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\sqrt{25a^2+144}+\sqrt{25b^2+144}+\sqrt{25c^2+144} \leq 5(a+b+c)+24.$$

2.76. If a, b are positive real numbers such that $ab+bc+ca = 3$, then

(a) $$\sqrt{a^2+3}+\sqrt{b^2+3}+\sqrt{b^2+3} \geq a+b+c+3;$$

(b) $$\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a} \geq \sqrt{4(a+b+c)+6}.$$

2.77. If a, b, c are nonnegative real numbers such that $a+b+c = 3$, then

$$\sqrt{(5a^2+3)(5b^2+3)}+\sqrt{(5b^2+3)(5c^2+3)}+\sqrt{(5c^2+3)(5a^2+3)} \geq 24.$$

2.78. If a, b, c are nonnegative real numbers such that $a+b+c = 3$, then

$$\sqrt{a^2+1}+\sqrt{b^2+1}+\sqrt{c^2+1} \geq \sqrt{\frac{4(a^2+b^2+c^2)+42}{3}}.$$

2.79. If a, b, c are nonnegative real numbers such that $a+b+c = 3$, then

(a) $$\sqrt{a^2+3}+\sqrt{b^2+3}+\sqrt{c^2+3} \geq \sqrt{2(a^2+b^2+c^2)+30};$$

(b) $$\sqrt{3a^2+1}+\sqrt{3b^2+1}+\sqrt{3c^2+1} \geq \sqrt{2(a^2+b^2+c^2)+30}.$$

2.80. If a, b, c are nonnegative real numbers such that $a+b+c = 3$, then

$$\sqrt{(32a^2+3)(32b^2+3)}+\sqrt{(32b^2+3)(32c^2+3)}+\sqrt{(32c^2+3)(32a^2+3)} \leq 105.$$

2.81. If a, b, c are positive real numbers, then

$$\left|\frac{b+c}{a}-3\right|+\left|\frac{c+a}{b}-3\right|+\left|\frac{a+b}{c}-3\right| \geq 2.$$

2.82. If a,b,c are real numbers such that $abc \neq 0$, then

$$\left|\frac{b+c}{a}\right| + \left|\frac{c+a}{b}\right| + \left|\frac{a+b}{c}\right| \geq 2.$$

2.83. Let a,b,c be nonnegative real numbers, no two of which are zero, and let

$$x = \frac{2a}{b+c}, \quad y = \frac{2b}{c+a}, \quad z = \frac{2c}{a+b}.$$

Prove that

(a) $$\sqrt{xy} + \sqrt{yz} + \sqrt{zx} \geq xyz + 2;$$

(b) $$x + y + z + \sqrt{xy} + \sqrt{yz} + \sqrt{zx} \geq 6;$$

(c) $$\sqrt{x} + \sqrt{y} + \sqrt{z} \geq \sqrt{8 + xyz};$$

(d) $$\frac{\sqrt{yz}}{x+2} + \frac{\sqrt{zx}}{y+2} + \frac{\sqrt{xy}}{z+2} \geq 1.$$

2.84. Let a,b,c be nonnegative real numbers, no two of which are zero, and let

$$x = \frac{2a}{b+c}, \quad y = \frac{2b}{c+a}, \quad z = \frac{2c}{a+b}.$$

Prove that

$$\sqrt{1+24x} + \sqrt{1+24y} + \sqrt{1+24z} \geq 15.$$

2.85. If a,b,c are positive real numbers, then

$$\sqrt{\frac{7a}{a+3b+3c}} + \sqrt{\frac{7b}{b+3c+3a}} + \sqrt{\frac{7c}{c+3a+3b}} \leq 3.$$

2.86. If a,b,c are positive real numbers such that $a+b+c=3$, then

$$\sqrt[3]{a^2(b^2+c^2)} + \sqrt[3]{b^2(c^2+a^2)} + \sqrt[3]{c^2(a^2+b^2)} \leq 3\sqrt[3]{2}.$$

2.87. If a,b,c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{1}{a+b+c} + \frac{2}{\sqrt{ab+bc+ca}}.$$

2.88. If $a, b \ge 1$, then

$$\frac{1}{\sqrt{3ab+1}} + \frac{1}{2} \ge \frac{1}{\sqrt{3a+1}} + \frac{1}{\sqrt{3b+1}}.$$

2.89. Let a, b, c be positive real numbers such that $a + b + c = 3$. If $k \ge \frac{1}{\sqrt{2}}$, then

$$(abc)^k(a^2 + b^2 + c^2) \le 3.$$

2.90. If $a, b, c \in [0, 4]$ and $ab + bc + ca = 4$, then

$$\sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a} \le 3 + \sqrt{5}.$$

2.91. Let

$$F(a,b,c) = \sqrt{\frac{a^2+b^2+c^2}{3}} - \frac{a+b+c}{3},$$

where a, b, c are positive real numbers such that

$$a^4bc \ge 1, \qquad a \le b \le c.$$

Then,

$$F(a,b,c) \ge F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

2.92. Let

$$F(a,b,c) = \sqrt{\frac{a^2+b^2+c^2}{3}} - \frac{a+b+c}{3},$$

where a, b, c are positive real numbers such that

$$a^2(b+c) \ge 1, \qquad a \le b \le c.$$

Then,

$$F(a,b,c) \ge F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

2.93. Let

$$F(a,b,c)=\sqrt{\frac{a^2+b^2+c^2}{3}}-\frac{a+b+c}{3},$$

where a,b,c are positive real numbers such that

$$a^4(b^2+c^2)\ge 2, \qquad a\le b\le c.$$

Then,

$$F(a,b,c)\ge F\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right).$$

2.94. Let

$$F(a,b,c)=\sqrt[3]{abc}-\frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}},$$

where a,b,c are positive real numbers such that

$$a^4b^7c^7\ge 1, \qquad a\ge b\ge c.$$

Then,

$$F(a,b,c)\ge F\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right).$$

2.95. Let

$$F(a,b,c,d)=\sqrt[4]{abcd}-\frac{4}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}},$$

where a,b,c,d are positive real numbers. If $ab\ge 1$ and $cd\ge 1$, then then

$$F(a,b,c,d)\ge F\left(\frac{1}{a},\frac{1}{b},\frac{1}{c},\frac{1}{d}\right).$$

2.96. Let a, b, c, d be nonnegative real numbers such that $a^2+b^2+c^2+d^2=1$. Prove that

$$\sqrt{1-a}+\sqrt{1-b}+\sqrt{1-c}+\sqrt{1-d}\ge\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}.$$

2.97. Let a, b, c, d be positive real numbers. Prove that

$$A+2\ge\sqrt{B+4},$$

where

$$A=(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)-16,$$

$$B=(a^2+b^2+c^2+d^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right)-16.$$

2.98. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1+a_2+\cdots+a_n = 1$. Prove that

$$\sqrt{3a_1+1}+\sqrt{3a_2+1}+\cdots+\sqrt{3a_n+1} \ge n+1.$$

2.99. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2\cdots a_n = 1$. Prove that

$$\frac{1}{\sqrt{1+(n^2-1)a_1}}+\frac{1}{\sqrt{1+(n^2-1)a_2}}+\cdots+\frac{1}{\sqrt{1+(n^2-1)a_n}} \ge 1.$$

2.100. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2\cdots a_n = 1$. Prove that

$$\sum_{i=1}^{n} \frac{1}{1+\sqrt{1+4n(n-1)a_i}} \ge \frac{1}{2}.$$

2.101. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1a_2\cdots a_n = 1$, then

$$a_1+a_2+\cdots+a_n \ge n-1+\sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}}.$$

2.102. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1a_2\cdots a_n = 1$, then

$$\sqrt{(n-1)(a_1^2+a_2^2+\cdots+a_n^2)}+n-\sqrt{n(n-1)} \ge a_1+a_2+\cdots+a_n.$$

2.103. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2\cdots a_n \ge 1$. If $k > 1$, then

$$\sum \frac{a_1^k}{a_1^k+a_2+\cdots+a_n} \ge 1.$$

2.104. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2\cdots a_n \ge 1$. If

$$\frac{-2}{n-2} \le k < 1,$$

then

$$\sum \frac{a_1^k}{a_1^k+a_2+\cdots+a_n} \le 1.$$

2.105. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1+a_2+\cdots+a_n \ge n$. If $1 < k \le n+1$, then

$$\sum \frac{a_1}{a_1^k + a_2 + \cdots + a_n} \le 1.$$

2.106. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \cdots a_n \ge 1$. If $k > 1$, then

$$\sum \frac{a_1}{a_1^k + a_2 + \cdots + a_n} \le 1.$$

2.107. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \cdots a_n \ge 1$. If

$$-1 - \frac{2}{n-2} \le k < 1,$$

then

$$\sum \frac{a_1}{a_1^k + a_2 + \cdots + a_n} \ge 1.$$

2.108. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \cdots a_n = 1$. If $k \ge 0$, then

$$\sum \frac{1}{a_1^k + a_2 + \cdots + a_n} \le 1.$$

2.109. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1+a_2+\cdots+a_n \le n$. If $0 \le k < 1$, then

$$\frac{1}{a_1^k + a_2 + \cdots + a_n} + \frac{1}{a_1 + a_2^k + \cdots + a_n} + \cdots + \frac{1}{a_1 + a_2 + \cdots + a_n^k} \ge 1.$$

2.110. Let $a_1, a_2, \ldots, a_n$ be positive real numbers. If $k > 1$, then

$$\sum \frac{a_2^k + a_3^k + \cdots + a_n^k}{a_2 + a_3 + \cdots + a_n} \le \frac{n(a_1^k + a_2^k + \cdots + a_n^k)}{a_1 + a_2 + \cdots + a_n}.$$

2.111. Let f be a convex function on the closed interval $[a, b]$, and let $a_1, a_2, \ldots, a_n \in [a, b]$ such that

$$a_1 + a_2 + \cdots + a_n = pa + qb,$$

where $p, q \ge 0$ such that $p + q = n$. Prove that

$$f(a_1) + f(a_2) + \cdots + f(a_n) \le pf(a) + qf(b).$$

2.2 Solutions

P 2.1. *If a, b are nonnegative real numbers such that $a^2+b^2 \le 1+\frac{2}{\sqrt{3}}$, then*

$$\frac{a}{2a^2+1}+\frac{b}{2b^2+1} \le \frac{\sqrt{2(a^2+b^2)}}{a^2+b^2+1}.$$

(Vasile Cîrtoaje, 2012)

Solution. With

$$s=\frac{a^2+b^2}{2}, \quad p=ab, \quad 0 \le p \le s \le \frac{1}{2}+\frac{1}{\sqrt{3}},$$

the inequality becomes as follows:

$$\frac{(2p+1)\sqrt{2(s+p)}}{4p^2+4s+1} \le \frac{2\sqrt{s}}{2s+1},$$

$$\sqrt{\frac{2s}{s+p}}-1 \ge \frac{(2p+1)(2s+1)}{4p^2+4s+1}-1,$$

$$\frac{s-p}{(s+p)\left(\sqrt{\frac{2s}{s+p}}+1\right)} \ge \frac{2(s-p)(2p-1)}{4p^2+4s+1}.$$

Thus, we need to show that

$$\frac{1}{(s+p)\left(\sqrt{\frac{2s}{s+p}}+1\right)} \ge \frac{2(2p-1)}{4p^2+4s+1}.$$

Since $\sqrt{\frac{2s}{s+p}} \ge 1$, it suffices to show that

$$\frac{1}{(s+p)\left(\sqrt{\frac{2s}{s+p}}+\sqrt{\frac{2s}{s+p}}\right)} \ge \frac{2(2p-1)}{4p^2+4s+1},$$

which is equivalent to

$$4p^2+4s+1 \ge 4(2p-1)\sqrt{2s(s+p)}.$$

For the nontrivial case $2p-1>0$, which involves $2s-1>0$, since $2\sqrt{2s(s+p)} \le 2s+(s+p)$, it suffices to show that

$$4p^2+4s+1 \ge 2(2p-1)(3s+p),$$

that is

$$10s+1 \ge 2p(6s-1).$$

We have

$$10s+1-2p(6s-1) \ge 10s+1-2s(6s-1) = 1+12s-12s^2 \ge 0.$$

The equality holds for $a = b$.

□

P 2.2. *If a, b, c are real numbers, then*

$$\sum \sqrt{a^2-ab+b^2} \le \sqrt{6(a^2+b^2+c^2)-3(ab+bc+ca)}.$$

Solution. By squaring, the inequality becomes as follows:

$$2(ab+bc+ca)+2\sum \sqrt{(a^2-ab+b^2)(a^2-ac+c^2)} \le 4(a^2+b^2+c^2),$$

$$\sum \left(\sqrt{a^2-ab+b^2}-\sqrt{a^2-ac+c^2}\right)^2 \ge 0.$$

The equality holds for $a = b = c$, and also for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 2.3. *If a, b, c are positive real numbers, then*

$$a\sqrt{b+c}+b\sqrt{c+a}+c\sqrt{a+b} \ge \frac{2bc}{\sqrt{b+c}}+\frac{2ca}{\sqrt{c+a}}+\frac{2ab}{\sqrt{a+b}}.$$

(Lorian Saceanu, 2015)

Solution. Use the SOS method. Write the inequality as follows:

$$\sum a\sqrt{b+c}-\sum \frac{2bc}{\sqrt{b+c}} \ge 0,$$

$$\sum \frac{a(b+c)-2bc}{\sqrt{b+c}} \ge 0,$$

$$\sum \frac{b(a-c)}{\sqrt{b+c}}+\sum \frac{c(a-b)}{\sqrt{b+c}} \ge 0,$$

$$\sum \frac{c(b-a)}{\sqrt{c+a}}+\sum \frac{c(a-b)}{\sqrt{b+c}} \ge 0,$$

$$\sum c(a-b)\left(\frac{1}{\sqrt{b+c}}-\frac{1}{\sqrt{c+a}}\right)\geq 0,$$

$$\sum \frac{c(a-b)^2}{\sqrt{(b+c)(c+a)}\left(\sqrt{b+c}+\sqrt{c+a}\right)}\geq 0.$$

The equality holds for $a=b=c$.

□

P 2.4. *If a,b,c are nonnegative real numbers, then*

$$\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}+\sqrt{c^2-ca+a^2}\leq 3\sqrt{\frac{a^2+b^2+c^2}{2}}.$$

Solution *(by Nguyen Van Quy).* Assume that $c=\min\{a,b,c\}$. Since

$$b^2-bc+c^2\leq b^2$$

and

$$c^2-ca+a^2\leq a^2,$$

it suffices to show that

$$\sqrt{a^2-ab+b^2}+b+a\leq 3\sqrt{\frac{a^2+b^2+c^2}{2}}.$$

Using the Cauchy-Schwarz inequality, we have

$$\sqrt{a^2-ab+b^2}+a+b\leq\sqrt{\left[(a^2-ab+b^2)+\frac{(a+b)^2}{k}\right](1+k)}$$

$$=\sqrt{\frac{(1+k)[(1+k)(a^2+b^2)+(2-k)ab]}{k}},\qquad k>0.$$

Choosing $k=2$, we get

$$\sqrt{a^2-ab+b^2}+a+b\leq 3\sqrt{\frac{a^2+b^2}{2}}\leq 3\sqrt{\frac{a^2+b^2+c^2}{2}}=3.$$

The equality holds for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 2.5. *If a,b,c are nonnegative real numbers, then*

$$\sqrt{a^2+b^2-\frac{2}{3}ab}+\sqrt{b^2+c^2-\frac{2}{3}bc}+\sqrt{c^2+a^2-\frac{2}{3}ca}\geq 2\sqrt{a^2+b^2+c^2}.$$

(Vasile Cîrtoaje, 2012)

First Solution. By squaring, the inequality becomes

$$2\sum\sqrt{(3a^2+3b^2-2ab)(3a^2+3c^2-2ac)} \ge 6(a^2+b^2+c^2)+2(ab+bc+ca),$$

$$6(a^2+b^2+c^2-ab-bc-ca) \ge \sum\left(\sqrt{3a^2+3b^2-2ab}-\sqrt{3a^2+3c^2-2ac}\right)^2,$$

$$3\sum(b-c)^2 \ge \sum\frac{(b-c)^2(3b+3c-2a)^2}{\left(\sqrt{3a^2+3b^2-2ab}+\sqrt{3a^2+3c^2-2ac}\right)^2},$$

$$\sum(b-c)^2\left[1-\frac{(3b+3c-2a)^2}{\left(\sqrt{9a^2+9b^2-6ab}+\sqrt{9a^2+9c^2-6ac}\right)^2}\right].$$

Since

$$\sqrt{9a^2+9b^2-6ab}=\sqrt{(3b-a)^2+8a^2} \ge |3b-a|,$$

$$\sqrt{9a^2+9c^2-6ac}=\sqrt{(3c-a)^2+8a^2} \ge |3c-a|,$$

it suffices to show that

$$\sum(b-c)^2\left[1-\left(\frac{|3b+3c-2a|}{|3b-a|+|3c-a|}\right)^2\right] \ge 0.$$

This is true since

$$|3b+3c-2a|=|(3b-a)+(3c-a)| \le |3b-a|+|3c-a|.$$

The equality holds for $a=b=c$, and also for $b=c=0$ (or any cyclic permutation).

Second Solution. Assume that $a \ge b \ge c$. Write the inequality as

$$\sqrt{(a+b)^2+2(a-b)^2}+\sqrt{(b+c)^2+2(b-c)^2}+\sqrt{(a+c)^2+2(a-c)^2} \ge$$

$$\ge 2\sqrt{3(a^2+b^2+c^2)}.$$

By Minkowski's inequality, it suffices to show that

$$\sqrt{[(a+b)+(b+c)+(a+c)]^2+2[(a-b)+(b-c)+(a-c)]^2} \ge 2\sqrt{3(a^2+b^2+c^2)},$$

which is equivalent to

$$\sqrt{(a+b+c)^2+2(a-c)^2} \ge \sqrt{3(a^2+b^2+c^2)}.$$

By squaring, the inequality turns into

$$(a-b)(b-c) \ge 0.$$

□

P 2.6. *If a,b,c are nonnegative real numbers, then*

$$\sum \sqrt{a^2+ab+b^2} \ge \sqrt{4(a^2+b^2+c^2)+5(ab+bc+ca)}.$$

(Vasile Cîrtoaje, 2009)

First Solution. By squaring, the inequality becomes

$$\sum \sqrt{(a^2+ab+b^2)(a^2+ac+c^2)} \ge (a+b+c)^2.$$

Using the Cauchy-Schwarz inequality, we get

$$\sum \sqrt{(a^2+ab+b^2)(a^2+ac+c^2)} = \sum \sqrt{\left[\left(a+\frac{b}{2}\right)^2+\frac{3b^2}{4}\right]\left[\left(a+\frac{c}{2}\right)^2+\frac{3c^2}{4}\right]}$$

$$\ge \sum \left[\left(a+\frac{b}{2}\right)\left(a+\frac{c}{2}\right)+\frac{3bc}{4}\right] = (a+b+c)^2.$$

The equality holds for $a=b=c$, and also for $b=c=0$ (or any cyclic permutation).

Second Solution. Assume that $a \ge b \ge c$. By Minkowski's inequality, we get

$$2\sum \sqrt{a^2+ab+b^2} = \sum \sqrt{3(a+b)^2+(a-b)^2}$$

$$\ge \sqrt{3[(a+b)+(b+c)+(c+a)]^2+[(a-b)+(b-c)+(a-c)]^2}$$

$$= 2\sqrt{3(a+b+c)^2+(a-c)^2}.$$

Therefore, it suffices to show that

$$3(a+b+c)^2+(a-c)^2 \ge 4(a^2+b^2+c^2)+5(ab+bc+ca),$$

which is equivalent to the obvious inequality

$$(a-b)(b-c) \ge 0.$$

Remark. Similarly, we can prove the following generalization.

- *Let a,b,c be nonnegative real numbers. If $|k| \le 2$, then*

$$\sum \sqrt{a^2+kab+b^2} \ge \sqrt{4(a^2+b^2+c^2)+(3k+2)(ab+bc+ca)},$$

with equality for $a=b=c$, and also for $b=c=0$ (or any cyclic permutation).

For $k=-2/3$ and $k=1$, we get the inequalities in P 2.5 and P 2.6, respectively. For $k=-1$ and $k=0$, we get the inequalities

$$\sum \sqrt{a^2-ab+b^2} \ge \sqrt{4(a^2+b^2+c^2)-ab-bc-ca},$$

$$\sum \sqrt{a^2+b^2} \ge \sqrt{4(a^2+b^2+c^2)+2(ab+bc+ca)}.$$

□

P 2.7. *If a, b, c are positive real numbers, then*

$$\sum \sqrt{a^2+ab+b^2} \le \sqrt{5(a^2+b^2+c^2)+4(ab+bc+ca)}.$$

(Michael Rozenberg, 2008)

First Solution *(by Vo Quoc Ba Can)*. Using the Cauchy-Schwarz inequality, we have

$$\left(\sum \sqrt{b^2+bc+c^2}\right)^2 \le \left[\sum(b+c)\right]\left(\sum \frac{b^2+bc+c^2}{b+c}\right)$$

$$=2(a+b+c)\left(\sum \frac{b^2+bc+c^2}{b+c}\right)=2\sum\left(1+\frac{a}{b+c}\right)(b^2+bc+c^2)$$

$$=4(a^2+b^2+c^2)+2(ab+bc+ca)+\sum \frac{2a(b^2+bc+c^2)}{b+c}$$

$$=4(a^2+b^2+c^2)+2(ab+bc+ca)+\sum 2a\left(b+c-\frac{bc}{b+c}\right)$$

$$=4(a^2+b^2+c^2)+6(ab+bc+ca)-2abc\sum \frac{1}{b+c}.$$

Thus, it suffices to prove that

$$4(a^2+b^2+c^2)+6(ab+bc+ca)-2abc\sum \frac{1}{b+c} \le 5(a^2+b^2+c^2)+4(ab+bc+ca),$$

which is equivalent to Schur's inequality

$$2(ab+bc+ca) \le a^2+b^2+c^2+2abc\sum \frac{1}{b+c}.$$

We can prove this inequality by writing it as follows:

$$(a+b+c)^2 \le 2\sum a\left(a+\frac{bc}{b+c}\right),$$

$$(a+b+c)^2 \le 2(ab+bc+ca)\sum \frac{a}{b+c},$$

$$(a+b+c)^2 \le \left[\sum a(b+c)\right]\sum \frac{a}{b+c}.$$

Clearly, the last inequality follows from the Cauchy-Schwarz inequality. The equality holds for $a=b=c$.

Second Solution. Use the SOS method. Let us denote

$$A=\sqrt{b^2+bc+c^2}, \quad B=\sqrt{c^2+ca+a^2}, \quad C=\sqrt{a^2+ab+b^2}.$$

Without loss of generality, assume that $a \ge b \ge c$. By squaring, the inequality becomes

$$2\sum BC \le 3\sum a^2+3\sum ab,$$

$$\sum a^2 - \sum ab \le \sum (B-C)^2,$$

$$\sum (b-c)^2 \le 2(a+b+c)^2 \sum \frac{(b-c)^2}{(B+C)^2}.$$

Since

$$(B+C)^2 \le 2(B^2+C^2) = 2(2a^2+b^2+c^2+ca+ab),$$

it suffices to show that

$$\sum (b-c)^2 \le (a+b+c)^2 \sum \frac{(b-c)^2}{2a^2+b^2+c^2+ca+ab},$$

which is equivalent to

$$\sum (b-c)^2 S_a \ge 0,$$

where

$$S_a = \frac{-a^2+ab+2bc+ca}{2a^2+b^2+c^2+ca+ab},$$

$$S_b = \frac{-b^2+bc+2ca+ab}{2b^2+c^2+a^2+ab+bc} \ge 0,$$

$$S_c = \frac{-c^2+ca+2ab+bc}{2c^2+a^2+b^2+bc+ca} \ge 0.$$

Since

$$\sum (b-c)^2 S_a \ge (b-c)^2 S_a + (a-c)^2 S_b \ge (b-c)^2 S_a + \frac{a^2}{b^2}(b-c)^2 S_b$$

$$\ge (b-c)^2 S_a + \frac{a}{b}(b-c)^2 S_b = a(b-c)^2\left(\frac{S_a}{a} + \frac{S_b}{b}\right),$$

we only need to prove that

$$\frac{S_a}{a} + \frac{S_b}{b} \ge 0,$$

which is equivalent to

$$\frac{-b^2+bc+2ca+ab}{b(2b^2+c^2+a^2+ab+bc)} \ge \frac{a^2-ab-2bc-ca}{a(2a^2+b^2+c^2+ca+ab)}.$$

Consider the nontrivial case where $a^2-ab-2bc-ca \ge 0$. Since

$$(2a^2+b^2+c^2+ca+ab) - (2b^2+c^2+a^2+ab+bc) = (a-b)(a+b+c) \ge 0,$$

it suffices to show that

$$\frac{-b^2+bc+2ca+ab}{b} \ge \frac{a^2-ab-2bc-ca}{a}.$$

Indeed,

$$a(-b^2+bc+2ca+ab) - b(a^2-ab-2bc-ca) = 2c(a^2+ab+b^2) > 0.$$

□

P 2.8. *If a, b, c are nonnegative real numbers, then*

$$\sum \sqrt{a^2+ab+b^2} \le 2\sqrt{a^2+b^2+c^2}+\sqrt{ab+bc+ca}.$$

(Vasile Cîrtoaje, 2010)

First Solution *(by Nguyen Van Quy).* Assume that $a = \max\{a, b, c\}$. Since

$$\sqrt{a^2+ab+b^2}+\sqrt{c^2+ca+a^2} \le \sqrt{2[(a^2+ab+b^2)+(c^2+ca+a^2)]},$$

it suffices to show that

$$2\sqrt{A}+\sqrt{b^2+bc+c^2} \le 2\sqrt{X}+\sqrt{Y},$$

where

$$A = a^2+\frac{1}{2}(b^2+c^2+ab+ac), \quad X = a^2+b^2+c^2, \quad Y = ab+bc+ca.$$

Write the desired inequality as follows:

$$2(\sqrt{A}-\sqrt{X}) \le \sqrt{Y}-\sqrt{b^2+bc+c^2},$$

$$\frac{2(A-X)}{\sqrt{A}+\sqrt{X}} \le \frac{Y-(b^2+bc+c^2)}{\sqrt{Y}+\sqrt{b^2+bc+c^2}},$$

$$\frac{b(a-b)+c(a-c)}{\sqrt{A}+\sqrt{X}} \le \frac{b(a-b)+c(a-c)}{\sqrt{Y}+\sqrt{b^2+bc+c^2}}.$$

Since $b(a-b)+c(a-c) \ge 0$, we only need to show that

$$\sqrt{A}+\sqrt{X} \ge \sqrt{Y}+\sqrt{b^2+bc+c^2}.$$

This inequality is true because $X \ge Y$ and

$$\sqrt{A} \ge \sqrt{b^2+bc+c^2}.$$

Indeed,

$$2(A-b^2-bc-c^2) = 2a^2+(b+c)a-(b+c)^2 = (2a-b-c)(a+b+c) \ge 0.$$

The equality holds for $a = b = c$, and also for $b = c = 0$ (or any cyclic permutation).

Second Solution. In the first solution of P 2.7, we have shown that

$$\left(\sum \sqrt{b^2+bc+c^2}\right)^2 \le 4(a^2+b^2+c^2)+6(ab+bc+ca)-2abc\sum \frac{1}{b+c}.$$

Thus, it suffices to prove that

$$4(a^2+b^2+c^2)+6(ab+bc+ca)-2abc\sum \frac{1}{b+c} \le \left(2\sqrt{a^2+b^2+c^2}+\sqrt{ab+bc+ca}\right)^2,$$

which is equivalent to

$$2abc\sum\frac{1}{b+c}+4\sqrt{(a^2+b^2+c^2)(ab+bc+ca)}\geq 5(ab+bc+ca).$$

Since

$$\sum\frac{1}{b+c}\geq\frac{9}{\sum(b+c)}=\frac{9}{2(a+b+c)},$$

it is enough to prove that

$$\frac{9abc}{a+b+c}+4\sqrt{(a^2+b^2+c^2)(ab+bc+ca)}\geq 5(ab+bc+ca),$$

which can be written as

$$\frac{9abc}{p}+4\sqrt{q(p^2-2q)}\geq 5q,$$

where

$$p=a+b+c,\quad q=ab+bc+ca.$$

For $p^2\geq 4q$, this inequality is true because $4\sqrt{q(p^2-2q)}\geq 5q$. Consider further

$$3q\leq p^2\leq 4q.$$

By Schur's inequality of third degree, we have

$$\frac{9abc}{p}\geq 4q-p^2.$$

Therefore, it suffices to show that

$$(4q-p^2)+4\sqrt{q(p^2-2q)}\geq 5q,$$

which is

$$4\sqrt{q(p^2-2q)}\geq p^2+q.$$

Indeed,

$$16q(p^2-2q)-(p^2+q)^2=(p^2-3q)(11q-p^2)\geq 0.$$

Third Solution. Let us denote

$$A=\sqrt{b^2+bc+c^2},\quad B=\sqrt{c^2+ca+a^2},\quad C=\sqrt{a^2+ab+b^2},$$

$$X=\sqrt{a^2+b^2+c^2},\quad Y=\sqrt{ab+bc+ca}.$$

By squaring, the inequality becomes

$$2\sum BC\leq 2\sum a^2+4XY,$$

$$\sum(B-C)^2\geq 2(X-Y)^2,$$

$$2(a+b+c)^2\sum\frac{(b-c)^2}{(B+C)^2}\ge\frac{\left[\sum(b-c)^2\right]^2}{(X+Y)^2}.$$

Since

$$B+C\le(c+a)+(a+b)=2a+b+c,$$

it suffices to show that

$$2(a+b+c)^2\sum\frac{(b-c)^2}{(2a+b+c)^2}\ge\frac{\left[\sum(b-c)^2\right]^2}{(X+Y)^2}.$$

According to the Cauchy-Schwarz inequality, we have

$$\sum\frac{(b-c)^2}{(2a+b+c)^2}\ge\frac{\left[\sum(b-c)^2\right]^2}{\sum(b-c)^2(2a+b+c)^2}.$$

Therefore, it is enough to prove that

$$\frac{2(a+b+c)^2}{\sum(b-c)^2(2a+b+c)^2}\ge\frac{1}{(X+Y)^2},$$

which is

$$(a+b+c)^2(X+Y)^2\ge\frac{1}{2}\sum(b-c)^2(2a+b+c)^2.$$

We see that

$$(a+b+c)^2(X+Y)^2\ge\left(\sum a^2+2\sum ab\right)\left(\sum a^2+\sum ab\right)$$

$$=\left(\sum a^2\right)^2+3\left(\sum ab\right)\left(\sum a^2\right)+2\left(\sum ab\right)^2$$

$$\ge\sum a^4+3\sum ab(a^2+b^2)+4\sum a^2b^2$$

and

$$\sum(b-c)^2(2a+b+c)^2=\sum(b-c)^2[4a^2+4a(b+c)+(b+c)^2]$$

$$=4\sum a^2(b-c)^2+4\sum a(b-c)(b^2-c^2)+\sum(b^2-c^2)^2$$

$$\le 8\sum a^2b^2+4\sum a(b^3+c^3)+2\sum a^4.$$

Thus, it suffices to show that

$$\sum a^4+3\sum ab(a^2+b^2)+4\sum a^2b^2\ge 4\sum a^2b^2+2\sum a(b^3+c^3)+\sum a^4,$$

which is equivalent to the obvious inequality

$$\sum ab(a^2+b^2)\ge 0.$$

$\square$

P 2.9. *If a, b, c are nonnegative real numbers, then*

$$\sqrt{a^2+2bc}+\sqrt{b^2+2ca}+\sqrt{c^2+2ab} \le \sqrt{a^2+b^2+c^2}+2\sqrt{ab+bc+ca}.$$

(Vasile Cîrtoaje and Nguyen Van Quy, 1989)

Solution *(by Nguyen Van Quy).* Let

$$X=\sqrt{a^2+b^2+c^2}, \quad Y=\sqrt{ab+bc+ca}.$$

Consider the nontrivial case when no two of a, b, c are zero and write the inequality as

$$\sum\left(X-\sqrt{a^2+2bc}\right) \ge 2(X-Y),$$

$$\sum \frac{(b-c)^2}{X+\sqrt{a^2+2bc}} \ge \frac{\sum(b-c)^2}{X+Y}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{(b-c)^2}{X+\sqrt{a^2+2bc}} \ge \frac{\left[\sum(b-c)^2\right]^2}{\sum(b-c)^2\left(X+\sqrt{a^2+2bc}\,\right)}.$$

Therefore, it suffices to show that

$$\frac{\sum(b-c)^2}{\sum(b-c)^2\left(X+\sqrt{a^2+2bc}\,\right)} \ge \frac{1}{X+Y},$$

which is equivalent to

$$\sum(b-c)^2\left(Y-\sqrt{a^2+2bc}\right) \ge 0.$$

From

$$\left(Y-\sqrt{a^2+2bc}\right)^2 \ge 0.$$

we get

$$Y-\sqrt{a^2+2bc} \ge \frac{Y^2-(a^2+2bc)}{2Y} = \frac{(a-b)(c-a)}{2Y}.$$

Thus,

$$\sum(b-c)^2\left(Y-\sqrt{a^2+2bc}\right) \ge \sum \frac{(b-c)^2(a-b)(c-a)}{2Y}$$

$$= \frac{(a-b)(b-c)(c-a)}{2Y}\sum(b-c) = 0.$$

The equality holds for $a=b$, or $b=c$, or $c=a$.

□

P 2.10. *If a,b,c are nonnegative real numbers, then*

$$\frac{1}{\sqrt{a^2+2bc}}+\frac{1}{\sqrt{b^2+2ca}}+\frac{1}{\sqrt{c^2+2ab}}\ge\frac{1}{\sqrt{a^2+b^2+c^2}}+\frac{2}{\sqrt{ab+bc+ca}}.$$

(Vasile Cîrtoaje, 1989)

Solution . Let

$$X=\sqrt{a^2+b^2+c^2},\quad Y=\sqrt{ab+bc+ca}.$$

Consider the nontrivial case when $Y>0$ and write the inequality as

$$\sum\left(\frac{1}{\sqrt{a^2+2bc}}-\frac{1}{X}\right)\ge 2\left(\frac{1}{Y}-\frac{1}{X}\right),$$

$$\sum\frac{(b-c)^2}{\sqrt{a^2+2bc}\left(X+\sqrt{a^2+2bc}\right)}\ge\frac{\sum(b-c)^2}{Y(X+Y)}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum\frac{(b-c)^2}{\sqrt{a^2+2bc}\left(X+\sqrt{a^2+2bc}\right)}\ge\frac{\left[\sum(b-c)^2\right]^2}{\sum(b-c)^2\sqrt{a^2+2bc}\left(X+\sqrt{a^2+2bc}\ \right)}.$$

Therefore, it suffices to show that

$$\frac{\sum(b-c)^2}{\sum(b-c)^2\sqrt{a^2+2bc}\left(X+\sqrt{a^2+2bc}\ \right)}\ge\frac{1}{Y(X+Y)},$$

which is equivalent to

$$\sum(b-c)^2[XY-X\sqrt{a^2+2bc}+(a-b)(c-a)]\ge 0.$$

Since

$$\sum(b-c)^2(a-b)(c-a)=(a-b)(b-c)(c-a)\sum(b-c)=0,$$

we can write the inequality as

$$\sum(b-c)^2X\left(Y-\sqrt{a^2+2bc}\right)\ge 0,$$

$$\sum(b-c)^2\left(Y-\sqrt{a^2+2bc}\right)\ge 0.$$

We have proved this inequality at the preceding problem P 2.9. The equality holds for $a=b$, or $b=c$, or $c=a$.

□

P 2.11. *If a,b,c are positive real numbers, then*

$$\sqrt{2a^2+bc}+\sqrt{2b^2+ca}+\sqrt{2c^2+ab}\le 2\sqrt{a^2+b^2+c^2}+\sqrt{ab+bc+ca}.$$

Solution. We will apply Lemma below for

$$X=2a^2+bc,\quad Y=2b^2+ca,\quad Z=2c^2+ab$$

and

$$A=B=a^2+b^2+c^2,\quad C=ab+bc+ca.$$

We have

$$X+Y+Z=A+B+C,\qquad A=B\ge C.$$

Without loss of generality, assume that

$$a\ge b\ge c,$$

which involves

$$X\ge Y\ge Z.$$

By Lemma below, it suffices to show that

$$\max\{X,Y,Z\}\ge A,\quad \min\{X,Y,Z\}\le C.$$

Indeed, we have

$$\max\{X,Y,Z\}-A=X-A=(a^2-b^2)+c(b-c)\ge 0,$$

$$\min\{X,Y,Z\}-C=Z-C=c(2c-a-b)\le 0.$$

Equality holds for $a=b=c$.

Lemma. *If X,Y,Z and A,B,C are positive real numbers such that*

$$X+Y+Z=A+B+C,$$

$$\max\{X,Y,Z\}\ge\max\{A,B,C\},\quad \min\{X,Y,Z\}\le\min\{A,B,C\},$$

then

$$\sqrt{X}+\sqrt{Y}+\sqrt{Z}\le\sqrt{A}+\sqrt{B}+\sqrt{C}.$$

Proof. On the assumption that $X\ge Y\ge Z$ and $A\ge B\ge C$, we have

$$X\ge A,\qquad Z\le C,$$

hence

$$\sqrt{X}+\sqrt{Y}+\sqrt{Z}-\sqrt{A}-\sqrt{B}-\sqrt{C}=(\sqrt{X}-\sqrt{A})+(\sqrt{Y}-\sqrt{B})+(\sqrt{Z}-\sqrt{C})$$
$$\le\frac{X-A}{2\sqrt{A}}+\frac{Y-B}{2\sqrt{B}}+\frac{Z-C}{2\sqrt{C}}\le\frac{X-A}{2\sqrt{B}}+\frac{Y-B}{2\sqrt{B}}+\frac{Z-C}{2\sqrt{C}}$$
$$=\frac{C-Z}{2\sqrt{B}}+\frac{Z-C}{2\sqrt{C}}=(C-Z)\left(\frac{1}{2\sqrt{B}}-\frac{1}{2\sqrt{C}}\right)\le 0.$$

Remark. This Lemma is a particular case of Karamata's inequality.

□

P 2.12. *Let a, b, c be nonnegative real numbers such that $a+b+c=3$. If $k=\sqrt{3}-1$, then*

$$\sum \sqrt{a(a+kb)(a+kc)} \le 3\sqrt{3}.$$

Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \sqrt{a(a+kb)(a+kc)} \le \sqrt{\left(\sum a\right)\left[\sum (a+kb)(a+kc)\right]}.$$

Thus, it suffices to show that

$$\sqrt{\sum (a+kb)(a+kc)} \le a+b+c,$$

which is an identity. The equality holds for $a=b=c=1$, and also for $a=3$ and $b=c=0$ (or any cyclic permutation).

□

P 2.13. *If a, b, c are nonnegative real numbers such that $a+b+c=3$, then*

$$\sum \sqrt{a(2a+b)(2a+c)} \ge 9.$$

Solution. Write the inequality as follows:

$$\sum \left[\sqrt{a(2a+b)(2a+c)} - a\sqrt{3(a+b+c)}\right] \ge 0,$$

$$\sum (a-b)(a-c)E_a \ge 0,$$

where

$$E_a = \frac{\sqrt{a}}{\sqrt{(2a+b)(2a+c)} + \sqrt{3a(a+b+c)}}.$$

Assume that $a \ge b \ge c$. Since $(c-a)(c-b)E_c \ge 0$, it suffices to show that

$$(a-c)E_a \ge (b-c)E_b,$$

which is equivalent to

$$(a-b)\sqrt{3ab(a+b+c)} + (a-c)\sqrt{a(2b+c)(2b+a)} \ge (b-c)\sqrt{b(2a+b)(2a+c)}.$$

This is true if

$$(a-c)\sqrt{a(2b+c)(2b+a)} \ge (b-c)\sqrt{b(2a+b)(2a+c)}.$$

For the nontrivial case $b > c$, we have

$$\frac{a-c}{b-c} \ge \frac{a}{b} \ge \frac{\sqrt{a}}{\sqrt{b}}.$$

Therefore, it is enough to show that

$$a^2(2b+c)(2b+a) \ge b^2(2a+b)(2a+c).$$

Write this inequality as

$$a^2(2ab+2bc+ca) \ge b^2(2ab+bc+2ca).$$

It is true if

$$a(2ab+2bc+ca) \ge b(2ab+bc+2ca).$$

Indeed,

$$a(2ab+2bc+ca) - b(2ab+bc+2ca) = (a-b)(2ab+bc+ca) \ge 0.$$

The equality holds for $a=b=c=1$, and also for $a=b=3/2$ and $c=0$ (or any cyclic permutation).

□

P 2.14. *Let a,b,c be nonnegative real numbers such that $a+b+c=3$. Prove that*

$$\sqrt{b^2+c^2+a(b+c)} + \sqrt{c^2+a^2+b(c+a)} + \sqrt{a^2+b^2+c(a+b)} \ge 6.$$

Solution. Denote

$$A = b^2+c^2+a(b+c), \quad B = c^2+a^2+b(c+a), \quad C = a^2+b^2+c(a+b),$$

and write the inequality in the homogeneous form

$$\sqrt{A}+\sqrt{B}+\sqrt{C} \ge 2(a+b+c).$$

Further, we use the SOS method.

First Solution. By squaring, the inequality becomes

$$2\sum\sqrt{BC} \ge 2\sum a^2 + 6\sum bc,$$

$$\sum(b-c)^2 \ge \sum\left(\sqrt{B}-\sqrt{C}\right)^2,$$

$$\sum(b-c)^2 S_a \ge 0,$$

where

$$S_a = 1 - \frac{(b+c-a)^2}{(\sqrt{B}+\sqrt{C})^2}.$$

Since

$$S_a \ge 1 - \frac{(b+c-a)^2}{B+C} = \frac{a(a+3b+3c)}{B+C} \ge 0, \quad S_b \ge 0, \quad S_c \ge 0,$$

the conclusion follows. The equality holds for $a = b = c = 1$, and also for $a = 3$ and $b = c = 0$ (or any cyclic permutation).

Second Solution. Write the desired inequality as follows:

$$\sum(\sqrt{A} - b - c) \ge 0,$$

$$\sum \frac{c(a-b) + b(a-c)}{\sqrt{A} + b + c} \ge 0,$$

$$\sum \frac{c(a-b)}{\sqrt{A} + b + c} + \sum \frac{c(b-a)}{\sqrt{B} + c + a} \ge 0,$$

$$\sum \frac{c(a-b)[a-b-(\sqrt{A}-\sqrt{B})]}{(\sqrt{A}+b+c)(\sqrt{B}+c+a)} \ge 0.$$

It suffices to show that

$$(a-b)[a-b+(\sqrt{B}-\sqrt{A})] \ge 0.$$

Indeed,

$$(a-b)[a-b+(\sqrt{B}-\sqrt{A})] = (a-b)^2\left(1 + \frac{a+b-c}{\sqrt{B}+\sqrt{A}}\right) \ge 0,$$

because, for the nontrivial case $a + b - c < 0$, we have

$$1 + \frac{a+b-c}{\sqrt{B}+\sqrt{A}} > 1 + \frac{a+b-c}{c+c} > 0.$$

Generalization. *Let a, b, c be nonnegative real numbers. If $0 < k \le \frac{16}{9}$, then*

$$\sum \sqrt{(b+c)^2 + k(ab - 2bc + ca)} \ge 2(a+b+c).$$

Notice that if $k = \frac{16}{9}$, then the equality holds for $a = b = c = 1$, for $a = 0$ and $b = c$ (or any cyclic permutation), and for $b = c = 0$ (or any cyclic permutation).

□

P 2.15. *Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. Prove that*

(a) $$\sqrt{a(3a^2+abc)} + \sqrt{b(3b^2+abc)} + \sqrt{c(3c^2+abc)} \ge 6;$$

(b) $$\sqrt{3a^2+abc} + \sqrt{3b^2+abc} + \sqrt{3c^2+abc} \ge 3\sqrt{3+abc}.$$

(Lorian Saceanu, 2015)

Solution. (a) Write the inequality in the homogeneous form

$$3\sum a\sqrt{(a+b)(a+c)} \ge 2(a+b+c)^2.$$

First Solution. Use the SOS method. Write the inequality as

$$\sum a^2 - \sum ab \ge \frac{3}{2}\sum a\left(\sqrt{a+b}-\sqrt{a+c}\right)^2,$$

$$\sum (b-c)^2 \ge 3\sum \frac{a(b-c)^2}{\left(\sqrt{a+b}+\sqrt{a+c}\right)^2},$$

$$\sum (b-c)^2 S_a \ge 0,$$

where

$$S_a = 1 - \frac{3a}{\left(\sqrt{a+b}+\sqrt{a+c}\right)^2}.$$

Since

$$S_a \ge 1 - \frac{3a}{\left(\sqrt{a}+\sqrt{a}\right)^2} > 0, \quad S_b > 0, \quad S_c > 0,$$

the inequality is true. The equality holds for $a = b = c = 1$.

Second Solution. By Hölder's inequality, we have

$$\left[\sum a\sqrt{(a+b)(a+c)}\right]^2 \ge \frac{(\sum a)^3}{\sum \frac{a}{(a+b)(a+c)}} = \frac{27}{\sum \frac{a}{(a+b)(a+c)}}.$$

Therefore, it suffices to show that

$$\sum \frac{a}{(a+b)(a+c)} \le \frac{3}{4}.$$

This inequality has the homogeneous form

$$\sum \frac{a}{(a+b)(a+c)} \le \frac{9}{4(a+b+c)},$$

which is equivalent to the obvious inequality

$$\sum a(b-c)^2 \ge 0.$$

(b) By squaring, the inequality becomes

$$3\sum a^2 + 2\sum \sqrt{(3b^2+abc)(3c^2+abc)} \ge 27 + 6abc.$$

According to the Cauchy-Schwarz inequality, we have

$$\sqrt{(3b^2+abc)(3c^2+abc)} \ge 3bc + abc.$$

Therefore, it suffices to show that

$$3\sum a^2+6\sum bc+6abc \ge 27+6abc,$$

which is an identity. The equality holds for $a=b=c=1$, and also for $a=0$, or $b=0$, or $c=0$.

□

P 2.16. *Let a,b,c be positive real numbers such that $ab+bc+ca=3$. Prove that*

$$a\sqrt{(a+2b)(a+2c)}+b\sqrt{(b+2c)(b+2a)}+c\sqrt{(c+2a)(c+2b)} \ge 9.$$

First Solution. Use the SOS method. Write the inequality as follows:

$$\sum a\sqrt{(a+2b)(a+2c)} \ge 3(ab+bc+ca),$$

$$\sum a^2-\sum ab \ge \frac{1}{2}\sum a\left(\sqrt{a+2b}-\sqrt{a+2c}\right)^2,$$

$$\sum(b-c)^2 \ge 4\sum \frac{a(b-c)^2}{\left(\sqrt{a+2b}+\sqrt{a+2c}\right)^2},$$

$$\sum(b-c)^2S_a \ge 0,$$

where

$$S_a=1-\frac{4a}{\left(\sqrt{a+2b}+\sqrt{a+2c}\right)^2}.$$

Since

$$S_a>1-\frac{4a}{\left(\sqrt{a}+\sqrt{a}\right)^2}=0, \quad S_b>0, \quad S_c>0,$$

the inequality is true. The equality holds for $a=b=c=1$.

Second Solution. We use the AM-GM inequality to get

$$\sum a\sqrt{(a+2b)(a+2c)}=\sum \frac{2a(a+2b)(a+2c)}{2\sqrt{(a+2b)(a+2c)}} \ge \sum \frac{2a(a+2b)(a+2c)}{(a+2b)+(a+2c)}$$
$$=\frac{1}{a+b+c}\sum a(a+2b)(a+2c).$$

Thus, it suffices to show that

$$\sum a(a+2b)(a+2c) \ge 9(a+b+c).$$

Write this inequality in the homogeneous form

$$\sum a(a+2b)(a+2c) \ge 3(a+b+c)(ab+bc+ca),$$

which is equivalent to Schur's inequality of degree three

$$a^3+b^3+c^3+3abc \ge ab(a+b)+bc(b+c)+ca(c+a).$$

□

P 2.17. *Let a, b, c be nonnegative real numbers such that $a+b+c=1$. Prove that*

$$\sqrt{a+(b-c)^2}+\sqrt{b+(c-a)^2}+\sqrt{c+(a-b)^2} \ge \sqrt{3}.$$

(Phan Thanh Nam, 2007)

Solution. By squaring, the inequality becomes

$$\sum \sqrt{[a+(b-c)^2][b+(c-a)^2]} \ge 3(ab+bc+ca).$$

Applying the Cauchy-Schwarz inequality, it suffices to show that

$$\sum \sqrt{ab}+\sum (b-c)(a-c) \ge 3(ab+bc+ca).$$

This is equivalent to the homogeneous inequality

$$\left(\sum a\right)\left(\sum \sqrt{ab}\right)+\sum a^2 \ge 4(ab+bc+ca).$$

Making the substitution $x=\sqrt{a}$, $y=\sqrt{b}$, $z=\sqrt{c}$, the inequality turns into

$$\left(\sum x^2\right)\left(\sum xy\right)+\sum x^4 \ge 4\sum x^2y^2,$$

which is equivalent to

$$\sum x^4+\sum xy(x^2+y^2)+xyz\sum x \ge 4\sum x^2y^2.$$

Since

$$4\sum x^2y^2 \le 2\sum xy(x^2+y^2),$$

it suffices to show that

$$\sum x^4+xyz\sum x \ge \sum xy(x^2+y^2),$$

which is just Schur's inequality of degree four. The equality holds for $a=b=c=\frac{1}{3}$, and for $a=0$ and $b=c=\frac{1}{2}$ (or any cyclic permutation).

□

P 2.18. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\sqrt{\frac{a(b+c)}{a^2+bc}}+\sqrt{\frac{b(c+a)}{b^2+ca}}+\sqrt{\frac{c(a+b)}{c^2+ab}} \ge 2.$$

(Vasile Cîrtoaje, 2006)

Solution. Using the AM-GM inequality gives

$$\sqrt{\frac{a(b+c)}{a^2+bc}} = \frac{a(b+c)}{\sqrt{(a^2+bc)(ab+ac)}} \ge \frac{2a(b+c)}{(a^2+bc)+(ab+ac)} = \frac{2a(b+c)}{(a+b)(a+c)}.$$

Therefore, it suffices to show that

$$\frac{a(b+c)}{(a+b)(a+c)}+\frac{b(c+a)}{(b+c)(b+a)}+\frac{c(a+b)}{(c+a)(c+b)} \ge 1,$$

which is equivalent to

$$a(b+c)^2+b(c+a)^2+c(a+b)^2 \ge (a+b)(b+c)(c+a),$$

$$4abc \ge 0.$$

The equality holds for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 2.19. *Let a, b, c be positive real numbers such that $abc=1$. Prove that*

$$\frac{1}{\sqrt[3]{a^2+25a+1}}+\frac{1}{\sqrt[3]{b^2+25b+1}}+\frac{1}{\sqrt[3]{c^2+25c+1}} \ge 1.$$

Solution. Replacing a, b, c by a^3, b^3, c^3, respectively, we need to show that $abc=1$ yields

$$\frac{1}{\sqrt[3]{a^6+25a^3+1}}+\frac{1}{\sqrt[3]{b^6+25b^3+1}}+\frac{1}{\sqrt[3]{c^6+25c^3+1}} \ge 1.$$

We first show that

$$\frac{1}{\sqrt[3]{a^6+25a^3+1}} \ge \frac{1}{a^2+a+1}.$$

This is equivalent to

$$(a^2+a+1)^3 \ge a^6+25a^3+1,$$

which is true since

$$(a^2+a+1)^3-(a^6+25a^3+1) = 3a(a-1)^2(a^2+4a+1) \ge 0.$$

Therefore, it suffices to prove that

$$\frac{1}{a^2+a+1}+\frac{1}{b^2+b+1}+\frac{1}{b^2+b+1}\geq 1.$$

Putting

$$a=\frac{yz}{x^2},\quad b=\frac{zx}{y^2},\quad c=\frac{xy}{z^2},\quad x,y,z>0$$

we need to show that

$$\sum\frac{x^4}{x^4+x^2yz+y^2z^2}\geq 1.$$

Indeed, the Cauchy-Schwarz inequality gives

$$\sum\frac{x^4}{x^4+x^2yz+y^2z^2}\geq\frac{\left(\sum x^2\right)^2}{\sum(x^4+x^2yz+y^2z^2)}=\frac{\sum x^4+2\sum y^2z^2}{\sum x^4+xyz\sum x+\sum y^2z^2}\geq 1.$$

The equality holds for $a=b=c=1$.

□

P 2.20. *If a,b,c are nonnegative real numbers, then*

$$\sqrt{a^2+bc}+\sqrt{b^2+ca}+\sqrt{c^2+ab}\leq\frac{3}{2}(a+b+c).$$

(Pham Kim Hung, 2005)

Solution. Without loss of generality, assume that $a\geq b\geq c$. Since the equality occurs for $a=b$ and $c=0$, we use the inequalities

$$\sqrt{a^2+bc}\leq a+\frac{c}{2}$$

and

$$\sqrt{b^2+ca}+\sqrt{c^2+ab}\leq\sqrt{2(b^2+ca)+2(c^2+ab)}.$$

Thus, it suffices to prove that

$$\sqrt{2(b^2+ca)+2(c^2+ab)}\leq\frac{a+3b+2c}{2}.$$

By squaring, this inequality becomes

$$a^2+b^2-4c^2-2ab+12bc-4ca\geq 0,$$

$$(a-b-2c)^2+8c(b-c)\geq 0.$$

The equality holds for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 2.21. *If a,b,c are nonnegative real numbers, then*

$$\sqrt{a^2+9bc}+\sqrt{b^2+9ca}+\sqrt{c^2+9ab}\ge 5\sqrt{ab+bc+ca}.$$

(Vasile Cîrtoaje, 2012)

Solution *(by Nguyen Van Quy)*. Assume that

$$c=\min\{a,b,c\}.$$

Since the equality occurs for $a=b$ and $c=0$, we use the inequality

$$\sqrt{c^2+9ab}\ge 3\sqrt{ab}.$$

On the other hand, by Minkowski's inequality, we have

$$\sqrt{a^2+9bc}+\sqrt{b^2+9ca}\ge\sqrt{(a+b)^2+9c\left(\sqrt{a}+\sqrt{b}\right)^2}.$$

Therefore, it suffices to show that

$$\sqrt{(a+b)^2+9c\left(\sqrt{a}+\sqrt{b}\right)^2}\ge 5\sqrt{ab+bc+ca}-3\sqrt{ab}.$$

By squaring, this inequality becomes

$$(a+b)^2+18c\sqrt{ab}+30\sqrt{ab(ab+bc+ca)}\ge 34ab+16c(a+b).$$

Since

$$ab(ab+bc+ca)-\left[ab+\frac{c(a+b)}{3}\right]^2=\frac{c(a+b)(3ab-ac-bc)}{9}\ge 0,$$

it suffices to show that $f(c)\ge 0$ for $0\le c\le\sqrt{ab}$, where

$$\begin{aligned}f(c)&=(a+b)^2+18c\sqrt{ab}+[30ab+10c(a+b)]-34ab-16c(a+b)\\&=(a+b)^2-4ab+6c(3\sqrt{ab}-a-b).\end{aligned}$$

Since $f(c)$ is a linear function, we only need to prove that $f(0)\ge 0$ and $f(\sqrt{ab})\ge 0$. We have

$$f(0)=(a-b)^2\ge 0,$$

$$f(\sqrt{ab})=(a+b)^2+14ab-6(a+b)\sqrt{ab}\ge(a+b)^2+9ab-6(a+b)\sqrt{ab}$$
$$=\left(a+b-3\sqrt{ab}\right)^2\ge 0.$$

The equality holds for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 2.22. *If a,b,c are nonnegative real numbers, then*

$$\sum\sqrt{(a^2+4bc)(b^2+4ca)}\ge 5(ab+ac+bc).$$

(Vasile Cîrtoaje, 2012)

Solution. Assume that

$$a\ge b\ge c.$$

First Solution (by *Michael Rozenberg*). Use the SOS method. For $b=c=0$, the inequality is trivial. Consider further that $b>0$ and write the inequality as follows:

$$\sum\left[\sqrt{(b^2+4ca)(c^2+4ab)}-(bc+2ab+2ac)\right]\ge 0,$$

$$\sum\frac{(b^2+4ca)(c^2+4ab)-(bc+2ab+2ac)^2}{\sqrt{(b^2+4ca)(c^2+4ab)}+bc+2a(b+c)}\ge 0,$$

$$\sum(b-c)^2S_a\ge 0,$$

where

$$S_a=\frac{a(b+c-a)}{A},\quad A=\sqrt{(b^2+4ca)(c^2+4ab)}+bc+2a(b+c),$$

$$S_b=\frac{b(c+a-b)}{B},\quad B=\sqrt{(c^2+4ab)(a^2+4bc)}+ca+2b(c+a),$$

$$S_c=\frac{c(a+b-c)}{C},\quad C=\sqrt{(a^2+4bc)(b^2+4ac)}+ab+2c(a+b).$$

Since $S_b\ge 0$ and $S_c\ge 0$, we have

$$\sum(b-c)^2S_a\ge(b-c)^2S_a+(a-c)^2S_b\ge(b-c)^2S_a+\frac{a^2}{b^2}(b-c)^2S_b$$

$$=\frac{a}{b}(b-c)^2\left(\frac{bS_a}{a}+\frac{aS_b}{b}\right).$$

Thus, it suffices to prove that

$$\frac{bS_a}{a}+\frac{aS_b}{b}\ge 0,$$

which is equivalent to

$$\frac{b(b+c-a)}{A}+\frac{a(c+a-b)}{B}\ge 0.$$

Since

$$\frac{b(b+c-a)}{A}+\frac{a(c+a-b)}{B}\ge\frac{b(b-a)}{A}+\frac{a(a-b)}{B}=\frac{(a-b)(aA-bB)}{AB},$$

it is enough to show that

$$aA - bB \ge 0.$$

Indeed,

$$aA - bB = \sqrt{c^2+4ab}\left[a\sqrt{b^2+4ca} - b\sqrt{a^2+4bc}\right] + 2(a-b)(ab+bc+ca)$$

$$= \frac{4c(a^3-b^3)\sqrt{c^2+4ab}}{a\sqrt{b^2+4ca}+b\sqrt{a^2+4bc}} + 2(a-b)(ab+bc+ca) \ge 0.$$

The equality holds for $a = b = c$, and also for $a = b$ and $c = 0$ (or any cyclic permutation).

Second Solution (by *Nguyen Van Quy*). Write the inequality as

$$\left(\sqrt{a^2+4bc}+\sqrt{b^2+4ca}+\sqrt{c^2+4ab}\right)^2 \ge a^2+b^2+c^2+14(ab+bc+ca),$$

$$\sqrt{a^2+4bc}+\sqrt{b^2+4ca}+\sqrt{c^2+4ab} \ge \sqrt{a^2+b^2+c^2+14(ab+bc+ca)}.$$

For $t = 2c$, the inequality (b) in Lemma below becomes

$$\sqrt{a^2+4bc}+\sqrt{b^2+4ca} \ge \sqrt{(a+b)^2+8(a+b)c}.$$

Thus, it suffices to show that

$$\sqrt{(a+b)^2+8(a+b)c}+\sqrt{c^2+4ab} \ge \sqrt{a^2+b^2+c^2+14(ab+bc+ca)}.$$

By squaring, this inequality becomes

$$\sqrt{[(a+b)^2+8(a+b)c](c^2+4ab)} \ge 4ab+3(a+b)c,$$

$$2(a+b)c^3 - 2(a+b)^2c^2 + 2ab(a+b)c + ab(a+b)^2 - 4a^2b^2 \ge 0,$$

$$2(a+b)(a-c)(b-c)c + ab(a-b)^2 \ge 0.$$

Lemma. *Let a, b and t be nonnegative numbers such that*

$$t \le 2(a+b).$$

Then,

(a) $$\sqrt{(a^2+2bt)(b^2+2at)} \ge ab+(a+b)t;$$

(b) $$\sqrt{a^2+2bt}+\sqrt{b^2+2at} \ge \sqrt{(a+b)^2+4(a+b)t}.$$

Proof. (a) By squaring, the inequality becomes

$$(a-b)^2t[2(a+b)-t] \ge 0,$$

which is clearly true.

(b) By squaring, this inequality turns into the inequality in (a).

□

P 2.23. *If a, b, c are nonnegative real numbers, then*

$$\sum \sqrt{(a^2+9bc)(b^2+9ca)} \ge 7(ab+ac+bc).$$

(Vasile Cîrtoaje, 2012)

***Solution** (by Nguyen Van Quy).* We see that the equality holds for $a=b$ and $c=0$. Without loss of generality, assume that

$$c = \min\{a, b, c\}.$$

For $t=4c$, the inequality (a) in Lemma from the preceding P 2.22 becomes

$$\sqrt{(a^2+8bc)(b^2+8ca)} \ge ab+4(a+b)c.$$

Thus, we have

$$\sqrt{(a^2+9bc)(b^2+9ca)} \ge ab+4(a+b)c$$

and

$$\sqrt{c^2+9ab}\left(\sqrt{a^2+9bc}+\sqrt{b^2+9ca}\right) \ge 3\sqrt{ab}\cdot 2\sqrt[4]{(a^2+9bc)(b^2+9ca)}$$

$$\ge 6\sqrt{ab}\cdot\sqrt{ab+4(a+b)c} = 3\sqrt{4a^2b^2+16abc(a+b)}$$

$$\ge 3\sqrt{4a^2b^2+4abc(a+b)+c^2(a+b)^2} = 3(2ab+bc+ca).$$

Therefore,

$$\sum \sqrt{(a^2+9bc)(b^2+9ca)} \ge (ab+4bc+4ca)+3(2ab+bc+ca)$$
$$= 7(ab+bc+ca).$$

The equality holds for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 2.24. *If a, b, c are nonnegative real numbers, then*

$$\sqrt{(a^2+b^2)(b^2+c^2)}+\sqrt{(b^2+c^2)(c^2+a^2)}+\sqrt{(c^2+a^2)(a^2+b^2)} \le (a+b+c)^2.$$

(Vasile Cîrtoaje, 2007)

***Solution**.* Without loss of generality, assume that

$$a = \min\{a, b, c\}.$$

Let us denote

$$y = \frac{a}{2}+b, \quad z = \frac{a}{2}+c.$$

Since

$$a^2+b^2 \le y^2, \quad b^2+c^2 \le y^2+z^2, \quad c^2+a^2 \le z^2,$$

it suffices to prove that

$$yz+(y+z)\sqrt{y^2+z^2} \le (y+z)^2.$$

This is true since

$$y^2+yz+z^2-(y+z)\sqrt{y^2+z^2} = \frac{y^2z^2}{y^2+yz+z^2+(y+z)\sqrt{y^2+z^2}} \ge 0.$$

The equality holds for $a=b=0$ (or any cyclic permutation).

□

P 2.25. *If a,b,c are nonnegative real numbers, then*

$$\sum \sqrt{(a^2+ab+b^2)(b^2+bc+c^2)} \ge (a+b+c)^2.$$

Solution. By the Cauchy-Schwarz inequality, we have

$$(a^2+ab+b^2)(a^2+ac+c^2) = \left[\left(a+\frac{b}{2}\right)^2+\frac{3b^2}{4}\right]\left[\left(a+\frac{c}{2}\right)^2+\frac{3c^2}{4}\right]$$

$$\ge \left(a+\frac{b}{2}\right)\left(a+\frac{c}{2}\right)+\frac{3bc}{4} = a^2+\frac{a(b+c)}{2}+bc.$$

Then,

$$\sum \sqrt{(a^2+ab+b^2)(a^2+ac+c^2)} \ge \sum \left[a^2+\frac{a(b+c)}{2}+bc\right] = (a+b+c)^2.$$

The equality holds for $a=b=c$, and also for $b=c=0$ (or any cyclic permutation).

□

P 2.26. *If a,b,c are nonnegative real numbers, then*

$$\sum \sqrt{(a^2+7ab+b^2)(b^2+7bc+c^2)} \ge 7(ab+ac+bc).$$

(Vasile Cîrtoaje, 2012)

First Solution. Without loss of generality, assume that

$$c = \min\{a, b, c\}.$$

We see that the equality holds for $a = b$ and $c = 0$. Since

$$\sqrt{(a^2 + 7ac + c^2)(b^2 + 7bc + c^2)} \geq (a + 2c)(b + 2c) \geq ab + 2c(a + b),$$

it suffices to show that

$$\sqrt{a^2 + 7ab + b^2}\left(\sqrt{a^2 + 7ac} + \sqrt{b^2 + 7bc}\right) \geq 6ab + 5c(a + b).$$

By Minkowski's inequality, we have

$$\sqrt{a^2 + 7ac} + \sqrt{b^2 + 7bc} \geq \sqrt{(a + b)^2 + 7c\left(\sqrt{a} + \sqrt{b}\right)^2}$$

$$\geq \sqrt{(a + b)^2 + 7c(a + b) + \frac{28abc}{a + b}}.$$

Therefore, it suffices to show that

$$(a^2 + 7ab + b^2)\left[(a + b)^2 + 7c(a + b) + \frac{28abc}{a + b}\right] \geq (6ab + 5bc + 5ca)^2.$$

Due to homogeneity, we may assume that $a + b = 1$. Let us denote $d = ab$, $d \leq \frac{1}{4}$. Since

$$c \leq \frac{2ab}{a + b} = 2d,$$

we need to show that $f(c) \geq 0$ for $0 \leq c \leq 2d \leq \frac{1}{2}$, where

$$f(c) = (1 + 5d)(1 + 7c + 28cd) - (6d + 5c)^2.$$

Since $f(c)$ is concave, it suffices to show that $f(0) \geq 0$ and $f(2d) \geq 0$. Indeed,

$$f(0) = 1 + 5d - 36d^2 = (1 - 4d)(1 + 9d) \geq 0$$

and

$$f(2d) = (1 + 5d)(1 + 14d + 56d^2) - 256d^2 \geq (1 + 4d)(1 + 14d + 56d^2) - 256d^2$$

$$= (1 - 4d)(1 + 22d - 56d^2) \geq d(1 - 4d)(22 - 56d) \geq 0.$$

The equality holds for $a = b$ and $c = 0$ (or any cyclic permutation).

Second Solution. We will use the inequality

$$\sqrt{x^2 + 7xy + y^2} \geq x + y + \frac{2xy}{x + y}, \quad x, y \geq 0,$$

which, by squaring, reduces to

$$xy(x-y)^2 \ge 0.$$

We have

$$\sum \sqrt{(a^2+7ab+b^2)(a^2+7ac+c^2)} \ge \sum \left(a+b+\frac{2ab}{a+b}\right)\left(a+c+\frac{2ac}{a+c}\right)$$

$$\ge \sum a^2 + 3\sum ab + \sum \frac{2a^2b}{a+b} + \sum \frac{2a^2c}{a+c} + \sum \frac{2abc}{a+b}.$$

Since

$$\sum \frac{2a^2b}{a+b} + \sum \frac{2a^2c}{a+c} = \sum \frac{2a^2b}{a+b} + \sum \frac{2b^2a}{b+a} = 2\sum ab$$

and

$$\sum \frac{2abc}{a+b} \ge \frac{18abc}{\sum(a+b)} = \frac{9abc}{a+b+c},$$

it suffices to show that

$$\sum a^2 + \frac{9abc}{a+b+c} \ge 2\sum ab,$$

which is just Schur's inequality of degree three.

□

P 2.27. *If a,b,c are nonnegative real numbers, then*

$$\sum \sqrt{\left(a^2+\frac{7}{9}ab+b^2\right)\left(b^2+\frac{7}{9}bc+c^2\right)} \le \frac{13}{12}(a+b+c)^2.$$

(Vasile Cîrtoaje, 2012)

Solution *(by Nguyen Van Quy)*. Without loss of generality, assume that

$$c = \min\{a,b,c\}.$$

It is easy to see that the equality holds for $a=b=1$ and $c=0$. By the AM-GM inequality, the following inequality holds for any $k>0$:

$$12\sqrt{a^2+\frac{7}{9}ab+b^2}\left(\sqrt{a^2+\frac{7}{9}ac+c^2}+\sqrt{b^2+\frac{7}{9}bc+c^2}\right) \le$$

$$\le \frac{36}{k}\left(a^2+\frac{7}{9}ab+b^2\right)+k\left(\sqrt{a^2+\frac{7}{9}ac+c^2}+\sqrt{b^2+\frac{7}{9}bc+c^2}\right)^2.$$

We can use this inequality to prove the original inequality only if

$$\frac{36}{k}\left(a^2+\frac{7}{9}ab+b^2\right)=k\left(\sqrt{a^2+\frac{7}{9}ac+c^2}+\sqrt{b^2+\frac{7}{9}bc+c^2}\right)^2$$

for $a=b=1$ and $c=0$. This condition if satisfied for $k=5$. Therefore, it suffices to show that

$$12\sqrt{\left(a^2+\frac{7}{9}ac+c^2\right)\left(b^2+\frac{7}{9}bc+c^2\right)}+\frac{36}{5}\left(a^2+\frac{7}{9}ab+b^2\right)+$$

$$+5\left(\sqrt{a^2+\frac{7}{9}ac+c^2}+\sqrt{b^2+\frac{7}{9}bc+c^2}\right)^2\le 13(a+b+c)^2.$$

which is equivalent to

$$22\sqrt{\left(a^2+\frac{7}{9}ac+c^2\right)\left(b^2+\frac{7}{9}bc+c^2\right)}\le\frac{4(a+b)^2+94ab}{5}+3c^2+\frac{199c(a+b)}{9}.$$

Since

$$\begin{aligned}2\sqrt{\left(a^2+\frac{7}{9}ac+c^2\right)\left(b^2+\frac{7}{9}bc+c^2\right)}&\le 2\sqrt{\left(a^2+\frac{16}{9}ac\right)\left(b^2+\frac{16}{9}bc\right)}\\&=2\sqrt{a\left(b+\frac{16}{9}c\right)\cdot b\left(a+\frac{16}{9}c\right)}\\&\le a\left(b+\frac{16}{9}c\right)+b\left(a+\frac{16}{9}c\right)\\&=2ab+\frac{16c(a+b)}{9},\end{aligned}$$

we only need to prove that

$$22\left[ab+\frac{8c(a+b)}{9}\right]\le\frac{4(a^2+b^2)+102ab}{5}+3c^2+\frac{199c(a+b)}{9}.$$

This reduces to the obvious inequality

$$\frac{4(a-b)^2}{5}+\frac{23c(a+b)}{9}+3c^2\ge 0.$$

Thus, the proof is completed. The equality holds for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 2.28. *If a, b, c are nonnegative real numbers, then*

$$\sum\sqrt{\left(a^2+\frac{1}{3}ab+b^2\right)\left(b^2+\frac{1}{3}bc+c^2\right)} \le \frac{61}{60}(a+b+c)^2.$$

(Vasile Cîrtoaje, 2012)

Solution *(by Nguyen Van Quy)*. Without loss of generality, assume that

$$c = \min\{a, b, c\}.$$

It is easy to see that the equality holds for $c = 0$ and $11(a^2+b^2) = 38ab$. By the AM-GM inequality, the following inequality holds for any $k > 0$:

$$60\sqrt{a^2+\frac{1}{3}ab+b^2}\left(\sqrt{a^2+\frac{1}{3}ac+c^2}+\sqrt{b^2+\frac{1}{3}bc+c^2}\right) \le$$

$$\le \frac{36}{k}\left(a^2+\frac{1}{3}ab+b^2\right)+25k\left(\sqrt{a^2+\frac{1}{3}ac+c^2}+\sqrt{b^2+\frac{1}{3}bc+c^2}\right)^2.$$

We can use this inequality to prove the original inequality only if the equality

$$\frac{36}{k}\left(a^2+\frac{1}{3}ab+b^2\right)=25k\left(\sqrt{a^2+\frac{1}{3}ac+c^2}+\sqrt{b^2+\frac{1}{3}bc+c^2}\right)^2$$

holds for $c = 0$ and $11(a^2+b^2) = 38ab$. This necessary condition if satisfied for $k = 1$. Therefore, it suffices to show that

$$60\sqrt{\left(a^2+\frac{1}{3}ab+b^2\right)\left(b^2+\frac{1}{3}bc+c^2\right)}+36\left(a^2+\frac{1}{3}ab+b^2\right)+$$

$$+25\left(\sqrt{a^2+\frac{1}{3}ac+c^2}+\sqrt{b^2+\frac{1}{3}bc+c^2}\right)^2 \le 61(a+b+c)^2,$$

which is equivalent to

$$10\sqrt{\left(a^2+\frac{1}{3}ac+c^2\right)\left(b^2+\frac{1}{3}bc+c^2\right)} \le 10ab+c^2+\frac{31c(a+b)}{3}.$$

Since

$$\begin{aligned}
2\sqrt{\left(a^2+\frac{1}{3}ac+c^2\right)\left(b^2+\frac{1}{3}bc+c^2\right)} &\le 2\sqrt{\left(a^2+\frac{4}{3}ac\right)\left(b^2+\frac{4}{3}bc\right)}\\
&= 2\sqrt{a\left(b+\frac{4}{3}c\right)\cdot b\left(a+\frac{4}{3}c\right)}\\
&\le a\left(b+\frac{4}{3}c\right)+b\left(a+\frac{4}{3}c\right)\\
&= 2ab+\frac{4c(a+b)}{3},
\end{aligned}$$

we only need to prove that

$$10\left[ab+\frac{2c(a+b)}{3}\right]\le 10ab+c^2+\frac{31c(a+b)}{3}.$$

This reduces to the obvious inequality

$$3c^2+11c(a+b)\ge 0.$$

Thus, the proof is completed. The equality holds for $11(a^2+b^2)=38ab$ and $c=0$ (or any cyclic permutation).

□

P 2.29. *If a,b,c are nonnegative real numbers, then*

$$\frac{a}{\sqrt{4b^2+bc+4c^2}}+\frac{b}{\sqrt{4c^2+ca+4a^2}}+\frac{c}{\sqrt{4a^2+ab+4b^2}}\ge 1.$$

(Pham Kim Hung, 2006)

Solution. By Hölder's inequality, we have

$$\left(\sum\frac{a}{\sqrt{4b^2+bc+4c^2}}\right)^2\ge\frac{\left(\sum a\right)^3}{\sum a(4b^2+bc+4c^2)}=\frac{\sum a^3+3\sum ab(a+b)+6abc}{4\sum ab(a+b)+3abc}.$$

Thus, it suffices to show that

$$\sum a^3+3abc\ge\sum ab(a+b),$$

which is Schur's inequality of degree three. The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 2.30. *If a,b,c are nonnegative real numbers, then*

$$\frac{a}{\sqrt{b^2+bc+c^2}}+\frac{b}{\sqrt{c^2+ca+a^2}}+\frac{c}{\sqrt{a^2+ab+b^2}}\ge\frac{a+b+c}{\sqrt{ab+bc+ca}}.$$

Solution. By Hölder's inequality, we have

$$\left(\sum\frac{a}{\sqrt{b^2+bc+c^2}}\right)^2\ge\frac{\left(\sum a\right)^3}{\sum a(b^2+bc+c^2)}=\frac{\left(\sum a\right)^2}{\sum ab},$$

from which the desired inequality follows. The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 2.31. *If a, b, c are nonnegative real numbers, then*

$$\frac{a}{\sqrt{a^2+2bc}}+\frac{b}{\sqrt{b^2+2ca}}+\frac{c}{\sqrt{c^2+2ab}} \le \frac{a+b+c}{\sqrt{ab+bc+ca}}.$$

(Ho Phu Thai, 2007)

Solution. Without loss of generality, assume that

$$a \ge b \ge c.$$

First Solution. Since

$$\frac{c}{\sqrt{c^2+2ab}} \le \frac{c}{\sqrt{ab+bc+ca}},$$

it suffices to show that

$$\frac{a}{\sqrt{a^2+2bc}}+\frac{b}{\sqrt{b^2+2ca}} \le \frac{a+b}{\sqrt{ab+bc+ca}},$$

which is equivalent to

$$\frac{a(\sqrt{a^2+2bc}-\sqrt{ab+bc+ca})}{\sqrt{a^2+2bc}} \ge \frac{b(\sqrt{ab+bc+ca}-\sqrt{b^2+2ca})}{\sqrt{b^2+2ca}}.$$

Since

$$\sqrt{a^2+2bc}-\sqrt{ab+bc+ca} \ge 0$$

and

$$\frac{a}{\sqrt{a^2+2bc}} \ge \frac{b}{\sqrt{b^2+2ca}},$$

it suffices to show that

$$\sqrt{a^2+2bc}-\sqrt{ab+bc+ca} \ge \sqrt{ab+bc+ca}-\sqrt{b^2+2ca},$$

which is equivalent to

$$\sqrt{a^2+2bc}+\sqrt{b^2+2ca} \ge 2\sqrt{ab+bc+ca}.$$

Using the AM-GM inequality, it suffices to show that

$$(a^2+2bc)(b^2+2ca) \ge (ab+bc+ca)^2,$$

which is equivalent to the obvious inequality

$$c(a-b)^2(2a+2b-c) \ge 0.$$

The equality holds for $a=b=c$, and also for $a=b$ and $c=0$ (or any cyclic permutation).

Second Solution. By the Cauchy-Schwarz inequality, we have

$$\left(\sum \frac{a}{\sqrt{a^2+2bc}}\right)^2 \le \left(\sum a\right)\left(\sum \frac{a}{a^2+2bc}\right).$$

Thus, it suffices to prove that

$$\sum \frac{a}{a^2+2bc} \le \frac{a+b+c}{ab+bc+ca}.$$

This is equivalent to

$$\sum a\left(\frac{1}{ab+bc+ca} - \frac{1}{a^2+2bc}\right) \ge 0,$$

$$\sum \frac{a(a-b)(a-c)}{a^2+2bc} \ge 0.$$

We have

$$\sum \frac{a(a-b)(a-c)}{a^2+2bc} \ge \frac{a(a-b)(a-c)}{a^2+2bc} + \frac{b(b-c)(b-a)}{b^2+2ca}$$
$$= \frac{c(a-b)^2[2a(a-c)+2b(b-c)+3ab]}{(a^2+2bc)(b^2+2ca)} \ge 0.$$

□

P 2.32. *If a,b,c are nonnegative real numbers, then*

$$a^3+b^3+c^3+3abc \ge a^2\sqrt{a^2+3bc}+b^2\sqrt{b^2+3ca}+c^2\sqrt{c^2+3ab}.$$

(Vo Quoc Ba Can, 2008)

Solution. For $a=0$, the inequality is an identity. Consider further that $a,b,c>0$, and write the inequality as follows:

$$\sum a^2(\sqrt{a^2+3bc}-a) \le 3abc,$$

$$\sum \frac{3a^2bc}{\sqrt{a^2+3bc}+a} \le 3abc,$$

$$\sum \frac{1}{\sqrt{1+3bc/a^2}+1} \le 1.$$

Using the notation

$$x = \frac{1}{\sqrt{1+3bc/a^2}+1}, \quad y = \frac{1}{\sqrt{1+3ca/b^2}+1}, \quad z = \frac{1}{\sqrt{1+3ab/c^2}+1},$$

implies

$$\frac{bc}{a^2}=\frac{1-2x}{3x^2},\quad \frac{ca}{b^2}=\frac{1-2y}{3y^2},\quad \frac{ab}{c^2}=\frac{1-2z}{3z^2},\quad 0<x,y,z<\frac{1}{2},$$

$$(1-2x)(1-2y)(1-2z)=27x^2y^2z^2.$$

We need to prove that

$$x+y+z\le 1$$

for $0<x,y,z<\frac{1}{2}$ such that $(1-2x)(1-2y)(1-2z)=27x^2y^2z^2$. To do it, we will use the contradiction method. Thus, assume that

$$x+y+z>1,\qquad 0<x,y,z<\frac{1}{2},$$

and show that

$$(1-2x)(1-2y)(1-2z)<27x^2y^2z^2.$$

We have

$$\begin{aligned}(1-2x)(1-2y)(1-2z)&<(x+y+z-2x)(x+y+z-2y)(x+y+z-2z)\\&<(y+z-x)(z+x-y)(x+y-z)(x+y+z)^3\\&\le 3(y+z-x)(z+x-y)(x+y-z)(x+y+z)(x^2+y^2+z^2)\\&=3(2x^2y^2+2y^2z^2+2z^2x^2-x^4-y^4-z^4)(x^2+y^2+z^2).\end{aligned}$$

Therefore, it suffices to show that

$$(2x^2y^2+2y^2z^2+2z^2x^2-x^4-y^4-z^4)(x^2+y^2+z^2)\le 9x^2y^2z^2,$$

which is equivalent to

$$x^6+y^6+z^5+3x^2y^2z^2\ge\sum y^2z^2(y^2+z^2).$$

Clearly, this is just Schur's inequality of degree three applied to x^2,y^2,z^2. So, the proof is completed. The equality holds for $a=b=c$, and also for $a=0$ or $b=0$ or $c=0$.

□

P 2.33. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a}{\sqrt{4a^2+5bc}}+\frac{b}{\sqrt{4b^2+5ca}}+\frac{c}{\sqrt{4c^2+5ab}}\le 1.$$

(Vasile Cîrtoaje, 2004)

First Solution *(by Vo Quoc Ba Can).* If one of a, b, c is zero, then the desired inequality is an equality. Consider next that $a, b, c > 0$ and denote

$$x = \frac{a}{\sqrt{4a^2+5bc}}, \quad y = \frac{b}{\sqrt{4b^2+5ca}}, \quad z = \frac{c}{\sqrt{4c^2+5ab}}, \quad x, y, z \in \left(0, \frac{1}{2}\right).$$

We have

$$\frac{bc}{a^2} = \frac{1-4x^2}{5x^2}, \quad \frac{ca}{b^2} = \frac{1-4y^2}{5y^2}, \quad \frac{ab}{c^2} = \frac{1-4z^2}{5z^2},$$

and

$$(1-4x^2)(1-4y^2)(1-4z^2) = 125x^2y^2z^2.$$

We use the contradiction method. For the sake of contradiction, assume that $x + y + z > 1$. Using the AM-GM inequality and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
x^2y^2z^2 &= \frac{1}{125}\prod(1-4x^2) < \frac{1}{125}\prod[(x+y+z)^2-4x^2] \\
&= \frac{1}{125}\prod(3x+y+z)\cdot\prod(y+z-x) \\
&\le \left(\frac{x+y+z}{3}\right)^3\prod(y+z-x) \\
&\le \frac{1}{9}(x^2+y^2+z^2)(x+y+z)\prod(y+z-x) \\
&= \frac{1}{9}(x^2+y^2+z^2)[2(x^2y^2+y^2z^2+z^2x^2)-x^4-y^4-z^4],
\end{aligned}$$

hence

$$9x^2y^2z^2 < (x^2+y^2+z^2)[2(x^2y^2+y^2z^2+z^2x^2)-x^4-y^4-z^4],$$

$$x^6+y^6+z^6+3x^2y^2z^2 < \sum x^2y^2(x^2+y^2).$$

The last inequality contradicts Schur's inequality

$$x^6+y^6+z^6+3x^2y^2z^2 \ge \sum x^2y^2(x^2+y^2).$$

Thus, the proof is completed. The equality holds for $a = b = c$, and also for $a = 0$ or $b = 0$ or $c = 0$.

Second Solution. Use the mixing variables method. In the nontrivial case when $a, b, c > 0$, setting $x = \dfrac{bc}{a^2}$, $y = \dfrac{ca}{b^2}$ and $z = \dfrac{ab}{c^2}$ (that implies $xyz = 1$), the desired inequality becomes $E(x, y, z) \le 1$, where

$$E(x,y,z) = \frac{1}{\sqrt{4+5x}} + \frac{1}{\sqrt{4+5y}} + \frac{1}{\sqrt{4+5z}}.$$

Without loss of generality, we may assume that

$$x \ge y \ge z, \qquad x \ge 1, \quad yz \le 1.$$

We will prove that

$$E(x,y,z) \le E(x,\sqrt{yz},\sqrt{yz}) \le 1.$$

The left inequality has the form

$$\frac{1}{\sqrt{4+5y}} + \frac{1}{\sqrt{4+5z}} \le \frac{1}{\sqrt{4+5\sqrt{yz}}}.$$

For the nontrivial case $y \ne z$, consider $y > z$ and denote

$$s = \frac{y+z}{2}, \quad p = \sqrt{yz},$$

$$q = \sqrt{(4+5y)(4+5z)}.$$

We have $s > p$, $p \le 1$ and

$$q = \sqrt{16+40s+25p^2} > \sqrt{16+40p+25p^2} = 4+5p.$$

By squaring, the desired inequality becomes in succession as follows:

$$\frac{1}{4+5y} + \frac{1}{4+5z} + \frac{2}{q} \le \frac{4}{4+5p},$$

$$\frac{1}{4+5y} + \frac{1}{4+5z} - \frac{2}{4+5p} \le \frac{2}{4+5p} - \frac{2}{q},$$

$$\frac{8+10s}{q^2} - \frac{2}{4+5p} \le \frac{2(q-4-5p)}{q(4+5p)},$$

$$\frac{(s-p)(5p-4)}{q^2(4+5p)} \le \frac{8(s-p)}{q(4+5p)(q+4+5p)},$$

$$\frac{5p-4}{q} \le \frac{8}{q+4+5p},$$

$$25p^2 - 16 \le (12-5p)q.$$

The last inequality is true since

$$\begin{aligned}(12-5p)q - 25p^2 + 16 &> (12-5p)(4+5p) - 25p^2 + 16 \\ &= 2(8-5p)(4+5p) > 0.\end{aligned}$$

In order to prove the right inequality, namely

$$\frac{1}{\sqrt{4+5x}} + \frac{2}{\sqrt{4+5\sqrt{yz}}} \le 1,$$

let us denote
$$\sqrt{4+5\sqrt{yz}}=3t, \qquad t\in(2/3,1].$$
Since
$$x=\frac{1}{yz}=\frac{25}{(9t^2-4)^2},$$
the inequality becomes
$$\frac{9t^2-4}{3\sqrt{36t^4-32t^2+21}}+\frac{2}{3t}\le 1,$$
$$(2-3t)\left(\sqrt{36t^4-32t^2+21}-3t^2-2t\right)\le 0.$$
Since $2-3t<0$, we still have to show that
$$\sqrt{36t^4-32t^2+21}\ge 3t^2+2t.$$
Indeed, we have
$$36t^4-32t^2+21-(3t^2+2t)^2=3(t-1)^2(9t^2+14t+7)\ge 0.$$

□

P 2.34. *Let a,b,c be nonnegative real numbers. Prove that*
$$a\sqrt{4a^2+5bc}+b\sqrt{4b^2+5ca}+c\sqrt{4c^2+5ab}\ge(a+b+c)^2.$$

(Vasile Cîrtoaje, 2004)

First Solution. Write the inequality as
$$\sum a\left(\sqrt{4a^2+5bc}-2a\right)\ge 2(ab+bc+ca)-a^2-b^2-c^2,$$
$$5abc\sum\frac{1}{\sqrt{4a^2+5bc}+2a}\ge 2(ab+bc+ca)-a^2-b^2-c^2.$$
Writing Schur's inequality
$$a^3+b^3+c^3+3abc\ge\sum ab(a^2+b^2)$$
in the form
$$\frac{9abc}{a+b+c}\ge 2(ab+bc+ca)-a^2-b^2-c^2,$$
it suffices to prove that
$$\sum\frac{5}{\sqrt{4a^2+5bc}+2a}\ge\frac{9}{a+b+c}.$$

Let $p = a+b+c$ and $q = ab+bc+ca$. By the AM-GM inequality, we have

$$\sqrt{4a^2+5bc} = \frac{2\sqrt{(16a^2+20bc)(3b+3c)^2}}{12(b+c)} \le \frac{(16a^2+20bc)+(3b+3c)^2}{12(b+c)}$$

$$\le \frac{16a^2+16bc+10(b+c)^2}{12(b+c)} = \frac{8a^2+5b^2+5c^2+18bc}{6(b+c)},$$

hence

$$\sum \frac{5}{\sqrt{4a^2+5bc}+2a} \ge \sum \frac{5}{\dfrac{8a^2+5b^2+5c^2+18bc}{6(b+c)}+2a}$$

$$= \sum \frac{30(b+c)}{8a^2+5b^2+5c^2+12ab+18bc+12ac} = \sum \frac{30(b+c)}{5p^2+2q+3a^2+6bc}.$$

Thus, it suffices to show that

$$\sum \frac{30(b+c)}{5p^2+2q+3a^2+6bc} \ge \frac{9}{p}.$$

By the Cauchy-Schwarz inequality, we get

$$\sum \frac{30(b+c)}{5p^2+2q+3a^2+6bc} \ge \frac{30\left[\sum(b+c)\right]^2}{\sum(b+c)(5p^2+2q+3a^2+6bc)}$$

$$= \frac{120p^2}{10p^3+4pq+9\sum bc(b+c)} = \frac{120p^2}{10p^3+13pq-27abc}.$$

Therefore, it is enough to show that

$$\frac{120p^2}{10p^3+13pq-27abc} \ge \frac{9}{p},$$

which is equivalent to

$$10p^3+81abc \ge 39pq.$$

From Schur's inequality $p^3+9abc \ge 4pq$ and the known inequality $pq \ge 9abc$, we have

$$10p^3+81abc-39pq = 10(p^3+9abc-4pq)+pq-9abc \ge 0.$$

This completes the proof. The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

Second Solution. By the Cauchy-Schwarz inequality, we have

$$\left(\sum a\sqrt{4a^2+5bc}\right)\left(\sum \frac{a}{\sqrt{4a^2+5bc}}\right) \ge (a+b+c)^2.$$

From this inequality and the inequality in P 2.33, namely

$$\sum \frac{a}{\sqrt{4a^2+5bc}} \le 1,$$

the desired inequality follows.

Remark. Using the same way as in the second solution, we can prove the following inequalities for $a,b,c>0$ satisfying $abc=1$:

$$a\sqrt{4a^2+5}+b\sqrt{4b^2+5}+c\sqrt{4c^2+5}\ge (a+b+c)^2;$$

$$\sqrt{4a^4+5}+\sqrt{4b^4+5}+\sqrt{4c^4+5}\ge (a+b+c)^2.$$

The first inequality is a consequence of the the Cauchy-Schwarz inequality

$$\left(\sum a\sqrt{4a^2+5}\right)\left(\sum \frac{a}{\sqrt{4a^2+5}}\right)\ge (a+b+c)^2$$

and the inequality

$$\sum \frac{a}{\sqrt{4a^2+5}}\le 1, \quad abc=1,$$

which follows from the inequality in P 2.33 by replacing bc/a^2, ca/b^2, ab/c^2 with $1/a^2$, $1/b^2$, $1/c^2$, respectively.

The second inequality is a consequence of the the Cauchy-Schwarz inequality

$$\left(\sum \sqrt{4a^4+5}\right)\left(\sum \frac{a^2}{\sqrt{4a^4+5}}\right)\ge (a+b+c)^2$$

and the inequality

$$\sum \frac{a^2}{\sqrt{4a^4+5}}\le 1, \quad abc=1,$$

which follows from the inequality in P 2.33 by replacing bc/a^2, ca/b^2, ab/c^2 with $1/a^4$, $1/b^4$, $1/c^4$, respectively.

□

P 2.35. *Let a,b,c be nonnegative real numbers. Prove that*

$$a\sqrt{a^2+3bc}+b\sqrt{b^2+3ca}+c\sqrt{c^2+3ab}\ge 2(ab+bc+ca).$$

(Vasile Cîrtoaje, 2005)

First Solution *(by Vo Quoc Ba Can).* Using the AM-GM inequality yields

$$\begin{aligned}\sum a\sqrt{a^2+3bc} &= \sum \frac{a(b+c)(a^2+3bc)}{\sqrt{(b+c)^2(a^2+3bc)}}\\ &\ge \sum \frac{2a(b+c)(a^2+3bc)}{(b+c)^2+(a^2+3bc)}.\end{aligned}$$

Thus, it suffices to prove that

$$\sum \frac{2a(b+c)(a^2+3bc)}{a^2+b^2+c^2+5bc} \ge \sum a(b+c).$$

We will use the SOS method. Write the inequality as follows:

$$\sum \frac{a(b+c)(a^2-b^2-c^2+bc)}{a^2+b^2+c^2+5bc} \ge 0,$$

$$\sum \frac{a^3(b+c)-a(b^3+c^3)}{a^2+b^2+c^2+5bc} \ge 0,$$

$$\sum \frac{ab(a^2-b^2)-ac(c^2-a^2)}{a^2+b^2+c^2+5bc} \ge 0,$$

$$\sum \frac{ab(a^2-b^2)}{a^2+b^2+c^2+5bc} - \sum \frac{ba(a^2-b^2)}{b^2+c^2+a^2+5ca} \ge 0,$$

$$\sum \frac{5abc(a+b)(a-b)^2}{(a^2+b^2+c^2+5bc)(a^2+b^2+c^2+5ac)} \ge 0.$$

The equality holds $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

Second Solution. Write the inequality as

$$\sum \left(a\sqrt{a^2+3bc}-a^2\right) \ge 2(ab+bc+ca)-a^2-b^2-c^2.$$

Due to homogeneity, we may assume that $a+b+c=3$. By the AM-GM inequality, we have

$$a\sqrt{a^2+3bc}-a^2 = \frac{3abc}{\sqrt{a^2+3bc}+a} = \frac{12abc}{2\sqrt{4(a^2+3bc)}+4a}$$
$$\ge \frac{12abc}{4+a^2+3bc+4a}.$$

Thus, it suffices to show that

$$12abc\sum \frac{1}{4+a^2+3bc+4a} \ge 2(ab+bc+ca)-a^2-b^2-c^2.$$

On the other hand, by Schur's inequality of degree three, we have

$$\frac{9abc}{a+b+c} \ge 2(ab+bc+ca)-a^2-b^2-c^2.$$

Therefore, it is enough to prove that

$$\sum \frac{1}{4+a^2+3bc+4a} \ge \frac{3}{4(a+b+c)}.$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\sum \frac{1}{4+a^2+3bc+4a} &\geq \frac{9}{\sum(4+a^2+3bc+4a)} = \frac{9}{24+\sum a^2+3\sum ab}\\ &= \frac{27}{8\left(\sum a\right)^2+3\sum a^2+9\sum ab}\\ &= \frac{9\sum a}{11\left(\sum a\right)^2+3\sum ab} \geq \frac{3}{4\sum a}.\end{aligned}$$

□

P 2.36. *Let a, b, c be nonnegative real numbers. Prove that*

$$a\sqrt{a^2+8bc}+b\sqrt{b^2+8ca}+c\sqrt{c^2+8ab} \leq (a+b+c)^2.$$

Solution. Multiplying by $a+b+c$, the inequality becomes

$$\sum a\sqrt{(a+b+c)^2(a^2+8bc)} \leq (a+b+c)^3.$$

Since

$$2\sqrt{(a+b+c)^2(a^2+8bc)} \leq (a+b+c)^2+(a^2+8bc),$$

it suffices to show that

$$\sum a[(a+b+c)^2+(a^2+8bc)] \leq 2(a+b+c)^3,$$

which can be written as

$$a^3+b^3+c^3+24abc \leq (a+b+c)^3.$$

This inequality is equivalent to

$$a(b-c)^2+b(c-a)^2+c(a-b)^2 \geq 0.$$

The equality holds for $a=b=c$, and also for $b=c=0$ (or any cyclic permutation).

□

P 2.37. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a^2+2bc}{\sqrt{b^2+bc+c^2}}+\frac{b^2+2ca}{\sqrt{c^2+ca+a^2}}+\frac{c^2+2ab}{\sqrt{a^2+ab+b^2}} \geq 3\sqrt{ab+bc+ca}.$$

(Michael Rozenberg and Marius Stanean, 2011)

Solution. By the AM-GM inequality, we have

$$\sum \frac{a^2+2bc}{\sqrt{b^2+bc+c^2}} = \sum \frac{2(a^2+2bc)\sqrt{ab+bc+ca}}{2\sqrt{(b^2+bc+c^2)(ab+bc+ca)}}$$
$$\geq \sqrt{ab+bc+ca}\sum \frac{2(a^2+2bc)}{(b^2+bc+c^2)+(ab+bc+ca)}$$
$$= \sqrt{ab+bc+ca}\sum \frac{2(a^2+2bc)}{(b+c)(a+b+c)}.$$

Thus, it suffices to show that

$$\frac{a^2+2bc}{b+c}+\frac{b^2+2ca}{c+a}+\frac{c^2+2ab}{a+b} \geq \frac{3}{2}(a+b+c).$$

This inequality is equivalent to

$$a^4+b^4+c^4+abc(a+b+c) \geq \frac{1}{2}\sum ab(a+b)^2.$$

We can prove this inequality by summing Schur's inequality of fourth degree

$$a^4+b^4+c^4+abc(a+b+c) \geq \sum ab(a^2+b^2)$$

and the obvious inequality

$$\sum ab(a^2+b^2) \geq \frac{1}{2}\sum ab(a+b)^2.$$

The equality holds for $a=b=c$.

□

P 2.38. *Let a,b,c be nonnegative real numbers, no two of which are zero. If $k \geq 1$, then*

$$\frac{a^{k+1}}{2a^2+bc}+\frac{b^{k+1}}{2b^2+ca}+\frac{c^{k+1}}{2c^2+ab} \leq \frac{a^k+b^k+c^k}{a+b+c}.$$

(Vasile Cîrtoaje and Vo Quoc Ba Can, 2011)

Solution. Write the inequality as follows:

$$\sum \left(\frac{a^k}{a+b+c} - \frac{a^{k+1}}{2a^2+bc} \right) \geq 0,$$

$$\sum \frac{a^k(a-b)(a-c)}{2a^2+bc} \geq 0.$$

Assume that $a \geq b \geq c$. Since $(c-a)(c-b) \geq 0$, it suffices to show that

$$\frac{a^k(a-b)(a-c)}{2a^2+bc}+\frac{b^k(b-a)(b-c)}{2b^2+ca} \geq 0.$$

This is true if

$$\frac{a^k(a-c)}{2a^2+bc}-\frac{b^k(b-c)}{2b^2+ca}\ge 0,$$

which is equivalent to

$$a^k(a-c)(2b^2+ca)\ge b^k(b-c)(2a^2+bc).$$

Since $a^k/b^k\ge a/b$, it remains to show that

$$a(a-c)(2b^2+ca)\ge b(b-c)(2a^2+bc),$$

which is equivalent to the obvious inequality

$$(a-b)c[a^2+3ab+b^2-c(a+b)]\ge 0.$$

The equality holds for $a=b=c$, and also for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 2.39. *If a,b,c are positive real numbers, then*

(a) $$\frac{a^2-bc}{\sqrt{3a^2+2bc}}+\frac{b^2-ca}{\sqrt{3b^2+2ca}}+\frac{c^2-ab}{\sqrt{3c^2+2ab}}\ge 0;$$

(b) $$\frac{a^2-bc}{\sqrt{8a^2+(b+c)^2}}+\frac{b^2-ca}{\sqrt{8b^2+(c+a)^2}}+\frac{c^2-ab}{\sqrt{8c^2+(a+b)^2}}\ge 0.$$

(Vasile Cîrtoaje, 2006)

***Solution*.** (a) Use the SOS technique. Let

$$A=\sqrt{3a^2+2bc},\quad B=\sqrt{3b^2+2ca},\quad C=\sqrt{3c^2+2ab}.$$

We have

$$\begin{aligned}2\sum\frac{a^2-bc}{A}&=\sum\frac{(a-b)(a+c)+(a-c)(a+b)}{A}\\&=\sum\frac{(a-b)(a+c)}{A}+\sum\frac{(b-a)(b+c)}{B}\\&=\sum(a-b)\left(\frac{a+c}{A}-\frac{b+c}{B}\right)\\&=\sum\frac{a-b}{AB}\cdot\frac{(a+c)^2B^2-(b+c)^2A^2}{(a+c)B+(b+c)A},\end{aligned}$$

hence

$$2\sum\frac{a^2-bc}{A}=\sum\frac{c(a-b)^2}{AB}\cdot\frac{2(a-b)^2+c(a+b+2c)}{(a+c)B+(b+c)A}\ge 0.$$

The equality holds for $a=b=c$.

(b) Let

$$A=\sqrt{8a^2+(b+c)^2},\quad B=\sqrt{8b^2+(c+a)^2},\quad C=\sqrt{8c^2+(a+b)^2b}.$$

As we have shown before,

$$2\sum\frac{a^2-bc}{A}=\sum\frac{a-b}{AB}\cdot\frac{(a+c)^2B^2-(b+c)^2A^2}{(a+c)B+(b+c)A},$$

hence

$$2\sum\frac{a^2-bc}{A}=\sum\frac{(a-b)^2}{AB}\cdot\frac{C_1}{(a+c)B+(b+c)A}\ge 0,$$

since

$$\begin{aligned}C_1&=[(a+c)+(b+c)][(a+c)^2+(b+c)^2]-8ac(b+c)-8bc(a+c)\\&\ge[(a+c)+(b+c)](4ac+4bc)-8ac(b+c)-8bc(a+c)\\&=4c(a-b)^2\ge 0.\end{aligned}$$

The equality holds for $a=b=c$.

□

P 2.40. *Let a,b,c be positive real numbers. If $0\le k\le 1+2\sqrt{2}$, then*

$$\frac{a^2-bc}{\sqrt{ka^2+b^2+c^2}}+\frac{b^2-ca}{\sqrt{kb^2+c^2+a^2}}+\frac{c^2-ab}{\sqrt{kc^2+a^2+b^2}}\ge 0.$$

Solution. Use the SOS method. Let

$$A=\sqrt{ka^2+b^2+c^2},\quad B=\sqrt{kb^2+c^2+a^2},\quad C=\sqrt{kc^2+a^2+b^2}.$$

As we have shown at the preceding problem,

$$2\sum\frac{a^2-bc}{A}=\sum\frac{a-b}{AB}\cdot\frac{(a+c)^2B^2-(b+c)^2A^2}{(a+c)B+(b+c)A};$$

therefore

$$2\sum\frac{a^2-bc}{A}=\sum\frac{(a-b)^2}{AB}\cdot\frac{C_1}{(a+c)B+(b+c)A},$$

where

$$C_1=(a^2+b^2+c^2)(a+b+2c)-(k-1)c(2ab+bc+ca).$$

It suffices to show that $C_1 \geq 0$. Putting $a+b=2x$, we have $a^2+b^2 \geq 2x^2$, $ab \leq x^2$, hence

$$\begin{aligned} C_1 &\geq (a^2+b^2+c^2)(a+b+2c) - 2\sqrt{2}\,c(2ab+bc+ca) \\ &\geq (2x^2+c^2)(2x+2c) - 2\sqrt{2}\,c(2x^2+2cx) \\ &= 2(x+c)(x\sqrt{2}-c)^2 \geq 0. \end{aligned}$$

The equality holds for $a=b=c$.

□

P 2.41. *If a,b,c are nonnegative real numbers, then*

$$(a^2-bc)\sqrt{b+c} + (b^2-ca)\sqrt{c+a} + (c^2-ab)\sqrt{a+b} \geq 0.$$

First Solution. Let us denote

$$x = \sqrt{\frac{b+c}{2}}, \quad y = \sqrt{\frac{c+a}{2}}, \quad z = \sqrt{\frac{a+b}{2}},$$

hence

$$a = y^2+z^2-x^2, \quad b = z^2+x^2-y^2, \quad c = x^2+y^2-z^2.$$

The inequality turns into

$$xy(x^3+y^3)+yz(y^3+z^3)+zx(z^3+x^3) \geq x^2y^2(x+y)+y^2z^2(y+z)+z^2x^2(z+x),$$

which is equivalent to the obvious inequality

$$xy(x+y)(x-y)^2 + yz(y+z)(y-z)^2 + zx(z+x)(z-x)^2 \geq 0.$$

The equality holds for $a=b=c$, and also for $b=c=0$ (or any cyclic permutation).

Second Solution. Use the SOS technique. Write the inequality as

$$A(a^2-bc)+B(b^2-ca)+C(c^2-ab) \geq 0,$$

where

$$A = \sqrt{b+c}, \quad B = \sqrt{c+a}, \quad C = \sqrt{a+b}.$$

We have

$$\begin{aligned} 2\sum A(a^2-bc) &= \sum A[(a-b)(a+c)+(a-c)(a+b)] \\ &= \sum A(a-b)(a+c) + \sum B(b-a)(b+c) \\ &= \sum (a-b)[A(a+c)-B(b+c)] \\ &= \sum (a-b)\cdot \frac{A^2(a+c)^2-B^2(b+c)^2}{A(a+c)+B(b+c)} \\ &= \sum \frac{(a-b)^2(a+c)(b+c)}{A(a+c)+B(b+c)} \geq 0. \end{aligned}$$

□

P 2.42. *If a,b,c are nonnegative real numbers, then*

$$(a^2-bc)\sqrt{a^2+4bc}+(b^2-ca)\sqrt{b^2+4ca}+(c^2-ab)\sqrt{c^2+4ab}\ge 0.$$

(Vasile Cîrtoaje, 2005)

Solution. If two of a,b,c are zero, then the inequality is clearly true. Otherwise, write the inequality as

$$AX+BY+CZ\ge 0,$$

where

$$A=\frac{\sqrt{a^2+4bc}}{b+c},\quad B=\frac{\sqrt{b^2+4ca}}{c+a},\quad C=\frac{\sqrt{c^2+4ab}}{a+b},$$

$$X=(a^2-bc)(b+c),\quad Y=(b^2-bc)(b+c),\quad Z=(c^2-ab)(a+b).$$

Without loss of generality, assume that

$$a\ge b\ge c.$$

We have

$$X\ge 0,\quad Z\le 0,\quad X+Y+Z=0.$$

In addition,

$$X-Y=ab(a-b)+2(a^2-b^2)c+(a-b)c^2\ge 0$$

and

$$A^2-B^2=\frac{a^4-b^4+2(a^3-c^3)c+(a^2-c^2)c^2+4abc(a-b)-4(a-b)c^3}{(b+c)^2(c+a)^2}$$

$$\ge\frac{4abc(a-b)-4(a-b)c^3}{(b+c)^2(c+a)^2}=\frac{4c(a-b)(ab-c^2)}{(b+c)^2(c+a)^2}\ge 0.$$

Since

$$\begin{aligned}2(AX+BY+CZ)&=(A-B)(X-Y)+(A+B)(X+Y)+2CZ\\&=(A-B)(X-Y)-(A+B-2C)Z,\end{aligned}$$

it suffices to show that

$$A+B-2C\ge 0.$$

This is true if $AB\ge C^2$. Using the Cauchy-Schwarz inequality gives

$$AB\ge\frac{ab+4c\sqrt{ab}}{(b+c)(c+a)}\ge\frac{ab+2c\sqrt{ab}+2c^2}{(b+c)(c+a)}.$$

Thus, it is enough to show that

$$(a+b)^2(ab+2c\sqrt{ab}+2c^2)\ge(b+c)(c+a)(c^2+4ab).$$

Write this inequality as

$$ab(a-b)^2+2c\sqrt{ab}(a+b)\left(\sqrt{a}-\sqrt{b}\right)^2+c^2[2(a+b)^2-5ab-c(a+b)-c^2]\geq 0.$$

It is true since

$$2(a+b)^2-5ab-c(a+b)-c^2=a(2a-b-c)+b(b-c)+b^2-c^2\geq 0.$$

The equality holds for $a=b=c$, and also for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 2.43. *If a,b,c are nonnegative real numbers, then*

$$\sqrt{\frac{a^3}{a^3+(b+c)^3}}+\sqrt{\frac{b^3}{b^3+(c+a)^3}}+\sqrt{\frac{c^3}{c^3+(a+b)^3}}\geq 1.$$

Solution. For $a=0$, the inequality reduces to the obvious inequality

$$\sqrt{b^3}+\sqrt{c^3}\geq\sqrt{b^3+c^3}.$$

For $a,b,c>0$, write the inequality as

$$\sum\sqrt{\frac{1}{1+\left(\frac{b+c}{a}\right)^3}}\geq 1.$$

For any $x\geq 0$, we have

$$\sqrt{1+x^3}=\sqrt{(1+x)(1-x+x^2)}\leq\frac{(1+x)+(1-x+x^2)}{2}=1+\frac{1}{2}x^2.$$

Therefore, we get

$$\sum\sqrt{\frac{1}{1+\left(\frac{b+c}{a}\right)^3}}\geq\sum\frac{1}{1+\frac{1}{2}\left(\frac{b+c}{a}\right)^2}$$

$$\geq\sum\frac{1}{1+\frac{b^2+c^2}{a^2}}=\sum\frac{a^2}{a^2+b^2+c^2}=1.$$

The equality holds for $a=b=c$, and also for $b=c=0$ (or any cyclic permutation).

□

P 2.44. *If a, b, c are positive real numbers, then*

$$\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \ge 1+\sqrt{1+\sqrt{(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}}.$$

(Vasile Cîrtoaje, 2002)

Solution. Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\left(\sum a\right)\left(\sum \frac{1}{a}\right) &= \sqrt{\left(\sum a^2+2\sum bc\right)\left(\sum \frac{1}{a^2}+2\sum \frac{1}{bc}\right)} \\ &\ge \sqrt{\left(\sum a^2\right)\left(\sum \frac{1}{a^2}\right)}+2\sqrt{\left(\sum bc\right)\left(\sum \frac{1}{bc}\right)} \\ &= \sqrt{\left(\sum a^2\right)\left(\sum \frac{1}{a^2}\right)}+2\sqrt{\left(\sum a\right)\left(\sum \frac{1}{a}\right)},\end{aligned}$$

hence

$$\left[\sqrt{\left(\sum a\right)\left(\sum \frac{1}{a}\right)}-1\right]^2 \ge 1+\sqrt{\left(\sum a^2\right)\left(\sum \frac{1}{a^2}\right)},$$

$$\sqrt{\left(\sum a\right)\left(\sum \frac{1}{a}\right)}-1 \ge \sqrt{1+\sqrt{\left(\sum a^2\right)\left(\sum \frac{1}{a^2}\right)}}.$$

The equality holds if and only if

$$\left(\sum a^2\right)\left(\sum \frac{1}{bc}\right)=\left(\sum \frac{1}{a^2}\right)\left(\sum bc\right),$$

which is equivalent to

$$(a^2-bc)(b^2-ca)(c^2-ab)=0.$$

Consequently, the equality occurs for $a^2=bc$ or $b^2=ca$ or $c^2=ab$.

□

P 2.45. *If a, b, c are positive real numbers, then*

$$5+\sqrt{2(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)-2} \ge (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

(Vasile Cîrtoaje, 2004)

Solution. Let us denote

$$x = \frac{a}{b} + \frac{b}{c} + \frac{c}{a}, \quad y = \frac{b}{a} + \frac{c}{b} + \frac{a}{c}.$$

From

$$\begin{aligned}
&2(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)-2 = \\
&= 2\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)+2\left(\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}\right)+4 \\
&= 2(x^2-2y)+2(y^2-2x)+4 \\
&= (x+y-2)^2+(x-y)^2 \\
&\ge (x+y-2)^2
\end{aligned}$$

and

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = x+y+3,$$

we get

$$\begin{aligned}
\sqrt{2(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)-2} &\ge x+y-2 \\
&= (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-5.
\end{aligned}$$

The equality occurs for $a = b$ or $b = c$ or $c = a$.

□

P 2.46. *If a, b, c are real numbers, then*

$$2(1+abc)+\sqrt{2(1+a^2)(1+b^2)(1+c^2)} \ge (1+a)(1+b)(1+c).$$

(Wolfgang Berndt, 2006)

First Solution. Denoting

$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc,$$

the inequality becomes

$$\sqrt{2(p^2+q^2+r^2-2pr-2q+1)} \ge p+q-r-1.$$

It suffices to show that

$$2(p^2+q^2+r^2-2pr-2q+1) \ge (p+q-r-1)^2,$$

which is equivalent to

$$p^2+q^2+r^2-2pq+2qr-2pr+2p-2q-2r+1\ge 0,$$

$$(p-q-r+1)^2\ge 0.$$

The equality holds for $p+1=q+r$ and $q\ge 1$. The last condition follows from $p+q-r-1\ge 0$.

Second Solution. Since

$$2(1+a^2)=(1+a)^2+(1-a)^2$$

and

$$(1+b^2)(1+c^2)=(b+c)^2+(bc-1)^2,$$

by the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\sqrt{2(1+a^2)(1+b^2)(1+c^2)}&\ge(1+a)(b+c)+(1-a)(bc-1)\\&=(1+a)(1+b)(1+c)-2(1+abc).\end{aligned}$$

□

P 2.47. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\sqrt{\frac{a^2+bc}{b^2+c^2}}+\sqrt{\frac{b^2+ca}{c^2+a^2}}+\sqrt{\frac{c^2+ab}{a^2+b^2}}\ge 2+\frac{1}{\sqrt{2}}.$$

(Vo Quoc Ba Can, 2006)

Solution. Assume that

$$a\ge b\ge c.$$

It suffices to show that

$$\sqrt{\frac{a^2+c^2}{b^2+c^2}}+\sqrt{\frac{b^2+c^2}{c^2+a^2}}+\sqrt{\frac{ab}{a^2+b^2}}\ge 2+\frac{1}{\sqrt{2}}.$$

Let us denote

$$x=\sqrt{\frac{a^2+c^2}{b^2+c^2}},\quad y=\sqrt{\frac{a}{b}}.$$

From

$$x^2-y^2=\frac{(a-b)(ab-c^2)}{b(b^2+c^2)}\ge 0,$$

it follows that

$$x\ge y\ge 1.$$

Also, from

$$x+\frac{1}{x}-\left(y+\frac{1}{y}\right)=\frac{(x-y)(xy-1)}{xy}\geq 0,$$

we have

$$\sqrt{\frac{a^2+c^2}{b^2+c^2}}+\sqrt{\frac{b^2+c^2}{c^2+a^2}}\geq\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}.$$

Therefore, it is enough to show that

$$\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}+\sqrt{\frac{ab}{a^2+b^2}}\geq 2+\frac{1}{\sqrt{2}},$$

which is equivalent to

$$\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}-2\geq\frac{1}{\sqrt{2}}-\sqrt{\frac{ab}{a^2+b^2}},$$

$$\frac{(\sqrt{a}-\sqrt{b})^2}{\sqrt{ab}}\geq\frac{(a-b)^2}{\sqrt{2(a^2+v^2}\left(\sqrt{a^2+b^2}+\sqrt{2ab}\right)}.$$

Since $2\sqrt{ab}\leq\sqrt{2(a^2+b^2)}$, it suffices to show that

$$2\geq\frac{(\sqrt{a}+\sqrt{b}\,)^2}{\sqrt{a^2+b^2}+\sqrt{2ab}}.$$

Indeed,

$$2\left(\sqrt{a^2+b^2}+\sqrt{2ab}\right)>\sqrt{2(a^2+b^2)}+2\sqrt{ab}\geq a+b+2\sqrt{ab}=\left(\sqrt{a}+\sqrt{b}\right)^2.$$

The equality holds for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 2.48. *If a,b,c are nonnegative real numbers, then*

$$\sqrt{a(2a+b+c)}+\sqrt{b(2b+c+a)}+\sqrt{c(2c+a+b)}\geq\sqrt{12(ab+bc+ca)}.$$

(Vasile Cîrtoaje, 2012)

Solution. By squaring, the inequality becomes

$$a^2+b^2+c^2+\sum\sqrt{bc(2b+c+a)(2c+a+b)}\geq 5(ab+bc+ca).$$

Using the Cauchy-Schwarz inequality yields

$$\sum\sqrt{bc(2b+c+a)(2c+a+b)}=\sum\sqrt{(2b^2+bc+ab)(2c^2+bc+ac)}$$

$$\geq \sum \left(2bc + bc + a\sqrt{bc}\right) = 3(ab+bc+ca) + \sum a\sqrt{bc}.$$

Therefore, it suffices to show that

$$a^2+b^2+c^2+\sum a\sqrt{bc} \geq 2(ab+bc+ca).$$

We can get this inequality by summing Schur's inequality

$$a^2+b^2+c^2+\sum a\sqrt{bc} \geq \sum \sqrt{ab}(a+b)$$

and

$$\sum \sqrt{ab}\,(a+b) \geq 2(ab+bc+ca).$$

The last inequality is equivalent to the obvious inequality

$$\sum \sqrt{ab}\,\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0.$$

The equality holds for $a = b = c$, and also for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 2.49. *Let a, b, c be nonnegative real numbers such that $a+b+c = 3$. Prove that*

$$a\sqrt{(4a+5b)(4a+5c)} + b\sqrt{(4b+5c)(4b+5a)} + c\sqrt{(4c+5a)(4c+5b)} \geq 27.$$

(Vasile Cîrtoaje, 2010)

Solution. Use the SOS technique. Assume that

$$a \geq b \geq c,$$

consider the nontrivial case $b > 0$, and write the inequality in the following equivalent homogeneous forms:

$$\sum a\sqrt{(4a+5b)(4a+5c)} \geq 3(a+b+c)^2,$$

$$2\left(\sum a^2 - \sum ab\right) \geq \sum a\left(\sqrt{4a+5b}-\sqrt{4a+5c}\right)^2,$$

$$\sum (b-c)^2 \geq \sum \frac{25a(b-c)^2}{\left(\sqrt{4a+5b}+\sqrt{4a+5c}\right)^2},$$

$$\sum (b-c)^2 S_a \geq 0,$$

where

$$S_a = 1 - \frac{25a}{\left(\sqrt{4a+5b}+\sqrt{4a+5c}\right)^2}.$$

Since

$$S_b = 1 - \frac{25b}{\left(\sqrt{4b+5c}+\sqrt{4b+5a}\right)^2} \ge 1 - \frac{25b}{\left(\sqrt{4b}+\sqrt{9b}\right)^2} = 0$$

and

$$S_c = 1 - \frac{25c}{\left(\sqrt{4c+5a}+\sqrt{4c+5b}\right)^2} \ge 1 - \frac{25c}{\left(\sqrt{9c}+\sqrt{9c}\right)^2} = 1 - \frac{25}{36} > 0,$$

we have

$$\sum (b-c)^2 S_a \ge (b-c)^2 S_a + (a-c)^2 S_b \ge (b-c)^2 S_a + \frac{a^2}{b^2}(b-c)^2 S_b$$

$$= \frac{a}{b}(b-c)^2\left(\frac{b}{a}S_a + \frac{a}{b}S_b\right).$$

Thus, it suffices to prove that

$$\frac{b}{a}S_a + \frac{a}{b}S_b \ge 0.$$

We have

$$S_a \ge 1 - \frac{25a}{\left(\sqrt{4a+5b}+\sqrt{4a}\right)^2} = 1 - \frac{a\left(\sqrt{4a+5b}-\sqrt{4a}\right)^2}{b^2},$$

$$S_b \ge 1 - \frac{25b}{\left(\sqrt{4b}+\sqrt{4b+5a}\right)^2} = 1 - \frac{b\left(\sqrt{4b+5a}-\sqrt{4b}\right)^2}{a^2},$$

hence

$$\frac{b}{a}S_a + \frac{a}{b}S_b \ge \frac{b}{a} - \frac{\left(\sqrt{4a+5b}-\sqrt{4a}\right)^2}{b} + \frac{a}{b} - \frac{\left(\sqrt{4b+5a}-\sqrt{4b}\right)^2}{a}$$

$$= 4\left(\sqrt{\frac{4a^2}{b^2}+\frac{5a}{b}} + \sqrt{\frac{4b^2}{a^2}+\frac{5b}{a}}\right) - 7\left(\frac{a}{b}+\frac{b}{a}\right) - 10$$

$$= 4\sqrt{4x^2+5x-8+2\sqrt{20x+41}} - 7x - 10,$$

where

$$x = \frac{a}{b} + \frac{b}{a} \ge 2.$$

To end the proof, we only need to show that $x \ge 2$ yields

$$4\sqrt{4x^2+5x-8+2\sqrt{20x+41}} \ge 7x + 10.$$

By squaring, this inequality becomes

$$15x^2 - 60x - 228 + 32\sqrt{20x+41} \ge 0.$$

Indeed,

$$15x^2-60x-228+32\sqrt{20x+41}\geq 15x^2-60x-228+32\sqrt{81}=15(x-2)^2\geq 0.$$

The equality holds for $a=b=c=1$, and also for $a=b=\dfrac{3}{2}$ and $c=0$ (or any cyclic permutation).

□

P 2.50. *Let a,b,c be nonnegative real numbers such that $ab+bc+ca=3$. Prove that*

$$a\sqrt{(a+3b)(a+3c)}+b\sqrt{(b+3c)(b+3a)}+c\sqrt{(c+3a)(c+3b)}\geq 12.$$

(Vasile Cîrtoaje, 2010)

Solution. Use the SOS method. Assume that $a\geq b\geq c\ \ (b>0)$, and write the inequality as

$$\sum a\sqrt{(a+3b)(a+3c)}\geq 4(ab+bc+ca),$$

$$2(\sum a^2-\sum ab)=\sum a\left(\sqrt{a+3b}-\sqrt{a+3c}\right)^2,$$

$$\sum(b-c)^2\geq\sum\frac{9a(b-c)^2}{\left(\sqrt{a+3b}+\sqrt{a+3c}\right)^2},$$

$$\sum(b-c)^2S_a\geq 0,$$

where

$$S_a=1-\frac{9a}{\left(\sqrt{a+3b}+\sqrt{a+3c}\right)^2}.$$

Since

$$S_b=1-\frac{9b}{\left(\sqrt{b+3c}+\sqrt{b+3a}\right)^2}\geq 1-\frac{9b}{\left(\sqrt{b}+\sqrt{4b}\right)^2}=0$$

and

$$S_c=1-\frac{9c}{\left(\sqrt{c+3a}+\sqrt{c+3b}\right)^2}\geq 1-\frac{9c}{\left(\sqrt{4c}+\sqrt{4c}\right)^2}=1-\frac{9}{16}>0,$$

we have

$$\sum(b-c)^2S_a\geq(b-c)^2S_a+(a-c)^2S_b\geq(b-c)^2S_a+\frac{a^2}{b^2}(b-c)^2S_b$$

$$=\frac{a}{b}(b-c)^2\left(\frac{b}{a}S_a+\frac{a}{b}S_b\right).$$

Thus, it suffices to prove that

$$\frac{b}{a}S_a + \frac{a}{b}S_b \ge 0.$$

We have

$$S_a \ge 1 - \frac{9a}{\left(\sqrt{a+3b}+\sqrt{a}\right)^2} = 1 - \frac{a\left(\sqrt{a+3b}-\sqrt{a}\right)^2}{b^2},$$

$$S_b \ge 1 - \frac{9b}{\left(\sqrt{b}+\sqrt{b+3a}\right)^2} = 1 - \frac{b\left(\sqrt{b+3a}-\sqrt{b}\right)^2}{a^2},$$

hence

$$\begin{aligned}\frac{b}{a}S_a + \frac{a}{b}S_b &\ge \frac{b}{a} - \frac{\left(\sqrt{a+3b}-\sqrt{a}\right)^2}{b} + \frac{a}{b} - \frac{\left(\sqrt{b+3a}-\sqrt{b}\right)^2}{a} \\ &= 2\left(\sqrt{\frac{a^2}{b^2}+\frac{3a}{b}} + \sqrt{\frac{b^2}{a^2}+\frac{3b}{a}}\right) - \left(\frac{a}{b}+\frac{b}{a}\right) - 6 \\ &= 2\sqrt{x^2+3x-2+2\sqrt{3x+10}} - x - 6,\end{aligned}$$

where

$$x = \frac{a}{b} + \frac{b}{a} \ge 2.$$

To end the proof, it remains to show that

$$2\sqrt{x^2+35x-2+2\sqrt{3x+10}} \ge x+6$$

for $x \ge 2$. By squaring, this inequality becomes

$$3x^2 - 44 + 8\sqrt{3x+10} \ge 0.$$

Indeed,

$$3x^2 - 44 + 8\sqrt{3x+10} \ge 12 - 44 + 32 = 0.$$

The equality holds for $a = b = c = 1$, and also for $a = b = \sqrt{3}$ and $c = 0$ (or any cyclic permutation).

□

P 2.51. *Let a, b, c be nonnegative real numbers such that $a^2+b^2+c^2 = 3$. Prove that*

$$\sqrt{2+7ab} + \sqrt{2+7bc} + \sqrt{2+7ca} \ge 3\sqrt{3(ab+bc+ca)}.$$

(Vasile Cîrtoaje, 2010)

Solution. Use the SOS method. Consider $a \ge b \ge c$. Since the inequality is trivial for $b = c = 0$, we may assume that $b > 0$. By squaring, the desired inequality becomes

$$6 + 2\sum \sqrt{(2+7ab)(2+7ac)} \ge 20(ab+bc+ca),$$

$$6(a^2+b^2+c^2-ab-bc-ca) \ge \sum \left(\sqrt{2+7ab} - \sqrt{2+7ac}\right)^2,$$

$$3\sum(b-c)^2 \ge \sum \frac{49a^2(b-c)^2}{\left(\sqrt{2+7ab}+\sqrt{2+7ac}\right)^2},$$

$$\sum (b-c)^2 S_a \ge 0,$$

where

$$S_a = 1 - \frac{49a^2}{\left(\sqrt{6+21ab}+\sqrt{6+21ac}\right)^2},$$

$$S_b = 1 - \frac{49b^2}{\left(\sqrt{6+21ab}+\sqrt{6+21bc}\right)^2},$$

$$S_c = 1 - \frac{49c^2}{\left(\sqrt{6+21ac}+\sqrt{6+21bc}\right)^2}.$$

Since $6 \ge 2(a^2+b^2) \ge 4ab$, we have

$$S_a \ge 1 - \frac{49a^2}{\left(\sqrt{4ab+21ab}+\sqrt{6}\right)^2} \ge 1 - \frac{49a^2}{\left(5\sqrt{ab}+2\sqrt{ab}\right)^2} = 1 - \frac{a}{b},$$

$$S_b \ge 1 - \frac{49b^2}{\left(\sqrt{4ab+21ab}+\sqrt{6}\right)^2} \ge 1 - \frac{49b^2}{\left(5\sqrt{ab}+2\sqrt{ab}\right)^2} = 1 - \frac{b}{a},$$

$$S_c \ge 1 - \frac{49c^2}{\left(\sqrt{4ab+21ac}+\sqrt{4ab+21bc}\right)^2} \ge 1 - \frac{49c^2}{(5c+5c)^2} = 1 - \frac{49}{100} > 0.$$

Therefore,

$$\begin{aligned}\sum (b-c)^2 S_a &\ge (b-c)^2 S_a + (c-a)^2 S_b \\ &\ge (b-c)^2\left(1-\frac{a}{b}\right) + (c-a)^2\left(1-\frac{b}{a}\right) \\ &= \frac{(a-b)^2(ab-c^2)}{ab} \ge 0.\end{aligned}$$

The equality holds for $a = b = c = 1$, and also for $a = b = \sqrt{3}$ and $c = 0$ (or any cyclic permutation).

□

P 2.52. *Let a,b,c be nonnegative real numbers such that $a^2+b^2+c^2=3$. Prove that*

$$\frac{a}{2a^2+1}+\frac{b}{2b^2+1}+\frac{c}{2c^2+1}\le 1.$$

(Vasile Cîrtoaje, 2006)

Solution. Assume that $a\le b\le c$ and denote

$$f(a,b,c)=\frac{a}{2a^2+1}+\frac{b}{2b^2+1}+\frac{c}{2c^2+1}.$$

We will show that

$$f(a,b,c)\le f(s,s,c)\le 1,$$

where

$$s=\sqrt{\frac{a^2+b^2}{2}},\qquad s\le 1.$$

The inequality $f(a,b,c)\le f(s,s,c)$ follows from P 2.1. The inequality $f(s,s,c)\le 1$ is equivalent to

$$\frac{2s}{2s^2+1}+\frac{c}{2c^2+1}\le 1,$$

where

$$2s^2+c^2=3,\qquad 0\le s\le 1\le c.$$

Write the requested inequality as follows:

$$\frac{1}{3}-\frac{c}{2c^2+1}\ge\frac{2s}{2s^2+1}-\frac{2}{3},$$

$$\frac{(c-1)(2c-1)}{2c^2+1}\ge\frac{2(1-s)(2s-1)}{2s^2+1},$$

$$\frac{(c^2-1)(2c-1)}{(c+1)(2c^2+1)}\ge\frac{2(1-s^2)(2s-1)}{(1+s)(2s^2+1)}.$$

Since

$$c^2-1=2(1-s^2)\ge 0,$$

we only need to show that

$$\frac{2c-1}{(c+1)(2c^2+1)}\ge\frac{2s-1}{(s+1)(2s^2+1)},$$

which is equivalent to $(c-s)A\ge 0$, where

$$A=2(s+c)^2+2(s+c)+3-6sc-4sc(s+c).$$

Substituting

$$x=\frac{s+c}{2},\qquad y=\sqrt{sc},\qquad x\ge y,$$

we need to show that $A(x,y) \geq 0$, where

$$A(x,y) = 8x^2 + 4x + 3 - 6y^2 - 8xy^2.$$

From

$$3 = 2s^2 + c^2 \geq 2\sqrt{2}sc = 2\sqrt{2}y^2,$$

we get

$$y \leq \sqrt{\frac{3}{2\sqrt{2}}}.$$

We will show that

$$A(x,y) \geq A(y,y) \geq 0.$$

We have

$$A(x,y) - A(y,y) = 4(x-y)(2x+2y+1-2y^2) \geq 4(x-y)[2y(2-y)+1] \geq 0$$

and

$$A(y,y) = 3 + 4y + 2y^2 - 8y^3.$$

From

$$A(y,y) = y^3\left(\frac{3}{y^3} + \frac{4}{y^2} + \frac{2}{y} - 8\right),$$

it follows that it suffices to show that $A(y,y) \geq 0$ for $y = \sqrt{\frac{3}{2\sqrt{2}}}$. Indeed, we have

$$A(y,y) = 3 + 2y^2 - 4(2y^2-1)y = 3 + \frac{3}{\sqrt{2}} - 4\left(\frac{3}{\sqrt{2}} - 1\right)y$$

$$= \frac{3\sqrt{2} + 3 - 4(3-\sqrt{2})y}{\sqrt{2}} = \frac{B}{\sqrt{2}[3\sqrt{2} + 3 + 4(3-\sqrt{2})y]},$$

where

$$B = (3\sqrt{2}+3)^2 - 16(3-\sqrt{2})^2y^2 = 9(\sqrt{2}+1)^2 - 12\sqrt{2}(3-\sqrt{2})^2$$

$$= 57(3-2\sqrt{2}) > 0.$$

The equality holds for $a = b = c = 1$.

Remark. The following more general statement is also valid.

- *If a,b,c,d are nonnegative real numbers such that $a^2+b^2+c^2+d^2 = 4$, then*

$$\frac{a}{2a^2+1} + \frac{b}{2b^2+1} + \frac{c}{2c^2+1} + \frac{d}{2d^2+1} \leq \frac{4}{3}.$$

□

P 2.53. *Let a, b, c be nonnegative real numbers such that $ab + bc + ca = 3$. Prove that*

(a)
$$\sum \sqrt{a(b+c)(a^2+bc)} \ge 6;$$

(b)
$$\sum a(b+c)\sqrt{a^2+2bc} \ge 6\sqrt{3};$$

(c)
$$\sum a(b+c)\sqrt{(a+2b)(a+2c)} \ge 18.$$

(Vasile Cîrtoaje, 2010)

Solution. Assume that

$$a \ge b \ge c, \quad b > 0.$$

(a) Write the inequality in the homogeneous form

$$\sum \sqrt{a(b+c)(a^2+bc)} \ge 2(ab+bc+ca).$$

First Solution. Write the homogeneous inequality as

$$\sum \sqrt{a(b+c)}\left[\sqrt{a^2+bc} - \sqrt{a(b+c)}\right] \ge 0,$$

$$\sum \frac{(a-b)(a-c)\sqrt{a(b+c)}}{\sqrt{a^2+bc}+\sqrt{a(b+c)}} \ge 0.$$

Since $(c-a)(c-b) \ge 0$, it suffices to show that

$$\frac{(a-b)(a-c)\sqrt{a(b+c)}}{\sqrt{a^2+bc}+\sqrt{a(b+c)}} + \frac{(b-c)(b-a)\sqrt{b(c+a)}}{\sqrt{b^2+ca}+\sqrt{b(c+a)}} \ge 0.$$

This is true if

$$\frac{(a-c)\sqrt{a(b+c)}}{\sqrt{a^2+bc}+\sqrt{a(b+c)}} \ge \frac{(b-c)\sqrt{b(c+a)}}{\sqrt{b^2+ca}+\sqrt{b(c+a)}}.$$

Since

$$\sqrt{a(b+c)} \ge \sqrt{b(c+a)},$$

it suffices to show that

$$\frac{a-c}{\sqrt{a^2+bc}+\sqrt{a(b+c)}} \ge \frac{b-c}{\sqrt{b^2+ca}+\sqrt{b(c+a)}}.$$

Moreover, since

$$\sqrt{a^2+bc} \ge \sqrt{a(b+c)}, \quad \sqrt{b^2+ca} \le \sqrt{b(c+a)},$$

it is enough to show that

$$\frac{a-c}{\sqrt{a^2+bc}} \ge \frac{b-c}{\sqrt{b^2+ca}}.$$

Indeed, we have

$$(a-c)^2(b^2+ca)-(b-c)^2(a^2+bc)=(a-b)(a^2+b^2+c^2+3ab-3bc-3ca)\geq 0,$$

because

$$a^2+b^2+c^2+3ab-3bc-3ca=(a^2-bc)+(b-c)^2+3a(b-c)\geq 0.$$

The equality holds for $a=b=c=1$, and also for $a=b=\sqrt{3}$ and $c=0$ (or any cyclic permutation).

Second Solution. By squaring, the homogeneous inequality becomes

$$\sum a(b+c)(a^2+bc)+2\sum\sqrt{bc(a+b)(a+c)(b^2+ca)(c^2+ab)}\geq 4(ab+bc+ca)^2.$$

Since

$$(b^2+ca)(c^2+ab)-bc(a+b)(a+c)=a(b+c)(b-c)^2\geq 0,$$

it suffices to show that

$$\sum a(b+c)(a^2+bc)+2\sum bc(a+b)(a+c)\geq 4(ab+bc+ca)^2,$$

which is equivalent to

$$\sum bc(b-c)^2\geq 0.$$

(b) Write the inequality as

$$\sum a(b+c)\sqrt{a^2+2bc}\geq 2(ab+bc+ca)\sqrt{ab+bc+ca},$$

$$\sum a(b+c)\left[\sqrt{a^2+2bc}-\sqrt{ab+bc+ca}\right]\geq 0,$$

$$\sum\frac{a(b+c)(a-b)(a-c)}{\sqrt{a^2+2bc}+\sqrt{ab+bc+ca}}\geq 0.$$

Since $(c-a)(c-b)\geq 0$, it suffices to show that

$$\frac{a(b+c)(a-b)(a-c)}{\sqrt{a^2+2bc}+\sqrt{ab+bc+ca}}+\frac{b(c+a)(b-c)(b-a)}{\sqrt{b^2+2ca}+\sqrt{ab+bc+ca}}\geq 0.$$

This is true if

$$\frac{a(b+c)(a-c)}{\sqrt{a^2+2bc}+\sqrt{ab+bc+ca}}\geq\frac{b(c+a)(b-c)}{\sqrt{b^2+2ca}+\sqrt{ab+bc+ca}}.$$

Since

$$(b+c)(a-c)\geq(c+a)(b-c),$$

it suffices to show that

$$\frac{a}{\sqrt{a^2+2bc}+\sqrt{ab+bc+ca}}\geq\frac{b}{\sqrt{b^2+2ca}+\sqrt{ab+bc+ca}}.$$

Moreover, since

$$\sqrt{a^2+2bc} \geq \sqrt{ab+bc+ca}, \qquad \sqrt{b^2+2ca} \leq \sqrt{ab+bc+ca},$$

it is enough to show that

$$\frac{a}{\sqrt{a^2+2bc}} \geq \frac{b}{\sqrt{b^2+2ca}}.$$

Indeed, we have

$$a^2(b^2+2ca)-b^2(a^2+2bc) = 2c(a^3-b^3) \geq 0.$$

The equality holds for $a=b=c=1$, and also for $a=b=\sqrt{3}$ and $c=0$ (or any cyclic permutation).

(c) Write the inequality as follows:

$$\sum a(b+c)\sqrt{(a+2b)(a+2c)} \geq 2(ab+bc+ca)\sqrt{3(ab+bc+ca)},$$

$$\sum a(b+c)\left[\sqrt{(a+2b)(a+2c)}-\sqrt{3(ab+bc+ca)}\right] \geq 0,$$

$$\sum \frac{a(b+c)(a-b)(a-c)}{\sqrt{(a+2b)(a+2c)}+\sqrt{3(ab+bc+ca)}} \geq 0.$$

Since $(c-a)(c-b) \geq 0$, it suffices to show that

$$\frac{a(b+c)(a-c)}{\sqrt{(a+2b)(a+2c)}+\sqrt{3(ab+bc+ca)}} \geq \frac{b(c+a)(b-c)}{\sqrt{(b+2c)(b+2a)}+\sqrt{3(ab+bc+ca)}}.$$

Since

$$(b+c)(a-c) \geq (c+a)(b-c),$$

it suffices to show that

$$\frac{a}{\sqrt{(a+2b)(a+2c)}+\sqrt{3(ab+bc+ca)}} \geq \frac{b}{\sqrt{(b+2c)(b+2a)}+\sqrt{3(ab+bc+ca)}}.$$

Moreover, since

$$\sqrt{(a+2b)(a+2c)} \geq \sqrt{3(ab+bc+ca)}, \quad \sqrt{(b+2c)(b+2a)} \leq \sqrt{3(ab+bc+ca)},$$

it is enough to show that

$$\frac{a}{\sqrt{(a+2b)(a+2c)}} \geq \frac{b}{\sqrt{(b+2c)(b+2a)}}.$$

This is true if

$$\frac{\sqrt{a}}{\sqrt{(a+2b)(a+2c)}} \geq \frac{\sqrt{b}}{\sqrt{(b+2c)(b+2a)}}.$$

Indeed, we have

$$a(b+2c)(b+2a)-b(a+2b)(a+2c)=(a-b)(ab+4bc+4ca)\geq 0.$$

The equality holds for $a=b=c=1$, and also for $a=b=\sqrt{3}$ and $c=0$ (or any cyclic permutation).

□

P 2.54. *Let a,b,c be nonnegative real numbers such that $ab+bc+ca=3$. Prove that*

$$a\sqrt{bc+3}+b\sqrt{ca+3}+c\sqrt{ab+3}\geq 6.$$

(Vasile Cîrtoaje, 2010)

Solution. Use the SOS method. Denote

$$A=\sqrt{ab+2bc+ca},\quad B=\sqrt{bc+2ca+ab},\quad C=\sqrt{ca+2ab+bc},$$

and write the inequality as follows:

$$\sum aA\geq 2(ab+bc+ca),$$

$$\sum a\,(A-b-c)\geq 0,$$

$$\sum \frac{a(ab+ac-b^2-c^2)}{A+b+c}\geq 0,$$

$$\sum \frac{ab(a-b)+ac(a-c)}{A+b+c}\geq 0,$$

$$\sum \frac{ab(a-b)}{A+b+c}+\sum \frac{ba(b-a)}{B+c+a}\geq 0,$$

$$\sum ab(a-b)\left(\frac{1}{A+b+c}-\frac{1}{B+c+a}\right)\geq 0,$$

$$\sum ab(a+b+C)(a-b)(a-b+B-A)\geq 0,$$

$$\sum ab(a+b+C)(a-b)^2\left(1+\frac{c}{A+B}\right)\geq 0.$$

The equality holds for $a=b=c=1$, and for $a=0$ and $b=c=\sqrt{3}$ (or any cyclic permutation).

□

P 2.55. *Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that*

(a) $$\sum(b+c)\sqrt{b^2+c^2+7bc} \ge 18;$$

(b) $$\sum(b+c)\sqrt{b^2+c^2+10bc} \le 12\sqrt{3}.$$

(Vasile Cîrtoaje, 2010)

Solution. Use the SOS technique.

(a) Write the inequality in the equivalent homogeneous forms

$$\sum(b+c)\sqrt{b^2+c^2+7bc} \ge 2(a+b+c)^2,$$

$$\sum\left[(b+c)\sqrt{b^2+c^2+7bc}-b^2-c^2-4bc\right] \ge 0,$$

$$\sum \frac{(b+c)^2(b^2+c^2+7bc)-(b^2+c^2+4bc)^2}{(b+c)\sqrt{b^2+c^2+7bc}+b^2+c^2+4bc} \ge 0,$$

$$\sum \frac{bc(b-c)^2}{(b+c)\sqrt{b^2+c^2+7bc}+b^2+c^2+4bc} \ge 0.$$

The equality holds for $a=b=c=1$, for $a=0$ and $b=c=\frac{3}{2}$ (or any cyclic permutation), and for $a=3$ and $b=c=0$ (or any cyclic permutation).

(b) Write the inequality as follows:

$$\sum(b+c)\sqrt{3(b^2+c^2+10bc)} \le 4(a+b+c)^2,$$

$$\sum\left[2b^2+2c^2+8bc-(b+c)\sqrt{3(b^2+c^2+10bc)}\right] \ge 0,$$

$$\sum \frac{4(b^2+c^2+4bc)^2-3(b+c)^2(b^2+c^2+10bc)}{2b^2+2c^2+8bc+(b+c)\sqrt{3(b^2+c^2+10bc)}} \ge 0,$$

$$\sum \frac{(b-c)^4}{2b^2+2c^2+8bc+(b+c)\sqrt{3(b^2+c^2+10bc)}} \ge 0.$$

The equality holds for $a=b=c=1$.

□

P 2.56. *Let a, b, c be nonnegative real numbers such then $a+b+c=2$. Prove that*

$$\sqrt{a+4bc}+\sqrt{b+4ca}+\sqrt{c+4ab} \ge 4\sqrt{ab+bc+ca}.$$

(Vasile Cîrtoaje, 2012)

Solution. Without loss of generality, assume that

$$c = \min\{a, b, c\}.$$

Using Minkowski's inequality gives

$$\sqrt{a+4bc} + \sqrt{b+4ca} \ge \sqrt{\left(\sqrt{a}+\sqrt{b}\right)^2 + 4c\left(\sqrt{a}+\sqrt{b}\right)^2} = \left(\sqrt{a}+\sqrt{b}\right)\sqrt{1+4c}.$$

Therefore, it suffices to show that

$$\left(\sqrt{a}+\sqrt{b}\right)\sqrt{1+4c} \ge 4\sqrt{ab+bc+ca} - \sqrt{c+4ab}.$$

By squaring, this inequality becomes

$$\left(a+b+2\sqrt{ab}\right)(1+4c) + 8\sqrt{(ab+bc+ca)(c+4ab)} \ge 16(ab+bc+ca) + c + 4ab.$$

According to Lemma below, it suffices to show that

$$\left(a+b+2\sqrt{ab}\right)(1+4c) + 8(2ab+bc+ca) \ge 16(ab+bc+ca) + c + 4ab,$$

which is equivalent to

$$a+b-c+2\sqrt{ab}+8c\sqrt{ab} \ge 4(ab+bc+ca).$$

Write this inequality in the homogeneous form

$$(a+b+c)\left(a+b-c+2\sqrt{ab}\right) + 16c\sqrt{ab} \ge 8(ab+bc+ca).$$

Due to homogeneity, we may assume that $a+b=1$. Let us denote

$$d = \sqrt{ab}, \quad 0 \le d \le \frac{1}{2}.$$

We need to show that $f(c) \ge 0$ for $0 \le c \le d$, where

$$\begin{aligned} f(c) &= (1+c)(1-c+2d) + 16cd - 8d^2 - 8c \\ &= (1-2d)(1+4d) + 2(9d-4)c - c^2. \end{aligned}$$

Since $f(c)$ is concave, it suffices to show that $f(0) \ge 0$ and $f(d) \ge 0$. Indeed,

$$f(0) = (1-2d)(1+4d) \ge 0,$$

$$f(d) = (3d-1)^2 \ge 0.$$

Thus, the proof is completed. The equality holds for $a = b = 1$ and $c = 0$ (or any cyclic permutation).

Lemma (*by Nguyen Van Quy*). *Let a,b,c be nonnegative real numbers such then*

$$c=\min\{a,b,c\},\quad a+b+c=2.$$

Then,

$$\sqrt{(ab+bc+ca)(c+4ab)}\ge 2ab+bc+ca.$$

Proof. By squaring, the inequality becomes

$$c[ab+bc+ca-c(a+b)^2]\ge 0.$$

We need to show that

$$(a+b+c)(ab+bc+ca)-2c(a+b)^2\ge 0.$$

We have

$$\begin{aligned}(a+b+c)(ab+bc+ca)-2c(a+b)^2&\ge(a+b)(b+c)(c+a)-2c(a+b)^2\\&=(a+b)(a-c)(b-c)\ge 0.\end{aligned}$$

□

P 2.57. *If a,b,c are nonnegative real numbers, then*

$$\sqrt{a^2+b^2+7ab}+\sqrt{b^2+c^2+7bc}+\sqrt{c^2+a^2+7ca}\ge 5\sqrt{ab+bc+ca}.$$

(Vasile Cîrtoaje, 2012)

Solution *(by Nguyen Van Quy).* Assume that

$$c=\min\{a,b,c\}.$$

Using Minkowski's inequality yields

$$\sqrt{b^2+c^2+7bc}+\sqrt{a^2+c^2+7ca}\ge\sqrt{(a+b)^2+4c^2+7c\left(\sqrt{a}+\sqrt{b}\right)^2}.$$

Therefore, it suffices to show that

$$\sqrt{(a+b)^2+4c^2+7c\left(\sqrt{a}+\sqrt{b}\right)^2}\ge 5\sqrt{ab+bc+ca}-\sqrt{a^2+b^2+7ab}.$$

By squaring, this inequality becomes

$$2c^2+7c\sqrt{ab}+5\sqrt{(a^2+b^2+7ab)(ab+bc+ca)}\ge 15ab+9c(a+b).$$

Due to homogeneity, we may assume that $a+b=1$, which implies $c \le \frac{1}{2}$. Let us denote $x = ab$. We need to show that $f(x) \ge 0$ for $c^2 \le x \le \frac{1}{4}$, where

$$f(x) = 2c^2 + 7c\sqrt{x} + 5\sqrt{(1+5x)(c+x)} - 15x - 9c.$$

Since

$$f''(x) = \frac{-7c}{4\sqrt{x^3}} - \frac{5(5c-1)^2}{4\sqrt{[5x^2+(5c+1)x+c]^3}} < 0$$

$f(c)$ is concave. Thus, it suffices to show that $f(c^2) \ge 0$ and $f\left(\frac{1}{4}\right) \ge 0$.
Write the inequality $f(c^2) \ge 0$ as

$$5\sqrt{(1+5c^2)(c+c^2)} \ge 6c^2 + 9c.$$

By squaring, this inequality turns into

$$c(89c^3 + 17c^2 - 56c + 25) \ge 0,$$

which is true since

$$89c^3 + 17c^2 - 56c + 25 \ge 12c^2 - 56c + 25 = (1-2c)(25-6c) \ge 0.$$

Write the inequality $f\left(\frac{1}{4}\right) \ge 0$ as

$$8c^2 - 22c + 15\left(\sqrt{4c+1} - 1\right) \ge 0.$$

Making the substitution $t = \sqrt{4c+1}$, $t \ge 1$, the inequality becomes

$$(t-1)(t^3 + t^2 - 12t + 18) \ge 0.$$

It is true since

$$t^3 + t^2 - 12t + 18 \ge 2t^2 - 12t + 18 = 2(t-3)^2 \ge 0.$$

Thus, the proof is completed. The equality holds for $a = b$ and $c = 0$ (or any cyclic permutation).

□

P 2.58. *If a, b, c are nonnegative real numbers, then*

$$\sqrt{a^2+b^2+5ab} + \sqrt{b^2+c^2+5bc} + \sqrt{c^2+a^2+5ca} \ge \sqrt{21(ab+bc+ca)}.$$

(Nguyen Van Quy, 2012)

Solution. Without loss of generality, assume that $c = \min\{a, b, c\}$. Using Minkowski's inequality, we have

$$\sqrt{(a+c)^2 + 3ac} + \sqrt{(b+c)^2 + 3bc} \ge \sqrt{(a+b+2c)^2 + 3c\left(\sqrt{a} + \sqrt{b}\right)^2}.$$

Therefore, it suffices to show that

$$\sqrt{(a+b+2c)^2 + 3c\left(\sqrt{a} + \sqrt{b}\right)^2} \ge \sqrt{21(ab+bc+ca)} - \sqrt{a^2+b^2+5ab}.$$

By squaring, this inequality becomes

$$2c^2 + 3c\sqrt{ab} + \sqrt{21(a^2+b^2+5ab)(ab+bc+ca)} \ge 12ab + 7c(a+b).$$

Due to homogeneity, we may assume that $a + b = 1$. Let us denote $x = ab$. We need to show that $f(x) \ge 0$ for $c^2 \le x \le \dfrac{1}{4}$, where

$$f(x) = 2c^2 + 3c\sqrt{x} + \sqrt{21(1+3x)(c+x)} - 12x - 7c.$$

Since

$$f''(x) = \frac{-3c}{4\sqrt{x^3}} - \frac{\sqrt{21}(3c-1)^2}{4\sqrt{[3x^2+(3c+1)x+c]^3}} < 0$$

$f(c)$ is concave. Thus, it suffices to show that $f(c^2) \ge 0$ and $f\left(\dfrac{1}{4}\right) \ge 0$.
Write the inequality $f(c^2) \ge 0$ as

$$\sqrt{21(1+3c^2)(c+c^2)} \ge 7(c+c^2).$$

By squaring, this inequality turns into

$$c(c+1)(1-2c)(3-c) \ge 0,$$

which is clearly true.
Write the inequality $f\left(\dfrac{1}{4}\right) \ge 0$ as

$$8c^2 - 22c + 7\sqrt{3(4c+1)} - 12 \ge 0.$$

Using the substitution $3t^2 = 4c + 1$, $t \ge \dfrac{1}{\sqrt{3}}$, the inequality becomes

$$(t-1)^2(3t^2 + 6t - 4) \ge 0.$$

This is true since

$$3t^2 + 6t - 4 \ge 1 + 2\sqrt{3} - 4 > 0.$$

Thus, the proof is completed. The equality holds for $a = b = c$.

□

P 2.59. *Let a,b,c be nonnegative real numbers such that $ab+bc+ca=3$. Prove that*

$$a\sqrt{a^2+5}+b\sqrt{b^2+5}+c\sqrt{c^2+5}\ge\sqrt{\frac{2}{3}}\,(a+b+c)^2.$$

(Vasile Cîrtoaje, 2010)

Solution. Write the inequality in the homogeneous form

$$\sum a\sqrt{3a^2+5(ab+bc+ca)}\ge\sqrt{2}\,(a+b+c)^2.$$

Due to homogeneity, we may assume that

$$ab+bc+ca=1.$$

By squaring, the inequality becomes

$$\sum a^4+2\sum bc\sqrt{(3b^2+5)(3c^2+5)}\ge 12\sum a^2b^2+19abc\sum a+3\sum ab(a^2+b^2).$$

Applying Lemma below for $x=3b^2$, $y=3c^2$ and $d=5$, we have

$$2\sqrt{(3b^2+5)(3c^2+5)}\ge 3(b^2+c^2)+10-\frac{9}{20}(b^2-c^2)^2,$$

hence

$$2bc\sqrt{(3b^2+5)(3c^2+5)}\ge 3bc(b^2+c^2)+10bc-\frac{9}{20}bc(b^2-c^2)^2,$$

$$2\sum bc\sqrt{(3b^2+5)(3c^2+5)}\ge 3\sum bc(b^2+c^2)+10\left(\sum bc\right)^2-\frac{9}{20}\sum bc(b^2-c^2)^2$$

$$=10\sum a^2b^2+20abc\sum a+3\sum ab(a^2+b^2)-\frac{9}{20}\sum bc(b^2-c^2)^2.$$

Therefore, it suffices to show that

$$\sum a^4+10\sum a^2b^2+20abc\sum a+3\sum ab(a^2+b^2)-\frac{9}{20}\sum bc(b^2-c^2)^2\ge$$

$$\ge 12\sum a^2b^2+19abc\sum a+3\sum ab(a^2+b^2),$$

which is equivalent to

$$\sum a^4-2\sum a^2b^2+abc\sum a-\frac{9}{20}\sum bc(b^2-c^2)^2\ge 0.$$

To prove this inequality, we use the SOS method. Since

$$2\left(\sum a^4-2\sum a^2b^2+abc\sum a\right)=2\left(\sum a^4-\sum a^2b^2\right)-2\left(\sum a^2b^2-abc\sum a\right)$$

$$= \sum (b^2 - c^2)^2 - \sum a^2(b-c)^2,$$

we can write the inequality as

$$\sum (b-c)^2 S_a \geq 0,$$

where

$$S_a = (b+c)^2 - a^2 - \frac{9}{10} bc(b+c)^2.$$

In addition, since

$$S_a \geq (b+c)^2 - a^2 - bc(b+c)^2 = (b+c)^2 - a^2 - \frac{bc(b+c)^2}{ab+bc+ca},$$

$$= \frac{a(b+c)^3 - a^2(ab+bc+ca)}{ab+bc+ca},$$

it is enough to show that

$$\sum (b-c)^2 E_a \geq 0,$$

where

$$E_a = a(b+c)^3 - a^2(ab+bc+ca).$$

Assume that

$$a \geq b \geq c, \quad b > 0$$

Since

$$E_b = b(c+a)^3 - b^2(ab+bc+ca) \geq b(c+a)^3 - b^2(c+a)(c+b)$$

$$\geq b(c+a)^3 - b^2(c+a)^2 = b(c+a)^2(c+a-b) \geq 0,$$

$$E_c = c(a+b)^3 - c^2(ab+bc+ca) \geq c(a+b)^3 - c^2(a+b)(b+c)$$

$$\geq c(a+b)^3 - c^2(a+b)^2 = c(a+b)^2(a+b-c) \geq 0$$

and

$$\frac{E_a}{a^2} + \frac{E_b}{b^2} = \frac{(b+c)^3}{a} + \frac{(c+a)^3}{b} - 2(ab+bc+ca)$$

$$\geq \frac{b^3 + 2b^2c}{a} + \frac{a^3 + 2a^2c}{b} - 2(ab+bc+ca)$$

$$= \frac{(a^2-b^2)^2 + 2c(a+b)(a-b)^2}{ab} \geq 0,$$

we get

$$\sum (b-c)^2 E_a \geq (b-c)^2 E_a + (a-c)^2 E_b \geq a^2(b-c)^2 \left(\frac{E_a}{a^2} + \frac{E_b}{b^2} \right) \geq 0.$$

The equality holds for $a = b = c = 1$, and also for $a = b = \sqrt{3}$ and $c = 0$ (or any cyclic permutation).

Lemma. *If $x \ge 0$, $y \ge 0$ and $d > 0$, then*

$$2\sqrt{(x+d)(y+d)} \ge x+y+2d-\frac{1}{4d}(x-y)^2.$$

Proof. We have

$$2\sqrt{(x+d)(y+d)}-2d=\frac{2xy+2d(x+y)}{\sqrt{(x+d)(y+d)}+d} \ge \frac{2xy+2d(x+y)}{\frac{(x+d)+(y+d)}{2}+d}$$

$$=\frac{4xy+4d(x+y)}{x+y+4d}=x+y-\frac{(x-y)^2}{x+y+4d} \ge x+y-\frac{(x-y)^2}{4d}.$$

□

P 2.60. *Let a,b,c be nonnegative real numbers such that $a^2+b^2+c^2=1$. Prove that*

$$a\sqrt{2+3bc}+b\sqrt{2+3ca}+c\sqrt{2+3ab} \ge (a+b+c)^2.$$

(Vasile Cîrtoaje, 2010)

Solution. Write the inequality as

$$\sum a\sqrt{2+3bc} \ge 1+2q,$$

where $q=ab+bc+ca$. By squaring, the inequality becomes

$$1+3abc\sum a+2\sum bc\sqrt{(2+3ab)(2+3ac)} \ge 4q+4q^2.$$

Applying Lemma from the preceding P 2.59 for $x=3ab$, $y=3ac$ and $d=2$, we have

$$2\sqrt{(2+3ab)(2+3ac)} \ge 3a(b+c)+4-\frac{9}{8}a^2(b-c)^2,$$

hence

$$2bc\sqrt{(2+3ab)(2+3ac)} \ge 3abc(b+c)+4-\frac{9}{8}a^2bc(b-c)^2,$$

$$2\sum bc\sqrt{(2+3ab)(2+3ac)} \ge 6abc\sum a+4q-\frac{9}{8}abc\sum a(b-c)^2.$$

Therefore, it suffices to show that

$$1+3abc\sum a+6abc\sum a+4q-\frac{9}{8}abc\sum a(b-c)^2 \ge 4q+4q^2,$$

which is equivalent to

$$1+9abc\sum a-4q^2 \ge \frac{9}{8}abc\sum a(b-c)^2.$$

Since

$$a^4+b^4+c^4=1-2(a^2b^2+b^2c^2+c^2a^2)=1-2q^2+4abc\sum a,$$

from Schur's inequality of fourth degree

$$a^4+b^4+c^4+2abc\sum a \ge \left(\sum a^2\right)\left(\sum ab\right),$$

we get

$$1 \ge 2q^2+q-6abc\sum a.$$

Thus, it is enough to prove that

$$\left(2q^2+q-6abc\sum a\right)+9abc\sum a-4q^2 \ge \frac{9}{8}abc\sum a(b-c)^2;$$

that is,

$$8\left(q-2q^2+3abc\sum a\right) \ge 9abc\sum a(b-c)^2.$$

Since

$$q-2q^2+3abc\sum a=\left(\sum a^2\right)\left(\sum ab\right)-2\left(\sum ab\right)^2+3abc\sum a$$

$$=\sum bc(b^2+c^2)-2\sum b^2c^2=\sum bc(b-c)^2,$$

we need to show that

$$\sum bc(8-9a^2)(b-c)^2 \ge 0.$$

Since

$$8-9a^2=8(b^2+c^2)-a^2 \ge b^2+c^2-a^2,$$

it suffices to prove the homogeneous inequality

$$\sum bc(b^2+c^2-a^2)(b-c)^2 \ge 0.$$

Assume that $a \ge b \ge c$. It is enough to show that

$$bc(b^2+c^2-a^2)(b-c)^2+ca(c^2+a^2-b^2)(c-a)^2 \ge 0.$$

This is true if

$$a(c^2+a^2-b^2)(a-c)^2 \ge b(a^2-b^2-c^2)(b-c)^2.$$

For the nontrivial case $a^2-b^2-c^2 \ge 0$, this inequality follows from

$$a \ge b, \quad c^2+a^2-b^2 \ge a^2-b^2-c^2, \quad (a-c)^2 \ge (b-c)^2.$$

The equality holds for $a=b=c=\dfrac{1}{\sqrt{3}}$, and for $a=0$ and $b=c=\dfrac{1}{\sqrt{2}}$ (or any cyclic permutation).

□

P 2.61. *Let a,b,c be nonnegative real numbers such that $a+b+c=3$. Prove that*

(a)
$$a\sqrt{\frac{2a+bc}{3}}+b\sqrt{\frac{2b+ca}{3}}+c\sqrt{\frac{2c+ab}{3}}\ge 3;$$

(b)
$$a\sqrt{\frac{a(1+b+c)}{3}}+b\sqrt{\frac{b(1+c+a)}{3}}+c\sqrt{\frac{c(1+a+b)}{3}}\ge 3.$$

(Vasile Cîrtoaje, 2010)

Solution. (a) If two of a,b,c are zero, then the inequality is trivial. Otherwise, by Hölder's inequality, we have

$$\left(\sum a\sqrt{\frac{2a+bc}{3}}\right)^2\ge \frac{\left(\sum a\right)^3}{\sum\frac{3a}{2a+bc}}=\frac{9}{\sum\frac{a}{2a+bc}}.$$

Therefore, it suffices to show that

$$\sum\frac{a}{2a+bc}\le 1.$$

Since

$$\frac{2a}{2a+bc}=1-\frac{bc}{2a+bc},$$

we can write this inequality as

$$\sum\frac{bc}{2a+bc}\ge 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum\frac{bc}{2a+bc}\ge\frac{\left(\sum bc\right)^2}{\sum bc(2a+bc)}=\frac{\left(\sum bc\right)^2}{2abc\sum a+\sum b^2c^2}=1.$$

The equality holds for $a=b=c=1$, and for $a=0$ and $b=c=\frac{3}{2}$ (or any cyclic permutation).

(b) Write the inequality in the homogeneous form

$$\sum a\sqrt{a(a+4b+4c)}\ge (a+b+c)^2.$$

By squaring, the inequality becomes

$$\sum bc\sqrt{bc(b+4c+4a)(c+4a+4b)}\ge 3\sum b^2c^2+6abc\sum a.$$

Applying the Cauchy-Schwarz inequality, we have

$$\sqrt{(b+4c+4a)(c+4a+4b)}=\sqrt{(4a+b+c+3c)(4a+b+c+3b)}$$

$$\geq 4a+b+c+3\sqrt{bc},$$

hence

$$bc\sqrt{bc(b+4c+4a)(c+4a+4b)} \geq (4a+b+c)bc\sqrt{bc}+3b^2c^2,$$

$$\sum bc\sqrt{bc(b+4c+4a)(c+4a+4b)} \geq \sum(4a+b+c)bc\sqrt{bc}+3\sum b^2c^2.$$

Thus, it is enough to show that

$$\sum(4a+b+c)bc\sqrt{bc} \geq 6abc\sum a.$$

Replacing a,b,c by a^2,b^2,c^2, respectively, this inequality becomes

$$\sum(4a^2+b^2+c^2)b^3c^3 \geq 6a^2b^2c^2\sum a^2,$$

$$\left(\sum a^2\right)\left(\sum b^3c^3\right)+3a^2b^2c^2\sum bc \geq 6a^2b^2c^2\sum a^2,$$

$$\left(\sum a^2\right)\left(\sum a^3b^3-3a^2b^2c^2\right) \geq 3a^2b^2c^2\left(\sum a^2-\sum ab\right).$$

Use next the SOS method. Since

$$\sum a^3b^3-3a^2b^2c^2=\left(\sum ab\right)\left(\sum a^2b^2-abc\sum a\right)=\frac{1}{2}\left(\sum ab\right)\sum a^2(b-c)^2,$$

and

$$\sum a^2-\sum ab=\frac{1}{2}\sum(b-c)^2,$$

we can write the inequality as

$$\sum(b-c)^2S_a \geq 0,$$

where

$$S_a=a^2\left(\sum a^2\right)\left(\sum ab\right)-3a^2b^2c^2.$$

Assume that $a \geq b \geq c$. Since $S_a \geq S_b \geq 0$ and

$$\begin{aligned}S_b+S_c&=(b^2+c^2)\left(\sum a^2\right)\left(\sum ab\right)-6a^2b^2c^2\\&\geq 2bc\left(\sum a^2\right)\left(\sum ab\right)-6a^2b^2c^2\\&\geq 2bca^2\left(\sum ab\right)-6a^2b^2c^2=2a^2bc(ab+ac-2bc)\geq 0,\end{aligned}$$

we get

$$\sum(b-c)^2S_a \geq (c-a)^2S_b+(a-b)^2S_c \geq (a-b)^2(S_b+S_c) \geq 0.$$

The equality holds for $a=b=c=1$, and also for $a=3$ and $b=c=0$ (or any cyclic permutation).

□

P 2.62. *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

$$\sqrt{8(a^2+bc)+9}+\sqrt{8(b^2+ca)+9}+\sqrt{8(c^2+ab)+9}\ge 15.$$

(Vasile Cîrtoaje, 2013)

Solution. Use the SOS technique. Let $q=ab+bc+ca$ and

$$A=(3a-b-c)^2+8q,\quad B=(3b-c-a)^2+8q,\quad C=(3c-a-b)^2+8q.$$

Since

$$\begin{aligned}8(a^2+bc)+9&=8(a^2+q)+9-8a(b+c)=8(a^2+q)+9-8a(3-a)\\&=(4a-3)^2+8q=(3a-b-c)^2+8q=A,\end{aligned}$$

we can rewrite the inequality as follows:

$$\sum\sqrt{A}\ge 15,$$

$$\sum[\sqrt{A}-(3a+b+c)]\ge 0,$$

$$\sum\frac{2bc-ca-ab}{\sqrt{A}+3a+b+c}\ge 0,$$

$$\sum\left[\frac{b(c-a)}{\sqrt{A}+3a+b+c}+\frac{c(b-a)}{\sqrt{A}+3a+b+c}\right]\ge 0,$$

$$\sum\frac{c(a-b)}{\sqrt{B}+3b+c+a}+\sum\frac{c(b-a)}{\sqrt{A}+3a+b+c}\ge 0,$$

$$\sum c(a-b)(\sqrt{C}+3c+a+b)[\sqrt{A}-\sqrt{B}+2(a-b)]\ge 0,$$

$$\sum c(a-b)^2(\sqrt{C}+3c+a+b)\left[\frac{4(a+b-c)}{\sqrt{A}+\sqrt{B}}+1\right]\ge 0.$$

Without loss of generality, assume that $a\ge b\ge c$. Since $a+b-c>0$, it suffices to show that

$$\begin{aligned}&b(a-c)^2(\sqrt{B}+3b+c+a)\left[\frac{4(c+a-b)}{\sqrt{A}+\sqrt{C}}+1\right]\ge\\&a(b-c)^2(\sqrt{A}+3a+b+c)\left[\frac{4(a-b-c)}{\sqrt{B}+\sqrt{C}}-1\right].\end{aligned}$$

This inequality follows from the inequalities

$$b^2(a-c)^2\ge a^2(b-c)^2,$$

$$a(\sqrt{B}+3b+c+a)\ge b(\sqrt{A}+3a+b+c),$$

$$\frac{4(c+a-b)}{\sqrt{A}+\sqrt{C}}+1 \ge \frac{4(a-b-c)}{\sqrt{B}+\sqrt{C}}-1.$$

Write the second inequality as

$$\frac{a^2B-b^2A}{a\sqrt{B}+b\sqrt{A}}+(a-b)(a+b+c) \ge 0.$$

Since

$$\begin{aligned} a^2B-b^2A &= (a-b)(a+b+c)(a^2+b^2-6ab+bc+ca)+8q(a^2-b^2) \\ &\ge (a-b)(a+b+c)(a^2+b^2-6ab) \ge -4ab(a-b)(a+b+c), \end{aligned}$$

it suffices to show that

$$\frac{-4ab}{a\sqrt{B}+b\sqrt{A}}+1 \ge 0.$$

Indeed, from $\sqrt{A} > \sqrt{8q} \ge 2\sqrt{ab}$ and $\sqrt{B} \ge \sqrt{8q} \ge 2\sqrt{ab}$, we get

$$a\sqrt{B}+b\sqrt{A}-4ab > 2(a+b)\sqrt{ab}-4ab = 2\sqrt{ab}(a+b-2\sqrt{ab}) \ge 0.$$

The third inequality holds if

$$1 \ge \frac{2(a-b-c)}{\sqrt{B}+\sqrt{C}}.$$

It suffices to show that $\sqrt{B} \ge a$ and $\sqrt{C} \ge a$. We have

$$B-a^2 = 8q-2a(3b-c)+(3b-c)^2 \ge 8ab-2a(3b-c) = 2a(b+c) \ge 0$$

and

$$C-a^2 = 8q-2a(3c-b)+(3c-b)^2 \ge 8ab-2a(3c-b) = 2a(5b-3c) \ge 0.$$

The equality holds for $a=b=c=1$, and also for $a=3$ and $b=c=0$ (or any cyclic permutation). $\square$

P 2.63. *Let a,b,c be nonnegative real numbers such that $a+b+c=3$. If $k \ge \frac{9}{8}$, then*

$$\sqrt{a^2+bc+k}+\sqrt{b^2+ca+k}+\sqrt{c^2+ab+k} \ge 3\sqrt{2+k}.$$

Solution. We will show that

$$\sum \sqrt{8(a^2+bc+k)} \ge \sum \sqrt{(3a+b+c)^2+8k-9} \ge 6\sqrt{2(k+2)}.$$

The right inequality is equivalent to

$$\sum \sqrt{(2a+3)^2+8k-9} \ge 6\sqrt{2(k+2)},$$

which follows immediately from Jensen's inequality applied to the convex function $f:[0,\infty)\to\mathbb{R}$ defined by

$$f(x)=\sqrt{(2x+3)^2+8k-9}.$$

To prove the left inequality, we use the SOS method. By means of the substitutions

$$A_1=8(a^2+bc+k), \quad B_1=8(b^2+ca+k), \quad C_1=8(c^2+ab+k),$$

$$A_2=(3a+b+c)^2+8k-9, \quad B_2=(3b+c+a)^2+8k-9, \quad C_2=(3c+a+b)^2+8k-9,$$

we can write the inequality as follows:

$$\frac{A_1-A_2}{\sqrt{A_1}+\sqrt{A_2}}+\frac{B_1-B_2}{\sqrt{B_1}+\sqrt{B_2}}+\frac{C_1-C_2}{\sqrt{C_1}+\sqrt{C_2}} \ge 0,$$

$$\frac{2bc-ca-ab}{\sqrt{A_1}+\sqrt{A_2}}+\frac{2ca-ab-bc}{\sqrt{B_1}+\sqrt{B_2}}+\frac{2ab-bc-ca}{\sqrt{C_1}+\sqrt{C_2}} \ge 0,$$

$$\sum\left[\frac{b(c-a)}{\sqrt{A_1}+\sqrt{A_2}}+\frac{c(b-a)}{\sqrt{A_1}+\sqrt{A_2}}\right] \ge 0,$$

$$\sum\frac{c(a-b)}{\sqrt{B_1}+\sqrt{B_2}}+\sum\frac{c(b-a)}{\sqrt{A_1}+\sqrt{A_2}} \ge 0,$$

$$\sum c(a-b)(\sqrt{C_1}+\sqrt{C_2})[(\sqrt{A_1}-\sqrt{B_1})+(\sqrt{A_2}-\sqrt{B_2})] \ge 0,$$

$$\sum c(a-b)^2(\sqrt{C_1}+\sqrt{C_2})\left[\frac{2(a+b-c)}{\sqrt{A_1}+\sqrt{B_1}}+\frac{2a+2b+c}{\sqrt{A_2}+\sqrt{B_2}}\right] \ge 0.$$

Without loss of generality, assume that $a \ge b \ge c$. Clearly, the desired inequality is true for $b+c \ge a$. Consider further the case $b+c<a$. Since $a+b-c>0$, it suffices to show that

$$a(b-c)^2(\sqrt{A_1}+\sqrt{A_2})\left[\frac{2(b+c-a)}{\sqrt{B_1}+\sqrt{C_1}}+\frac{2b+2c+a}{\sqrt{B_2}+\sqrt{C_2}}\right]+$$
$$+b(a-c)^2(\sqrt{B_1}+\sqrt{B_2})\left[\frac{2(c+a-b)}{\sqrt{C_1}+\sqrt{A_1}}+\frac{2c+2a+b}{\sqrt{C_2}+\sqrt{AC_2}}\right] \ge 0.$$

Since

$$b^2(a-c)^2 \ge a^2(b-c)^2,$$

it suffices to show that

$$b(\sqrt{A_1}+\sqrt{A_2})\left[\frac{2(b+c-a)}{\sqrt{B_1}+\sqrt{C_1}}+\frac{2b+2c+a}{\sqrt{B_2}+\sqrt{C_2}}\right]+$$
$$+a(\sqrt{B_1}+\sqrt{B_2})\left[\frac{2(c+a-b)}{\sqrt{C_1}+\sqrt{A_1}}+\frac{2c+2a+b}{\sqrt{C_2}+\sqrt{A_2}}\right]\geq 0.$$

From

$$a^2B_1-b^2A_1=8c(a^3-b^3)+8k(a^2-b^2)\geq 0$$

and

$$a^2B_2-b^2A_2=(a-b)(a+b+c)(a^2+b^2+6ab+bc+ca)+(8k-9)(a^2-b^2)\geq 0,$$

we get $a\sqrt{B_1}\geq b\sqrt{A_1}$ and $a\sqrt{B_2}\geq b\sqrt{A_2}$, hence

$$a(\sqrt{B_1}+\sqrt{B_2})\geq b(\sqrt{A_1}+\sqrt{A_2}).$$

Therefore, it is enough to show that

$$\frac{2(b+c-a)}{\sqrt{B_1}+\sqrt{C_1}}+\frac{2b+2c+a}{\sqrt{B_2}+\sqrt{C_2}}+\frac{2(c+a-b)}{\sqrt{C_1}+\sqrt{A_1}}+\frac{2c+2a+b}{\sqrt{C_2}+\sqrt{A_2}}\geq 0.$$

This is true if

$$\frac{2b}{\sqrt{B_1}+\sqrt{C_1}}+\frac{-2b}{\sqrt{C_1}+\sqrt{A_1}}\geq 0$$

and

$$\frac{-2a}{\sqrt{B_1}+\sqrt{C_1}}+\frac{2a}{\sqrt{C_1}+\sqrt{A_1}}+\frac{2a}{\sqrt{C_2}+\sqrt{A_2}}\geq 0.$$

The first inequality is true because $A_1-B_1=8(a-b)(a+b-c)\geq 0$. The second inequality can be written as

$$\frac{1}{\sqrt{C_1}+\sqrt{A_1}}+\frac{1}{\sqrt{C_2}+\sqrt{A_2}}\geq\frac{1}{\sqrt{B_1}+\sqrt{C_1}}.$$

Since

$$\frac{1}{\sqrt{C_1}+\sqrt{A_1}}+\frac{1}{\sqrt{C_2}+\sqrt{A_2}}\geq\frac{4}{\sqrt{C_1}+\sqrt{A_1}+\sqrt{C_2}+\sqrt{A_2}},$$

it suffices to show that

$$4\sqrt{B_1}+3\sqrt{C_1}\geq\sqrt{A_1}+\sqrt{A_2}+\sqrt{C_2}.$$

Taking account of

$$C_1-C_2=4(2ab-bc-ca)\geq 0,$$
$$C_1-B_1=8(b-c)(a-b-c)\geq 0,$$

$$A_2 - A_1 = 4(ab - 2bc + ca) \ge 0,$$

we have

$$\begin{aligned} 4\sqrt{B_1} + 3\sqrt{C_1} - \sqrt{A_1} - \sqrt{A_2} - \sqrt{C_2} &\ge 4\sqrt{B_1} + 2\sqrt{C_1} - \sqrt{A_1} - \sqrt{A_2} \\ &\ge 4\sqrt{B_1} + 2\sqrt{B_1} - \sqrt{A_2} - \sqrt{A_2} \\ &= 2(3\sqrt{B_1} - \sqrt{A_2}). \end{aligned}$$

In addition,

$$\begin{aligned} 9B_1 - A_2 &= 64k - 8a^2 + 72b^2 - 4ab + 68ac \\ &\ge 72 - 8a^2 + 72b^2 - 4ab + 68ac \\ &= 8(a+b+c)^2 - 8a^2 + 72b^2 - 4ab + 68ac \\ &= 4(20b^2 + 2c^2 + 3ab + 4bc + 21ac) \ge 0. \end{aligned}$$

Thus, the proof is completed. The equality holds for $a = b = c = 1$. If $k = 9/8$, then the equality holds also for $a = 3$ and $b = c = 0$ (or any cyclic permutation).

□

P 2.64. *If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then*

$$\sqrt{a^3 + 2bc} + \sqrt{b^3 + 2ca} + \sqrt{c^3 + 2ab} \ge 3\sqrt{3}.$$

(Nguyen Van Quy, 2013)

Solution. Since

$$(a^3 + 2bc)(a + 2bc) \ge (a^2 + 2bc)^2,$$

it suffices to prove that

$$\sum \frac{a^2 + 2bc}{\sqrt{a + 2bc}} \ge 3\sqrt{3}.$$

By Hölder's inequality, we have

$$\left(\sum \frac{a^2 + 2bc}{\sqrt{a + 2bc}}\right)^2 \sum (a^2 + 2bc)(a + 2bc) \ge \left[\sum (a^2 + 2bc)\right]^3 = (a + b + c)^6.$$

Therefore, it suffices to show that

$$(a + b + c)^6 \ge 27 \sum (a^2 + 2bc)(a + 2bc).$$

which is equivalent to

$$(a + b + c)^4 \ge \sum (a^2 + 2bc)(a^2 + 6bc + ca + ab).$$

Indeed,

$$(a + b + c)^4 - \sum (a^2 + 2bc)(a^2 + 6bc + ca + ab) = 3\sum ab(a - b)^2 \ge 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 3$ and $b = c = 0$ (or any cyclic permutation).

□

P 2.65. *If a, b, c are positive real numbers, then*

$$\frac{\sqrt{a^2+bc}}{b+c}+\frac{\sqrt{b^2+ca}}{c+a}+\frac{\sqrt{c^2+ab}}{a+b}\geq\frac{3\sqrt{2}}{2}.$$

(Vasile Cîrtoaje, 2006)

Solution. According to the well-known inequality

$$(x+y+z)^2\geq 3(xy+yz+zx), \quad x,y,z\geq 0,$$

it suffices to show that

$$\sum\frac{\sqrt{(b^2+ca)(c^2+ab)}}{(c+a)(a+b)}\geq\frac{3}{2}.$$

Replacing a, b, c by a^2, b^2, c^2, respectively, the inequality becomes

$$2\sum(b^2+c^2)\sqrt{(b^4+c^2a^2)(c^4+a^2b^2)}\geq 3(a^2+b^2)(b^2+c^2)(c^2+a^2).$$

Multiplying the Cauchy-Schwarz inequalities

$$\sqrt{(b^2+c^2)(b^4+c^2a^2)}\geq b^3+ac^2,$$

$$\sqrt{(c^2+b^2)(c^4+a^2b^2)}\geq c^3+ab^2,$$

we get

$$(b^2+c^2)\sqrt{(b^4+c^2a^2)(c^4+a^2b^2)}\geq(b^3+ac^2)(c^3+ab^2)$$
$$=b^3c^3+a(b^5+c^5)+a^2b^2c^2.$$

Therefore, it suffices to show that

$$2\sum b^3c^3+2\sum a(b^5+c^5)+6a^2b^2c^2\geq 3(a^2+b^2)(b^2+c^2)(c^2+a^2).$$

This inequality is equivalent to

$$2\sum b^3c^3+2\sum bc(b^4+c^4)\geq 3\sum b^2c^2(b^2+c^2),$$

$$\sum bc[2b^2c^2+2(b^4+c^4)-3bc(b^2+c^2)]\geq 0,$$

$$\sum bc(b-c)^2(2b^2+bc+2c^2)\geq 0.$$

The equality holds for $a=b=c$.

□

P 2.66. *If a,b,c are nonnegative real numbers, no two of which are zero,then*

$$\frac{\sqrt{bc+4a(b+c)}}{b+c}+\frac{\sqrt{ca+4b(c+a)}}{c+a}+\frac{\sqrt{ab+4c(a+b)}}{a+b}\ge\frac{9}{2}.$$

(Vasile Cîrtoaje, 2006)

Solution. Let us denote

$$A=4ab+bc+4ca,\quad B=4ab+4bc+ca,\quad C=ab+4bc+4ca.$$

By squaring, the inequality becomes

$$\sum\frac{A}{(b+c)^2}+2\sum\frac{\sqrt{BC}}{(c+a)(a+b)}\ge\frac{81}{4}.$$

According to the known inequality Iran-1996, namely

$$\sum\frac{ab+bc+ca}{(b+c)^2}\ge\frac{9}{4}$$

(see Remark from the proof of P 1.72), we have

$$\sum\frac{A}{(b+c)^2}=\sum\frac{ab+bc+ca}{(b+c)^2}+3\sum\frac{a}{b+c}\ge\frac{9}{4}+3\sum\frac{a}{b+c}.$$

On the other hand, from Lemma below, we have

$$\sqrt{BC}\ge 2ab+4bc+2ca+\frac{2abc}{b+c},$$

$$\sqrt{BC}\ge\frac{2a(b^2+c^2)+4bc(b+c)+6abc}{b+c},$$

$$2\sum\frac{\sqrt{BC}}{(c+a)(a+b)}\ge\frac{4\sum a(b^2+c^2)+8\sum bc(b+c)+36abc}{(a+b)(b+c)c+a)},$$

$$2\sum\frac{\sqrt{BC}}{(c+a)(a+b)}\ge\frac{12\sum bc(b+c)+36abc}{(a+b)(b+c)c+a)}.$$

Thus, it suffices to show that

$$3\sum\frac{a}{b+c}+\frac{12\sum bc(b+c)+36abc}{(a+b)(b+c)c+a)}\ge 18.$$

This is equivalent to Schur's inequality of degree three

$$\sum a^3+3abc\ge\sum bc(b+c).$$

The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

Lemma. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

$$\sqrt{(4ab+4bc+ca)(ab+4bc+4ca)} \geq 2ab+4bc+2ca+\frac{2abc}{b+c},$$

with equality for $b=c$, and also for $abc=0$.

Proof. We use the AM-GM inequality as follows:

$$\sqrt{(4ab+4bc+ca)(ab+4bc+4ca)}-2ab-4bc-2ca=$$

$$=\frac{abc(9a+4b+4c)}{\sqrt{(4ab+4bc+ca)(ab+4bc+4ca)}+2ab+4bc+2ca}$$

$$\geq \frac{2abc(9a+4b+4c)}{(4ab+4bc+ca)+(ab+4bc+4ca)+4ab+8bc+4ca}$$

$$=\frac{2abc(9a+4b+4c)}{9ab+16bc+9ca}.$$

Thus, it suffices to show that

$$\frac{9a+4b+4c}{9ab+16bc+9ca} \geq \frac{1}{b+c}.$$

Indeed,

$$(9a+4b+4c)(b+c)-(9ab+16bc+9ca)=4(b-c)^2 \geq 0.$$

□

P 2.67. *If a, b, c are nonnegative real numbers, no two of which are zero,then*

$$\frac{a\sqrt{a^2+3bc}}{b+c}+\frac{b\sqrt{b^2+3ca}}{c+a}+\frac{c\sqrt{c^2+3ab}}{a+b} \geq a+b+c.$$

(Cezar Lupu, 2006)

Solution. Using the AM-GM inequality, we have

$$\frac{a\sqrt{a^2+3bc}}{b+c}=\frac{2a(a^2+3bc)}{2\sqrt{(b+c)^2(a^2+3bc)}} \geq \frac{2a(a^2+3bc)}{(b+c)^2+(a^2+3bc)}=\frac{2a^3+6abc}{S+5bc},$$

where $S=a^2+b^2+c^2$. Thus, it suffices to show that

$$\sum \frac{2a^3+6abc}{S+5bc} \geq a+b+c.$$

Write this inequality as

$$\sum a\left(\frac{2a^2+6bc}{S+5bc}-1\right)\geq 0,$$

or, equivalently,

$$AX+BY+XZ\geq 0,$$

where

$$A=\frac{1}{S+5bc},\quad B=\frac{1}{S+5ca},\quad C=\frac{1}{S+5ab},$$

$$X=a^3+abc-a(b^2+c^2),\quad Y=b^3+abc-b(c^2+a^2),\quad Z=c^3+abc-c(a^2+b^2).$$

Without loss of generality, assume that $a\geq b\geq c$. We have

$$A\geq B\geq C,$$

$$X=a(a^2-b^2)+ac(b-c)\geq 0,\quad Z=c(c^2-b^2)+ac(b-a)\leq 0$$

and, according to Schur's inequality of third degree,

$$X+Y+Z=\sum a^3+3abc-\sum a(b^2+c^2)\geq 0.$$

Therefore,

$$AX+BY+CZ\geq BX+BY+BZ=B(X+Y+Z)\geq 0.$$

The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

Remark. We can also prove the inequality $AX+BY+XZ\geq 0$ by the SOS procedure. Write this inequality as follows:

$$\sum\frac{a(a^2+bc-b^2-c^2)}{S+5bc}\geq 0,$$

$$\sum\frac{a(a^2b+a^2c-b^3-c^3)}{(b+c)(S+5bc)}\geq 0,$$

$$\sum\frac{ab(a^2-b^2)+ac(a^2-c^2)}{(b+c)(S+5bc)}\geq 0,$$

$$\sum\frac{ab(a^2-b^2)}{(b+c)(S+5bc)}+\sum\frac{ba(b^2-a^2)}{(c+a)(S+5ca)}\geq 0,$$

$$\sum\frac{ab(a+b)(a-b)^2[S+5c(a+b+c)]}{(b+c)(c+a)(S+5bc)(S+5ca)}\geq 0.$$

□

P 2.68. *If a, b, c are nonnegative real numbers, no two of which are zero,then*

$$\sqrt{\frac{2a(b+c)}{(2b+c)(b+2c)}}+\sqrt{\frac{2b(c+a)}{(2c+a)(c+2a)}}+\sqrt{\frac{2c(a+b)}{(2a+b)(a+2b)}}\ge 2.$$

(Vasile Cîrtoaje, 2006)

Solution. Making the substitution

$$x=\sqrt{a},\quad y=\sqrt{b},\quad z=\sqrt{c},$$

the inequality becomes

$$\sum x\sqrt{\frac{2(y^2+z^2)}{(2y^2+z^2)(y^2+2z^2)}}\ge 2.$$

We claim that

$$\sqrt{\frac{2(y^2+z^2)}{(2y^2+z^2)(y^2+2z^2)}}\ge \frac{y+z}{y^2+yz+z^2}.$$

Indeed, be squaring and direct calculation, this inequality reduces to

$$y^2z^2(y-z)^2\ge 0.$$

Thus, it suffices to show that

$$\sum\frac{x(y+z)}{y^2+yz+z^2}\ge 2,$$

which is just the inequality in P 1.69. The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 2.69. *If a, b, c are nonnegative real numbers such that $ab+bc+ca=3$, then*

$$\sqrt{\frac{bc}{3a^2+6}}+\sqrt{\frac{ca}{3b^2+6}}+\sqrt{\frac{ab}{3c^2+6}}\le 1\le\sqrt{\frac{bc}{6a^2+3}}+\sqrt{\frac{ca}{6b^2+3}}+\sqrt{\frac{ab}{6c^2+3}}.$$

(Vasile Cîrtoaje, 2011)

Solution. By the Cauchy-Schwarz inequality, we have

$$\left(\sum\sqrt{\frac{bc}{3a^2+6}}\right)^2\le\left(\sum\frac{1}{3a^2+6}\right)\left(\sum bc\right),$$

hence

$$\left(\sum\sqrt{\frac{bc}{3a^2+6}}\right)^2\le\sum\frac{1}{a^2+2}.$$

Therefore, to prove the original left inequality, it suffices to show that

$$\sum \frac{1}{a^2+2} \le 1.$$

This inequality is equivalent to

$$\sum \frac{a^2}{a^2+2} \ge 1.$$

By the Cauchy-Schwarz inequality, we get

$$\sum \frac{a^2}{a^2+2} \ge \frac{(a+b+c)^2}{\sum(a^2+2)} = \frac{(a+b+c)^2}{\sum a^2+6} = 1.$$

The equality occurs for $a = b = c = 1$.

To prove the original right inequality we apply Hölder's inequality as follows:

$$\left(\sum \sqrt{\frac{bc}{6a^2+3}}\right)^2 \left[\sum b^2c^2(6a^2+3)\right] \ge \left(\sum bc\right)^3.$$

Thus, it suffices to show that

$$(ab+bc+ca)^3 \ge \sum b^2c^2(6a^2+ab+bc+ca),$$

which is equivalent to

$$(ab+bc+ca)\left[(ab+bc+ca)^2 - \sum b^2c^2\right] \ge 18a^2b^2c^2,$$

$$2abc(ab+bc+ca)(a+b+c) \ge 18a^2b^2c^2,$$

$$2abc\sum a(b-c)^2 \ge 0.$$

The equality occurs for $a = b = c = 1$, and for $a = 0$ and $bc = 3$ (or any cyclic permutation).

□

P 2.70. *Let a, b, c be nonnegative real numbers such that $ab + bc + ca = 3$. If $k > 1$, than*

$$a^k(b+c) + b^k(c+a) + c^k(a+b) \ge 6.$$

Solution. Let

$$E = a^k(b+c) + b^k(c+a) + c^k(a+b).$$

We consider two cases.

Case 1: $k \geq 2$. Applying Jensen's inequality to the convex function $f(x) = x^{k-1}$, $x \geq 0$, we get

$$\begin{aligned} E &= (ab+ac)a^{k-1} + (bc+ba)b^{k-1} + (ca+cb)c^{k-1} \\ &\geq 2(ab+bc+ca)\left[\frac{(ab+ac)a+(bc+ba)b+(ca+cb)c}{2(ab+bc+ca)}\right]^{k-1} \\ &= 6\left[\frac{a^2(b+c)+b^2(c+a)+c^2(a+b)}{6}\right]^{k-1}. \end{aligned}$$

Thus, it suffices to show that

$$a^2(b+c)+b^2(c+a)+c^2(a+b) \geq 6.$$

Write this inequality as

$$(ab+bc+ca)(a+b+c)-3abc \geq 6,$$

$$a+b+c \geq 2+abc.$$

It is true since

$$a+b+c \geq \sqrt{3(ab+bc+ca)} = 3$$

and

$$abc \leq \left(\frac{a+b+c}{3}\right)^3 = 1.$$

Case 2: $1 < k < 2$. We have

$$E = a^{k-1}(3-bc)+b^{k-1}(3-ca)+c^{k-1}(3-ab)$$

$$= 3(a^{k-1}+b^{k-1}+c^{k-1}) - a^{k-1}b^{k-1}c^{k-1}\left[(ab)^{2-k}+(bc)^{2-k}+(ca)^{2-k}\right].$$

Since $0 < 2-k < 1$, $f(x) = x^{2-k}$ is concave for $x \geq 0$. Thus, by Jensen's inequality, we have

$$(ab)^{2-k}+(bc)^{2-k}+(ca)^{2-k} \leq 3\left(\frac{ab+bc+ca}{3}\right)^{2-k} = 3,$$

hence

$$E \geq 3(a^{k-1}+b^{k-1}+c^{k-1}) - 3a^{k-1}b^{k-1}c^{k-1}.$$

Consequently, it suffices to show that

$$a^{k-1}+b^{k-1}+c^{k-1} \geq a^{k-1}b^{k-1}c^{k-1}+2.$$

Due to symmetry, we may assume that

$$a \geq b \geq c,$$

which involves

$$ab \geq \frac{1}{3}(ab + bc + ca) \geq 1.$$

Let

$$x = \sqrt{a^{k-1}b^{k-1}}, \quad x \geq 1.$$

From

$$2 \geq 3 - ab = bc + ca \geq 2c\sqrt{ab},$$

we get

$$c \leq \frac{1}{\sqrt{ab}},$$

hence

$$c^{k-1} \leq \frac{1}{x}.$$

Write the required inequality as

$$a^{k-1} + b^{k-1} - 2 \geq \left(a^{k-1}b^{k-1} - 1\right)c^{k-1}.$$

It suffices to show that

$$a^{k-1} + b^{k-1} - 2 \geq \frac{a^{k-1}b^{k-1} - 1}{x}.$$

Since

$$a^{k-1} + b^{k-1} \geq 2\sqrt{a^{k-1}b^{k-1}} = 2x,$$

we only need to prove that

$$2x - 2 \geq \frac{x^2 - 1}{x}.$$

Indeed,

$$2x - 2 - \frac{x^2 - 1}{x} = \frac{(x-1)^2}{x} \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 2.71. *Let a, b, c be nonnegative real numbers such that $a + b + c = 2$. If*

$$2 \leq k \leq 3,$$

than

$$a^k(b + c) + b^k(c + a) + c^k(a + b) \leq 2.$$

Solution. Denote by $E_k(a,b,c)$ the left hand side of the inequality, assume that

$$a \le b \le c,$$

and show that

$$E_k(a,b,c) \le E_k(0,a+b,c) \le 2.$$

The left inequality is equivalent to

$$\frac{ab}{c}(a^{k-1}+b^{k-1}) \le (a+b)^k - a^k - b^k.$$

Clearly, it suffices to consider $c = b$, when the inequality becomes

$$2a^k + b^{k-1}(a+b) \le (a+b)^k.$$

Since $2a^k \le a^{k-1}(a+b)$, it remains to show that

$$a^{k-1}+b^{k-1} \le (a+b)^{k-1},$$

which is true since

$$\frac{a^{k-1}+b^{k-1}}{(a+b)^{k-1}} = \left(\frac{a}{a+b}\right)^{k-1} + \left(\frac{b}{a+b}\right)^{k-1} \le \frac{a}{a+b} + \frac{b}{a+b} = 1.$$

Using the notation $d = a+b$, we can write the right inequality $E_k(0,a+b,c) \le 2$ in the form

$$cd(c^{k-1}+d^{k-1}) \le 2,$$

where $c+d=2$. By the Power-Mean inequality , we have

$$\left(\frac{c^{k-1}+d^{k-1}}{2}\right)^{1/(k-1)} \le \left(\frac{c^2+d^2}{2}\right)^{1/2},$$

$$c^{k-1}+d^{k-1} \le 2\left(\frac{c^2+d^2}{2}\right)^{(k-1)/2}.$$

Thus, it suffices to show that

$$cd\left(\frac{c^2+d^2}{2}\right)^{(k-1)/2} \le 1,$$

which is equivalent to

$$cd(2-cd)^{(k-1)/2} \le 1.$$

Since $2-cd \ge 1$, we have

$$cd(2-cd)^{(k-1)/2} \le cd(2-cd) = 1-(1-cd)^2 \le 1.$$

The equality holds for $a=0$ and $b=c=1$ (or any cyclic permutation).

□

P 2.72. *Let a, b, c be nonnegative real numbers, no two of which are zero. If*

$$m > n \geq 0,$$

than

$$\frac{b^m + c^m}{b^n + c^n}(b + c - 2a) + \frac{c^m + a^m}{c^n + a^n}(c + a - 2b) + \frac{a^m + b^m}{a^n + b^n}(a + b - 2c) \geq 0.$$

(Vasile Cîrtoaje, 2006)

Solution. Write the inequality as

$$AX + BY + CZ \geq 0,$$

where

$$A = \frac{b^m + c^m}{b^n + c^n}, \quad B = \frac{c^m + a^m}{c^n + a^n}, \quad C = \frac{a^m + b^m}{a^n + b^n},$$

$$X = b + c - 2a, \quad Y = c + a - 2b, \quad Z = a + b - 2c, \quad X + Y + Z = 0.$$

Without loss of generality, assume that

$$a \leq b \leq c,$$

which involves $X \geq Y \geq Z$ and $X \geq 0$. Since

$$\begin{aligned} 2(AX + BY + CZ) &= (2A - B - C)X + (B + C)X + 2(BY + CZ) \\ &= (2A - B - C)X - (B + C)(Y + Z) + 2(BY + CZ) \\ &= (2A - B - C)X + (B - C)(Y - Z), \end{aligned}$$

it suffices to show that $B \geq C$ and $2A - B - C \geq 0$. The inequality $B \geq C$ can be written as

$$b^n c^n (c^{m-n} - b^{m-n}) + a^n (c^m - b^m) - a^m (c^n - b^n) \geq 0,$$

$$b^n c^n (c^{m-n} - b^{m-n}) + a^n [c^m - b^m - a^{m-n}(c^n - b^n)] \geq 0.$$

This is true since $c^{m-n} \geq b^{m-n}$ and

$$c^m - b^m - a^{m-n}(c^n - b^n) \geq c^m - b^m - b^{m-n}(c^n - b^n) = c^n(c^{m-n} - b^{m-n}) \geq 0.$$

The inequality $2A - B - C \geq 0$ follows from

$$2A \geq b^{m-n} + c^{m-n}, \quad b^{m-n} \geq C, \quad c^{m-n} \geq B.$$

Indeed, we have

$$2A - b^{m-n} - c^{m-n} = \frac{(b^n - c^n)(b^{m-n} - c^{m-n})}{b^n + c^n} \geq 0,$$

$$b^{m-n} - C = \frac{a^n(b^{m-n} - a^{m-n})}{a^n + b^n} \geq 0,$$

$$c^{m-n} - B = \frac{a^n(c^{m-n} - a^{m-n})}{c^n + a^n} \geq 0.$$

The equality holds for $a = b = c$, and also for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 2.73. *Let a,b,c be positive real numbers such that $abc=1$. Prove that*

$$\sqrt{a^2-a+1}+\sqrt{b^2-b+1}+\sqrt{c^2-c+1}\geq a+b+c.$$

(Vasile Cîrtoaje, 2012)

First Solution. Among $a-1$, $b-1$ and $c-1$ there are two with the same sign. Let $(b-1)(c-1)\geq 0$, that is,

$$t\leq \frac{1}{a},\quad t=b+c-1.$$

By Minkowsky's inequality, we have

$$\sqrt{b^2-b+1}+\sqrt{c^2-c+1}=\sqrt{\left(b-\frac{1}{2}\right)^2+\frac{3}{4}}+\sqrt{\left(c-\frac{1}{2}\right)^2+\frac{3}{4}}\geq\sqrt{t^2+3}.$$

Thus, it suffices to show that

$$\sqrt{a^2-a+1}+\sqrt{t^2+3}\geq a+b+c,$$

which is equivalent to

$$\sqrt{a^2-a+1}+f(t)\geq a+1,$$

where

$$f(t)=\sqrt{t^2+3}-t.$$

Clearly, $f(t)$ is decreasing for $t\leq 0$. Since

$$f(t)=\frac{3}{\sqrt{t^2+3}+t},$$

$f(t)$ is also decreasing for $t\geq 0$. Then, $f(t)\geq f\left(\frac{1}{a}\right)$, and it suffices to show that

$$\sqrt{a^2-a+1}+f\left(\frac{1}{a}\right)\geq a+1,$$

which is equivalent to

$$\sqrt{a^2-a+1}+\sqrt{\frac{1}{a^2}+3}\geq a+\frac{1}{a}+1.$$

By squaring, this inequality becomes

$$2\sqrt{(a^2-a+1)\left(\frac{1}{a^2}+3\right)}\geq 3a+\frac{2}{a}-1.$$

Indeed, by the Cauchy-Schwarz inequality, we have

$$2\sqrt{(a^2-a+1)\left(\frac{1}{a^2}+3\right)} = \sqrt{[(2-a)^2+3a^2]\left(\frac{1}{a^2}+3\right)}$$
$$\geq \frac{2-a}{a}+3a = 3a+\frac{2}{a}-1.$$

The equality holds for $a=b=c$.

Second Solution. If the inequality

$$\sqrt{x^2-x+1}-x \geq \frac{1}{2}\left(\frac{3}{x^2+x+1}-1\right)$$

holds for all $x>0$, then it suffices to prove that

$$\frac{1}{a^2+a+1}+\frac{1}{b^2+b+1}+\frac{1}{c^2+c+1} \geq 1,$$

which is just the known inequality in P 1.45. The above inequality in x is equivalent to

$$\frac{1-x}{\sqrt{x^2-x+1}+x} \geq \frac{(1-x)(2+x)}{2(x^2+x+1)},$$
$$(x-1)\left[(x+2)\sqrt{x^2-x+1}-x^2-2\right] \geq 0,$$
$$\frac{3x^2(x-1)^2}{(x+2)\sqrt{x^2-x+1}+x^2+2} \geq 0.$$

□

P 2.74. *Let a,b,c be positive real numbers such that $abc=1$. Prove that*

$$\sqrt{16a^2+9}+\sqrt{16b^2+9}+\sqrt{16b^2+9} \geq 4(a+b+c)+3.$$

(MEMO, 2012)

First Solution *(by Vo Quoc Ba Can).* Since

$$\sqrt{16a^2+9}-4a = \frac{9}{\sqrt{16a^2+9}+4a},$$

the inequality is equivalent to

$$\sum \frac{1}{\sqrt{16a^2+9}+4a} \geq \frac{1}{3}.$$

By the AM-GM inequality, we have

$$2\sqrt{16a^2+9} \leq \frac{16a^2+9}{2a+3}+2a+3,$$

$$2(\sqrt{16a^2+9}+4a) \le \frac{16a^2+9}{2a+3} + 10a + 3 = \frac{18(2a^2+2a+1)}{2a+3}.$$

Thus, it suffices to show that

$$\sum \frac{2a+3}{2a^2+2a+1} \ge 3.$$

If the inequality

$$\frac{2a+3}{2a^2+2a+1} \ge \frac{3}{a^{8/5}+a^{4/5}+1}$$

holds for all $a > 0$, then it suffices to show that

$$\sum \frac{1}{a^{8/5}+a^{4/5}+1} \ge 1,$$

which follows immediately from the inequality in P 1.45. Therefore, using the substitution $x = a^{1/5}$, $x > 0$, we need to show that

$$\frac{2x^5+3}{2x^{10}+2x^5+1} \ge \frac{3}{x^8+x^4+1},$$

which is equivalent to

$$2x^4(x^5-3x^2+x+1) + x^4 - 4x + 3 \ge 0.$$

This is true since, by the AM-GM inequality, we have

$$x^5 + x + 1 \ge 3\sqrt[3]{x^5 \cdot x \cdot 1} = 3x^2$$

and

$$x^4 + 3 = x^4 + 1 + 1 + 1 \ge 4\sqrt[4]{x^4 \cdot 1 \cdot 1 \cdot 1} = 4x.$$

The equality holds for $a = b = c = 1$.

Second Solution. Making the substitution

$$x = \sqrt{16a^2+9} - 4a, \quad y = \sqrt{16b^2+9} - 4b, \quad z = \sqrt{16c^2+9} - 4c, \quad x, y, z > 0,$$

which involves

$$a = \frac{9-x^2}{8x}, \quad b = \frac{9-y^2}{8y}, \quad c = \frac{9-z^2}{8z},$$

we need to show that

$$(9-x^2)(9-y^2)(9-z^2) = 512xyz$$

yields

$$x + y + z \ge 3.$$

Use the contradiction method. Assume that

$$x + y + z < 3,$$

and show that

$$(9 - x^2)(9 - y^2)(9 - z^2) > 512xyz.$$

According to the AM-GM inequality, we get

$$3 + x = 1 + 1 + 1 + x \ge 4\sqrt[4]{x}, \quad 3 + y \ge 4\sqrt[4]{y}, \quad 3 + z \ge 4\sqrt[4]{z},$$

hence

$$(3 + x)(3 + y)(3 + z) \ge 64\sqrt[4]{xyz}.$$

Therefore, it suffices to prove that

$$(3 - x)(3 - y)(3 - z) > 8\sqrt[4]{x^3y^3z^3}.$$

Since

$$1 > \left(\frac{x + y + z}{3}\right)^3 \ge xyz,$$

we have

$$\begin{aligned}(3 - x)(3 - y)(3 - z) &= 9(3 - x - y - z) + 3(xy + yz + zx) - xyz \\ &> 3(xy + yz + zx) - xyz \ge 9(xyz)^{2/3} - xyz \\ &> 8(xyz)^{2/3} > 8(xyz)^{3/4}.\end{aligned}$$

□

P 2.75. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\sqrt{25a^2 + 144} + \sqrt{25b^2 + 144} + \sqrt{25c^2 + 144} \le 5(a + b + c) + 24.$$

(Vasile Cîrtoaje, 2012)

First Solution. Since

$$\sqrt{25a^2 + 144} - 5a = \frac{144}{\sqrt{25a^2 + 144} + 5a},$$

the inequality is equivalent to

$$\sum \frac{1}{\sqrt{25a^2 + 144} + 5a} \le \frac{1}{6}.$$

If the inequality

$$\frac{1}{\sqrt{25a^2 + 144} + 5a} \le \frac{1}{6\sqrt{5a^{18/13} + 4}}$$

holds for all $a > 0$, then it suffices to show that

$$\sum \frac{1}{\sqrt{5a^{18/13}+4}} \le 1,$$

which follows immediately from P 2.33. Using the substitution $x = a^{1/13}$, $x > 0$, we only need to show that

$$\sqrt{25x^{26}+144} + 5x^{13} \ge 6\sqrt{5x^{18}+4}.$$

By squaring, the inequality becomes

$$10x^{13}(\sqrt{25x^{26}+144} + 5x^{13} - 18x^5) \ge 0.$$

This is true if

$$25x^{26} + 144 \ge (18x^5 - 5x^{13})^2,$$

which is equivalent to

$$5x^{18} + 4 \ge 9x^{10}.$$

By the AM-GM inequality, we have

$$\begin{aligned} 5x^{18} + 4 &= x^{18} + x^{18} + x^{18} + x^{18} + x^{18} + 1 + 1 + 1 + 1 \\ &\ge 9\sqrt[9]{x^{18} \cdot x^{18} \cdot x^{18} \cdot x^{18} \cdot x^{18} \cdot 1 \cdot 1 \cdot 1 \cdot 1} = 9x^{10}. \end{aligned}$$

The equality holds for $a = b = c = 1$.

Second Solution. Making the substitution

$$8x = \sqrt{25a^2+144} - 5a, \quad 8y = \sqrt{25b^2+144} - 5b, \quad 8z = \sqrt{25c^2+144} - 5c,$$

which involves

$$a = \frac{9-4x^2}{5x}, \quad b = \frac{9-4y^2}{5y}, \quad c = \frac{9-4z^2}{5z}, \quad x, y, z \in \left(0, \frac{3}{2}\right),$$

we need to show that

$$(9-4x^2)(9-4y^2)(9-4z^2) = 125xyz$$

involves

$$x + y + z \le 3.$$

Use the contradiction method. Assume that

$$x + y + z > 3,$$

and show that

$$(9-4x^2)(9-4y^2)(9-4z^2) < 125xyz.$$

Since

$$9-4x^2 < 3(x+y+z)-\frac{12x^2}{x+y+z} = \frac{3(y+z-x)(y+z+3x)}{x+y+z},$$

it suffices to prove the homogeneous inequality

$$27AB \le 125xyz(x+y+z)^3,$$

where

$$A=(y+z-x)(z+x-y)(x+y-z),$$
$$B=(y+z+3x)(z+x+3y)(x+y+3z).$$

Consider the nontrivial case $A \ge 0$. By the AM-GM inequality, we have

$$B \le \frac{125}{27}(x+y+z)^3.$$

Therefore, it suffices to show that

$$A \le xyz,$$

which is a well known inequality (equivalent to Schur's inequality of degree three).

□

P 2.76. *If a,b are positive real numbers such that $ab+bc+ca=3$, then*

(a) $$\sqrt{a^2+3}+\sqrt{b^2+3}+\sqrt{b^2+3} \ge a+b+c+3;$$

(b) $$\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a} \ge \sqrt{4(a+b+c)+6}.$$

(Lee Sang Hoon, 2007)

Solution. (a) ***First Solution*** (by *Pham Thanh Hung*). By squaring, the inequality becomes

$$\sum \sqrt{(b^2+3)(c^2+3)} \ge 3(1+a+b+c).$$

Since

$$(b^2+3)(c^2+3) = (b+c)(b+a)(c+a)(c+b) = (b+c)^2(a^2+3)$$
$$\ge \frac{1}{4}(b+c)^2(a+3)^2,$$

we have

$$\sum \sqrt{(b^2+3)(c^2+3)} \ge \frac{1}{2}\sum(b+c)(a+3) = \frac{1}{2}\left(6\sum a+2\sum bc\right)$$
$$= 3(a+b+c+1).$$

The equality holds for $a = b = c = 1$.

Second Solution. Use the SOS method. Write the inequality as follows:

$$\sqrt{(a+b)(a+c)} + \sqrt{(b+c)(b+a)} + \sqrt{(c+a)(c+b)} \ge a+b+c+3,$$

$$2\left[a+b+c-\sqrt{3(ab+bc+ca)}\right] \ge \sum\left(\sqrt{a+b}-\sqrt{a+c}\right)^2,$$

$$\frac{1}{a+b+c+\sqrt{3(ab+bc+ca)}}\sum(b-c)^2 \ge \sum\frac{(b-c)^2}{\left(\sqrt{a+b}+\sqrt{a+c}\right)^2},$$

$$\sum\frac{S_a(b-c)^2}{\left(\sqrt{a+b}+\sqrt{a+c}\right)^2} \ge 0,$$

where

$$S_a = \left(\sqrt{a+b}+\sqrt{a+c}\right)^2 - a - b - c - \sqrt{3(ab+bc+ca)}.$$

The inequality is true since

$$S_a = 3(a+b+c) + 2\sqrt{(a+b)(a+c)} - \sqrt{3(ab+bc+ca)}$$

$$> 2\sqrt{a^2+(ab+bc+ca)} - \sqrt{3(ab+bc+ca)} > 0.$$

Third Solution. Use the substitution

$$x = \sqrt{a^2+3}-a, \quad y = \sqrt{b^2+3}-b, \quad z = \sqrt{c^2+3}-c, \quad x, y, z > 0.$$

We need to show that

$$x+y+z \ge 3.$$

We have

$$\sum yz = \sum\left[\sqrt{(b+a)(b+c)}-b\right]\left[\sqrt{(c+a)(c+b)}-c\right]$$

$$= \sum(b+c)\sqrt{(a+b)(a+c)} - \sum b\sqrt{(c+a)(c+b)} - \sum c\sqrt{(b+a)(b+c)} + \sum bc$$

$$= \sum(b+c)\sqrt{(a+b)(a+c)} - \sum c\sqrt{(a+b)(a+c)} - \sum b\sqrt{(a+c)(a+b)} + \sum bc$$

$$= \sum bc = 3.$$

Thus, we get

$$x+y+z \ge \sqrt{3(xy+yz+zx)} = 3.$$

(b) By squaring, we get the inequality in (a).

□

P 2.77. *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

$$\sqrt{(5a^2+3)(5b^2+3)}+\sqrt{(5b^2+3)(5c^2+3)}+\sqrt{(5c^2+3)(5a^2+3)}\ge 24.$$

(Nguyen Van Quy, 2012)

Solution. Assume that

$$a\ge b\ge c,\qquad 1\le a\le 3,\quad b+c\le 2.$$

Using the notation

$$A=5a^2+3,\quad B=5b^2+3,\quad C=5c^2+3,$$

we can write the inequality as follows:

$$\sqrt{A}\left(\sqrt{B}+\sqrt{C}\right)+\sqrt{BC}\ge 24,$$

$$\sqrt{A\left(B+C+2\sqrt{BC}\right)}\ge 24-\sqrt{BC}.$$

Consider the nontrivial case $\sqrt{BC}<24$. The inequality is true if

$$A\left(B+C+2\sqrt{BC}\right)\ge\left(24-\sqrt{BC}\right)^2,$$

which is equivalent to

$$A(A+B+C+48)\ge\left(A+24-\sqrt{BC}\right)^2.$$

Applying Lemma below for $k=5/3$ and $m=4/15$ yields

$$5\sqrt{BC}\ge 25bc+15+4(b-c)^2.$$

Therefore, it suffices to show that

$$25A(A+B+C+48)\ge[5A+120-25bc-15-4(b-c)^2]^2,$$

which is equivalent to

$$25(5a^2+3)[5(a^2+b^2+c^2)+57]\ge[25a^2+120-25bc-4(b-c)^2]^2.$$

Since

$$5(a^2+b^2+c^2)+57=5a^2+5(b+c)^2-10bc+57=2(5a^2-15a+51-5bc)$$

and

$$\begin{aligned}25a^2+120-25bc-4(b-c)^2&=25a^2+120-4(b+c)^2-9bc\\&=3(7a^2+8a+28-3bc),\end{aligned}$$

we need to show that

$$50(5a^2+3)(5a^2-15a+51-5bc) \geq 9(7a^2+8a+28-3bc)^2.$$

From $bc \leq (b+c)^2/4$ and $(a-b)(a-c) \geq 0$, we get

$$bc \leq \frac{(3-a)^2}{4}, \quad bc \geq a(b+c)-a^2 = 3a-2a^2.$$

Consider a fixed, $a \geq 1$, and denote $x = bc$. So, we only need to prove that $f(x) \geq 0$ for

$$3a-2a^2 \leq x \leq \frac{a^2-6a+9}{4},$$

where

$$f(x) = 50(5a^2+3)(5a^2-15a+51-5x)-9(7a^2+8a+28-3x)^2.$$

Since f is concave, it suffices to show that $f(3a{-}2a^2) \geq 0$ and $f\left(\frac{a^2-6a+9}{4}\right) \geq 0$. Indeed, we have

$$\begin{aligned} f(3a-2a^2) &= 3(743a^4-2422a^3+2813a^2-1332a+198) \\ &= 3(a-1)^2[(a-1)(743a-193)+5] \geq 0, \end{aligned}$$

$$\begin{aligned} f\left(\frac{a^2-6a+9}{4}\right) &= \frac{375}{16}(25a^4-140a^3+286a^2-252a+81) \\ &= \frac{375}{16}(a-1)^2(5a-9)^2 \geq 0. \end{aligned}$$

Thus, the proof is completed. The equality holds for $a = b = c = 1$, and also for $a = 9/5$ and $b = c = 3/5$ (or any cyclic permutation).

Lemma. *Let $b, c \geq 0$ such that $b+c \leq 2$. If $k > 0$ and $0 \leq m \leq \frac{k}{2k+2}$, then*

$$\sqrt{(kb^2+1)(kc^2+1)} \geq kbc+1+m(b-c)^2.$$

Proof. By squaring, the inequality becomes

$$(b-c)^2[k-2m-2kmbc-m^2(b-c)^2] \geq 0.$$

This is true since

$$\begin{aligned} k-2m-2kmbc-m^2(b-c)^2 &= k-2m-2m(k-2m)bc-m^2(b+c)^2 \\ &\geq k-2m-\frac{m(k-2m)}{2}(b+c)^2-m^2(b+c)^2 \\ &= k-2m-\frac{km}{2}(b+c)^2 \geq k-2m-2km \geq 0. \end{aligned}$$

$\square$

P 2.78. *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

$$\sqrt{a^2+1}+\sqrt{b^2+1}+\sqrt{c^2+1}\ge\sqrt{\frac{4(a^2+b^2+c^2)+42}{3}}.$$

(Vasile Cîrtoaje, 2014)

Solution. Assume that

$$a\ge b\ge c,\qquad a\ge 1,\quad b+c\le 2.$$

By squaring, the inequality becomes

$$\sqrt{A}\left(\sqrt{B}+\sqrt{C}\right)+\sqrt{BC}\ge\frac{a^2+b^2+c^2+33}{6},$$

$$\sqrt{A\left(B+C+2\sqrt{BC}\right)}+\sqrt{BC}\ge\frac{a^2+b^2+c^2+33}{6},$$

where

$$A=a^2+1,\quad B=b^2+1,\quad C=c^2+1.$$

Applying Lemma from the preceding problem P 2.77 for $k=1$ and $m=1/4$ gives

$$\sqrt{BC}\ge bc+1+\frac{1}{4}(b-c)^2.$$

Therefore, it suffices to show that

$$\sqrt{A\left[B+C+2bc+2+\frac{1}{2}(b-c)^2\right]}+bc+1+\frac{1}{4}(b-c)^2\ge\frac{a^2+b^2+c^2+33}{6},$$

which is equivalent to

$$6\sqrt{2(a^2+1)[3(b+c)^2+8-4bc]}\ge 2a^2-(b+c)^2+54-4bc,$$

$$6\sqrt{2(a^2+1)(3a^2-18a+35-4bc)}\ge a^2+6a+45-4bc.$$

From $bc\le(b+c)^2/4$ and $(a-b)(a-c)\ge 0$, we get

$$bc\le\frac{(3-a)^2}{4},\qquad bc\ge a(b+c)-a^2=3a-2a^2.$$

Consider a fixed, $a\ge 1$, and denote $x=bc$. So, we only need to prove that $f(x)\ge 0$ for

$$3a-2a^2\le x\le\frac{a^2-6a+9}{4},$$

where

$$f(x)=72(a^2+1)(3a^2-18a+35-4x)-(a^2+6a+45-4x)^2.$$

Since f is concave, it suffices to show that $f(3a–2a^2) \ge 0$ and $f\left(\frac{a^2-6a+9}{4}\right) \ge 0$. Indeed,

$$\begin{aligned} f(3a-2a^2) &= 9(79a^4-228a^3+274a^2-180a+55) \\ &= 9(a-1)^2(79a^2-70a+55 \ge 0, \end{aligned}$$

$$\begin{aligned} f\left(\frac{a^2-6a+9}{4}\right) &= 144(a^4-6a^3+13a^2-12a+4) \\ &= 144(a-1)^2(a-2)^2 \ge 0. \end{aligned}$$

The equality holds for $a=b=c=1$, and also for $a=2$ and $b=c=1/2$ (or any cyclic permutation).

□

P 2.79. *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

(a) $$\sqrt{a^2+3}+\sqrt{b^2+3}+\sqrt{c^2+3} \ge \sqrt{2(a^2+b^2+c^2)+30};$$

(b) $$\sqrt{3a^2+1}+\sqrt{3b^2+1}+\sqrt{3c^2+1} \ge \sqrt{2(a^2+b^2+c^2)+30}.$$

(Vasile Cîrtoaje, 2014)

Solution. Assume that

$$a \ge b \ge c, \qquad a \ge 1, \quad b+c \le 2.$$

(a) By squaring, the inequality becomes

$$\sqrt{A}\left(\sqrt{B}+\sqrt{C}\right)+\sqrt{BC} \ge \frac{a^2+b^2+c^2+21}{2},$$

$$\sqrt{A\left(B+C+2\sqrt{BC}\right)}+\sqrt{BC} \ge \frac{a^2+b^2+c^2+21}{2},$$

where

$$A=a^2+3, \quad B=b^2+3, \quad C=c^2+3.$$

Applying Lemma from problem P 2.77 for $k=1/3$ and $m=1/9$ gives

$$\sqrt{BC} \ge bc+3+\frac{1}{3}(b-c)^2.$$

Therefore, it suffices to show that

$$\sqrt{A\left[B+C+2bc+6+\frac{2}{3}(b-c)^2\right]}+bc+3+\frac{1}{3}(b-c)^2 \ge \frac{a^2+b^2+c^2+21}{2},$$

which is equivalent to

$$2\sqrt{3(a^2+3)[5(b+c)^2+36-8bc]} \ge 3a^2+(b+c)^2+45-4bc,$$

$$\sqrt{3(a^2+3)(5a^2-30a+81-8bc)} \ge 2a^2-3a+27-2bc.$$

From $bc \le (b+c)^2/4$ and $(a-b)(a-c) \ge 0$, we get

$$bc \le \frac{(3-a)^2}{4}, \quad bc \ge a(b+c)-a^2 = 3a-2a^2.$$

Consider a fixed, $a \ge 1$, and denote $x = bc$. So, we only need to prove that $f(x) \ge 0$ for

$$3a-2a^2 \le x \le \frac{a^2-6a+9}{4},$$

where

$$f(x) = 3(a^2+3)(5a^2-30a+81-8x)-(2a^2-3a+27-2x)^2.$$

Since f is concave, it suffices to show that $f(3a-2a^2) \ge 0$ and $f\left(\frac{a^2-6a+9}{4}\right) \ge 0$. Indeed,

$$f(3a-2a^2) = 27a^2(a-1)^2 \ge 0,$$

$$\begin{aligned} f\left(\frac{a^2-6a+9}{4}\right) &= \frac{27}{4}(a^4-8a^3+22a^2-24a+9) \\ &= \frac{27}{4}(a-1)^2(a-3)^2 \ge 0. \end{aligned}$$

The equality holds for $a = b = c = 1$, and also for $a = 3$ and $b = c = 0$ (or any cyclic permutation).

(b) By squaring, the inequality becomes

$$\sqrt{A}\left(\sqrt{B}+\sqrt{C}\right)+\sqrt{BC} \ge \frac{27-a^2-b^2-c^2}{2},$$

$$\sqrt{A\left(B+C+2\sqrt{BC}\right)}+\sqrt{BC} \ge \frac{27-a^2-b^2-c^2}{2},$$

where

$$A = 3a^2+1, \quad B = 3b^2+1, \quad C = 3c^2+1.$$

Applying Lemma from problem P 2.77 for $k = 3$ and $m = 1/3$ gives

$$\sqrt{BC} \ge 3bc+1+\frac{1}{3}(b-c)^2.$$

Therefore, it suffices to show that

$$\sqrt{A\left[B+C+6bc+2+\frac{2}{3}(b-c)^2\right]}+3bc+1+\frac{1}{3}(b-c)^2 \geq \frac{27-a^2-b^2-c^2}{2},$$

which is equivalent to

$$2\sqrt{3(3a^2+1)[11(b+c)^2+12-8bc]} \geq 75-3a^2-5(b+c)^2-4bc,$$

$$\sqrt{3(3a^2+1)(11a^2-66a+111-8bc)} \geq 15+15a-4a^2-2bc.$$

From $bc \leq (b+c)^2/4$ and $(a-b)(a-c) \geq 0$, we get

$$bc \leq \frac{(3-a)^2}{4}, \quad bc \geq a(b+c)-a^2 = 3a-2a^2.$$

Consider a fixed, $a \geq 1$, and denote $x = bc$. So, we only need to prove that $f(x) \geq 0$ for

$$3a-2a^2 \leq x \leq \frac{a^2-6a+9}{4},$$

where

$$f(x) = 3(3a^2+1)(11a^2-66a+111-8x)-(15+15a-4a^2-2x)^2.$$

Since f is concave, it suffices to show that $f(3a-2a^2) \geq 0$ and $f\left(\frac{a^2-6a+9}{4}\right) \geq 0$. Indeed,

$$f(3a-2a^2) = 27(a-1)^2(3a-2)^2 \geq 0,$$

$$\begin{aligned} f\left(\frac{a^2-6a+9}{4}\right) &= \frac{27}{4}(9a^4-48a^3+94a^2-80a+25) \\ &= \frac{27}{4}(a-1)^2(3a-5)^2 \geq 0. \end{aligned}$$

The equality holds for $a = b = c = 1$, and also for $a = 5/3$ and $b = c = 2/3$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization.

- *Let a, b, c be nonnegative real numbers such that $a+b+c=3$. If $k>0$, then*

$$\sqrt{ka^2+1}+\sqrt{kb^2+1}+\sqrt{kc^2+1} \geq \sqrt{\frac{8k(a^2+b^2+c^2)+3(9k^2+10k+9)}{3(k+1)}},$$

with equality for $a = b = c = 1$, and also for $a = \frac{3k+1}{2k}$ and $b = c = \frac{3k-1}{4k}$ (or any cyclic permutation).

□

P 2.80. *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

$$\sqrt{(32a^2+3)(32b^2+3)}+\sqrt{(32b^2+3)(32c^2+3)}+\sqrt{(32c^2+3)(32a^2+3)}\le 105.$$

(Vasile Cîrtoaje, 2014)

Solution. Assume that

$$a\le b\le c,\qquad a\le 1,\quad b+c\ge 2.$$

Denote

$$A=32a^2+3,\quad B=32b^2+3,\quad C=32c^2+3,$$

and write the inequality as follows:

$$\sqrt{A}\left(\sqrt{B}+\sqrt{C}\right)+\sqrt{BC}\le 105,$$

$$\sqrt{A}\cdot\sqrt{B+C+2\sqrt{BC}}\le 105-\sqrt{BC}.$$

By Lemma below, we have

$$\sqrt{BC}\le 5(b+c)^2+12bc+3\le 8(b+c)^2+3\le 8(a+b+c)^2+3=75<105.$$

Therefore, we can write the desired inequality as

$$A\left(B+C+2\sqrt{BC}\right)\le\left(105-\sqrt{BC}\right)^2,$$

which is equivalent to

$$A(A+B+C+210)\le(A+105-\sqrt{BC})^2.$$

According to Lemma below, it suffices to show that

$$A(A+B+C+210)\le[A+105-5(b^2+c^2)-22bc-3]^2,$$

which is equivalent to

$$[32a^2+105-5(b^2+c^2)-22bc]^2\ge(32a^2+3)[32(a^2+b^2+c^2)+219].$$

Since

$$32(a^2+b^2+c^2)+219=32a^2+32(b+c)^2-64bc+219=64a^2-192a+507-64bc$$

and

$$32a^2+105-5(b^2+c^2)-22bc=32a^2+105-5(b+c)^2-12bc=3(9a^2+10a+20-4bc),$$

we need to show that

$$9(9a^2+10a+20-4bc)^2\ge(32a^2+3)(64a^2-192a+507-64bc).$$

From $bc \le (b+c)^2/4$, we get

$$bc \le \frac{(3-a)^2}{4}.$$

Consider a fixed, $0 \le a \le 1$, and denote $x = bc$. So, we only need to prove that $f(x) \ge 0$ for

$$0 \le x \le \frac{a^2-6a+9}{4},$$

where

$$f(x) = 9(9a^2+10a+20-4x)^2 - (32a^2+3)(64a^2-192a+507-64x).$$

Since

$$\begin{aligned} f'(x) &= 72(4x-9a^2-10a-20)+64(32a^2+3) \\ &\le 72[(a^2-6a+9)-9a^2-10a-20)+64(32a^2+3) \\ &= 8[184a(a-1)+(44a-75)] < 0, \end{aligned}$$

f is decreasing, hence $f(x) \ge f\left(\dfrac{a^2-6a+9}{4}\right)$. Therefore, it suffices to show that $f\left(\dfrac{a^2-6a+9}{4}\right) \ge 0$. We have

$$\begin{aligned} f\left(\frac{a^2-6a+9}{4}\right) =& 9[9a^2+10a+20-(a^2-6a+9)]^2 \\ & -(32a^2+3)[64a^2-192a+507-16(a^2-6a+9)] \\ =& 9(8a^2+16a+11)^2-(32a^2+3)(48a^2-96a+363) \\ =& 192a(a-1)^2(18-5a) \ge 0. \end{aligned}$$

Thus, the proof is completed. The equality holds for $a = b = c = 1$, and also for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

Lemma. *If $b, c \ge 0$ such that $b+c \ge 2$, then*

$$\sqrt{(32b^2+3)(32c^2+3)} \le 5(b^2+c^2)+22bc+3.$$

Proof. By squaring, the inequality becomes

$$(5b^2+5c^2+22bc)^2 - 32^2b^2c^2 \ge 96(b^2+c^2) - 6(5b^2+5c^2+22bc),$$

$$5(b-c)^2(5b^2+5c^2+54bc) \ge 66(b-c)^2.$$

It suffices to show that

$$5(5b^2+5c^2+10bc) \ge 100,$$

which is equivalent to the obvious inequality $(b+c)^2 \ge 4$.

□

P 2.81. *If a, b, c are positive real numbers, then*

$$\left|\frac{b+c}{a}-3\right|+\left|\frac{c+a}{b}-3\right|+\left|\frac{a+b}{c}-3\right| \geq 2.$$

(Vasile Cîrtoaje, 2012)

Solution. Without loss of generality, assume that $a \geq b \geq c$.

Case 1: $a > b+c$. We have

$$\left|\frac{b+c}{a}-3\right|+\left|\frac{a+b}{c}-3\right|+\left|\frac{c+a}{b}-3\right| \geq \left|\frac{b+c}{a}-3\right| = 3-\frac{b+c}{a} > 2.$$

Case 2: $a \leq b+c$. We have

$$\left|\frac{b+c}{a}-3\right|+\left|\frac{a+b}{c}-3\right|+\left|\frac{c+a}{b}-3\right| \geq \left|\frac{b+c}{a}-3\right|+\left|\frac{c+a}{b}-3\right|$$

$$=\left(3-\frac{b+c}{a}\right)+\left(3-\frac{c+a}{b}\right) \geq 6-\frac{b+b}{a}-\frac{b+a}{b} = 2+\frac{(a-b)(2b-a)}{ab} \geq 2.$$

Thus, the proof is completed. The equality holds for $\frac{a}{2} = b = c$ (or any cyclic permutation).

□

P 2.82. *If a, b, c are real numbers such that $abc \neq 0$, then*

$$\left|\frac{b+c}{a}\right|+\left|\frac{c+a}{b}\right|+\left|\frac{a+b}{c}\right| \geq 2.$$

First Solution. Let

$$|a| = \max\{|a|, |b|, |c|\}.$$

We have

$$\left|\frac{b+c}{a}\right|+\left|\frac{c+a}{b}\right|+\left|\frac{a+b}{c}\right| \geq \left|\frac{b+c}{a}\right|+\left|\frac{c+a}{a}\right|+\left|\frac{a+b}{a}\right|$$

$$\geq \frac{|(-b-c)+(c+a)+(a+b)|}{|a|} = 2.$$

The equality holds for $a = 1$, $b = -1$ and $|c| \leq 1$ (or any permutation).

Second Solution. Since the inequality remains unchanged by replacing a, b, c with $-a, -b, -c$, it suffices to consider two cases: $a, b, c > 0$, and $a < 0$, $b, c > 0$.

Case 1: $a, b, c > 0$. We have

$$\left|\frac{b+c}{a}\right|+\left|\frac{c+a}{b}\right|+\left|\frac{a+b}{c}\right| = \left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{c}{a}+\frac{a}{c}\right) \geq 6.$$

Case 2: $a < 0$ and $b, c > 0$. Replacing a by $-a$, we need to show that

$$\frac{b+c}{a} + \frac{|a-c|}{b} + \frac{|a-b|}{c} \geq 2$$

for all $a, b, c > 0$. Without loss of generality, assume that $b \geq c$. There are three case to consider: $b \geq c \geq a$, $b \geq a \geq c$ and $a \geq b \geq c$.
For $b \geq c \geq a$, we have

$$\frac{b+c}{a} + \frac{|a-c|}{b} + \frac{|a-b|}{c} \geq \frac{b+c}{a} \geq 2.$$

For $b \geq a \geq c$, we have

$$\frac{b+c}{a} + \frac{|a-c|}{b} + \frac{|a-b|}{c} - 2 \geq \frac{b+c}{a} + \frac{a-c}{b} - 2 = \frac{(a-b)^2 + c(b-a)}{ab} \geq 0.$$

For $a \geq b \geq c$, we have

$$\frac{b+c}{a} + \frac{|a-c|}{b} + \frac{|a-b|}{c} - 2 = \frac{b+c}{a} + \frac{a-c}{b} + \frac{a-b}{c} - 2$$

$$= \left(\frac{a}{b} + \frac{b}{a} - 2\right) + \frac{a-b}{c} + c\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{(a-b)^2}{ab} + \frac{(a-b)(ab-c^2)}{abc} \geq 0.$$

Third Solution. Using the substitution

$$x = \frac{b+c}{a}, \quad y = \frac{c+a}{b}, \quad z = \frac{a+b}{c},$$

we need to show that

$$x + y + z + 2 = xyz, \quad x, y, z \in \mathbb{R},$$

involves

$$|x| + |y| + |z| \geq 2.$$

If $xyz \leq 0$, then

$$-x - y - z = 2 - xyz \geq 2,$$

hence

$$|x| + |y| + |z| \geq |x + y + z| = |-x - y - z| \geq -x - y - z \geq 2.$$

If $xyz > 0$, then either $x, y, z > 0$ or only one of x, y, z is positive (for instance, $x > 0$ and $y, z < 0$).

Case 1: $x, y, z > 0$. We need to show that $x + y + z \geq 2$. We have

$$xyz = x + y + z + 2 > 2$$

and, by the AM-GM inequality, we get

$$x+y+z \ge 3\sqrt[3]{xyz} > 3\sqrt[3]{2} > 2,$$

Case 2: $x > 0$ and $y,z < 0$. Replacing y,z by $-y,-z$, we need to prove that

$$x-y-z+2 = xyz$$

involves

$$x+y+z \ge 2$$

for all $x,y,z > 0$. Since

$$x+y+z-2 = x+y+z-(xyz-x+y+z) = x(2-yz),$$

we need to show that $yz \le 2$. Indeed, we have

$$x+2 = y+z+xyz \ge 2\sqrt{yz}+xyz,$$

$$x(1-yz)+2(1-\sqrt{yz}\,) \ge 0,$$

$$(1-\sqrt{yz}\,)[x(1+\sqrt{yz}\,)+2] \ge 0,$$

hence

$$yz \le 1 < 2.$$

□

P 2.83. *Let a,b,c be nonnegative real numbers, no two of which are zero, and let*

$$x = \frac{2a}{b+c}, \quad y = \frac{2b}{c+a}, \quad z = \frac{2c}{a+b}.$$

Prove that

(a) $$\sqrt{xy}+\sqrt{yz}+\sqrt{zx} \ge xyz+2;$$

(b) $$x+y+z+\sqrt{xy}+\sqrt{yz}+\sqrt{zx} \ge 6;$$

(c) $$\sqrt{x}+\sqrt{y}+\sqrt{z} \ge \sqrt{8+xyz};$$

(d) $$\frac{\sqrt{yz}}{x+2}+\frac{\sqrt{zx}}{y+2}+\frac{\sqrt{xy}}{z+2} \ge 1.$$

Solution. (a) Since

$$\sqrt{yz} = \frac{2\sqrt{bc(a+b)(c+a)}}{(a+b)(c+a)} \ge \frac{2\sqrt{bc}\,(a+\sqrt{bc})}{(a+b)(c+a)}$$
$$= \frac{2a(b+c)\sqrt{bc}+2bc(b+c)}{(a+b)(b+c)(c+a)} \ge \frac{4abc+2bc(b+c)}{(a+b)(b+c)(c+a)},$$

we have

$$\sum \sqrt{yz} \ge \frac{12abc+2\sum bc(b+c)}{(a+b)(b+c)(c+a)}$$
$$= \frac{8abc}{(a+b)(b+c)(c+a)} + 2 = xyz + 2.$$

The equality holds for $a = b = c$, and also for $a = 0$ or $b = 0$ or $c = 0$.

(b) **First Solution.** Taking into account the inequality (a), it suffices to show that

$$x + y + z + xyz \ge 4,$$

which is equivalent to Schur's inequality of degree three

$$a^3 + b^3 + c^3 + 3abc \ge \sum ab(a+b).$$

The equality holds for $a = b = c$, and also for $a = 0$ and $b = c$ (or any cyclic permutation).

Second Solution. We use the SOS technique. Write the inequality as

$$4\sum(x-1) \ge \sum\left(\sqrt{y}-\sqrt{z}\right)^2.$$

Since

$$\sum(x-1) = \sum \frac{(a-b)+(a-c)}{b+c} = \sum \frac{a-b}{b+c} + \sum \frac{b-a}{c+a}$$
$$= \sum \frac{(a-b)^2}{(b+c)(c+a)} = \sum \frac{(b-c)^2}{(a+b)(a+c)}$$

and

$$\left(\sqrt{y}-\sqrt{z}\right)^2 = \frac{(y-z)^2}{\left(\sqrt{y}+\sqrt{z}\right)^2} = \frac{2(b-c)^2(a+b+c)^2}{(a+b)(a+c)\left(\sqrt{b^2+ab}+\sqrt{c^2+ac}\right)^2},$$

we can write the inequality as

$$\sum(b-c)^2 S_a \ge 0,$$

where

$$S_a = (b+c)\left[2 - \frac{(a+b+c)^2}{\left(\sqrt{b^2+ab}+\sqrt{c^2+ac}\right)^2}\right].$$

By Minkowski's inequality, we have

$$\begin{aligned}\left(\sqrt{b^2+ab}+\sqrt{c^2+ac}\right)^2 &\ge (b+c)^2 + a\left(\sqrt{b}+\sqrt{c}\right)^2\\ &\ge (b+c)^2 + a(b+c) = (b+c)(a+b+c),\end{aligned}$$

hence

$$S_a \ge (b+c)\left(2 - \frac{a+b+c}{b+c}\right) = b+c-a.$$

Thus, it suffices to show that

$$\sum (b-c)^2(b+c-a) \ge 0,$$

which is just Schur's inequality of third degree.

Third Solution. Using the Cauchy-Schwarz inequality yields

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \ge \frac{(a+b+c)^2}{a(b+c)+b(c+a)+c(a+b)} = \frac{(a+b+c)^2}{2(ab+bc+ca)}.$$

Also, using Hölder's inequality, we have

$$\left(\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}\right)^2 \ge \frac{(a+b+c)^3}{a^2(b+c)+b^2(c+a)+c^2(a+b)}.$$

Thus, it suffices to prove that

$$\frac{(a+b+c)^2}{ab+bc+ca} + \frac{2(a+b+c)^3}{a^2(b+c)+b^2(c+a)+c^2(a+b)} \ge 12.$$

Due to homogeneity, we may assume that $a+b+c=1$. Substituting

$$q = ab+bc+ca, \qquad 3q \le 1,$$

the inequality becomes

$$\frac{1}{q} + \frac{2}{q-3abc} \ge 12.$$

The fourth degree Schur's inequality

$$6abcp \ge (p^2-q)(4q-p^2), \quad p = a+b+c,$$

gives

$$6abc \ge (1-q)(4q-1).$$

Therefore,

$$\frac{1}{q}+\frac{2}{q-3abc}-12\geq\frac{1}{q}+\frac{4}{2q-(1-q)(4q-1)}-12=\frac{(1-3q)(1-4q)^2}{q(4q^2-3q+1)}\geq 0.$$

(c) By squaring, the inequality becomes

$$x+y+z+2\sqrt{xy}+2\sqrt{yz}+2\sqrt{zx}\geq 8+xyz.$$

Based on the inequality in (a), it suffices to show that

$$x+y+z+2(xyz+2)\geq 8+xyz,$$

which is equivalent to

$$x+y+z+xyz\geq 4,$$

$$a^3+b^3+c^3+3abc\geq\sum ab(a+b).$$

The last form is just Schur's inequality of third degree. The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

(d) Write the inequality as

$$\sum(b+c)\sqrt{yz}\geq 2(a+b+c).$$

First Solution. Since

$$\begin{aligned}\sqrt{yz}&=\frac{2\sqrt{bc(a+b)(c+a)}}{(a+b)(c+a)}\geq\frac{2\sqrt{bc}\,(a+\sqrt{bc})}{(a+b)(c+a)}\\&=\frac{2a(b+c)\sqrt{bc}+2bc(b+c)}{(a+b)(b+c)(c+a)}\geq\frac{4abc+2bc(b+c)}{(a+b)(b+c)(c+a)},\end{aligned}$$

it suffices to show that

$$\sum(b+c)[2abc+bc(b+c)]\geq(a+b+c)(a+b)(b+c)(c+a),$$

which is an identity. The equality holds for $a=b=c$, and also for $a=0$ or $b-0$ or $c=0$.

Second Solution. Let

$$q=ab+bc+ca.$$

Since

$$\sqrt{yz}=\sqrt{\frac{2b}{a+b}\cdot\frac{2c}{c+a}}\geq\frac{2\cdot\frac{2b}{a+b}\cdot\frac{2c}{c+a}}{\frac{2b}{a+b}+\frac{2c}{c+a}}=\frac{4bc}{bc+q},$$

we can write the inequality as follows:

$$\sum\frac{2bc(b+c)}{bc+q}\geq a+b+c,$$

$$\sum\left[\frac{2bc(b+c)}{bc+q}-a\right]\ge 0,$$

$$\sum\frac{bc(b-a)+bc(c-a)+b(c^2-a^2)+c(b^2-a^2)}{bc+q}\ge 0,$$

$$\sum\frac{c(b-a)(2b+a)+b(c-a)(2c+a)}{bc+q}\ge 0,$$

$$\sum\frac{c(b-a)(2b+a)}{bc+q}+\sum\frac{c(a-b)(2a+b)}{ca+q}\ge 0,$$

$$\sum c(a-b)\left[\frac{2a+b}{ca+q}-\frac{2b+a}{bc+q}\right]\ge 0,$$

$$\sum\frac{c(a-b)[q(a-b)-c(a^2-b^2)]}{(ca+q)(bc+q)}\ge 0,$$

$$abc\sum\frac{(a-b)^2}{(ca+q)(bc+q)}\ge 0.$$

□

P 2.84. *Let a,b,c be nonnegative real numbers, no two of which are zero, and let*

$$x=\frac{2a}{b+c},\quad y=\frac{2b}{c+a},\quad z=\frac{2c}{a+b}.$$

Prove that

$$\sqrt{1+24x}+\sqrt{1+24y}+\sqrt{1+24z}\ge 15.$$

(Vasile Cîrtoaje, 2005)

Solution *(by Vo Quoc Ba Can).* Assume that $c=\min\{a,b,c\}$, hence $z\le 1$. By Hölder's inequality

$$\left(\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}\right)^2\left[a^2(b+c)+b^2(c+a)\right]\ge (a+b)^3,$$

we get

$$\begin{aligned}\left(\sqrt{x}+\sqrt{y}\right)^2 &\ge \frac{2(a+b)^3}{c(a^2+b^2)+ab(a+b)}=\frac{2(a+b)^3}{c(a+b)^2+ab(a+b-2c)}\\ &\ge \frac{2(a+b)^3}{c(a+b)^2+\frac{1}{4}(a+b)^2(a+b-2c)}=\frac{8(a+b)}{a+b+2c}=\frac{8}{1+z}.\end{aligned}$$

Using this result and Minkowski's inequality, we have

$$\sqrt{1+24x}+\sqrt{1+24y}\ge\sqrt{(1+1)^2+24(\sqrt{x}+\sqrt{y})^2}\ge 2\sqrt{1+\frac{48}{1+z}}.$$

Therefore, it suffices to show that

$$2\sqrt{1+\frac{48}{1+z}}+\sqrt{1+24z}\ge 15.$$

Using the substitution

$$\sqrt{1+24z}=5t, \quad \frac{1}{5}\le t\le 1,$$

the inequality turns into

$$2\sqrt{\frac{t^2+47}{25t^2+23}}\ge 3-t.$$

By squaring, this inequality becomes

$$25t^4-150t^3+244t^2-138t+19\le 0,$$

which is equivalent to the obvious inequality

$$(t-1)^2(5t-1)(5t-19)\le 0.$$

The equality holds for $a=b=c$, and also for $a=b$ and $c=0$ (or any cyclic permutation).

□

P 2.85. *If a,b,c are positive real numbers, then*

$$\sqrt{\frac{7a}{a+3b+3c}}+\sqrt{\frac{7b}{b+3c+3a}}+\sqrt{\frac{7c}{c+3a+3b}}\le 3.$$

(Vasile Cîrtoaje, 2005)

First Solution. Using the substitution

$$x=\sqrt{\frac{7a}{a+3b+3c}}, \quad y=\sqrt{\frac{7b}{b+3c+3a}}, \quad z=\sqrt{\frac{7c}{c+3a+3b}},$$

we have

$$\begin{cases} (x^2-7)a+3x^2b+3x^2c=0 \\ 3y^2a+(y^2-7)b+3y^2c=0 \\ 3z^2a+3z^2b+(z^2-7)c=0 \end{cases},$$

which involves

$$\begin{vmatrix} x^2-7 & 3x^2 & 3x^2 \\ 3y^2 & y^2-7 & 3y^2 \\ 3z^2 & 3z^2 & z^2-7 \end{vmatrix}=0 \;;$$

that is,

$$F(x,y,z)=0,$$

where

$$F(x,y,z)=4x^2y^2z^2+8\sum x^2y^2+7\sum x^2-49.$$

We need to show that $F(x,y,z)=0$ involves $x+y+z\le 3$, where $x,y,z>0$. To do this, we use the contradiction method. Assume that $x+y+z>3$ and show that $F(x,y,z)>0$. Since $F(x,y,z)$ is strictly increasing in each of its arguments, it is enough to prove that $x+y+z=3$ involves $F(x,y,z)\ge 0$. We will use the mixing variables technique. Assume that $x=\max\{x,y,z\}$ and denote

$$t=\frac{y+z}{2},\quad 0<t\le 1\le x.$$

We will show that

$$F(x,y,z)\ge F(x,t,t)\ge 0.$$

We have

$$\begin{aligned}F(x,y,z)-F(x,t,t)&=(8x^2+7)(y^2+z^2-2t^2)-4(x^2+2)(t^4-y^2z^2)\\&=\frac{1}{2}(8x^2+7)(y-z)^2-(x^2+2)(t^2+yz)(y-z)^2\\&\ge\frac{1}{2}(8x^2+7)(y-z)^2-2(x^2+2)t^2(y-z)^2\\&=\frac{1}{2}(4x^2-1)(y-z)^2\ge 0\end{aligned}$$

and

$$F(x,t,t)=F\left(x,\frac{3-x}{2},\frac{3-x}{2}\right)=\frac{1}{4}(x-1)^2(x-2)^2(x^2-6x+23)\ge 0.$$

The equality holds for $a=b=c$, and also for $\frac{a}{8}=b=c$ (or any cyclic permutation).

Second Solution. Due to homogeneity, we may assume that $a+b+c=3$, when the inequality becomes

$$\sum\sqrt{\frac{7a}{9-2a}}\le 3.$$

Using the substitution

$$x=\sqrt{\frac{7a}{9-2a}},\quad y=\sqrt{\frac{7b}{9-2b}},\quad z=\sqrt{\frac{7c}{9-2c}},$$

we need to show that if x,y,z are positive real numbers such that

$$\sum\frac{1}{2x^2+7}=\frac{1}{3},$$

then

$$x + y + z \le 3.$$

For the sake of contradiction, assume that $x+y+z > 3$ and show that $F(x,y,z) < 0$, where

$$F(x,y,z) = \sum \frac{1}{2x^2+7} - \frac{1}{3}.$$

Since $F(x,y,z)$ is strictly decreasing in each of its arguments, it is enough to prove that $x+y+z=3$ involves $F(x,y,z) \le 0$. This is just the inequality in P 1.33.

□

P 2.86. *If a,b,c are positive real numbers such that $a+b+c=3$, then*

$$\sqrt[3]{a^2(b^2+c^2)} + \sqrt[3]{b^2(c^2+a^2)} + \sqrt[3]{c^2(a^2+b^2)} \le 3\sqrt[3]{2}.$$

(Michael Rozenberg, 2013)

Solution. By Hölder's inequality, we have

$$\left[\sum \sqrt[3]{a^2(b^2+c^2)}\right]^3 \le \left[\sum a(b+c)\right]^2 \cdot \sum \frac{b^2+c^2}{(b+c)^2}.$$

Therefore, it suffices to show that

$$\sum \frac{b^2+c^2}{(b+c)^2} \le \frac{27}{2(ab+bc+ca)^2},$$

which is equivalent to the homogeneous inequalities

$$\sum \left[\frac{b^2+c^2}{(b+c)^2} - 1\right] \le \frac{p^4}{6q^2} - 3,$$

$$\sum \frac{2bc}{(b+c)^2} + \frac{p^4}{6q^2} \ge 3,$$

where

$$p = a+b+c, \quad q = ab+bc+ca.$$

According to P 1.62, the following inequality holds

$$\sum \frac{2bc}{(b+c)^2} + \frac{p^2}{q} \ge \frac{9}{2}.$$

Thus, it is enough to show that

$$\frac{9}{2} - \frac{p^2}{q} + \frac{p^4}{6q^2} \ge 3,$$

which is equivalent to

$$\left(\frac{p^2}{q} - 3\right)^2 \ge 0.$$

The equality holds for $a=b=c=1$.

□

P 2.87. *If a,b,c are nonnegative real numbers, no two of which are zero, then*

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \ge \frac{1}{a+b+c}+\frac{2}{\sqrt{ab+bc+ca}}.$$

(Vasile Cîrtoaje, 2005)

Solution. Using the notation

$$p=a+b+c, \quad q=ab+bc+ca, \quad r=abc,$$

we can write the inequality as

$$\frac{p^2+q}{pq-r} \ge \frac{1}{p}+\frac{2}{\sqrt{q}}.$$

According to P 3.57-(a) in Volume 1, for fixed p and q, the product $r=abc$ is minimum when two of a,b,c are equal or one of a,b,c is zero. Therefore, it suffices to prove the inequality for $b=c=1$ and for $a=0$. For $a=0$, the inequality reduces to

$$\frac{1}{b}+\frac{1}{c} \ge \frac{2}{\sqrt{bc}},$$

which is obvious. For $b=c=1$, the inequality becomes as follows:

$$\frac{1}{2}+\frac{2}{a+1} \ge \frac{1}{a+2}+\frac{2}{\sqrt{2a+1}},$$

$$\frac{1}{2}-\frac{1}{a+2} \ge \frac{2}{\sqrt{2a+1}}-\frac{2}{a+1},$$

$$\frac{a}{2(a+2)} \ge \frac{2\left(a+1-\sqrt{2a+1}\right)}{(a+1)\sqrt{2a+1}},$$

$$\frac{a}{2(a+2)} \ge \frac{2a^2}{(a+1)\sqrt{2a+1}\left(a+1+\sqrt{2a+1}\right)}.$$

So, we need to show that

$$\frac{1}{2(a+2)} \ge \frac{2a}{(a+1)\sqrt{2a+1}\left(a+1+\sqrt{2a+1}\right)}.$$

Consider two cases: $0 \le a \le 1$ and $a > 1$.

Case 1: $0 \le a \le 1$. Since

$$\sqrt{2a+1}(a+1+\sqrt{2a+1}) \ge \sqrt{2a+1}\left(\sqrt{2a+1}+\sqrt{2a+1}\right)=2(2a+1),$$

it suffices to prove that

$$\frac{1}{2(a+2)} \ge \frac{a}{(a+1)(2a+1)},$$

which is equivalent to $1-a \ge 0$.

Case 2: $a > 1$. Write the desired inequality as

$$\frac{1}{2(a+2)} \ge \frac{2a}{(a+1)\left[(a+1)\sqrt{2a+1}+2a+1\right]}.$$

First, we will show that

$$(a+1)\sqrt{2a+1} > 3a.$$

Indeed, by squaring, we get the obvious inequality

$$a^3 + a(a-2)^2 + 1 > 0.$$

Therefore, it suffices to show that

$$\frac{1}{2(a+2)} \ge \frac{2a}{(a+1)(3a+2a+1)},$$

which is equivalent to $(a-1)^2 \ge 0$.

The equality holds for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 2.88. *If $a, b \ge 1$, then*

$$\frac{1}{\sqrt{3ab+1}} + \frac{1}{2} \ge \frac{1}{\sqrt{3a+1}} + \frac{1}{\sqrt{3b+1}}.$$

Solution. Using the substitution

$$x = \frac{2}{\sqrt{3a+1}}, \quad y = \frac{2}{\sqrt{3b+1}}, \quad x, y \in (0,1],$$

the desired inequality can be written as

$$xy\sqrt{\frac{3}{x^2y^2 - x^2 - y^2 + 4}} \ge x + y - 1.$$

Consider the nontrivial case $x + y - 1 \ge 0$, and denote

$$t = x + y - 1, \quad p = xy.$$

We have

$$1 \ge p \ge t \ge 0.$$

Since

$$x^2 + y^2 = (x+y)^2 - 2xy = (t+1)^2 - 2p,$$

we need to prove that

$$p\sqrt{\frac{3}{p^2+2p-t^2-2t+3}} \geq t.$$

By squaring, we get the inequality

$$(p-t)[(3-t^2)p+t(1-t)(3+t)] \geq 0,$$

which is clearly true. The equality holds for $a=b=1$.

□

P 2.89. *Let a,b,c be positive real numbers such that $a+b+c=3$. If $k \geq \dfrac{1}{\sqrt{2}}$, then*

$$(abc)^k(a^2+b^2+c^2) \leq 3.$$

(Vasile Cîrtoaje, 2006)

Solution. Since

$$abc \leq \left(\frac{a+b+c}{3}\right)^3 = 1,$$

it suffices to prove the desired inequality for $k=1/\sqrt{2}$. Write the inequality in the homogeneous form

$$(abc)^k(a^2+b^2+c^2) \leq 3\left(\frac{a+b+c}{3}\right)^{3k+2}.$$

According to P 3.57-(a) in Volume 1, for fixed $a+b+c$ and $ab+bc+ca$, the product abc is maximum when two of a,b,c are equal. Therefore, it suffices to prove the homogeneous inequality for $b=c=1$; that is, $f(a) \geq 0$, where

$$f(a) = (3k+2)\ln(a+2)-(3k+1)\ln 3 - k\ln a - \ln(a^2+2).$$

From

$$\begin{aligned} f'(a) =& \frac{3k+2}{a+2} - \frac{k}{a} - \frac{2a}{a^2+2} = \frac{2(a-1)(ka^2-2a+2k)}{a(a+2)(a^2+2)} \\ =& \frac{\sqrt{2}(a-1)(a-\sqrt{2})^2}{a(a+2)(a^2+2)}, \end{aligned}$$

it follows that f is decreasing on $(0,1]$ and increasing on $[1,\infty)$; therefore, $f(a) \geq f(1)=0$. This completes the proof. The equality holds for $a=b=c=1$.

□

P 2.90. *If $a,b,c \in [0,4]$ and $ab+bc+ca=4$, then*

$$\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a} \le 3+\sqrt{5}.$$

(Vasile Cîrtoaje, 2019)

Solution. Assume that $a \ge b \ge c$, $1 \le a \le 4$, and write the inequality as follows

$$\sqrt{b+c}+\sqrt{(a+b)+(a+c)+2\sqrt{(a+b)(a+c)}} \le 3+\sqrt{5},$$

$$\sqrt{b+c}+\sqrt{2a+b+c+2\sqrt{a^2+4}} \le 3+\sqrt{5},$$

From $4-a(b+c)=bc \ge 0$, we get

$$b+c \le \frac{4}{a}.$$

Thus, it suffices to show that

$$\frac{2}{\sqrt{a}}+\sqrt{2a+\frac{4}{a}+2\sqrt{a^2+4}} \le 3+\sqrt{5},$$

which is equivalent to

$$\frac{2}{\sqrt{a}}+\frac{a+\sqrt{a^2+4}}{\sqrt{a}} \le 3+\sqrt{5},$$

$$a-3\sqrt{a}+2 \le \sqrt{5a}-\sqrt{a^2+4},$$

$$(\sqrt{a}-1)(\sqrt{a}-2) \le \frac{(a-1)(4-a)}{\sqrt{5a}+\sqrt{a^2+4}}.$$

This is true if

$$1 \le \frac{(\sqrt{a}+1)(\sqrt{a}+2)}{\sqrt{5a}+\sqrt{a^2+4}},$$

that can be written in the obvious form

$$(a+2-\sqrt{a^2+4})+(3-\sqrt{5})\sqrt{a} \ge 0.$$

The equality occurs for $a=4$, $b=1$ and $c=0$ (or any permutation).

□

P 2.91. *Let*

$$F(a,b,c)=\sqrt{\frac{a^2+b^2+c^2}{3}}-\frac{a+b+c}{3},$$

where a,b,c are positive real numbers such that

$$a^4bc \ge 1, \qquad a \le b \le c.$$

Then,

$$F(a,b,c) \ge F\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right).$$

(Vasile Cîrtoaje and Vasile Mircea Popa, 2020)

Solution. Write the inequality as $E(a,b,c) \ge 0$, where

$$E(a,b,c) = \sqrt{3(a^2+b^2+c^2)} - (a+b+c) - \sqrt{3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)} + \frac{1}{a}+\frac{1}{b}+\frac{1}{c},$$

and show that

$$E(a,b,c) \ge E(a,x,x) \ge 0,$$

where

$$x = \sqrt{bc} \ge a, \quad a^2x \ge 1, \quad x \ge 1.$$

Write the inequality $E(a,b,c) \ge E(a,x,x)$ it in the form

$$A - C \ge B - D,$$

where

$$A = \sqrt{3(a^2+b^2+c^2)} - \sqrt{3(a^2+2x^2)} = \frac{3(b-c)^2}{\sqrt{3(a^2+b^2+c^2)}+\sqrt{3(a^2+2x^2)}}$$

$$\ge \frac{3(b-c)^2}{\sqrt{3(x^2+b^2+c^2)}+3x},$$

$$B = (a+b+c)-(a+2x) = \left(\sqrt{b}-\sqrt{c}\right)^2,$$

$$C = \sqrt{3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)} - \sqrt{3\left(\frac{1}{a^2}+\frac{2}{x^2}\right)}$$

$$= \frac{3}{x^4}\cdot\frac{(b-c)^2}{\sqrt{3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}+\sqrt{3\left(\frac{1}{a^2}+\frac{2}{x^2}\right)}}$$

$$\le \frac{3}{x^4}\cdot\frac{(b-c)^2}{\sqrt{3\left(\frac{1}{x^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}+\frac{3}{x}} = \frac{3}{x^2}\cdot\frac{(b-c)^2}{\sqrt{3(x^2+c^2+b^2)}+3x},$$

$$D = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{1}{a}-\frac{2}{x} = \frac{\left(\sqrt{b}-\sqrt{c}\right)^2}{x^2}.$$

Thus, we need to show that

$$3\left(\sqrt{b}+\sqrt{c}\right)^2\left[\frac{1}{\sqrt{3(x^2+b^2+c^2)}+3x}-\frac{1}{x^2}\cdot\frac{1}{\sqrt{3(x^2+c^2+b^2)}+3x}\right]\ge\frac{x^2-1}{x^2}.$$

This inequality is true if

$$3\left(\sqrt{b}+\sqrt{c}\right)^2\ge\sqrt{3(x^2+b^2+c^2)}+3x,$$

that is equivalent to

$$\sqrt{3}\,\left(b+c+\sqrt{bc}\right)\ge\sqrt{bc+b^2+c^2},$$

which is true.

Write now the inequality $E(a,x,x)\ge 0$ in the form

$$\sqrt{3(a^2+2x^2)}-(a+2x)\ge\sqrt{3\left(\frac{1}{a^2}+\frac{2}{x^2}\right)}-\frac{1}{a}-\frac{2}{x}.$$

Since both sides of the inequality are nonnegative and $a^2x\ge 1$, it suffices to prove the homogeneous inequality

$$\sqrt{3(a^2+2x^2)}-(a+2x)\ge(a^2x)^{2/3}\left[\sqrt{3\left(\frac{1}{a^2}+\frac{2}{x^2}\right)}-\frac{1}{a}-\frac{2}{x}\right].$$

Due to homogeneity, we may set $x=1$. Thus, we need to show that $a\le x=1$ yields

$$\sqrt{3(a^2+2)}-a-2\ge a^{1/3}\left[\sqrt{3(1+2a^2)}-1-2a\right],$$

which is equivalent to

$$\frac{2(a-1)^2}{\sqrt{3(a^2+2)}+a+2}\ge a^{1/3}\frac{2(a-1)^2}{\sqrt{3(1+2a^2)}+1+2a}.$$

It is true if

$$\sqrt{3(1+2a^2)}+1+2a\ge a^{1/3}\left[\sqrt{3(a^2+2)}+a+2\right].$$

For $t=a^{1/3}$, $t\in(0,1]$, the inequality becomes

$$\sqrt{3(1+2t^6)}+1+2t^3\ge\sqrt{3(t^8+2t^2)}+t^4+2t,$$

which is true because

$$1+2t^6-(t^8+2t^2)=(1-t^4)(1-t^2)^2\ge 0,$$

$$1+2t^3-(t^4+2t)=(1-t^2)(1-t)^2\ge 0.$$

The equality occurs for $a=b=c\ge 1$.

Remark. The inequality is true in the particular case $a,b,c\ge 1$, which implies $a^4bc\ge 1$.

□

P 2.92. *Let*

$$F(a,b,c)=\sqrt{\frac{a^2+b^2+c^2}{3}}-\frac{a+b+c}{3},$$

where a,b,c are positive real numbers such that

$$a^2(b+c)\geq 2, \qquad a\leq b\leq c.$$

Then,

$$F(a,b,c)\geq F\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right).$$

(Vasile Cîrtoaje, 2020)

Solution. The proof follows the same way as the proof of the preceding P 2.91. Write the inequality as $E(a,b,c)\geq 0$, where

$$E(a,b,c)=\sqrt{a^2+b^2+c^2}-\frac{a+b+c}{\sqrt{3}}-\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}+\frac{1}{\sqrt{3}}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right),$$

and show that

$$E(a,b,c)\geq E(a,x,x)\geq 0,$$

where

$$x=\frac{b+c}{2}\geq b, \quad a^2x\geq 1, \quad x\geq 1.$$

Write the inequality $E(a,b,c)\geq E(a,x,x)$ it in the form

$$A+B\geq C,$$

where

$$A=\sqrt{a^2+b^2+c^2}-\sqrt{a^2+2x^2}$$
$$=\frac{(b-c)^2}{2}\cdot\frac{1}{\sqrt{a^2+b^2+c^2}+\sqrt{a^2+2x^2}}$$
$$\geq\frac{(b-c)^2}{2}\cdot\frac{1}{\sqrt{2b^2+c^2}+\sqrt{b^2+2x^2}},$$

$$B=\frac{1}{\sqrt{3}}\left(\frac{1}{b}+\frac{1}{c}-\frac{2}{x}\right)=\frac{(b-c)^2}{\sqrt{3}bc(b+c)},$$

$$C=\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}-\sqrt{\frac{1}{a^2}+\frac{2}{x^2}}$$
$$=\frac{(b-c)^2(b^2+4bc+c^2)}{b^2c^2(b+c)^2}\cdot\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}+\sqrt{\frac{1}{a^2}+\frac{2}{x^2}}}$$

$$\leq \frac{(b-c)^2(b^2+4bc+c^2)}{b^2c^2(b+c)^2} \cdot \frac{1}{\sqrt{\frac{2}{b^2}+\frac{1}{c^2}}+\sqrt{\frac{1}{b^2}+\frac{2}{x^2}}}.$$

Thus, we need to show that

$$\frac{1}{2} \cdot \frac{1}{\sqrt{2b^2+c^2}+\sqrt{b^2+2x^2}} + \frac{1}{\sqrt{3}bc(b+c)} \geq \frac{b^2+4bc+c^2}{b^2c^2(b+c)^2} \cdot \frac{1}{\sqrt{\frac{2}{b^2}+\frac{1}{c^2}}+\sqrt{\frac{1}{b^2}+\frac{2}{x^2}}}.$$

Since

$$b^2+4bc+c^2 = 4bc+(b^2+c^2),$$

it suffices to show that

$$\frac{1}{\sqrt{3}bc(b+c)} \geq \frac{4bc}{b^2c^2(b+c)^2} \cdot \frac{1}{\sqrt{\frac{2}{b^2}+\frac{1}{c^2}}+\sqrt{\frac{1}{b^2}+\frac{2}{x^2}}}$$

and

$$\frac{1}{2} \cdot \frac{1}{\sqrt{2b^2+c^2}+\sqrt{b^2+2x^2}} \geq \frac{b^2+c^2}{b^2c^2(b+c)^2} \cdot \frac{1}{\sqrt{\frac{2}{b^2}+\frac{1}{c^2}}+\sqrt{\frac{1}{b^2}+\frac{2}{x^2}}}.$$

Write the first inequality as

$$(b+c)\left[\sqrt{\frac{2}{b^2}+\frac{1}{c^2}}+\sqrt{\frac{1}{b^2}+\frac{2}{x^2}}\right) \geq 4\sqrt{3}.$$

Since

$$\sqrt{\frac{2}{b^2}+\frac{1}{c^2}}+\sqrt{\frac{1}{b^2}+\frac{2}{x^2}} \geq \frac{1}{\sqrt{3}}\left(\frac{2}{b}+\frac{1}{c}\right)+\frac{1}{\sqrt{3}}\left(\frac{1}{b}+\frac{2}{x}\right)$$
$$\geq \frac{1}{\sqrt{3}}\left(\frac{2}{b}+\frac{1}{c}\right)+\frac{1}{\sqrt{3}}\left(\frac{1}{c}+\frac{2}{x}\right) = \frac{2}{\sqrt{3}}\left(\frac{1}{b}+\frac{1}{c}+\frac{2}{b+c}\right)$$
$$\geq \frac{2}{\sqrt{3}}\left(\frac{4}{b+c}+\frac{2}{b+c}\right) = \frac{4\sqrt{3}}{b+c},$$

the inequality is proved.

The second inequality reduces to

$$bc(b+c)^2 \geq 2(b^2+c^2).$$

It is true if the following homogeneous inequality is true:

$$bc(b+c)^2 \geq 2(b^2+c^2)\left[\frac{b^2(b+c)}{2}\right]^{2/3}.$$

Due to homogeneity, we may set $b=1$, hence $c \geq 1$, when the inequality becomes

$$c(c+1)^2 \geq 2(c^2+1)\left(\frac{c+1}{2}\right)^{2/3}.$$

It is true if

$$c^3(c+1)^4 \geq 2(c^2+1)^3,$$

that is

$$c^7 + 2c^6 + 6c^5 - 2c^4 + c^3 - 6c^2 - 2 \geq 0,$$

$$(c^7 + c^3 - 2) + 2c^4(c^2 - 1) + 6c^2(c^3 - 1) \geq 0.$$

To complete the proof, we need to show that $E(a,x,x) \geq 0$ for $a^2x \geq 1$, $x \geq a$. This inequality was proved at the preceding P 2.91.

The equality occurs for $a = b = c \geq 1$.

Remark. Since $a^4bc \geq 1$ yields $a^2(b+c) \geq 2$, the inequality in P 2.91 follows from the inequality in P 2.92.

□

P 2.93. *Let*

$$F(a,b,c) = \sqrt{\frac{a^2+b^2+c^2}{3}} - \frac{a+b+c}{3},$$

where a,b,c are positive real numbers such that

$$a^4(b^2+c^2) \geq 2, \qquad a \leq b \leq c.$$

Then,

$$F(a,b,c) \geq F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

(Vasile Cîrtoaje, 2020)

Solution. The proof follows the same way as the proof of the preceding P 2.92. Write the inequality as $E(a,b,c) \geq 0$, where

$$E(a,b,c) = \sqrt{a^2+b^2+c^2} - \frac{a+b+c}{\sqrt{3}} - \sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}} + \frac{1}{\sqrt{3}}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right),$$

and show that

$$E(a,b,c) \geq E(a,x,x) \geq 0,$$

where

$$x = \sqrt{\frac{b^2+c^2}{2}} \geq b, \quad a^2x \geq 1, \quad x \geq 1.$$

Write the inequality $E(a,b,c) \geq E(a,x,x)$ it in the form

$$A + B \geq C,$$

where

$$A = \frac{2x-b-c}{\sqrt{3}} = \frac{(b-c)^2}{\sqrt{3}\,(2x+b+c)}\ ,$$

$$B = \frac{1}{\sqrt{3}}\left(\frac{1}{b}+\frac{1}{c}-\frac{2}{x}\right) = \frac{(b-c)^2(b^2+c^2+4bc)}{2\sqrt{3}\ b^2c^2x^2\left(\frac{1}{b}+\frac{1}{c}+\frac{2}{x}\right)}\ ,$$

$$C = \sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}} - \sqrt{\frac{1}{a^2}+\frac{2}{x^2}}$$

$$= \frac{(b^2-c^2)^2}{2b^2c^2x^2}\cdot\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}+\sqrt{\frac{1}{a^2}+\frac{2}{x^2}}}$$

$$\le \frac{\sqrt{3}(b^2-c^2)^2}{2b^2c^2x^2}\cdot\frac{1}{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\left(\frac{1}{a}+\frac{2}{x}\right)}$$

$$\le \frac{\sqrt{3}(b^2-c^2)^2}{2b^2c^2x^2}\cdot\frac{1}{\frac{3}{b}+\frac{1}{c}+\frac{2}{x}}\ .$$

Thus, we need to show that

$$\frac{1}{2x+b+c}+\frac{b^2+c^2+4bc}{2b^2c^2x^2\left(\frac{1}{b}+\frac{1}{c}+\frac{2}{x}\right)} \ge \frac{3(b+c)^2}{2b^2c^2x^2}\cdot\frac{1}{\frac{3}{b}+\frac{1}{c}+\frac{2}{x}}\ .$$

Since

$$b^2x \ge a^2x \ge 1,$$

it suffices to prove the homogeneous inequality

$$\frac{1}{(b^2x)^{3/2}(2x+b+c)}+\frac{b^2+c^2+4bc}{2b^2c^2x^2\left(\frac{1}{b}+\frac{1}{c}+\frac{2}{x}\right)} \ge \frac{3(b+c)^2}{2b^2c^2x^2}\cdot\frac{1}{\frac{3}{b}+\frac{1}{c}+\frac{2}{x}}\ .$$

Since

$$2\left(\frac{3}{b}+\frac{1}{c}+\frac{2}{x}\right)-3\left(\frac{1}{b}+\frac{1}{c}+\frac{2}{x}\right) = \frac{3}{b}-\frac{2}{c}-\frac{2}{x} \ge 0,$$

it is enough to show that

$$\frac{1}{(b^2x)^{2/3}(2x+b+c)}+\frac{b^2+c^2+4bc}{2b^2c^2x^2\left(\frac{1}{b}+\frac{1}{c}+\frac{2}{x}\right)} \ge \frac{2(b+c)^2}{2b^2c^2x^2}\cdot\frac{1}{\frac{1}{b}+\frac{1}{c}+\frac{2}{x}}\ ,$$

that is

$$\frac{1}{(b^2x)^{2/3}(2x+b+c)} \ge \frac{1}{b^2c^2}\cdot\frac{1}{\frac{1}{b}+\frac{1}{c}+\frac{2}{x}}\ ,$$

$$c\left(b+c+\frac{2bc}{x}\right) \ge b^{1/3}\left[2x^{5/3}+(b+c)x^{2/3}\right].$$

Since $x \le c$, it suffices to show that

$$c\left(b+c+\frac{2bc}{c}\right) \ge b^{1/3}\left[2cx^{2/3}+(b+c)x^{2/3}\right],$$

that is

$$c(3b+c) \geq (b+3c)(bx^2)^{1/3}.$$

Due to homogeneity, we may set $c = 1$, when $0 < b \leq 1$ and

$$x = \sqrt{\frac{b^2+1}{2}}.$$

Thus, we need to show that

$$3b+1 \geq (b+3)\sqrt[3]{\frac{b^3+b}{2}},$$

which is true if

$$2(3b+1)^3 \geq b(b^2+1)(b+3)^3.$$

Since

$$(b+3)^3 = b^3+39b^2+27b+27 \leq 37b+27 \leq 32(b+1),$$

it suffices to sow that

$$(3b+1)^3 \geq 16(b^2+1)(b+1),$$

which is equivalent to

$$1-7b+11b^2+11b^3-16b^4 \geq 0,$$

$$(1-b)(1-6b+5b^2+16b^3) \geq 0.$$

This is true because

$$1-6b+5b^2+16b^3 = (1-4b)^2+b(2-11b+16b^2) > 0.$$

To complete the proof, we need to show that $E(a,x,x) \geq 0$ for $a^2x \geq 1$, $x \geq a$. This inequality was proved at P 2.91.

The equality occurs for $a = b = c \geq 1$.

Remark. Since $a^2(b+c) \geq 1$ yields $a^4(b^2+c^2) \geq 2$, the inequality in P 2.92 follows from the inequality in P 2.93.

□

P 2.94. *Let*

$$F(a,b,c) = \sqrt[3]{abc} - \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}},$$

where a,b,c are positive real numbers such that

$$a^4b^7c^7 \geq 1, \qquad a \geq b \geq c.$$

Then,

$$F(a,b,c) \geq F\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right).$$

(Vasile Cîrtoaje and Vasile Mircea Popa, 2019)

Solution. By the AM-GM inequality, both sides of the inequality are nonnegative. Denote

$$x = \sqrt{bc}.$$

We have

$$a \ge 1, \quad x \le a, \quad a^2x^7 \ge 1.$$

From

$$x \ge \frac{1}{a^{2/7}} \ge \frac{1}{a^{1/2}},$$

it follows that

$$a \ge \frac{1}{x^2}.$$

Write the inequality as $E(a,b,c) \ge 0$, where

$$E(a,b,c) = \sqrt[3]{abc} - \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} - \frac{1}{\sqrt[3]{abc}} + \frac{3}{a+b+c},$$

and prove that

$$E(a,b,c) \ge E(a,x,x) \ge 0.$$

We will show that the left inequality is true for $a \ge 1$ and $a \ge \dfrac{1}{x^2}$. Write the inequality as follows

$$\frac{1}{a+b+c} - \frac{1}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \ge \frac{1}{a+2\sqrt{bc}} - \frac{1}{\frac{1}{a}+\frac{2}{\sqrt{bc}}},$$

$$\frac{1}{\frac{1}{a}+\frac{2}{\sqrt{bc}}} - \frac{1}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \ge \frac{1}{a+2\sqrt{bc}} - \frac{1}{a+b+c},$$

$$\frac{\left(\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{c}}\right)^2}{\left(\frac{1}{a}+\frac{2}{\sqrt{bc}}\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \ge \frac{(\sqrt{b}-\sqrt{c})^2}{(a+2\sqrt{bc})(a+b+c)}.$$

After dividing by $(\sqrt{b}-\sqrt{c})^2$, we need to show that

$$(a+2x)(a+b+c) \ge x^2\left(\frac{1}{a}+\frac{2}{x}\right)\left(\frac{1}{a}+\frac{b+c}{x^2}\right). \tag{*}$$

Write this inequality as

$$A(b+c)+B \ge 0,$$

where

$$A = a+2x-\frac{1}{a}-\frac{2}{x}, \quad B = a^2+2ax-\frac{x^2}{a^2}-\frac{2x}{a}.$$

Clearly, $A \ge 0$ for $x \ge 1$. Also,$A \ge 0$ for $x \le 1$, because

$$A \ge \frac{1}{x^2}+2x-x^2-\frac{2}{x} = \frac{(1-x)^3(1+x)}{x^2} \ge 0.$$

Since $A \ge 0$ and $b+c \ge 2\sqrt{bc}$, it suffices to replace $b+c$ in (*) with $2x$. So, we need to show that

$$(a+2x)(a+2x) \ge x^2\left(\frac{1}{a}+\frac{2}{x}\right)\left(\frac{1}{a}+\frac{2}{x}\right) ,$$

which is equivalent to

$$a+2x \ge x\left(\frac{1}{a}+\frac{2}{x}\right),$$

$$a+2x \ge \frac{x}{a}+2.$$

For $x \ge 1$, we have

$$a+2x-\frac{x}{a}-2 = a-2+\left(2-\frac{1}{a}\right)x \ge a-2+\left(2-\frac{1}{a}\right) = a-\frac{1}{a} \ge 0 ,$$

and for $x \le 1$, we have

$$a+2x-\frac{x}{a}-2 \ge \frac{1}{x^2}+2x-x^3-2 = \frac{(1-x)(1+x-x^2+x^3+x^4)}{x^2} \ge 0 .$$

Write the right inequality $E(a,x,x) \ge 0$, as follows

$$\sqrt[3]{ax^2}-\frac{3ax}{2a+x} \ge \frac{1}{\sqrt[3]{ax^2}}-\frac{3}{a+2x} .$$

Since $a^{4/7}x^2 \ge 1$, it suffices to prove the homogeneous inequality

$$\sqrt[3]{ax^2}-\frac{3ax}{2a+x} \ge \left(a^{4/7}x^2\right)^{7/9}\left(\frac{1}{\sqrt[3]{ax^2}}-\frac{3}{a+2x}\right) .$$

Setting $x=1$ and substituting

$$a = d^9, \qquad d \ge 1,$$

the inequality becomes

$$d^3-\frac{3d^9}{2d^9+1} \ge d^4\left(\frac{1}{d^3}-\frac{3}{d^9+2}\right) ,$$

$$\frac{d^2(d^3-1)^2(2d^3+1)}{2d^9+1} \ge \frac{(d^3-1)^2(d^3+2)}{d^9+2} .$$

Thus, we need to show that

$$d^2(2d^3+1)(d^9+2) \ge (d^3+2)(2d^9+1) ,$$

that is

$$2(d^{12}+1)(d^2-1)+d^3(d^8-1)-4d^5(d^4-1) \ge 0 ,$$

$$(d^2-1)A \ge 0,$$

where

$$A = 2(d^{12}+1)+d^3(d^6+d^4+d^2+1)-4d^5(d^2+1)$$
$$= 2d^7(d^5-1)+d^7(d^2-1)-3d^3(d^2-1)-2(d^3-1)$$
$$\ge 2d(d^5-1)+(d^2-1)-3d^3(d^2-1)-2(d^3-1) = (d-1)B\ ,$$

where

$$B = 2d(d^4+d^3+d^2+d+1)+(d+1)-3d^3(d+1)-2(d^2+d+1)$$

$$= 2d^5-d^4-d^3+d-1 = (d-1)(2d^4+d^3+1) \ge 0\ .$$

The equality holds for $a = b = c \ge 1$.

Remark. The inequality is true in the particular case $a, b, c \ge 1$, which implies $a^4b^7c^7 \ge 1$.

□

P 2.95. *Let*

$$F(a,b,c,d) = \sqrt[4]{abcd} - \frac{4}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}},$$

where a, b, c, d are positive real numbers. If $ab \ge 1$ and $cd \ge 1$, then then

$$F(a,b,c,d) \ge F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}\right).$$

(Vasile Cîrtoaje, 2019)

Solution. Write the inequality as $E(a,b,c,d) \ge 0$, where

$$E(a,b,c,d) = \sqrt[4]{abcd} - \frac{4}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}} - \frac{1}{\sqrt[4]{abcd}} + \frac{4}{a+b+c+d},$$

assume that

$$ab \ge cd \ge 1,$$

and show that

$$E(a,b,c,d) \ge E(a,b,\sqrt{cd},\sqrt{cd}) \ge E(\sqrt{ab},\sqrt{ab},\sqrt{cd},\sqrt{cd}) \ge 0.$$

Since

$$1 - \frac{\sqrt{cd}}{ab} \ge 1 - \frac{cd}{ab} \ge 0$$

and

$$\sqrt{cd} - 1 \ge 0,$$

the left inequality $E(a,b,c,d) \ge E(a,b,\sqrt{cd},\sqrt{cd})$ follows from Lemma below, point (a). The inequality $E(a,b,\sqrt{cd},\sqrt{cd}) \ge E(\sqrt{ab},\sqrt{ab},\sqrt{cd},\sqrt{cd})$ follows also from Lemma below by replacing c and d with $\sqrt{cd}$. We only need to show that

$$(\sqrt{cd}+\sqrt{cd})\left(1-\frac{\sqrt{ab}}{cd}\right)+2(\sqrt{ab}-1)\ge 0,$$

which is equivalent to the obvious inequality

$$(\sqrt{cd}-1)\left(\sqrt{\frac{ab}{cd}}+1\right)\ge 0.$$

The inequality $E(\sqrt{ab},\sqrt{ab},\sqrt{cd},\sqrt{cd})\ge 0$, is true if the inequality $E(a,b,c,d)\ge 0$ holds for $a=b=x^2$ and $c=d=y^2$, where $x\ge 1$, $y\ge 1$. We need to show that

$$xy-\frac{4}{\frac{2}{x^2}+\frac{2}{y^2}}\ge \frac{1}{xy}-\frac{4}{2x^2+2y^2},$$

that is

$$(x^2y^2-1)(x-y)^2\ge 0.$$

This completes the proof. The equality holds for $a=b=c=d\ge 1$, and for $ab=cd=1$.

Lemma. *Let*

$$E(a,b,c,d)=\sqrt[4]{abcd}-\frac{4}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}}-\frac{1}{\sqrt[4]{abcd}}+\frac{4}{a+b+c+d},$$

where a,b,c,d are positive real numbers such that $ab\ge 1$ and $cd\ge 1$.

(a) *If*

$$(a+b)\left(1-\frac{\sqrt{cd}}{ab}\right)+2(\sqrt{cd}-1)\ge 0,$$

then

$$E(a,b,c,d)\ge E(a,b,\sqrt{cd},\sqrt{cd}).$$

(b) *If*

$$(c+d)\left(1-\frac{\sqrt{ab}}{cd}\right)+2(\sqrt{ab}-1)\ge 0,$$

then

$$E(a,b,c,d)\ge E(\sqrt{ab},\sqrt{ab},c,d).$$

Proof. (a) Write the inequality $E(a,b,c,d)\ge E(a,b,\sqrt{cd},\sqrt{cd})$ as follows:

$$\frac{1}{a+b+c+d}-\frac{1}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}}\ge \frac{1}{a+b+2\sqrt{cd}}-\frac{1}{\frac{1}{a}+\frac{1}{b}+\frac{2}{\sqrt{cd}}},$$

$$\frac{1}{\frac{1}{a}+\frac{1}{b}+\frac{2}{\sqrt{cd}}}-\frac{1}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}} \geq \frac{1}{a+b+2\sqrt{cd}}-\frac{1}{a+b+c+d},$$

$$\frac{(\sqrt{c}-\sqrt{d})^2}{cd\left(\frac{1}{a}+\frac{1}{b}+\frac{2}{\sqrt{cd}}\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)} \geq \frac{(\sqrt{c}-\sqrt{d})^2}{(a+b+2\sqrt{cd})(a+b+c+d)},$$

After dividing by $(\sqrt{c}-\sqrt{d})^2$, we need to show that

$$(a+b+2\sqrt{cd})(a+b+c+d) \geq \left(\frac{a+b}{ab}cd+2\sqrt{cd}\right)\left(\frac{a+b}{ab}+\frac{c+d}{cd}\right), \qquad (*)$$

that is

$$A(c+d)+B \geq 0,$$

where

$$A = a+b+\sqrt{cd}-\frac{a+b}{ab}-\frac{2}{\sqrt{cd}} = (a+b)\left(1-\frac{1}{ab}\right)+2\left(\sqrt{cd}-\frac{1}{\sqrt{cd}}\right) \geq 0,$$

$$B = (a+b)\left[\frac{a+b}{a^2b^2}cd+\frac{2\sqrt{cd}}{ab}-a-b-2\sqrt{cd}\right].$$

Since

$$A(c+d)+B \geq 2A\sqrt{cd}+b,$$

we need to show that $2A\sqrt{cd}+b \geq 0$. This is equivalent to (*) if the sum $c+d$ is replaced by $2\sqrt{cd}$:

$$(a+b+2\sqrt{cd})(a+b+2\sqrt{cd}) \geq \left(\frac{a+b}{ab}cd+2\sqrt{cd}\right)\left(\frac{a+b}{ab}+\frac{2\sqrt{cd}}{cd}\right),$$

that is

$$(a+b+2\sqrt{cd})^2 \geq \left(\frac{a+b}{ab}cd+2\sqrt{cd}\right)^2,$$

$$a+b+2\sqrt{cd} \geq \frac{a+b}{ab}cd+2\sqrt{cd},$$

$$(a+b)\left(1-\frac{\sqrt{cd}}{ab}\right)+2(\sqrt{cd}-1) \geq 0.$$

The last inequality is true by hypothesis.

(b) Due to symmetry, this follows from (a).

Remark. The inequality is true in the particular case $a,b,c,d \geq 1$, which implies $ab \geq 1$ and $cd \geq 1$.

□

P 2.96. *Let a, b, c, d be positive real numbers such that $a^2+b^2+c^2+d^2=1$. Prove that*

$$\sqrt{1-a}+\sqrt{1-b}+\sqrt{1-c}+\sqrt{1-d} \ge \sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}.$$

(Vasile Cîrtoaje, 2007)

First Solution. We can obtain the desired inequality by summing the inequalities

$$\sqrt{1-a}+\sqrt{1-b} \ge \sqrt{c}+\sqrt{d},$$

$$\sqrt{1-c}+\sqrt{1-d} \ge \sqrt{a}+\sqrt{b}.$$

Since

$$\sqrt{1-a}+\sqrt{1-b} \ge 2\sqrt[4]{(1-a)(1-b)}$$

and

$$\sqrt{c}+\sqrt{d} \le 2\sqrt{\frac{c+d}{2}} \le 2\sqrt[4]{\frac{c^2+d^2}{2}},$$

the former inequality holds if

$$(1-a)(1-b) \ge \frac{c^2+d^2}{2}.$$

Indeed,

$$2(1-a)(1-b)-c^2-d^2 = 2(1-a)(1-b)+a^2+b^2-1 = (a+b-1)^2 \ge 0.$$

Similarly, we can prove the second inequality. The equality holds for

$$a=b=c=d=\frac{1}{2}.$$

Second Solution. We can obtain the desired inequality by summing the inequalities

$$\sqrt{1-a}-\sqrt{a} \ge \frac{1}{2\sqrt{2}}(1-4a^2), \qquad \sqrt{1-b}-\sqrt{b} \ge \frac{1}{2\sqrt{2}}(1-4b^2),$$

$$\sqrt{1-c}-\sqrt{c} \ge \frac{1}{2\sqrt{2}}(1-4c^2), \qquad \sqrt{1-d}-\sqrt{d} \ge \frac{1}{2\sqrt{2}}(1-4d^2).$$

To prove the first inequality, we write it as

$$\frac{1-2a}{\sqrt{1-a}+\sqrt{a}} \ge \frac{1}{2\sqrt{2}}(1-2a)(1+2a).$$

Case 1: $0 < a \le \frac{1}{2}$. We need to show that

$$2\sqrt{2} \ge (1+2a)(\sqrt{1-a}+\sqrt{a}).$$

Since $\sqrt{1-a}+\sqrt{a} \le \sqrt{2[(1-a)+a]} = \sqrt{2}$, we have

$$2\sqrt{2}-(1+2a)(\sqrt{1-a}+\sqrt{a}) \ge \sqrt{2}(1-2a) \ge 0.$$

Case 2: $\dfrac{1}{2} \le a < 1$. We need to show that

$$2\sqrt{2} \le (1+2a)(\sqrt{1-a}+\sqrt{a}).$$

Since $1+2a \ge 2\sqrt{2a}$, it suffices to prove that

$$1 \le \sqrt{a(1-a)}+a.$$

Indeed,

$$1-a-\sqrt{a(1-a)} = \sqrt{1-a}\,(\sqrt{1-a}-\sqrt{a}) = \frac{\sqrt{1-a}\,(1-2a)}{\sqrt{1-a}+\sqrt{a}} \le 0.$$

□

P 2.97. *Let a, b, c, d be positive real numbers. Prove that*

$$A+2 \ge \sqrt{B+4},$$

where

$$A = (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)-16,$$
$$B = (a^2+b^2+c^2+d^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right)-16.$$

(Vasile Cîrtoaje, 2004)

Solution. By squaring, the inequality becomes

$$A^2+4A \ge B.$$

Let us denote

$$f(x,y,z) = \frac{x}{y}+\frac{y}{z}+\frac{z}{x}-3, \quad F(x,y,z) = \frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}-3,$$

where $x,y,z > 0$. By the AM-GM inequality, it follows that

$$f(x,y,z) \ge 0.$$

We can check that

$$\begin{aligned} A &= f(a,b,c)+f(b,d,c)+f(c,d,a)+f(d,b,a) \\ &= f(a,c,b)+f(b,c,d)+f(c,a,d)+f(d,a,b) \end{aligned}$$

and

$$B = F(a,b,c) + F(b,d,c) + F(c,d,a) + F(d,b,a).$$

Since

$$\begin{aligned} F(x,y,z) &= [f(x,y,z)+3]^2 - 2[f(x,z,y)+3] - 3 \\ &= f^2(x,y,z) + 6f(x,y,z) - 2f(x,z,y), \end{aligned}$$

we get

$$B = f^2(a,b,c) + f^2(b,d,c) + f^2(c,d,a) + f^2(d,b,a) + 4A,$$

$$4A - B = -f^2(a,b,c) - f^2(b,d,c) - f^2(c,d,a) - f^2(d,b,a).$$

Therefore,

$$\begin{aligned} A^2 + 4A - B &= [f(a,b,c) + f(b,d,c) + f(c,d,a) + f(d,b,a)]^2 \\ &\quad - f^2(a,b,c) - f^2(b,d,c) - f^2(c,d,a) - f^2(d,b,a) \ge 0. \end{aligned}$$

The equality holds for $a = b = c = d$.

□

P 2.98. *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = 1$. Prove that*

$$\sqrt{3a_1+1} + \sqrt{3a_2+1} + \cdots + \sqrt{3a_n+1} \ge n+1.$$

First Solution. Without loss of generality, assume that $a_1 = \max\{a_1, a_2, \ldots, a_n\}$. Write the inequality as follows:

$$(\sqrt{3a_1+1} - 2) + (\sqrt{3a_2+1} - 1) + \cdots + (\sqrt{3a_n+1} - 1) \ge 0,$$

$$\frac{a_1 - 1}{\sqrt{3a_1+1}+2} + \frac{a_2}{\sqrt{3a_2+1}+1} + \cdots + \frac{a_n}{\sqrt{3a_n+1}+1} \ge 0,$$

$$\frac{a_2}{\sqrt{3a_2+1}+1} + \cdots + \frac{a_n}{\sqrt{3a_n+1}+1} \ge \frac{a_2 + \cdots + a_n}{\sqrt{3a_1+1}+2},$$

$$a_2\left(\frac{1}{\sqrt{3a_2+1}+1} - \frac{1}{\sqrt{3a_1+1}+2}\right) + \cdots + a_n\left(\frac{1}{\sqrt{3a_n+1}+1} - \frac{1}{\sqrt{3a_1+1}+2}\right) \ge 0.$$

The last inequality is clearly true. The equality holds for $a_1 = 1$ and $a_2 = \cdots = a_n = 0$ (or any cyclic permutation).

Second Solution. We use the induction method. For $n = 1$, the inequality is an equality. We claim that

$$\sqrt{3a_1+1} + \sqrt{3a_n+1} \ge \sqrt{3(a_1+a_n)+1} + 1.$$

By squaring, this inequality becomes

$$\sqrt{(3a_1+1)(a_n+1)} \geq \sqrt{3(a_1+a_n)+1},$$

which is equivalent to $a_1a_n \geq 0$. Thus, to prove the original inequality, it suffices to show that

$$\sqrt{3(a_1+a_n)+1}+\sqrt{3a_2+1}+\cdots+\sqrt{3a_{n-1}+1} \geq n.$$

Using the substitution $b_1 = a_1 + a_n$ and $b_2 = a_2, \ldots, b_{n-1} = a_{n-1}$, this inequality turns into

$$\sqrt{3b_1+1}+\sqrt{3b_2+1}+\cdots+\sqrt{3b_{n-1}+1} \geq n$$

for $b_1 + b_2 + \cdots + b_{n-1} = 1$. Clearly, this is true by the induction hypothesis.

□

P 2.99. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2\cdots a_n = 1$. Prove that*

$$\frac{1}{\sqrt{1+(n^2-1)a_1}}+\frac{1}{\sqrt{1+(n^2-1)a_2}}+\cdots+\frac{1}{\sqrt{1+(n^2-1)a_n}} \geq 1.$$

First Solution. For the sake of contradiction, assume that

$$\frac{1}{\sqrt{1+(n^2-1)a_1}}+\frac{1}{\sqrt{1+(n^2-1)a_2}}+\cdots+\frac{1}{\sqrt{1+(n^2-1)a_n}} < 1.$$

It suffices to show that $a_1a_2\cdots a_n > 1$. Let

$$x_i = \frac{1}{\sqrt{1+(n^2-1)a_i}}, \quad 0 < x_i < 1, \quad i = 1, 2, \cdots, n.$$

Since $a_i = \dfrac{1-x_i^2}{(n^2-1)x_i^2}$ for all i, we need to show that

$$x_1 + x_2 + \cdots + x_n < 1$$

implies

$$(1-x_1^2)(1-x_2^2)\cdots(1-x_n^2) > (n^2-1)^n x_1^2x_2^2\cdots x_n^2.$$

Using the AM-GM inequality gives

$$\prod(1-x_1^2) > \prod\left[\left(\sum x_1\right)^2 - x_1^2\right] = \prod(x_2+\cdots+x_n)(2x_1+x_2+\cdots+x_n)$$

$$\geq (n^2-1)^n \prod\left(\sqrt[n-1]{x_2\cdots x_n}\cdot\sqrt[n+1]{x_1^2x_2\cdots x_n}\right) = (n^2-1)^n x_1^2x_2^2\cdots x_n^2.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Second Solution. We will show that

$$\frac{1}{\sqrt{1+(n^2-1)x}} \ge \frac{1}{1+(n-1)x^k}$$

for $x > 0$ and $k = \dfrac{n+1}{2n}$. By squaring, the inequality becomes

$$(n-1)x^{2k-1} + 2x^{k-1} \ge n+1.$$

Applying the AM-GM inequality, we get

$$(n-1)x^{2k-1} + 2x^{k-1} \ge (n+1)\sqrt[n+1]{x^{(n-1)(2k-1)} \cdot x^{2(k-1)}} = n+1.$$

Using this result, it suffices to show that

$$\frac{1}{1+(n-1)a_1^k} + \frac{1}{1+(n-1)a_2^k} + \cdots + \frac{1}{1+(n-1)a_n^k} \ge 1.$$

Since $a_1^k a_2^k \cdots a_n^k = 1$, this inequality follows immediately from P 1.200-(a).

□

P 2.100. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \cdots a_n = 1$. Prove that*

$$\sum_{i=1}^{n} \frac{1}{1+\sqrt{1+4n(n-1)a_i}} \ge \frac{1}{2}.$$

First Solution. Write the inequality as follows:

$$\sum_{i=1}^{n} \frac{\sqrt{1+4n(n-1)a_i}-1}{a_i} \ge 2n(n-1),$$

$$\sum_{i=1}^{n} \sqrt{\frac{1}{a_i^2} + \frac{4n(n-1)}{a_i}} \ge 2n(n-1) + \sum \frac{1}{a_i}.$$

By squaring, the inequality becomes

$$\sum_{1\le i<j\le n} \sqrt{\left[\frac{1}{a_i^2} + \frac{4n(n-1)}{a_i}\right]\left[\frac{1}{a_j^2} + \frac{4n(n-1)}{a_j}\right]} \ge 2n^2(n-1)^2 + \sum_{1\le i<j\le n} \frac{1}{a_i a_j}.$$

The Cauchy-Schwarz inequality gives

$$\sqrt{\left[\frac{1}{a_i^2} + \frac{4n(n-1)}{a_i}\right]\left[\frac{1}{a_j^2} + \frac{4n(n-1)}{a_j}\right]} \ge \frac{1}{a_i a_j} + \frac{4n(n-1)}{\sqrt{a_i a_j}}.$$

Thus, it suffices to show that

$$\sum_{1\le i<j\le n} \frac{1}{\sqrt{a_ia_j}} \ge \frac{n(n-1)}{2},$$

which follows immediately from the AM-GM inequality. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Second Solution. For the sake of contradiction, assume that

$$\sum_{i=1}^{n} \frac{1}{1+\sqrt{1+4n(n-1)a_i}} < \frac{1}{2}.$$

It suffices to show that $a_1a_2\cdots a_n > 1$. Using the substitution

$$\frac{x_i}{2n} = \frac{1}{1+\sqrt{1+4n(n-1)a_i}}, \quad i = 1, 2, \cdots, n,$$

which yields

$$a_i = \frac{n-x_i}{(n-1)x_i^2}, \quad 0 < x_i < n, \quad i = 1, 2, \cdots, n,$$

we need to show that

$$x_1 + x_2 + \cdots + x_n < n$$

implies

$$(n-x_1)(n-x_2)\cdots(n-x_n) > (n-1)^n x_1^2x_2^2\cdots x_n^2.$$

By the AM-GM inequality, we have

$$x_1x_2\cdots x_n \le \left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^n < 1$$

and

$$n - x_i > (x_1+x_2+\cdots+x_n) - x_i \ge (n-1)\sqrt[n-1]{\frac{x_1x_2\cdots x_n}{x_i}}, \quad i = 1, 2, \cdots, n.$$

Therefore, we get

$$(n-x_1)(n-x_2)\cdots(n-x_n) > (n-1)^n x_1x_2\cdots x_n > (n-1)^n x_1^2x_2^2\cdots x_n^2.$$

□

P 2.101. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1a_2\cdots a_n = 1$, then*

$$a_1 + a_2 + \cdots + a_n \ge n - 1 + \sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}}.$$

Solution. Let us denote

$$a=\frac{a_1+a_2+\cdots+a_n}{n}, \quad b=\sqrt{\frac{2\sum_{1\le i<j\le n}a_ia_j}{n(n-1)}},$$

where $a\ge 1$ and $b\ge 1$ (by the AM-GM inequality). We need to show that

$$na-n+1\ge\sqrt{\frac{n^2a^2-n(n-1)b^2}{n}}.$$

By squaring, this inequality becomes

$$(n-1)[n(a-1)^2+b^2-1]\ge 0,$$

which is clearly true. The equality holds for $a_1=a_2=\cdots=a_n=1$.

□

P 2.102. *If $a_1,a_2,\ldots,a_n$ are positive real numbers such that $a_1a_2\cdots a_n=1$, then*

$$\sqrt{(n-1)(a_1^2+a_2^2+\cdots+a_n^2)}+n-\sqrt{n(n-1)}\ge a_1+a_2+\cdots+a_n.$$

(Vasile Cîrtoaje, 2006)

Solution. We use the induction method. For $n=2$, the inequality is equivalent to the obvious inequality

$$a_1+\frac{1}{a_1}\ge 2.$$

Assume now that the inequality holds for $n-1$ numbers, $n\ge 3$, and prove that it holds also for n numbers. Let $a_1=\min\{a_1,a_2,\ldots,a_n\}$, and denote

$$x=\frac{a_2+a_3+\cdots+a_n}{n-1}, \quad y=\sqrt[n-1]{a_2a_3\cdots a_n},$$

$$f(a_1,a_2,\ldots,a_n)=\sqrt{(n-1)(a_1^2+a_2^2+\cdots+a_n^2)}+n-\sqrt{n(n-1)}-(a_1+a_2+\cdots+a_n).$$

By the AM-GM inequality, we have $x\ge y$. We will show that

$$f(a_1,a_2,\ldots,a_n)\ge f(a_1,y,\cdots,y)\ge 0. \tag{*}$$

Write the left inequality as

$$\sqrt{a_1^2+a_2^2+\cdots+a_n^2}-\sqrt{a_1^2+(n-1)y^2}\ge\sqrt{n-1}\,(x-y).$$

To prove this inequality, we use the induction hypothesis, written in the homogeneous form

$$\sqrt{(n-2)(a_2^2+a_3^2+\cdots+a_n^2)}+\left[n-1-\sqrt{(n-1)(n-2)}\right]y\ge(n-1)x,$$

which is equivalent to

$$a_2^2 + \cdots + a_n^2 \ge (n-1)A^2,$$

where

$$A = kx - (k-1)y, \quad k = \sqrt{\frac{n-1}{n-2}}.$$

So, we need to prove that

$$\sqrt{a_1^2 + (n-1)A^2} - \sqrt{a_1^2 + (n-1)y^2} \ge \sqrt{n-1}\,(x-y).$$

Write this inequality as

$$\frac{A^2 - y^2}{\sqrt{a_1^2 + (n-1)A^2} + \sqrt{a_1^2 + (n-1)y^2}} \ge \frac{x-y}{\sqrt{n-1}}.$$

Since $x \ge y$ and

$$A^2 - y^2 = k(x-y)[kx - (k-2)y] = k(x-y)(A+y),$$

we need to show that

$$\frac{k(A+y)}{\sqrt{a_1^2 + (n-1)A^2} + \sqrt{a_1^2 + (n-1)y^2}} \ge \frac{1}{\sqrt{n-1}}.$$

In addition, since $a_1 \le y$, it suffices to show that

$$\frac{k(A+y)}{\sqrt{y^2 + (n-1)A^2} + \sqrt{n}y} \ge \frac{1}{\sqrt{n-1}}.$$

From

$$kA - y = k^2x - (k^2 - k + 1)y \ge k^2y - (k^2 - k + 1)y = (k-1)y > 0,$$

it follows that

$$y^2 + (n-1)A^2 < k^2A^2 + (n-1)^2 = (n-1)k^2A^2.$$

Therefore, it is enough to prove that

$$\frac{k(A+y)}{\sqrt{n-1}\,kA + \sqrt{n}\,y} \ge \frac{1}{\sqrt{n-1}},$$

which is equivalent to

$$\left(k\sqrt{n-1} - \sqrt{n}\right)y \ge 0.$$

This is true since

$$k\sqrt{n-1} - \sqrt{n} = \frac{n-1}{\sqrt{n-2}} - \sqrt{n} = \frac{1}{n-1+\sqrt{n(n-2)}} > 0.$$

The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 2.103. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2\cdots a_n \ge 1$. If $k > 1$, then*

$$\sum \frac{a_1^k}{a_1^k + a_2 + \cdots + a_n} \ge 1.$$

(Vasile Cîrtoaje, 2006)

First Solution. Let us denote $r = \sqrt[n]{a_1a_2\cdots a_n}$ and $b_i = a_i/r$ for $i = 1, 2, \cdots, n$. Note that $r \ge 1$ and $b_1b_2\cdots b_n = 1$. The desired inequality becomes

$$\sum \frac{b_1^k}{b_1^k + (b_2 + \cdots + b_n)/r^{k-1}} \ge 1,$$

and we see that it suffices to prove it for $r = 1$; that is, for $a_1a_2\cdots a_n = 1$. On this hypothesis, we will show that there exists a positive number p, $1 < p < k$, such that

$$\frac{a_1^k}{a_1^k + a_2 + \cdots + a_n} \ge \frac{a_1^p}{a_1^p + a_2^p + \cdots + a_n^p}.$$

Clearly, by adding this inequality and the analogous inequalities for $a_2, \ldots, a_n$, we get the desired inequality. Write the claimed inequality as

$$a_2^p + \cdots + a_n^p \ge (a_2\cdots a_n)^{k-p}(a_2 + \cdots + a_n).$$

Based on the AM-GM inequality

$$a_2\cdots a_n \le \left(\frac{a_2 + \cdots + a_n}{n-1}\right)^{n-1},$$

it suffices to show that

$$a_2^p + \cdots + a_n^p \ge (n-1)\left(\frac{a_2 + \cdots + a_n}{n-1}\right)^{(n-1)(k-p)+1}.$$

Choosing

$$p = \frac{(n-1)k+1}{n}, \quad 1 < p < k,$$

the inequality becomes

$$a_2^p + \cdots + a_n^p \ge (n-1)\left(\frac{a_2 + \cdots + a_n}{n-1}\right)^p,$$

which is just Jensen's inequality applied to the convex function $f(x) = x^p$. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Second Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a_1^k}{a_1^k + a_2 + \cdots + a_n} \ge \frac{\left(\sum a_1^{\frac{k+1}{2}}\right)^2}{\sum a_1(a_1^k + a_2 + \cdots + a_n)} = \frac{\sum a_1^{k+1} + 2\sum_{1\le i<j\le n}(a_ia_j)^{\frac{k+1}{2}}}{\sum a_1^{k+1} + 2\sum_{1\le i<j\le n} a_ia_j}.$$

Thus, it suffices to show that

$$\sum_{1\le i<j\le n}(a_ia_j)^{\frac{k+1}{2}} \ge \sum_{1\le i<j\le n} a_ia_j.$$

Jensen's inequality applied to the convex function $f(x)=x^{\frac{k+1}{2}}$ yields

$$\sum_{1\le i<j\le n}(a_ia_j)^{\frac{k+1}{2}} \ge \frac{n(n-1)}{2}\left(\frac{2\sum_{1\le i<j\le n}a_ia_j}{n(n-1)}\right)^{\frac{k+1}{2}}.$$

On the other hand, by the AM-GM inequality, we get

$$\frac{2}{n(n-1)}\sum_{1\le i<j\le n}a_ia_j \ge (a_1a_2\cdots a_n)^{\frac{2}{n}} \ge 1.$$

Therefore,

$$\left(\frac{2\sum_{1\le i<j\le n}a_ia_j}{n(n-1)}\right)^{\frac{k+1}{2}} = \left(\frac{2\sum_{1\le i<j\le n}a_ia_j}{n(n-1)}\right)^{\frac{k-1}{2}}\cdot\frac{2\sum_{1\le i<j\le n}a_ia_j}{n(n-1)} \ge \frac{2\sum_{1\le i<j\le n}a_ia_j}{n(n-1)},$$

hence

$$\sum_{1\le i<j\le n}(a_ia_j)^{\frac{k+1}{2}} \ge \frac{n(n-1)}{2}\cdot\frac{2\sum_{1\le i<j\le n}a_ia_j}{n(n-1)} = \sum_{1\le i<j\le n}a_ia_j.$$

□

P 2.104. *Let $a_1,a_2,\ldots,a_n$ be positive real numbers such that $a_1a_2\cdots a_n\ge 1$. If*

$$\frac{-2}{n-2}\le k<1,$$

then

$$\sum\frac{a_1^k}{a_1^k+a_2+\cdots+a_n}\le 1.$$

(Vasile Cîrtoaje, 2006)

Solution. Let us denote $r=\sqrt[n]{a_1a_2\cdots a_n}$ and $b_i=a_i/r$ for $i=1,2,\cdots,n$. Clearly, $r\ge 1$ and $b_1b_2\cdots b_n=1$. The desired inequality becomes

$$\sum\frac{b_1^k}{b_1^k+(b_2+\cdots+b_n)r^{1-k}}\le 1,$$

and we see that it suffices to prove it for $r=1$; that is, for $a_1a_2\cdots a_n=1$. On this hypothesis, we will show that there exists a real number p such that

$$\frac{a_1^k}{a_1^k+a_2+\cdots+a_n}\le\frac{a_1^p}{a_1^p+a_2^p+\cdots+a_n^p}.$$

By adding this inequality and the analogous inequalities for $a_2, \ldots, a_n$, we get the desired inequality. Write the claimed inequality as

$$a_2 + \cdots + a_n \geq (a_2^p + \cdots + a_n^p)a_1^{k-p},$$

$$a_2 + \cdots + a_n \geq (a_2^p + \cdots + a_n^p)(a_2 \cdots a_n)^{p-k}.$$

This inequality is homogeneous when $1 = p + (n-1)(p-k)$; that is, for

$$p = \frac{(n-1)k+1}{n}, \quad \frac{-1}{n-2} \leq p < 1.$$

Rewrite the homogeneous inequality as

$$a_2 + \cdots + a_n \geq (a_2^p + \cdots + a_n^p)(a_2 \cdots a_n)^{\frac{1-p}{n-1}}. \tag{*}$$

To prove it, we use the weighted AM-GM inequality

$$ma_2 + a_3 + \cdots + a_n \geq (m+n-2)a_2^{\frac{m}{m+n-2}}(a_3 \cdots a_n)^{\frac{1}{m+n-2}}, \quad m \geq 0,$$

which can be rewritten as

$$ma_2 + a_3 + \cdots + a_n \geq (m+n-2)a_2^{\frac{m-1}{m+n-2}}(a_2 \cdots a_n)^{\frac{1}{m+n-2}}.$$

Choosing m such that $\dfrac{m-1}{m+n-2} = p$, i.e.

$$m = \frac{1+(n-2)p}{1-p} \geq 0,$$

we get

$$\frac{1+(n-2)p}{1-p}a_2 + a_3 + \cdots + a_n \geq \frac{n-1}{1-p}a_2^p(a_2a_3 \cdots a_n)^{\frac{1-p}{n-1}}.$$

Adding this inequality and the analogous inequalities for $a_3, \cdots, a_n$ yields the inequality (*). Thus, the proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

$\square$

P 2.105. *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1+a_2+\cdots+a_n \geq n$. If $1 < k \leq n+1$, then*

$$\sum \frac{a_1}{a_1^k + a_2 + \cdots + a_n} \leq 1.$$

(Vasile Cîrtoaje, 2006)

Solution. Using the substitutions

$$s = \frac{a_1 + a_2 + \cdots + a_n}{n},$$

and

$$x_1 = \frac{a_1}{s}, \quad x_2 = \frac{a_2}{s}, \quad \cdots, \quad x_n = \frac{a_n}{s},$$

the desired inequality becomes

$$\frac{x_1}{s^{k-1}x_1^k + x_2 + \cdots + x_n} + \cdots + \frac{x_n}{x_1 + x_2 + \cdots + s^{k-1}x_n^k} \le 1,$$

where $s \ge 1$ and $x_1 + x_2 + \cdots + x_n = n$. Clearly, if this inequality holds for $s = 1$, then it holds for any $s \ge 1$. Therefore, we only need to consider the case $s = 1$, when $a_1 + a_2 + \cdots + a_n = n$, and the desired inequality is equivalent to

$$\frac{a_1}{a_1^k - a_1 + n} + \frac{a_2}{a_2^k - a_2 + n} + \cdots + \frac{a_n}{a_n^k - a_n + n} \le 1.$$

By Bernoulli's inequality, we have

$$a_1^k - a_1 + n \ge 1 + k(a_1 - 1) - a_1 + n = n - k + 1 + (k-1)a_1 \ge 0.$$

Consequently, it suffices to prove that

$$\sum_{i=1}^{n} \frac{a_i}{n - k + 1 + (k-1)a_i} \le 1.$$

For $k = n+1$, this inequality is an equality. Otherwise, for $1 < k < n+1$, we rewrite the inequality as

$$\sum_{i=1}^{n} \frac{1}{n - k + 1 + (k-1)a_i} \ge 1,$$

which follows from the AM-HM inequality as follows:

$$\sum_{i=1}^{n} \frac{1}{n - k + 1 + (k-1)a_i} \ge \frac{n^2}{\sum_{i=1}^{n}[n - k + 1 + (k-1)a_i]} = 1.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 2.106. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \cdots a_n \ge 1$. If $k > 1$, then*

$$\sum \frac{a_1}{a_1^k + a_2 + \cdots + a_n} \le 1.$$

(Vasile Cîrtoaje, 2006)

Solution. Consider two cases: $1 < k \le n+1$ and $k \ge n - \dfrac{1}{n-1}$.

Case 1: $1 < k \le n+1$. By the AM-GM inequality, we have

$$a_1 + a_2 + \cdots + a_n \ge n\sqrt[n]{a_1 a_2 \cdots a_n} \ge n.$$

Thus, the desired inequality follows from the preceding P 2.105.

Case 2: $k \ge n - \dfrac{1}{n-1}$. Let $r = \sqrt[n]{a_1 a_2 \cdots a_n}$ and $b_i = a_i/r$ for $i = 1, 2, \cdots, n$. Note that $r \ge 1$ and $b_1 b_2 \cdots b_n = 1$. The desired inequality can be rewritten as

$$\sum \frac{b_1}{r^{k-1} b_1^k + b_2 + \cdots + b_n} \le 1.$$

Obviously, it suffices to prove this inequality for $r = 1$; that is, for

$$a_1 a_2 \cdots a_n = 1.$$

On this hypothesis, it suffices to show that there exists a real p such that

$$\frac{(n-1)a_1}{a_1^k + a_2 + \cdots + a_n} + \frac{a_1^p}{a_1^p + a_2^p + \cdots + a_n^p} \le 1.$$

Then, adding this inequality and the analogous inequalities for $a_2, \cdots, a_n$ yields the desired inequality. Let us denote $t = \sqrt[n-1]{a_2 \cdots a_n}$. By the AM-GM inequality, we have

$$a_2 + \cdots + a_n \ge (n-1)t, \quad a_2^p + \cdots + a_n^p \ge (n-1)t^p.$$

Thus, it suffices to show that

$$\frac{(n-1)a_1}{a_1^k + (n-1)t} + \frac{a_1^p}{a_1^p + (n-1)t^p} \le 1.$$

Since $a_1 = 1/t^{n-1}$, this inequality is equivalent to

$$(n-1)t^q(t^n - 1) - (t^{q-np} - 1) \ge 0,$$

where

$$q = (n-1)(k-1).$$

Choose p such that $(n-1)n = q - np$, i.e.

$$p = \frac{(n-1)(k-n-1)}{n}.$$

The inequality becomes as follows:

$$(n-1)t^q(t^n - 1) - \left[t^{n(n-1)} - 1\right] \ge 0,$$

$$(n-1)t^q(t^n-1)-(t^n-1)(t^{n^2-2n}+t^{n^2-3n}+\cdots+1)\geq 0,$$
$$(t^n-1)[(t^q-t^{n^2-2n})+(t^q-t^{n^2-3n})+\cdots+(t^q-1)]\geq 0.$$

The last inequality is clearly true for $q\geq n^2-2n$; that is, for $k\geq n-\dfrac{1}{n-1}$.

The equality holds for $a_1=a_2=\cdots=a_n=1$.

□

P 2.107. *Let $a_1,a_2,\ldots,a_n$ be positive real numbers such that $a_1a_2\cdots a_n\geq 1$. If*

$$-1-\frac{2}{n-2}\leq k<1,$$

then

$$\sum\frac{a_1}{a_1^k+a_2+\cdots+a_n}\geq 1.$$

(Vasile Cîrtoaje, 2006)

Solution. Let us denote $r=\sqrt[n]{a_1a_2\cdots a_n}$ and $b_i=a_i/r$ for $i=1,2,\cdots,n$. Note that $r\geq 1$ and $b_1b_2\cdots b_n=1$. The desired inequality becomes

$$\sum\frac{b_1}{b_1^k/r^{1-k}+b_2+\cdots+b_n}\geq 1,$$

and we see that it suffices to prove it for $r=1$; that is, for $a_1a_2\cdots a_n=1$. On this hypothesis, by the Cauchy-Schwarz inequality, we have

$$\sum\frac{a_1}{a_1^k+a_2+\cdots+a_n}\geq\frac{\left(\sum a_1\right)^2}{\sum a_1(a_1^k+a_2+\cdots+a_n)}=\frac{\left(\sum a_1\right)^2}{\sum a_1^{1+k}+\left(\sum a_1\right)^2-\sum a_1^2}.$$

Thus, we still have to show that

$$\sum a_1^2\geq\sum a_1^{1+k}.$$

Case 1: $-1\leq k<1$. Using Chebyshev's inequality and the AM-GM inequality yields

$$\sum a_1^2\geq\frac{1}{n}\left(\sum a_1^{1-k}\right)\left(\sum a_1^{1+k}\right)\geq(a_1a_2\cdots a_n)^{\frac{1-k}{n}}\sum a_1^{1+k}=\sum a_1^{1+k}.$$

Case 2: $-1-\dfrac{2}{n-1}\leq k<-1$. It is convenient to replace $a_1,a_2,\cdots,a_n$ by

$$a_1^{(n-1)/2},a_2^{(n-1)/2},\cdots,a_n^{(n-1)/2},$$

respectively. Thus, we need to show that $a_1a_2\cdots a_n=1$ involves

$$\sum a_1^{n-1}\geq\sum a_1^q,$$

where
$$q=\frac{(n-1)(1+k)}{2}, \quad -1\le q<0.$$
By the AM-GM inequality, we get
$$\sum a_1^{n-1}=\sum\frac{a_2^{n-1}+\cdots+a_n^{n-1}}{n-1}\ge\sum a_2\cdots a_n=\sum\frac{1}{a_1}.$$
Thus, it suffice to show that
$$\sum\frac{1}{a_1}\ge\sum a_1^q.$$
By Chebyshev's inequality and the AM-GM inequality, we have
$$\sum\frac{1}{a_1}\ge\frac{1}{n}\left(\sum a_1^{-1-q}\right)\left(\sum a_1^q\right)\ge(a_1a_2\cdots a_n)^{-(1+q)/n}\ \left(\sum a_1^q\right)=\sum a_1^q.$$
Thus, the proof is completed. The equality holds for $a_1=a_2=\cdots=a_n=1$.

□

P 2.108. *Let $a_1,a_2,\ldots,a_n$ be positive real numbers such that $a_1a_2\cdots a_n=1$. If $k\ge 0$, then*
$$\sum\frac{1}{a_1^k+a_2+\cdots+a_n}\le 1.$$
(Vasile Cîrtoaje, 2006)

Solution. Consider two cases: $0\le k\le 1$ and $k\ge 1$.

Case 1: $0\le k\le 1$. By the Cauchy-Schwarz inequality and the AM-GM inequality, we have
$$\frac{1}{a_1^k+a_2+\cdots+a_n}\le\frac{a_1^{1-k}+1+\cdots+1}{\left(\sqrt{a_1}+\sqrt{a_2}+\cdots+\sqrt{a_n}\right)^2}$$
$$=\frac{a_1^{1-k}+n-1}{\sum a_1+2\sum_{1\le i<j\le n}\sqrt{a_ia_j}}\le\frac{a_1^{1-k}+n-1}{\sum a_1+n(n-1)},$$
hence
$$\sum\frac{1}{a_1^k+a_2+\cdots+a_n}\le\frac{\sum a_1^{1-k}+n(n-1)}{\sum a_1+n(n-1)}.$$
Therefore, it suffices to show that
$$\sum a_1^{1-k}\le\sum a_1.$$
Indeed, by Chebyshev's inequality and the AM-GM inequality, we have
$$\sum a_1=\sum a_1^k\cdot a_1^{1-k}\ge\frac{1}{n}\left(\sum a_1^k\right)\left(\sum a_1^{1-k}\right)\ge(a_1a_2\cdots a_n)^{k/n}\left(\sum a_1^{1-k}\right)=\sum a_1^{1-k}.$$

Case 2: $k > 1$. Write the inequality as

$$\sum\left(\frac{n-1}{a_1^k + a_2 + \cdots + a_n} + \frac{a_1^p}{a_1^p + a_2^p + \cdots + a_n^p} - 1\right) \le 0,$$

where $p > 0$. It suffices to show that there exists a positive number p such that

$$\frac{n-1}{a_1^k + a_2 + \cdots + a_n} + \frac{a_1^p}{a_1^p + a_2^p + \cdots + a_n^p} \le 1.$$

Let

$$x = \sqrt[n-1]{a_1}, \quad x > 0.$$

By the AM-GM inequality, we have

$$a_2 + \cdots + a_n \ge (n-1)\sqrt[n-1]{a_2 \cdots a_n} = \frac{n-1}{\sqrt[n-1]{a_1}} = \frac{n-1}{x}$$

and

$$a_2^p + \cdots + a_n^p \ge \sqrt[n-1]{(a_2 \cdots a_n)^p} = \frac{n-1}{\sqrt[n-1]{a_1^p}} = \frac{n-1}{x^p}.$$

Thus, it is enough to show that

$$\frac{n-1}{x^{(n-1)k} + \frac{n-1}{x}} + \frac{x^{(n-1)p}}{x^{(n-1)p} + \frac{n-1}{x^p}} \le 1,$$

which is equivalent to

$$\frac{x}{x^{(n-1)k+1} + n - 1} \le \frac{1}{x^{np} + n - 1},$$

$$x^{(n-1)k+1} - x^{np+1} - (n-1)(x-1) \ge 0,$$

$$x^{np+1}\left[(x^{(n-1)k-np} - 1\right] - (n-1)(x-1) \ge 0.$$

Choose p such that $(n-1)k - np = n - 1$, i.e.

$$p = \frac{(k-1)(n-1)}{n} > 0.$$

The inequality becomes as follows:

$$x^{np+1}\left[(x^{n-1} - 1\right] - (n-1)(x-1) \ge 0,$$

$$(x-1)\left[(x^{np+n-1} - 1) + (x^{np+n-2} - 1) + \cdots + (x^{np+1} - 1)\right] \ge 0.$$

Since the last inequality is obvious true, the proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 2.109. *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1+a_2+\cdots+a_n \le n$. If $0 \le k < 1$, then*

$$\frac{1}{a_1^k+a_2+\cdots+a_n}+\frac{1}{a_1+a_2^k+\cdots+a_n}+\cdots+\frac{1}{a_1+a_2+\cdots+a_n^k} \ge 1.$$

Solution. By the AM-HM inequality

$$\sum \frac{1}{a_1^k+a_2+\cdots+a_n} \ge \frac{n^2}{\sum(a_1^k+a_2+\cdots+a_n)} = \frac{n^2}{\sum a_1^k+(n-1)\sum a_1}$$

and Jensen's inequality

$$\sum a_1^k \le n\left(\frac{1}{n}\sum a_1\right)^k,$$

we get

$$\sum \frac{1}{a_1^k+a_2+\cdots+a_n} \ge \frac{n^2}{n\left(\frac{1}{n}\sum a_1\right)^k+(n-1)\sum a_1} \ge 1.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 2.110. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers. If $k > 1$, then*

$$\sum \frac{a_2^k+a_3^k+\cdots+a_n^k}{a_2+a_3+\cdots+a_n} \le \frac{n(a_1^k+a_2^k+\cdots+a_n^k)}{a_1+a_2+\cdots+a_n}.$$

(Wolfgang Berndt and Vasile Cîrtoaje, 2006)

Solution. Due to homogeneity, we may assume that $a_1+a_2+\ldots+a_n = 1$. Write the inequality as follows:

$$\sum\left(1+\frac{a_1}{a_2+a_3+\cdots+a_n}\right)(a_2^k+a_3^k+\cdots+a_n^k) \le n(a_1^k+a_2^k+\cdots+a_n^k);$$

$$\sum \frac{a_1(a_2^k+a_3^k+\cdots+a_n^k)}{a_2+a_3+\cdots+a_n} \le a_1^k+a_2^k+\cdots+a_n^k;$$

$$\sum a_1\left(a_1^{k-1}-\frac{a_2^k+a_3^k+\cdots+a_n^k}{a_2+a_3+\cdots+a_n}\right) \ge 0;$$

$$\sum \frac{a_1a_2(a_1^{k-1}-a_2^{k-1})+a_1a_3(a_1^{k-1}-a_3^{k-1})+\cdots+a_1a_n(a_1^{k-1}-a_n^{k-1})}{a_2+a_3+\cdots+a_n} \ge 0;$$

$$\sum_{1\le i<j\le n} a_ia_j\left(\frac{a_i^{k-1}-a_j^{k-1}}{1-a_i}+\frac{a_j^{k-1}-a_i^{k-1}}{1-a_j}\right) \ge 0;$$

$$\sum_{1\le i<j\le n} \frac{a_ia_j(a_i^{k-1}-a_j^{k-1})(a_i-a_j)}{(1-a_i)(1-a_j)} \ge 0.$$

Since the last inequality is true for $k>1$, the proof is completed. The equality holds for $a_1=a_2=\cdots=a_n$.

P 2.111. *Let f be a convex function on the closed interval $[a,b]$, and let $a_1,a_2,\ldots,a_n\in[a,b]$ such that*

$$a_1+a_2+\cdots+a_n=pa+qb,$$

where $p,q\ge 0$ such that $p+q=n$. Prove that

$$f(a_1)+f(a_2)+\cdots+f(a_n)\le pf(a)+qf(b).$$

(Vasile Cîrtoaje, 2009)

Solution. Consider the nontrivial case $a<b$. Since $a_1,a_2,\ldots,a_n\in[a,b]$, there exist $\lambda_1,\lambda_2,\ldots,\lambda_n\in[0,1]$ such that

$$a_i=\lambda_ia+(1-\lambda_i)b,\quad i=1,2,\ldots,n.$$

From

$$\lambda_i=\frac{a_i-b}{a-b},\quad i=1,2,\ldots,n,$$

we have

$$\sum_{i=1}^n\lambda_i=\frac{1}{a-b}\left(\sum_{i=1}^n a_i-nb\right)=\frac{(pa+qb)-(p+q)b}{a-b}=p.$$

Since f is convex on $[a,b]$, we get

$$\begin{aligned}\sum_{i=1}^n f(a_i)&\le\sum_{i=1}^n[\lambda_if(a)+(1-\lambda_i)f(b)]\\&=\left(\sum_{i=1}^n\lambda_i\right)[f(a)-f(b)]+nf(b)\\&=p[f(a)-f(b)]+(p+q)f(b)\\&=pf(a)+qf(b).\end{aligned}$$

Chapter 3

Symmetric Power-Exponential Inequalities

3.1 Applications

3.1. If a, b are positive real numbers such that $a + b = a^4 + b^4$, then

$$a^a b^b \le 1 \le a^{a^3} b^{b^3}.$$

3.2. If a, b are positive real numbers, then

$$a^{2a} + b^{2b} \ge a^{a+b} + b^{a+b}.$$

3.3. If a, b are positive real numbers, then

$$a^a + b^b \ge a^b + b^a.$$

3.4. If a, b are positive real numbers, then

$$a^{2a} + b^{2b} \ge a^{2b} + b^{2a}.$$

3.5. If a, b are nonnegative real numbers such that $a + b = 2$, then

(a) $$a^b + b^a \le 1 + ab;$$

(b) $$a^{2b} + b^{2a} \le 1 + ab.$$

3.6. If a, b are nonnegative real numbers such that $\frac{2}{3} \le a + b \le 2$, then

$$a^{2b} + b^{2a} \le 1 + ab.$$

3.7. If a, b are nonnegative real numbers such that $a^2 + b^2 = 2$, then

$$a^{2b} + b^{2a} \le 1 + ab.$$

3.8. If a, b are nonnegative real numbers such that $a^2 + b^2 = \frac{1}{4}$, then

$$a^{2b} + b^{2a} \le 1 + ab.$$

3.9. If a, b are positive real numbers, then

$$a^a b^b \le (a^2 - ab + b^2)^{\frac{a+b}{2}}.$$

3.10. If $a, b \in (0, 1]$, then

$$a^a b^b \le 1 - ab + a^2 b^2.$$

3.11. If a, b are positive real numbers such that $a + b \le 2$, then

$$\left(\frac{a}{b}\right)^b + \left(\frac{b}{a}\right)^a \le 2.$$

3.12. If a, b are positive real numbers such that $a + b = 2$, then

$$2a^a b^b \ge a^{2b} + b^{2a} + \frac{3}{4}(a - b)^2.$$

3.13. If $a, b \in (0, 1]$ or $a, b \in [1, \infty)$, then

$$2a^a b^b \ge a^2 + b^2.$$

3.14. If a, b are positive real numbers, then

$$2a^a b^b \geq a^2 + b^2.$$

3.15. If $a \geq 1 \geq b > 0$, then

$$2a^a b^b \geq a^{2b} + b^{2a}.$$

3.16. If $a \geq e \geq b > 0$, then

$$2a^a b^b \geq a^{2b} + b^{2a}.$$

3.17. If a, b are positive real numbers, then

$$a^a b^b \geq \left(\frac{a^2 + b^2}{2}\right)^{\frac{a+b}{2}}.$$

3.18. If a, b are positive real numbers such that $a^2 + b^2 = 2$, then

$$2a^a b^b \geq a^{2b} + b^{2a} + \frac{1}{2}(a-b)^2.$$

3.19. If $a, b \in (0, 1]$, then

$$(a^2 + b^2)\left(\frac{1}{a^{2a}} + \frac{1}{b^{2b}}\right) \leq 4.$$

3.20. If a, b are positive real numbers such that $a + b = 2$, then

$$a^b b^a + 2 \geq 3ab.$$

3.21. Let a, b be positive real numbers such that $a + b = 2$. If $k \geq \dfrac{1}{2}$, then

$$a^{a^{kb}} b^{b^{ka}} \geq 1.$$

3.22. If a, b are positive real numbers such that $a + b = 2$, then

$$a^{\sqrt{a}} b^{\sqrt{b}} \ge 1.$$

3.23. If a, b are positive real numbers such that $a + b = 2$, then

$$a^{a+1} b^{b+1} \le 1 - \frac{1}{48}(a-b)^4.$$

3.24. If a, b are positive real numbers such that $a + b = 2$, then

$$a^{-a} + b^{-b} \le 2.$$

3.25. If $a, b \in [0, 1]$, then

$$a^{b-a} + b^{a-b} + (a-b)^2 \le 2.$$

3.26. If a, b are nonnegative real numbers such that $a + b \le 2$, then

$$a^{b-a} + b^{a-b} + \frac{7}{16}(a-b)^2 \le 2.$$

3.27. If a, b are nonnegative real numbers such that $a + b \le 4$, then

$$a^{b-a} + b^{a-b} \le 2.$$

3.28. If a, b are nonnegative real numbers such that $a + b = 2$, then

$$a^{2b} + b^{2a} \ge a^b + b^a \ge a^2 b^2 + 1.$$

3.29. If a, b are positive real numbers such that $a + b = 2$, then

$$a^{3b} + b^{3a} \le 2.$$

3.30. If a, b are nonnegative real numbers such that $a + b = 2$, then

$$a^{3b} + b^{3a} + \left(\frac{a-b}{2}\right)^4 \le 2.$$

3.31. If a, b are positive real numbers such that $a + b = 2$, then

$$a^{\frac{2}{a}} + b^{\frac{2}{b}} \le 2.$$

3.32. If a, b are positive real numbers such that $a + b = 2$, then

$$a^{\frac{3}{a}} + b^{\frac{3}{b}} \ge 2.$$

3.33. If a, b are positive real numbers such that $a + b = 2$, then

$$a^{5b^2} + b^{5a^2} \le 2.$$

3.34. If a, b are positive real numbers such that $a + b = 2$, then

$$a^{2\sqrt{b}} + b^{2\sqrt{a}} \le 2.$$

3.35. If a, b are nonnegative real numbers such that $a + b = 2$, then

$$\frac{ab(1-ab)^2}{2} \le a^{b+1} + b^{a+1} - 2 \le \frac{ab(1-ab)^2}{3}.$$

3.36. If a, b are nonnegative real numbers such that $a + b = 1$, then

$$a^{2b} + b^{2a} \le 1.$$

3.37. If a, b are positive real numbers such that $a + b = 1$, then

$$2a^a b^b \ge a^{2b} + b^{2a}.$$

3.38. If a, b are positive real numbers such that $a + b = 1$, then

$$a^{-2a} + b^{-2b} \le 4.$$

3.39. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 a_2 \cdots a_n = 1$, then

$$\left(1-\frac{1}{n}\right)^{a_1} + \left(1-\frac{1}{n}\right)^{a_2} + \cdots + \left(1-\frac{1}{n}\right)^{a_n} \le n-1.$$

3.2 Solutions

P 3.1. *If a, b are positive real numbers such that $a+b=a^4+b^4$, then*

$$a^a b^b \le 1 \le a^{a^3} b^{b^3}.$$

(Vasile Cîrtoaje, 2008)

Solution. We will use the inequality

$$\ln x \le x-1, \quad x>0.$$

To prove this inequality, let us denote

$$f(x)=x-1-\ln x, \quad x>0.$$

From

$$f'(x)=\frac{x-1}{x},$$

it follows that $f(x)$ is decreasing on $(0,1]$ and increasing on $[1,\infty)$. Therefore,

$$f(x) \ge f(1)=0.$$

Using this inequality, we have

$$\ln a^a b^b = a\ln a + b\ln b \le a(a-1)+b(b-1)=a^2+b^2-(a+b).$$

Therefore, the left inequality $a^a b^b \le 1$ is true if $a^2+b^2 \le a+b$. We write this inequality in the homogeneous form

$$(a^2+b^2)^3 \le (a+b)^2(a^4+b^4),$$

which is equivalent to the obvious inequality

$$ab(a-b)(a^3-b^3) \ge 0.$$

Taking now $x=\dfrac{1}{a}$ in the inequality $\ln x \le x-1$ yields

$$a\ln a \ge a-1.$$

Similarly,

$$b\ln b \ge b-1,$$

hence

$$\ln a^{a^3} b^{b^3} = a^3\ln a + b^3 \ln b \ge a^2(a-1)+b^2(b-1)=a^3+b^3-(a^2+b^2).$$

Thus, to prove the right inequality $a^{a^3}b^{b^3} \geq 1$, it suffices to show that $a^3 + b^3 \geq a^2 + b^2$, which is equivalent to the homogeneous inequality

$$(a+b)(a^3+b^3)^3 \geq (a^4+b^4)(a^2+b^2)^3.$$

We can write this inequality as

$$A - 3B \geq 0,$$

where

$$A = (a+b)(a^9+b^9) - (a^4+b^4)(a^6+b^6),$$

$$B = a^2b^2(a^2+b^2)(a^4+b^4) - a^3b^3(a+b)(a^3+b^3).$$

Since

$$A = ab(a^3-b^3)(a^5-b^5), \quad B = a^2b^2(a-b)(a^5-b^5),$$

we get

$$A - 3B = ab(a-b)^3(a^5-b^5) \geq 0.$$

Both inequalities become equalities for $a = b = 1$.

□

P 3.2. *If a, b are positive real numbers, then*

$$a^{2a} + b^{2b} \geq a^{a+b} + b^{a+b}.$$

(Vasile Cîrtoaje, 2010)

Solution. Assume that $a \geq b$ and consider the following two cases.

Case 1: $a \geq 1$. Write the inequality as

$$a^{a+b}(a^{a-b} - 1) \geq b^{2b}(b^{a-b} - 1).$$

For $b \leq 1$, we have

$$a^{a+b}(a^{a-b} - 1) \geq 0 \geq b^{2b}(b^{a-b} - 1).$$

For $b \geq 1$, the inequality is also true since

$$a^{a+b} \geq a^{2b} \geq b^{2b}, \quad a^{a-b} - 1 \geq b^{a-b} - 1 \geq 0.$$

Case 2: $a \leq 1$. Since

$$a^{2a} + b^{2b} \geq 2a^a b^b,$$

it suffices to show that

$$2a^a b^b \geq a^{a+b} + b^{a+b},$$

which can be written as

$$\left(\frac{a}{b}\right)^b + \left(\frac{b}{a}\right)^a \leq 2.$$

By Bernoulli's inequality, we get

$$\left(\frac{a}{b}\right)^b+\left(\frac{b}{a}\right)^a=\left(1+\frac{a-b}{b}\right)^b+\left(1+\frac{b-a}{a}\right)^a\le 1+\frac{b(a-b)}{b}+1+\frac{a(b-a)}{a}=2.$$

The equality holds for $a=b$.

Conjecture 1. *If a,b are positive real numbers, then*

$$a^{4a}+b^{4b}\ge a^{2a+2b}+b^{2a+2b}.$$

Conjecture 2. *If a,b,c are positive real numbers, then*

$$a^{3a}+b^{3b}+c^{3c}\ge a^{a+b+c}+b^{a+b+c}+c^{a+b+c}.$$

Conjecture 3. *If a,b,c,d are positive real numbers, then*

$$a^{4a}+b^{4b}+c^{4c}+d^{4d}\ge a^{a+b+c+d}+b^{a+b+c+d}+c^{a+b+c+d}+d^{a+b+c+d}.$$

□

P 3.3. *If a,b are positive real numbers, then*

$$a^a+b^b\ge a^b+b^a.$$

(M. Laub, Israel, 1985, AMM)

Solution. Assume that $a\ge b$. We will show that if $a\ge 1$, then the inequality is true. From

$$a^{a-b}\ge b^{a-b},$$

we get

$$b^b\ge\frac{a^bb^a}{a^a}.$$

Therefore,

$$a^a+b^b-a^b-b^a\ge a^a+\frac{a^bb^a}{a^a}-a^b-b^a=\frac{(a^a-a^b)(a^a-b^a)}{a^a}\ge 0.$$

Consider further the case $0<b\le a<1$.

First Solution. Denoting

$$c=a^b,\quad d=b^b,\quad k=\frac{a}{b},$$

where $c\ge d$ and $k\ge 1$, the inequality becomes

$$c^k-d^k\ge c-d.$$

Since the function $f(x)=x^k$ is convex for $x\ge 0$, from the well-known inequality

$$f(c)-f(d)\ge f'(d)(c-d),$$

we get

$$c^k-d^k\ge kd^{k-1}(c-d).$$

Thus, it suffices to show that

$$kd^{k-1}\ge 1,$$

which is equivalent to

$$b^{1-a+b}\le a.$$

Indeed, since $0<1-a+b\le 1$, by Bernoulli's inequality, we get

$$b^{1-a+b}=[1+(b-1)]^{1-a+b}\le 1+(1-a+b)(b-1)=a-b(a-b)\le a.$$

The equality holds for $a=b$.

Second Solution. Denoting

$$c=\frac{b^a}{a^b+b^a},\quad d=\frac{a^b}{a^b+b^a},\quad k=\frac{a}{b},$$

where $c+d=1$ and $k\ge 1$, the inequality becomes

$$ck^a+dk^{-b}\ge 1.$$

By the weighted AM-GM inequality, we have

$$ck^a+dk^{-b}\ge k^{ac}\cdot k^{-bd}=k^{ac-bd}.$$

Thus, it suffices to show that $ac\ge bd$; that is,

$$a^{1-b}\ge b^{1-a},$$

which is equivalent to $f(a)\ge f(b)$, where

$$f(x)=\frac{\ln x}{1-x}.$$

It is enough to prove that $f(x)$ is an increasing function. Since

$$f'(x)=\frac{g(x)}{(1-x)^2},\quad g(x)=\frac{1}{x}-1+\ln x.$$

we need to show that $g(x)\ge 0$ for $x\in(0,1)$. Indeed, from

$$g'(x)=\frac{x-1}{x^2}<0,$$

it follows that $g(x)$ is strictly decreasing, hence $g(x)>g(1)=0$.

□

P 3.4. *If a, b are positive real numbers, then*

$$a^{2a} + b^{2b} \ge a^{2b} + b^{2a}.$$

Solution. Without loss of generality, assume that $a > b$. We have two cases to consider: $a \ge 1$ and $0 < b < a < 1$.

Case 1: $a \ge 1$. From

$$a^{2(a-b)} \ge b^{2(a-b)},$$

we get

$$b^{2b} \ge \frac{a^{2b}b^{2a}}{a^{2a}}.$$

Therefore,

$$a^{2a} + b^{2b} - a^{2b} - b^{2a} \ge a^{2a} + \frac{a^{2b}b^{2a}}{a^{2a}} - a^{2b} - b^{2a} = \frac{(a^{2a} - a^{2b})(a^{2a} - b^{2a})}{a^{2a}} \ge 0$$

because $a^{2a} \ge a^{2b}$ and $a^{2a} \ge b^{2a}$.

Case 2: $0 < b < a < 1$. Denoting

$$c = a^b, \quad d = b^b, \quad k = \frac{a}{b},$$

where $c > d$ and $k > 1$, the inequality becomes

$$c^{2k} - d^{2k} \ge c^2 - d^2.$$

We will show that

$$c^{2k} - d^{2k} > k(cd)^{k-1}(c^2 - d^2) > c^2 - d^2.$$

The left inequality follows from Lemma below for $x = (c/d)^2$. The right inequality is equivalent to

$$k(cd)^{k-1} > 1,$$

$$(ab)^{a-b} > \frac{b}{a},$$

$$\frac{1+a-b}{1-a+b}\ln a > \ln b.$$

For fixed a, let us define

$$f(b) = \frac{1+a-b}{1-a+b}\ln a - \ln b.$$

If $f'(b) < 0$, then $f(b)$ is strictly decreasing, and hence $f(b) > f(a) = 0$. Since

$$f'(b) = \frac{-2}{(1-a+b)^2}\ln a - \frac{1}{b},$$

we need to show that $g(a) > 0$, where

$$g(a) = 2\ln a + \frac{(1-a+b)^2}{b}.$$

From

$$g'(a) = \frac{2}{a} - \frac{2(1-a+b)}{b} = \frac{2(a-b)(a-1)}{ab} < 0,$$

it follows that $g(a)$ is strictly decreasing, therefore $g(a) > g(1) = b > 0$. This completes the proof. The equality holds for $a = b$.

Lemma. *Let k and x be positive real numbers. If either $k > 1$ and $x \ge 1$, or $0 < k < 1$ and $0 < x \le 1$, then*

$$x^k - 1 \ge kx^{\frac{k-1}{2}}(x-1).$$

Proof. We need to show that $f(x) \ge 0$, where

$$f(x) = x^k - 1 - kx^{\frac{k-1}{2}}(x-1).$$

We have

$$f'(x) = \frac{1}{2}kx^{\frac{k-3}{2}}g(x), \quad g(x) = 2x^{\frac{k+1}{2}} - (k+1)x + k - 1.$$

Since

$$g'(x) = (k+1)\left(x^{\frac{k-1}{2}} - 1\right) \ge 0,$$

$g(x)$ is increasing. If $x \ge 1$, then $g(x) \ge g(1) = 0$, $f(x)$ is increasing, hence $f(x) \ge f(1) = 0$. If $0 < x \le 1$, then $g(x) \le g(1) = 0$, $f(x)$ is decreasing, hence $f(x) \ge f(1) = 0$. The equality holds for $x = 1$.

Remark. The following more general results are valid (*Vasile Cîrtoaje*, 2006):

- *Let $0 < k \le e$.*
  (a) *If $a, b > 0$, then*

$$a^{ka} + b^{kb} \ge a^{kb} + b^{ka};$$

  (b) *If $a, b \in (0, 1]$, then*

$$2\sqrt{a^{ka}b^{kb}} \ge a^{kb} + b^{ka}.$$

Notice that these inequalities are known as the first and the second *Vasc's power exponential inequalities*.

Conjecture 1. *If $0 < k \le e$ and either $a, b \in (0, 4]$ or $0 < a \le 1 \le b$, then*

$$2\sqrt{a^{ka}b^{kb}} \ge a^{kb} + b^{ka}.$$

Conjecture 2. *If $0 < a \le 1 \le b$, then*

$$2\sqrt{a^{3a}b^{3b}} \ge a^{3b} + b^{3a}.$$

Conjecture 3. *If $a, b \in (0, 5]$, then*

$$2a^a b^b \geq a^{2b} + b^{2a}.$$

Conjecture 4. *If $a, b \in [0, 5]$, then*

$$\left(\frac{a^2 + b^2}{2}\right)^{\frac{a+b}{2}} \geq a^{2b} + b^{2a}.$$

□

P 3.5. *If a, b are nonnegative real numbers such that $a + b = 2$, then*

(a) $$a^b + b^a \leq 1 + ab;$$

(b) $$a^{2b} + b^{2a} \leq 1 + ab.$$

Solution. Without loss of generality, assume that $a \geq b$. Since

$$0 \leq b \leq 1, \qquad 0 \leq a - 1 \leq 1,$$

by Bernoulli's inequality, we have

$$a^b \leq 1 + b(a - 1) = 1 + b - b^2$$

and

$$b^a = b \cdot b^{a-1} \leq b[1 + (a - 1)(b - 1)] = b^2(2 - b).$$

(a) We have

$$a^b + b^a - 1 - ab \leq (1 + b - b^2) + b^2(2 - b) - 1 - (2 - b)b = -b(b - 1)^2 \leq 0.$$

The equality holds for $a = b = 1$, for $a = 2$ and $b = 0$, and for $a = 0$ and $b = 2$.

(b) We have

$$\begin{aligned} a^{2b} + b^{2a} - 1 - ab &\leq (1 + b - b^2)^2 + b^4(2 - b)^2 - 1 - (2 - b)b \\ &= b^3(b - 1)^2(b - 2) = -ab^3(b - 1)^2 \leq 0. \end{aligned}$$

The equality holds for $a = b = 1$, for $a = 2$ and $b = 0$, and for $a = 0$ and $b = 2$.

□

P 3.6. *If a, b are nonnegative real numbers such that $\frac{2}{3} \leq a + b \leq 2$, then*

$$a^{2b} + b^{2a} \leq 1 + ab.$$

(Vasile Cîrtoaje, 2007)

Solution. Assume that

$$a \ge b.$$

From $2\sqrt{ab} \le a+b \le 2$, we get

$$ab \le 1.$$

There are two cases to consider: $a+b \le 1$ and $a+b \ge 1$.

Case 1: $\frac{2}{3} \le a+b \le 1$. Since $2b \le 1$, by Bernoulli's inequality, we have

$$a^{2b} \le 1+2b(a-1) = 1+2ab-2b.$$

Therefore, it suffices to show that

$$(1+2ab-2b)+b^{2a} \le 1+ab,$$

which is equivalent to

$$ab+b^{2a} \le 2b.$$

For $2a \ge 1$, this inequality is true since

$$ab \le b, \qquad b^{2a} \le b.$$

For $2a \le 1$, by Bernoulli's inequality, we have

$$b^{2a} \le 1+2a(b-1) = 1+2ab-2a.$$

Therefore, it suffices to show that

$$(1+2ab-2b)+(1+2ab-2a) \le 1+ab,$$

which is equivalent to

$$1+3ab \le 2(a+b).$$

Indeed, we have

$$\begin{aligned} 4+12ab-8(a+b) &\le 4+3(a+b)^2-8(a+b) \\ &= (a+b-2)[3(a+b)-2] \le 0. \end{aligned}$$

Case 2: $1 \le a+b \le 2$. For $a, b \le 1$, by Bernoulli's inequality, we have

$$a^{2b} = (a^2)^b \le 1+b(a^2-1) = 1-b+a^2b,$$

$$b^{2a} = (b^2)^a \le 1+a(b^2-1) = 1-a+ab^2,$$

hence

$$\begin{aligned} a^{2b}+b^{2a}-1-ab &\le (1-b+a^2b)+(1-a+ab^2)-1-ab \\ &= (1-ab)(1-a-b) \le 0. \end{aligned}$$

Consider further that $a \ge 1 \ge b$. By Bernoulli's inequality, we have

$$a^b \le 1 + b(a-1) = ab + 1 - b,$$

$$\begin{aligned} b^{2a} = b^{a-1} \cdot b^{a+1} \le b^{a+1} = b^2 \cdot b^{a-1} &\le b^2[1 + (a-1)(b-1)] \\ &= b^2(ab + 2 - a - b). \end{aligned}$$

Therefore, it suffices to show that

$$(ab + 1 - b)^2 + b^2(ab + 2 - a - b) \le 1 + ab,$$

which can be written as

$$1 + ab - (ab + 1 - b)^2 \ge b^2(ab + 2 - a - b).$$

Since

$$1 + ab - (ab + 1 - b)^2 = bB,$$

where

$$B = (2 - a - b) + 2ab - a^2b \ge 2ab - a^2b = ab(2 - a),$$

it is enough to prove that

$$ab^2(2 - a) \ge b^2(ab + 2 - a - b),$$

which is equivalent to the obvious inequality

$$b^2(a - 1)(2 - a - b) \ge 0.$$

The equality holds for $a = 0$ or $b = 0$. If $a + b = 2$, then the equality holds also for $a = b = 1$.

Remark. Actually, the following extension is valid:

- *If a, b are nonnegative real numbers such that*

$$\frac{1}{2} \le a + b \le 2,$$

then

$$a^{2b} + b^{2a} \le 1 + ab.$$

$\square$

P 3.7. *If a, b are nonnegative real numbers such that $a^2 + b^2 = 2$, then*

$$a^{2b} + b^{2a} \le 1 + ab.$$

(Vasile Cîrtoaje, 2007)

Solution. Without loss of generality, assume that $a \ge 1 \ge b$. Applying Bernoulli's inequality gives

$$a^b \le 1 + b(a-1),$$

hence

$$a^{2b} \le (1+ab-b)^2.$$

Also, since $0 \le b \le 1$ and $2a \ge 2$, we have

$$b^{2a} \le b^2.$$

Therefore, it suffices to show that

$$(1+ab-b)^2 + b^2 \le 1 + ab,$$

which can be written as

$$b(2+2ab-a-2b-a^2b) \ge 0.$$

So, we need to show that

$$2+2ab-a-2b-a^2b \ge 0,$$

which is equivalent to

$$4(1-a)(1-b) + a(2-2ab) \ge 0,$$

$$4(1-a)(1-b) + a(a-b)^2 \ge 0.$$

Since $a \ge 1$, it suffices to show that

$$4(1-a)(1-b) + (a-b)^2 \ge 0.$$

Indeed,

$$\begin{aligned} 4(1-a)(1-b) + (a-b)^2 &= -4(a-1)(1-b) + [(a-1)+(1-b)]^2 \\ &= [(a-1)-(1-b)]^2 = (a+b-2)^2 \ge 0. \end{aligned}$$

The equality holds for $a = b = 1$, for $a = \sqrt{2}$ and $b = 0$, and for $a = 0$ and $b = \sqrt{2}$. □

P 3.8. *If a, b are nonnegative real numbers such that $a^2 + b^2 = \frac{1}{4}$, then*

$$a^{2b} + b^{2a} \le 1 + ab.$$

(Vasile Cîrtoaje, 2007)

Solution. From $a^2+b^2=\dfrac{1}{4}$, it follows that

$$a,b \le \frac{1}{2},$$

$$ab=\frac{1}{2}(a+b)^2-\frac{1}{8},$$

$$a+b \ge \sqrt{a^2+b^2}=\frac{1}{2},$$

$$a+b \le \sqrt{2(a^2+b^2)}=\frac{1}{\sqrt{2}}.$$

Applying Bernoulli's inequality gives

$$a^{2b} \le 1+2b(a-1)=1-2b+2ab,$$

$$b^{2a} \le 1+2a(b-1)=1-2a+2ab.$$

Thus, it suffices to show that

$$(1-2b+2ab)+(1-2a+2ab) \le 1+ab,$$

$$1+3ab \le 2(a+b),$$

$$1+\frac{3}{2}(a+b)^2-\frac{3}{8} \le 2(a+b),$$

$$\left(a+b-\frac{1}{2}\right)\left(a+b-\frac{5}{6}\right) \le 0.$$

The left inequality is true since

$$a+b \le \frac{1}{\sqrt{2}} < \frac{5}{6}.$$

The equality holds for $a=0$ and $b=\dfrac{1}{2}$, and for $a=\dfrac{1}{2}$ and $b=0$.

Remark. Actually, the following extended result is valid:

- *If a,b are nonnegative real numbers such that*

$$\frac{1}{4} \le a^2+b^2 \le 2,$$

then

$$a^{2b}+b^{2a} \le 1+ab.$$

This inequality is a consequence of Remark from P 3.6 (since $\dfrac{1}{4} \le a^2+b^2 \le 2$ involves $\dfrac{1}{2} \le a+b \le 2$).

$\square$

P 3.9. *If a, b are positive real numbers, then*

$$a^a b^b \le (a^2 - ab + b^2)^{\frac{a+b}{2}}.$$

Solution. By the weighted AM-GM inequality, we have

$$a \cdot a + b \cdot b \ge (a+b) a^{\frac{a}{a+b}} b^{\frac{b}{a+b}},$$

$$\left(\frac{a^2+b^2}{a+b}\right)^{a+b} \ge a^a b^b.$$

Thus, it suffices to show that

$$a^2 - ab + b^2 \ge \left(\frac{a^2+b^2}{a+b}\right)^2,$$

which is equivalent to

$$(a+b)(a^3+b^3) \ge (a^2+b^2)^2,$$

$$ab(a-b)^2 \ge 0.$$

The equality holds for $a = b$.

□

P 3.10. *If $a, b \in (0,1]$, then*

$$a^a b^b \le 1 - ab + a^2 b^2.$$

(Vasile Cîrtoaje, 2010)

Solution. We claim that

$$x^x \le 1 - x + x^2$$

for all $x \in (0,1]$. If this is true, then

$$\begin{aligned} 1 - ab + a^2b^2 - a^a b^b &\ge 1 - ab + a^2b^2 - (1-a+a^2)(1-b+b^2) \\ &= (a+b)(1-a)(1-b) \ge 0. \end{aligned}$$

Thus, it suffices to show that $f(x) \le 0$ for $x \in (0,1]$, where

$$f(x) = x \ln x - \ln(x^2 - x + 1).$$

We have

$$f'(x) = \ln x + 1 - \frac{2x-1}{x^2-x+1},$$

$$f''(x) = \frac{(1-x)(1-2x-x^2-x^4)}{x(x^2-x+1)^2}.$$

Let $x_1 \in (0,1)$ be the positive root of the equation $x^4+x^2+2x = 1$. Then, $f''(x) > 0$ for $x \in (0, x_1)$ and $f''(x) < 0$ for $x \in (x_1, 1)$, hence f' is strictly increasing on $(0, x_1]$ and strictly decreasing on $[x_1, 1]$. Since $\lim_{x\to 0} f'(x) = -\infty$ and $f'(1) = 0$, there is $x_2 \in (0, x_1)$ such that $f'(x_2) = 0$, $f'(x) < 0$ for $x \in (0, x_2)$ and $f'(x) > 0$ for $x \in (x_2, 1)$. Therefore, f is decreasing on $(0, x_2]$ and increasing on $[x_2, 1]$. Since $\lim_{x\to 0} f(x) = 0$ and $f(1) = 0$, it follows that $f(x) \le 0$ for $x \in (0,1]$. The proof is completed. The equality holds for $a = b = 1$.

□

P 3.11. *If a, b are positive real numbers such that $a + b \le 2$, then*

$$\left(\frac{a}{b}\right)^b + \left(\frac{b}{a}\right)^a \le 2.$$

(Vasile Cîrtoaje, 2010)

Solution. Using the substitution $a = tc$ and $b = td$, where c, d, t are positive real numbers such that $c + d = 2$ and $t \le 1$, we need to show that

$$\left(\frac{c}{d}\right)^{td} + \left(\frac{d}{c}\right)^{tc} \le 2.$$

Write this inequality as

$$f(t) \le 2,$$

where

$$f(t) = A^t + B^t, \qquad A = \left(\frac{c}{d}\right)^d, \ \ B = \left(\frac{d}{c}\right)^c.$$

Since $f(t)$ is a convex function, we have

$$f(t) \le \max\{f(0), f(1)\} = \max\{2, f(1)\}.$$

Therefore, we only need to show that $f(1) \le 2$; that is,

$$2c^c d^d \ge c^2 + d^2.$$

Setting $c = 1 + x$ and $d = 1 - x$, where $0 \le x < 1$, this inequality turns into

$$(1+x)^{1+x}(1-x)^{1-x} \ge 1 + x^2,$$

which is equivalent to $f(x) \ge 0$, where

$$f(x) = (1+x)\ln(1+x) + (1-x)\ln(1-x) - \ln(1+x^2).$$

We have

$$f'(x) = \ln(1+x) - \ln(1-x) - \frac{2x}{1+x^2},$$

$$f''(x) = \frac{1}{1+x} + \frac{1}{1-x} - \frac{2(1-x^2)}{(1+x^2)^2} = \frac{8x^2}{(1-x^2)(1+x^2)^2}.$$

Since $f''(x) \ge 0$ for $x \in [0,1)$, it follows that f' is increasing, $f'(x) \ge f'(0) = 0$, $f(x)$ is increasing, hence $f(x) \ge f(0) = 0$. The proof is completed. The equality holds for $a = b$.

□

P 3.12. *If a, b are positive real numbers such that $a + b = 2$, then*

$$2a^a b^b \ge a^{2b} + b^{2a} + \frac{3}{4}(a-b)^2.$$

(Vasile Cîrtoaje, 2010)

Solution. According to the inequalities in P 3.5-(b) and P 3.11 (for $a + b = 2$), we have

$$a^{2b} + b^{2a} \le 1 + ab$$

and

$$2a^a b^b \ge a^2 + b^2.$$

Therefore, it suffices to show that

$$a^2 + b^2 \ge 1 + ab + \frac{3}{4}(a-b)^2.$$

which is an identity. The equality holds for $a = b = 1$.

□

P 3.13. *If $a, b \in (0,1]$ or $a, b \in [1,\infty)$, then*

$$2a^a b^b \ge a^2 + b^2.$$

Solution. For $a = x$ and $b = 1$, the desired inequality becomes

$$2x^x \ge x^2 + 1, \quad x > 0.$$

If this inequality is true, then

$$4a^a b^b - 2(a^2 + b^2) \ge (a^2+1)(b^2+1) - 2(a^2+b^2) = (a^2-1)(b^2-1) \ge 0.$$

To prove the inequality $2x^x \ge x^2+1$, we show that $f(x) \ge 0$, where

$$f(x) = \ln 2 + x\ln x - \ln(x^2+1).$$

We have

$$f'(x) = \ln x + 1 - \frac{2x}{x^2+1},$$
$$f''(x) = \frac{x^2(x+1)^2+(x-1)^2}{x(x^2+1)^2}.$$

Since $f''(x) > 0$ for $x > 0$, f' is strictly increasing. Since $f'(1) = 0$, it follows that $f'(x) < 0$ for $x \in (0,1)$ and $f'(x) > 0$ for $x \in (1,\infty)$. Therefore, f is decreasing on $(0,1]$ and increasing on $[1,\infty)$, hence $f(x) \ge f(1) = 0$ for $x > 0$. This completes the proof. The equality holds for $a = b = 1$.

□

P 3.14. *If a,b are positive real numbers, then*

$$2a^ab^b \ge a^2+b^2.$$

(Vasile Cîrtoaje, 2014)

Solution. By Lemma below, it suffices to show that

$$(a^4-2a^3+4a^2-2a+3)(b^4-2b^3+4b^2-2b+3) \ge 8(a^2+b^2),$$

which is equivalent to $A \ge 0$, where

$$\begin{aligned} A = & a^4b^4 - 2a^3b^3(a+b) + 4a^2b^2(a^2+b^2+ab) - [2ab(a^3+b^3) + 8a^2b^2(a+b)] \\ & + [3(a^4+b^4) + 4ab(a^2+b^2) + 16a^2b^2] - [6(a^3+b^3) + 8ab(a+b)] \\ & + 4(a^2+b^2+ab) - 6(a+b) + 9. \end{aligned}$$

We can check that

$$A = [a^2b^2 - ab(a+b) + a^2 + b^2 - 1]^2 + B,$$

where

$$\begin{aligned} B = & a^2b^2(a+b)^2 - 6a^2b^2(a+b) + [2(a^4+b^4) + 4ab(a^2+b^2) + 16a^2b^2] \\ & - [6(a^3+b^3) + 10ab(a+b)] + [6(a^2+b^2) + 4ab] - 6(a+b) + 8. \end{aligned}$$

Also, we have

$$B = [ab(a+b) - 3ab + 1]^2 + C,$$

where

$$\begin{aligned} C = & [2(a^4+b^4) + 4ab(a^2+b^2) + 7a^2b^2] - [6(a^3+b^3) + 12ab(a+b)] \\ & + [6(a^2+b^2) + 10ab] - 6(a+b) + 7, \end{aligned}$$

and

$$C = (ab-1)^2 + 2D,$$

where

$$\begin{aligned} D =& [a^4 + b^4 + 2ab(a^2+b^2) + 3a^2b^2] - [3(a^3+b^3) + 6ab(a+b)] \\ &+ 3(a+b)^2 - 3(a+b) + 3, \end{aligned}$$

It suffices to show that $D \ge 0$. Indeed,

$$\begin{aligned} D =& [(a+b)^4 - 2ab(a+b)^2 + a^2b^2] - 3[(a+b)^3 - ab(a+b)] \\ &+ 3(a+b)^2 - 3(a+b) + 3 \\ =& [(a+b)^2 - ab]^2 - 3(a+b)[(a+b)^2 - ab] + 3(a+b)^2 - 3(a+b) + 3 \\ =& \left[(a+b)^2 - ab - \frac{3}{2}(a+b)\right]^2 + 3\left(\frac{a+b}{2} - 1\right)^2 \ge 0. \end{aligned}$$

This completes the proof. The equality holds for $a = b = 1$.

Lemma. *If $x > 0$, then*

$$x^x \ge x + \frac{1}{4}(x-1)^2(x^2+3).$$

Proof. We need to show that $f(x) \ge 0$ for $x > 0$, where

$$f(x) = \ln 4 + x \ln x - \ln g(x), \quad g(x) = x^4 - 2x^3 + 4x^2 - 2x + 3.$$

We have

$$f'(x) = 1 + \ln x - \frac{2(2x^3 - 3x^2 + 4x - 1)}{g(x)},$$

$$f''(x) = \frac{x^8 + 6x^4 - 32x^3 + 48x^2 - 32x + 9}{g^2(x)} = \frac{(x-1)^2 h(x)}{g^2(x)},$$

where

$$h(x) = x^6 + 2x^5 + 3x^4 + 4x^3 + 11x^2 - 14x + 9.$$

Since

$$h(x) > 7x^2 - 14x + 7 = 7(x-1)^2 \ge 0,$$

we have $f''(x) \ge 0$, hence f' is strictly increasing on $(0, \infty)$. Since $f'(1) = 0$, it follows that $f'(x) < 0$ for $x \in (0,1)$ and $f'(x) > 0$ for $x \in (1, \infty)$. Therefore, f is decreasing on $(0,1]$ and increasing on $[1, \infty)$, hence $f(x) \ge f(1) = 0$ for $x > 0$.

□

P 3.15. *If $a \ge 1 \ge b > 0$, then*

$$2a^a b^b \ge a^{2b} + b^{2a}.$$

Solution. Taking into account the inequality $2a^ab^b \geq a^2 + b^2$ from the preceding P 3.14, it suffices to show that

$$a^2 + b^2 \geq a^{2b} + b^{2a}.$$

This inequality follows immediately from $a^2 \geq a^{2b}$ and $b^2 \geq b^{2a}$. The equality holds for $a = b = 1$.

$\square$

P 3.16. *If $a \geq e \geq b > 0$, then*

$$2a^ab^b \geq a^{2b} + b^{2a}.$$

Solution. It suffices to show that $a^ab^b \geq a^{2b}$ and $a^ab^b \geq b^{2a}$. Write the first inequality as

$$a^{a-b} \geq \left(\frac{a}{b}\right)^b,$$

$$a^{x-1} \geq x, \qquad x = \frac{a}{b} \geq 1.$$

Since $a^{x-1} \geq e^{x-1}$, we only need to show that

$$e^{x-1} \geq x,$$

which is equivalent to $f(x) \geq 0$ for $x \geq 1$, where

$$f(x) = x - 1 - \ln x.$$

From

$$f'(x) = 1 - \frac{1}{x} \geq 0,$$

it follows that f is increasing on $[1, \infty)$, therefore $f(x) \geq f(1) = 0$.

Write the second inequality as

$$\left(\frac{b}{a}\right)^a b^{a-b} \leq 1,$$

$$xb^{1-x} \leq 1, \qquad x = \frac{b}{a} \leq 1.$$

Since $b^{1-x} \leq e^{1-x}$, we only need to show that

$$xe^{1-x} \leq 1,$$

which is equivalent to $f(x) \leq 0$ for $x \leq 1$, where

$$f(x) = \ln x + 1 - x.$$

Since
$$f'(x) = \frac{1}{x} - 1 \ge 0,$$
f is increasing on $(0,1]$, therefore $f(x) \le f(1) = 0$. This completes the proof. The equality holds for $a = b = e$.

□

P 3.17. *If a, b are positive real numbers, then*
$$a^a b^b \ge \left(\frac{a^2+b^2}{2}\right)^{\frac{a+b}{2}}.$$

First Solution. Using the substitution $a = bx$, where $x > 0$, the inequality becomes as follows:
$$(bx)^{bx} b^b \ge \left(\frac{b^2x^2+b^2}{2}\right)^{\frac{bx+b}{2}},$$
$$(bx)^x b \ge \left(\frac{b^2x^2+b^2}{2}\right)^{\frac{x+1}{2}},$$
$$b^{x+1}x^x \ge b^{x+1}\left(\frac{x^2+1}{2}\right)^{\frac{x+1}{2}},$$
$$x^x \ge \left(\frac{x^2+1}{2}\right)^{\frac{x+1}{2}}.$$
It is true if $f(x) \ge 0$ for all $x > 0$, where
$$f(x) = \frac{x}{x+1}\ln x - \frac{1}{2}\ln\frac{x^2+1}{2}.$$
We have
$$f'(x) = \frac{1}{(x+1)^2}\ln x + \frac{1}{x+1} - \frac{x}{x^2+1} = \frac{g(x)}{(x+1)^2},$$
where
$$g(x) = \ln x - \frac{x^2-1}{x^2+1}.$$
Since
$$g'(x) = \frac{(x^2-1)^2}{x(x^2+1)^2} \ge 0,$$
g is strictly increasing, therefore $g(x) < 0$ for $x \in (0,1)$, $g(1) = 0$, $g(x) > 0$ for $x \in (1,\infty)$. Thus, f is decreasing on $(0,1]$ and increasing on $[1,\infty)$, hence $f(x) \ge f(1) = 0$. This completes the proof. The equality holds for $a = b$.

Second Solution. Write the inequality in the form

$$a \ln a + b \ln b \geq \frac{a+b}{2} \ln \frac{a^2+b^2}{2}.$$

Without loss of generality, consider $a+b=2k$, $k>0$, and denote

$$a = k+x, \quad b = k-x, \quad 0 \leq x < k.$$

We need to show that $f(x) \geq 0$, where

$$f(x) = (k+x)\ln(k+x) + (k-x)\ln(k-x) - k\ln(x^2+k^2).$$

We have

$$f'(x) = \ln(k+x) - \ln(k-x) - \frac{2kx}{x^2+k^2},$$

$$\begin{aligned} f''(x) &= \frac{1}{k+x} + \frac{1}{k-x} + \frac{2k(x^2-k^2)}{(x^2+k^2)^2} \\ &= \frac{8k^2x^2}{(k^2-x^2)(x^2+k^2)^2}. \end{aligned}$$

Since $f''(x) \geq 0$ for $x \geq 0$, f' is increasing, hence $f'(x) \geq f'(0) = 0$. Therefore, f is increasing on $[0,k)$, hence $f(x) \geq f(0) = 0$.

Remark. For $a+b=2$, this inequality can be rewritten as

$$2a^a b^b \geq a^2 + b^2,$$

$$2 \geq \left(\frac{a}{b}\right)^b + \left(\frac{b}{a}\right)^a.$$

Also, for $a+b=1$, the inequality becomes

$$2a^{2a} b^{2b} \geq a^2 + b^2,$$

$$2 \geq \left(\frac{a}{b}\right)^{2b} + \left(\frac{b}{a}\right)^{2a}.$$

□

P 3.18. *If a, b are positive real numbers such that $a^2+b^2=2$, then*

$$2a^a b^b \geq a^{2b} + b^{2a} + \frac{1}{2}(a-b)^2.$$

(Vasile Cîrtoaje, 2010)

Solution. According to the inequalities in P 3.7 and P 3.17, we have

$$a^{2b}+b^{2a}\le 1+ab$$

and

$$a^a b^b \ge 1.$$

Therefore, it suffices to show that

$$2\ge 1+ab+\frac{1}{2}(a-b)^2,$$

which is an identity. The equality holds for $a=b=1$.

□

P 3.19. *If $a,b\in(0,1]$, then*

$$(a^2+b^2)\left(\frac{1}{a^{2a}}+\frac{1}{b^{2b}}\right)\le 4.$$

(Vasile Cîrtoaje, 2014)

Solution. For $a=x$ and $b=1$, the desired inequality becomes

$$x^{2x}\ge\frac{1+x^2}{3-x^2},\quad x\in(0,1].$$

If this inequality is true, it suffices to show that

$$(a^2+b^2)\left(\frac{3-a^2}{1+a^2}+\frac{3-b^2}{1+b^2}\right)\le 4,$$

which is equivalent to

$$a^2b^2(2+a^2+b^2)+2-(a^2+b^2)-(a^2+b^2)^2\ge 0,$$

$$(2+a^2+b^2)(1-a^2)(1-b^2)\ge 0.$$

To prove the inequality $x^{2x}\ge\dfrac{1+x^2}{3-x^2}$, we show that $f(x)\ge 0$, where

$$f(x)=x\ln x+\frac{1}{2}\ln(3-x^2)-\frac{1}{2}\ln(1+x^2),\qquad x\in(0,1].$$

We have

$$f'(x)=1+\ln x-\frac{x}{3-x^2}-\frac{x}{1+x^2},$$

$$f''(x) = \frac{1}{x} - \frac{3+x^2}{(3-x^2)^2} - \frac{1-x^2}{(1+x^2)^2}$$
$$= \frac{(1-x)(9+6x-x^3)}{x(3-x)^2} - \frac{1-x^2}{(1+x^2)^2}.$$

We will show that $f''(x) > 0$ for $0 < x < 1$. This is true if

$$\frac{9+6x-x^3}{x(3-x)^2} - \frac{1+x}{(1+x^2)^2} > 0.$$

Indeed,

$$\frac{9+6x-x^3}{x(3-x)^2} - \frac{1+x}{(1+x^2)^2} > \frac{9}{9x} - \frac{1+x}{x(1+x)^2} = \frac{1}{1+x} > 0.$$

Since $f''(x) > 0$, f' is strictly increasing on $(0,1]$. Since $f'(1) = 0$, it follows that $f'(x) < 0$ for $x \in (0,1)$, f is strictly decreasing on $(0,1]$, hence $f(x) \ge f(1) = 0$. This completes the proof. The equality holds for $a = b = 1$.

□

P 3.20. *If a, b are positive real numbers such that $a + b = 2$, then*

$$a^b b^a + 2 \ge 3ab.$$

(Vasile Cîrtoaje, 2010)

Solution. Setting

$$a = 1 + x, \quad b = 1 - x, \quad 0 \le x < 1,$$

the inequality is equivalent to

$$(1+x)^{1-x}(1-x)^{1+x} \ge 1 - 3x^2.$$

Consider further the nontrivial case $0 \le x < \dfrac{1}{\sqrt{3}}$, and write the desired inequality as $f(x) \ge 0$, where

$$f(x) = (1-x)\ln(1+x) + (1+x)\ln(1-x) - \ln(1-3x^2).$$

We have

$$f'(x) = -\ln(1+x) + \ln(1-x) + \frac{1-x}{1+x} - \frac{1+x}{1-x} + \frac{6x}{1-3x^2},$$

$$\frac{1}{2}f''(x) = \frac{-1}{1-x^2} - \frac{2(x^2+1)}{(1-x^2)^2} + \frac{3(3x^2+1)}{(1-3x^2)^2}.$$

Making the substitution

$$t = x^2, \quad 0 \le t < \frac{1}{3},$$

we get

$$\frac{1}{2}f''(x) = \frac{3(3t+1)}{(3t-1)^2} - \frac{t+3}{(t-1)^2} = \frac{4t(5-9t)}{(t-1)^2(3t-1)^2} > 0.$$

Therefore, $f'(x)$ is strictly increasing, $f'(x) \ge f'(0) = 0$, $f(x)$ is strictly increasing, hence $f(x) \ge f(0) = 0$. This completes the proof. The equality holds for $a = b = 1$.

□

P 3.21. *Let a, b be positive real numbers such that $a + b = 2$. If $k \ge \frac{1}{2}$, then*

$$a^{a^{kb}} b^{b^{ka}} \ge 1.$$

(Vasile Cîrtoaje, 2010)

Solution. Setting

$$a = 1 + x, \quad b = 1 - x, \quad 0 \le x < 1,$$

the inequality can be written as

$$(1+x)^{k(1-x)} \ln(1+x) + (1-x)^{k(1+x)} \ln(1-x) \ge 0.$$

Consider further the nontrivial case $0 < x < 1$, and write the desired inequality as $f(x) \ge 0$, where

$$f(x) = k(1-x)\ln(1+x) - k(1+x)\ln(1-x) + \ln\ln(1+x) - \ln(-\ln(1-x)).$$

It suffices to show that $f'(x) > 0$. Indeed, if this is true, then $f(x)$ is strictly increasing, hence

$$f(x) > \lim_{x \to 0} f(x) = 0.$$

We have

$$\begin{aligned}
f'(x) &= \frac{2k(1+x^2)}{1-x^2} - k\ln(1-x^2) + \frac{1}{(1+x)\ln(1+x)} + \frac{1}{(1-x)\ln(1-x)} \\
&> \frac{2k}{1-x^2} + \frac{1}{(1+x)\ln(1+x)} + \frac{1}{(1-x)\ln(1-x)} \\
&\ge \frac{1}{1-x^2} + \frac{1}{(1+x)\ln(1+x)} + \frac{1}{(1-x)\ln(1-x)} \\
&= \frac{g(x)}{(1-x^2)\ln(1+x)\ln(1-x)},
\end{aligned}$$

where

$$g(x) = \ln(1+x)\ln(1-x) + (1+x)\ln(1+x) + (1-x)\ln(1-x).$$

It is enough to how that $g(x) < 0$. We have

$$g'(x) = \frac{-x}{1-x^2}h(x),$$

where

$$h(x) = (1+x)\ln(1+x) + (1-x)\ln(1-x).$$

Since

$$h'(x) = \ln\frac{1+x}{1-x} > 0,$$

$h(x)$ is strictly increasing, $h(x) > h(0) = 0$, $g'(x) < 0$, $g(x)$ is strictly decreasing, and hence $g(x) < g(0) = 0$. This completes the proof. The equality holds for $a = b = 1$.

□

P 3.22. *If a, b are positive real numbers such that $a + b = 2$, then*

$$a^{\sqrt{a}} b^{\sqrt{b}} \geq 1.$$

(Vasile Cîrtoaje, 2010)

Solution. Assume that $a > 1 > b$. Taking logarithms of both sides, the inequality becomes in succession:

$$\sqrt{a}\ln a + \sqrt{b}\ln b \geq 0,$$

$$\sqrt{a}\ln a \geq \sqrt{b}(-\ln b),$$

$$\frac{1}{2}\ln a + \ln\ln a \geq \frac{1}{2}\ln b + \ln(-\ln b).$$

Substituting

$$a = 1+x, \quad b = 1-x, \quad 0 < x < 1,$$

we need to show that $f(x) \geq 0$, where

$$f(x) = \frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x) + \ln\ln(1+x) - \ln(-\ln(1-x)).$$

We have

$$f'(x) = \frac{1}{1-x^2} + \frac{1}{(1+x)\ln(1+x)} + \frac{1}{(1-x)\ln(1-x)}.$$

As shown in the proof of the preceding P 3.21, we have $f'(x) > 0$. Therefore, $f(x)$ is strictly increasing, therefore

$$f(x) > \lim_{x\to 0} f(x) = 0.$$

The equality holds for $a = b = 1$.

□

P 3.23. *If a, b are positive real numbers such that $a+b=2$, then*

$$a^{a+1}b^{b+1} \le 1-\frac{1}{48}(a-b)^4.$$

(Vasile Cîrtoaje, 2010)

Solution. Putting

$$a=1+x, \quad b=1-x, \quad 0\le x<1,$$

the inequality becomes

$$(1+x)^{2+x}(1-x)^{2-x} \le 1-\frac{1}{3}x^4.$$

Write this inequality as $f(x)\le 0$, where

$$f(x)=(2+x)\ln(1+x)+(2-x)\ln(1-x)-\ln\left(1-\frac{1}{3}x^4\right).$$

We have

$$f'(x)=\ln(1+x)-\ln(1-x)-\frac{2x}{1-x^2}+\frac{4x^3}{3-x^4},$$

$$f''(x)=\frac{2}{1-x^2}-\frac{2(1+x^2)}{(1-x^2)^2}+\frac{4x^2(x^4+9)}{(3-x^4)^2}$$

$$=\frac{-4x^2}{(1-x^2)^2}+\frac{4x^2(x^4+9)}{(3-x^4)^2}=\frac{-8x^4[x^4+1+8(1-x^2)]}{(1-x^2)^2(3-x^4)^2}\le 0.$$

Therefore, $f'(x)$ is decreasing, $f'(x)\le f'(0)=0$, $f(x)$ is decreasing, $f(x)\le f(0)=0$. The equality holds for $a=b=1$.

□

P 3.24. *If a, b are positive real numbers such that $a+b=2$, then*

$$a^{-a}+b^{-b}\le 2.$$

(Vasile Cîrtoaje, 2010)

Solution. Consider $a\ge b$, when we have

$$0<b\le 1\le a<2,$$

and write the inequality as

$$\frac{a^a-1}{a^a}+\frac{b^b-1}{b^b}\ge 0.$$

According to Lemma from the proof of P 3.4, we have

$$a^a - 1 \ge a^{\frac{a+1}{2}}(a-1), \qquad b^b - 1 \ge b^{\frac{b+1}{2}}(b-1).$$

Therefore, it suffices to show that

$$a^{\frac{1-a}{2}}(a-1) + b^{\frac{1-b}{2}}(b-1) \ge 0,$$

which is equivalent to

$$a^{\frac{1-a}{2}} \ge b^{\frac{1-b}{2}},$$

$$(ab)^{\frac{1-b}{2}} \le 1,$$

$$ab \le 1,$$

$$(a-b)^2 \ge 0.$$

The equality holds for $a = b = 1$.

□

P 3.25. *If* $a, b \in [0,1]$, *then*

$$a^{b-a} + b^{a-b} + (a-b)^2 \le 2.$$

(Vasile Cîrtoaje, 2010)

Solution *(by Vo Quoc Ba Can).* Without loss of generality, assume that $a \ge b$. Using the substitution

$$c = a - b,$$

we need to show that

$$(b+c)^{-c} + b^c + c^2 \le 2$$

for

$$0 \le b \le 1 - c, \quad 0 \le c \le 1.$$

If $c = 1$, then $b = 0$, and the inequality is an equality. Also, for $c = 0$, the inequality is an equality. Consider further that

$$0 < c < 1.$$

We need to show that $f(x) \le 0$, where

$$f(x) = (x+c)^{-c} + x^c + c^2 - 2, \qquad x \in [0, 1-c].$$

We claim that $f'(x) > 0$ for $x > 0$. On this assumption, $f(x)$ is strictly increasing on $[0, 1-c]$, hence

$$f(x) \le f(1-c) = (1-c)^c - (1-c^2).$$

By Bernoulli's inequality, we have

$$f(x) \le 1 + c(-c) - (1 - c^2) = 0.$$

Since

$$f'(x) = \frac{c[(x+c)^{1+c} - x^{1-c}]}{(x+c)^{1+c}x^{1-c}},$$

the inequality $f'(x) > 0$ holds for $x > 0$ if and only if

$$x + c > x^{\frac{1-c}{1+c}}.$$

For any $d > 0$, using the weighted AM-GM inequality yields

$$x + c = x + d \cdot \frac{c}{d} \ge (1+d)x^{\frac{1}{1+d}} \left(\frac{c}{d}\right)^{\frac{d}{1+d}}.$$

Choosing

$$d = \frac{2c}{1-c},$$

we get

$$x + c \ge \frac{1+c}{2} \left(\frac{1-c}{2}\right)^{\frac{c-1}{1+c}} x^{\frac{1-c}{1+c}}.$$

Thus, it suffices to show that

$$\frac{1+c}{2} \ge \left(\frac{1-c}{2}\right)^{\frac{1-c}{1+c}}.$$

Indeed, using Bernoulli's inequality, we get

$$\left(\frac{1-c}{2}\right)^{\frac{1-c}{1+c}} = \left(1 - \frac{1+c}{2}\right)^{\frac{1-c}{1+c}} \le 1 - \frac{1-c}{1+c} \cdot \frac{1+c}{2} = \frac{1+c}{2}.$$

The equality holds for $a = b$, for $a = 1$ and $b = 0$, and for $a = 0$ and $b = 1$.

□

P 3.26. *If a, b are nonnegative real numbers such that $a + b \le 2$, then*

$$a^{b-a} + b^{a-b} + \frac{7}{16}(a-b)^2 \le 2.$$

(Vasile Cîrtoaje, 2010)

Solution. Assume that $a \ge b$. Using the substitution

$$c = a - b,$$

we need to show that

$$a^{-c} + (a-c)^c + \frac{7}{16}c^2 \le 2$$

for

$$0 \le c \le 2, \quad c \le a \le 1 + \frac{c}{2}.$$

For $c = 0$ and $c = 2$ (which involves $a = 2$), the inequality is an equality. Therefore, we only need to show that $f(x) \le 0$ for $0 < c < 2$, where

$$f(x) = x^{-c} + (x-c)^c + \frac{7}{16}c^2 - 2, \qquad x \in \left[c, 1 + \frac{c}{2}\right].$$

In the case $c = 1$, we need to show that $f(x) \le 0$ for $x \in \left[1, \frac{3}{2}\right]$; indeed, we have

$$f(x) = \frac{1}{x} + x - \frac{41}{16} \le \frac{2}{3} + \frac{3}{2} - \frac{41}{16} = \frac{-19}{48} < 0.$$

Consider next that

$$c \in (0,1) \cup (1,2).$$

The derivative

$$f'(x) = \frac{c[x^{1+c} - (x-c)^{1-c}]}{x^{1+c}(x-c)^{1-c}}$$

has the same sign as

$$g(x) = (1+c)\ln x - (1-c)\ln(x-c).$$

We have

$$g'(x) = \frac{c(2x-1-c)}{x(x-c)}.$$

Case 1: $0 < c < 1$. We claim that $g(x) > 0$ for $x \in \left(c, 1 + \frac{c}{2}\right]$. On this assumption, f is strictly increasing on $\left[c, 1 + \frac{c}{2}\right]$, hence

$$f(x) \le f\left(1 + \frac{c}{2}\right).$$

Thus, we need to show that $f\left(1 + \frac{c}{2}\right) \le 0$, which is just the inequality in Lemma 4 below.

From the expression of $g'(x)$, it follows that $g(x)$ is decreasing on $\left(c, \frac{1+c}{2}\right]$, and increasing on $\left[\frac{1+c}{2}, 1+\frac{c}{2}\right]$. Then, to show that $g(x) > 0$ for $x \in \left(c, 1+\frac{c}{2}\right]$, it suffices to prove that

$$g\left(\frac{1+c}{2}\right) > 0,$$

which is equivalent to

$$\left(\frac{1+c}{2}\right)^{1+c} > \left(\frac{1-c}{2}\right)^{1-c}.$$

This inequality follows from Bernoulli's inequality, as follows:

$$\left(\frac{1+c}{2}\right)^{1+c} = \left(1-\frac{1-c}{2}\right)^{1+c} > 1-\frac{(1+c)(1-c)}{2} = \frac{1+c^2}{2}$$

and

$$\left(\frac{1-c}{2}\right)^{1-c} = \left(1-\frac{1+c}{2}\right)^{1-c} < 1-\frac{(1-c)(1+c)}{2} = \frac{1+c^2}{2}.$$

Case 2: $1 < c < 2$. Since

$$2x-1-c \geq 2c-1-c = c-1 > 0,$$

it follows that $g'(x) > 0$, hence $g(x)$ is strictly increasing. For $x \to c$, we have $g(x) \to -\infty$. If $g(1+c/2) \leq 0$, then $g(x) \leq 0$, hence f is decreasing. If $g(1+c/2) > 0$, then there exists $x_1 \in (c, 1+c/2)$ such that $g(x_1) = 0$, $g(x) < 0$ for $x \in [c, x_1)$ and $g(x) > 0$ for $x \in (x_1, 1+c/2]$, hence f is decreasing on $[c, x_1]$ and increasing on $[x_1, 1+c/2]$. Therefore, it suffices to show that $f(c) \leq 0$ and $f\left(1+\frac{c}{2}\right) \leq 0$. These inequalities follow respectively from Lemma 1 and Lemma 4 below.

The proof is completed. The equality holds for $a = b$, for $a = 2$ and $b = 0$, and for $a = 0$ and $b = 2$.

Lemma 1. *If* $1 \leq c \leq 2$*, then*

$$c^{-c} + \frac{7}{16}c^2 \leq 2,$$

with equality for $c = 2$*.*

Proof. The desired inequality is equivalent to $h(c) \geq 0$, where

$$h(c) = c\ln c + \ln\left(2-\frac{7}{16}c^2\right), \qquad c \in [1,2].$$

We have

$$h'(c) = 1+\ln c - \frac{14c}{32-7c^2},$$

$$h''(c) = \frac{1}{c} - \frac{14(32+7c^2)}{(32-7c^2)^2}.$$

Since h'' is strictly decreasing, $h''(1) = 79/625$ and $h''(2) = -52$, there exists $c_1 \in (1,2)$ such that $h''(c_1) = 0$, $h''(c) > 0$ for $c \in [1, c_1)$ and $h''(c) < 0$ for $c \in (c_1, 2]$, hence h' is strictly increasing on $[1, c_1]$ and strictly decreasing on $[c_1, 2]$. Since $h'(1) = 11/25$ and $h'(2) = \ln 2 - 6 < 0$, there exists $c_2 \in (1,2)$ such that $h'(c_2) = 0$, $h'(c) > 0$ for $c \in [1, c_2)$ and $h'(c) < 0$ for $c \in (c_2, 2]$, hence h is strictly increasing on $[1, c_2]$ and strictly decreasing on $[c_2, 2]$. Thus, it suffices to show that $h(1) \ge 0$ and $h(2) \ge 0$. Indeed, $h(1) = \ln 25 - \ln 16 > 0$ and $h(2) = 0$.

Lemma 2. *If $0 \le x \le 2$, then*

$$\left(1 + \frac{x}{2}\right)^{-x} + \frac{3}{16}x^2 \le 1,$$

with equality for $x = 0$ and $x = 2$.

Proof. We need to show that $f(x) \le 0$, where

$$f(x) = -x \ln\left(1 + \frac{x}{2}\right) - \ln\left(1 - \frac{3}{16}x^2\right), \qquad x \in [0,2].$$

We have

$$f'(x) = -\ln\left(1 + \frac{x}{2}\right) + \frac{x(3x^2 + 6x - 4)}{(2+x)(16-3x^2)},$$

$$f''(x) = \frac{g(x)}{(2+x)^2(16-3x^2)^2},$$

where

$$g(x) = -9x^5 - 18x^4 + 168x^3 + 552x^2 + 128x - 640.$$

Since $g(x_1) = 0$ for $x_1 \approx 0,88067$, $g(x) < 0$ for $x \in [0, x_1)$ and $g(x) > 0$ for $x \in (x_1, 2]$, f' is strictly decreasing on $[0, x_1]$ and strictly increasing on $[x_1, 2]$. Since $f'(0) = 0$ and $f'(2) = -\ln 2 + \dfrac{5}{2} > 0$, there is $x_2 \in (x_1, 2)$ such that $f'(x_2) = 0$, $f'(x) < 0$ for $x \in (0, x_2)$, and $f'(x) > 0$ for $x \in (x_2, 2]$. Therefore, f is decreasing on $[0, x_2]$ and increasing on $[x_2, 2]$. Since $f(0) = f(2) = 0$, it follows that $f(x) \le 0$ for $x \in [0, 2]$.

Lemma 3. *If $0 \le x \le 2$, then*

$$\left(1 - \frac{x}{2}\right)^{x} + \frac{1}{4}x^2 \le 1,$$

with equality for $x = 0$ and $x = 2$.

Proof. We need to show that $f(x) \le 0$, where

$$f(x) = x \ln\left(1 - \frac{x}{2}\right) - \ln\left(1 - \frac{1}{4}x^2\right), \qquad x \in [0,2).$$

We have

$$f'(x) = \ln\left(1 - \frac{x}{2}\right) - \frac{x^2}{4-x^2},$$

$$f''(x) = \frac{-1}{2-x} - \frac{8x}{(4-x^2)^2}.$$

Since $f'' < 0$ for $x \in [0,2)$, f' is strictly decreasing, hence $f'(x) \le f'(0) = 0$, f is strictly decreasing, therefore $f(x) \le f(0) = 0$ for $x \in [0,2)$.

Lemma 4. *If $0 \le x \le 2$, then*

$$\left(1+\frac{x}{2}\right)^{-x} + \left(1-\frac{x}{2}\right)^{x} + \frac{7}{16}x^2 \le 2,$$

with equality for $x = 0$ and $x = 2$.

Proof. By Lemma 2 and Lemma 3, we have

$$\left(1+\frac{x}{2}\right)^{-x} + \frac{3}{16}x^2 \le 1$$

and

$$\left(1-\frac{x}{2}\right)^{x} + \frac{1}{4}x^2 \le 1.$$

The desired inequality follows by adding up these inequalities.

Conjecture. *If a, b are nonnegative real numbers such that $a + b = \frac{1}{4}$, then*

$$a^{2(b-a)} + b^{2(a-b)} \le 2.$$

□

P 3.27. *If a, b are nonnegative real numbers such that $a + b \le 4$, then*

$$a^{b-a} + b^{a-b} \le 2.$$

(Vasile Cîrtoaje, 2010)

Solution. Without loss of generality, assume that $a \ge b$. Consider first that $a - b \ge 2$. We have

$$a \ge a - b \ge 2,$$

and from

$$4 \ge a + b = (a-b) + 2b \ge 2 + 2b,$$

we get $b \le 1$. Clearly, the desired inequality is true because

$$a^{b-a} < 1, \quad b^{a-b} \le 1.$$

Since the case $a - b = 0$ is trivial, consider further that $0 < a - b < 2$ and use the substitution

$$c = a - b.$$

So, we need to show that

$$a^{-c} + (a-c)^c \le 2$$

for

$$0 < c < 2, \quad c \le a \le 2 + \frac{c}{2}.$$

Equivalently, we need to show that $f(x) \le 0$ for $0 < c < 2$, where

$$f(x) = x^{-c} + (x-c)^c - 2, \qquad x \in \left[c, 2 + \frac{c}{2}\right].$$

The derivative

$$f'(x) = \frac{c[x^{1+c} - (x-c)^{1-c}]}{x^{1+c}(x-c)^{1-c}}$$

has the same sign as

$$g(x) = (1+c)\ln x - (1-c)\ln(x-c).$$

We have

$$g'(x) = \frac{c(2x-1-c)}{x(x-c)}.$$

Case 1: $c = 1$. We need to show that $x^2 - 3x + 1 \le 0$ for $x \in \left[1, \frac{5}{2}\right]$. Indeed, we have

$$2(x^2 - 3x + 1) = (x-1)(2x-5) + (x-3) < 0.$$

Case 2: $0 < c < 1$. We will show that $g(x) > 0$ for $x \in \left(c, 2 + \frac{c}{2}\right]$. From

$$g'(x) = \frac{c(2x-1-c)}{x(x-c)},$$

it follows that $g(x)$ is decreasing on $\left(c, \frac{1+c}{2}\right]$ and increasing on $\left[\frac{1+c}{2}, 2 + \frac{c}{2}\right]$. Then, to show that $g(x) > 0$ for $x \in \left(c, 1 + \frac{c}{2}\right]$, it suffices to prove that

$$g\left(\frac{1+c}{2}\right) > 0,$$

which is equivalent to

$$\left(\frac{1+c}{2}\right)^{1+c} > \left(\frac{1-c}{2}\right)^{1-c}.$$

This inequality follows from Bernoulli's inequality, as follows:

$$\left(\frac{1+c}{2}\right)^{1+c} = \left(1 - \frac{1-c}{2}\right)^{1+c} > 1 - \frac{(1+c)(1-c)}{2} = \frac{1+c^2}{2}$$

and
$$\left(\frac{1-c}{2}\right)^{1-c}=\left(1-\frac{1+c}{2}\right)^{1-c}<1-\frac{(1-c)(1+c)}{2}=\frac{1+c^2}{2}.$$
Since $g(x)>0$ involves $f'(x)>0$, it follows that $f(x)$ is strictly increasing on $\left[c,2+\frac{c}{2}\right]$, and hence
$$f(x)\le f\left(2+\frac{c}{2}\right).$$
So, we need to show that $f\left(2+\frac{c}{2}\right)\le 0$ for $0<c<1$, which follows immediately from Lemma 3 below.

Case 3: $1<c<2$. Since
$$2x-1-c\ge 2c-1-c>0,$$
we have $g'(x)>0$, hence $g(x)$ is strictly increasing. Since $g(x)\to-\infty$ when $x\to c$ and
$$\begin{aligned}g\left(2+\frac{c}{2}\right)&=(1+c)\ln\left(2+\frac{c}{2}\right)+(c-1)\ln\left(2-\frac{c}{2}\right)\\&>(c-1)\ln\left(2-\frac{c}{2}\right)>0,\end{aligned}$$
there exists $x_1\in\left(c,2+\frac{c}{2}\right)$ such that $g(x_1)=0$, $g(x)<0$ for $x\in(c,x_1)$ and $g(x)>0$ for $x\in\left(x_1,2+\frac{c}{2}\right)$. Thus, $f(x)$ is decreasing on $[c,x_1]$ and increasing on $\left[x_1,2+\frac{c}{2}\right]$. Then, it suffices to show that $f(c)\le 0$ and $f\left(2+\frac{c}{2}\right)\le 0$. The first inequality is true because
$$f(c)=c^{-c}-2<1-2<0,$$
while the second inequality follows immediately from Lemma 3 below.

The proof is completed. The equality holds for $a=b$.

Lemma 1. *If* $x<4$*, then*
$$xh(x)\le 0,$$
where
$$h(x)=\ln\left(2-\frac{x}{2}\right)-\left(\ln 2-\frac{x}{4}-\frac{1}{32}x^2\right).$$
Proof. From
$$h'(x)=\frac{-x^2}{16(4-x)}\le 0,$$
it follows that $h(x)$ is decreasing. Since $h(0)=0$, we have $h(x)\ge 0$ for $x\le 0$, and $h(x)\le 0$ for $x\in[0,4)$; that is, $xh(x)\le 0$ for $x<4$.

Lemma 2. *If*

$$-2 \le x \le 2,$$

then

$$\left(2-\frac{x}{2}\right)^x \le 1+x\ln 2-\frac{x^3}{9}.$$

Proof. We have

$$\ln 2 \approx 0.693 < 7/9.$$

If $x \in [0,2]$, then

$$1+x\ln 2-\frac{x^3}{9} \ge 1-\frac{x^3}{9} \ge 1-\frac{8}{9} > 0.$$

Also, if $x \in [-2,0]$, then

$$1+x\ln 2-\frac{x^3}{9} \ge 1+\frac{7x}{9}-\frac{x^3}{9} > \frac{8+7x-x^3}{9}$$
$$= \frac{2(x+2)^2+(-x)(x+1)^2}{9} > 0.$$

So, we can write the desired inequality as $f(x) \ge 0$, where

$$f(x) = \ln\left(1+dx-\frac{x^3}{9}\right)-x\ln\left(2-\frac{x}{2}\right), \quad d = \ln 2.$$

We have

$$f'(x) = \frac{9d-3x^2}{9+9dx-x^3}+\frac{x}{4-x}-\ln\left(2-\frac{x}{2}\right).$$

Since $f(0)=0$, it suffices to show that $f'(x) \le 0$ for $x \in [-2,0]$, and $f'(x) \ge 0$ for $x \in [0,2]$; that is, $xf'(x) \ge 0$ for $x \in [-2,2]$. We have

$$f'(x) = g(x)-h(x),$$

where

$$g(x) = \frac{9d-3x^2}{9+9dx-x^3}+\frac{x}{4-x}-\left(d-\frac{x}{4}-\frac{1}{32}x^2\right),$$
$$h(x) = \ln\left(2-\frac{x}{2}\right)-\left(d-\frac{x}{4}-\frac{1}{32}x^2\right).$$

According to Lemma 1,

$$xf'(x) = xg(x)-xh(x) \ge xg(x).$$

Therefore, to show that $xf'(x) \ge 0$, it suffices to prove that $xg(x) \ge 0$. We have

$$g(x) = \left(\frac{9d-3x^2}{9+9dx-x^3}-d\right)+\left(\frac{x}{4-x}+\frac{x}{4}+\frac{1}{32}x^2\right)$$
$$= x\left[\frac{dx^2-3x-9d^2}{9+9dx-x^3}+\frac{64-4x-x^2}{32(4-x)}\right],$$

hence

$$xg(x) = \frac{x^2 g_1(x)}{32(4-x)(9+9dx-x^3)},$$

where

$$\begin{aligned} g_1(x) =& 32(4-x)(dx^2-3x-9d^2)+(64-4x-x^2)(9+9dx-x^3) \\ =& x^5+4x^4-(64+41d)x^3+(87+92d)x^2+12(24d^2+48d-35)x \\ &+576(1-2d^2). \end{aligned}$$

Since $g_1(x) \ge 0$ for $x \in [a_1, b_1]$, where $a_1 \approx -12.384$ and $b_1 = \approx 2.652$, we have $g_1(x) \ge 0$ for $x \in [-2, 2]$.

Lemma 3. *If* $0 \le c \le 2$*, then*

$$\left(2+\frac{c}{2}\right)^{-c} + \left(2-\frac{c}{2}\right)^{c} \le 2.$$

Proof. According to Lemma 2, the following inequalities hold for $c \in [0, 2]$:

$$\left(2+\frac{c}{2}\right)^{-c} \le 1 - c\ln 2 + \frac{c^3}{9},$$

$$\left(2-\frac{c}{2}\right)^{c} \le 1 + c\ln 2 - \frac{c^3}{9}.$$

Summing these inequalities, the desired inequality follows.

□

P 3.28. *If* a, b *are nonnegative real numbers such that* $a + b = 2$*, then*

$$a^{2b} + b^{2a} \ge a^b + b^a \ge a^2 b^2 + 1.$$

(Vasile Cîrtoaje, 2010)

Solution. Since $a, b \in [0, 2]$ and

$$(1-a)(1-b) = -(1-a)^2 \le 0,$$

from Lemma below, we have

$$a^b - 1 \ge \frac{b(ab+3-a-b)(a-1)}{2} = \frac{b(ab+1)(a-1)}{2}$$

and

$$b^a - 1 \ge \frac{a(ab+1)(b-1)}{2}.$$

Based on these inequalities, we get

$$\begin{aligned} a^b + b^a - a^2b^2 - 1 &= (a^b - 1) + (b^a - 1) + 1 - a^2b^2 \\ &\geq \frac{b(ab+1)(a-1)}{2} + \frac{a(ab+1)(b-1)}{2} + 1 - a^2b^2 \\ &= (ab+1)(ab-1) + 1 - a^2b^2 = 0 \end{aligned}$$

and

$$\begin{aligned} a^{2b} + b^{2a} - a^b - b^a &= a^b(a^b - 1) + b^a(b^a - 1) \\ &\geq \frac{a^b b(ab+1)(a-1)}{2} + \frac{b^a a(ab+1)(b-1)}{2} \\ &= \frac{ab(ab+1)(a-b)(a^{b-1} - b^{a-1})}{4}. \end{aligned}$$

Under the assumption that $a \geq b$, we only need to show that $a^{b-1} \geq b^{a-1}$, which is equivalent to

$$a^{\frac{b-a}{2}} \geq b^{\frac{a-b}{2}}, \quad 1 \geq (ab)^{\frac{a-b}{2}}, \quad 1 \geq ab, \quad (a-b)^2 \geq 0.$$

For both inequalities, the equality holds when $a = b = 1$, when $a = 0$ and $b = 2$, and when $a = 2$ and $b = 0$.

Lemma. *If $x, y \in [0,2]$ such that $(1-x)(1-y) \leq 0$, then*

$$x^y - 1 \geq \frac{y(xy + 3 - x - y)(x-1)}{2},$$

with equality for $x = 1$, and also for $y = 0$, $y = 1$ and $y = 2$.

Proof. For $y = 0$, $y = 1$ and $y = 2$, the inequality is an identity. For fixed

$$y \in (0,1) \cup (1,2),$$

let us define

$$f(x) = x^y - 1 - \frac{y(xy + 3 - x - y)(x-1)}{2}.$$

We have

$$f'(x) = y\left[x^{y-1} - \frac{xy + 3 - x - y}{2} - \frac{(x-1)(y-1)}{2}\right],$$

$$f''(x) = y(y-1)(x^{y-2} - 1).$$

Since $x^{y-2} - 1$ has the same sign as $1 - x$, it follows that $f''(x) \geq 0$ for $x \in (0,2]$, therefore f' is increasing. There are two cases to consider.

Case 1: $x \geq 1 > y$. We have $f'(x) \geq f'(1) = 0$, $f(x)$ is increasing, hence

$$f(x) \geq f(1) = 0.$$

Case 2: $y > 1 \geq x$. We have $f'(x) \leq f'(1) = 0$, $f(x)$ is decreasing, hence

$$f(x) \geq f(1) = 0.$$

□

P 3.29. *If a, b are positive real numbers such that $a+b=2$, then*

$$a^{3b}+b^{3a}\le 2.$$

(Vasile Cîrtoaje, 2007)

Solution. Without loss of generality, assume that $a\ge b$. Using the substitution

$$a=1+x,\quad b=1-x,\qquad 0\le x<1,$$

we can write the inequality as

$$e^{3(1-x)\ln(1+x)}+e^{3(1+x)\ln(1-x)}\le 2.$$

Applying Lemma below, it suffices to show that $f(x)\le 2$, where

$$f(x)=e^{3(1-x)\left(x-\frac{x^2}{2}+\frac{x^3}{3}\right)}+e^{-3(1+x)\left(x+\frac{x^2}{2}+\frac{x^3}{3}\right)}.$$

Since $f(0)=2$, it suffices to show that $f'(x)\le 0$ for $x\in[0,1)$. From

$$f'(x)=\left(3-9x+\frac{15}{2}x^2-4x^3\right)e^{3x-\frac{9x^2}{2}+\frac{5x^3}{2}-x^4}$$
$$-\left(3+9x+\frac{15}{2}x^2+4x^3\right)e^{-3x-\frac{9x^2}{2}-\frac{5x^3}{2}-x^4},$$

it follows that $f'(x)\le 0$ is equivalent to

$$e^{-6x-5x^3}\ge\frac{6-18x+15x^2-8x^3}{6+18x+15x^2+8x^3}.$$

For the nontrivial case $6-18x+15x^2-8x^3>0$, we rewrite this inequality as $g(x)\ge 0$, where

$$g(x)=-6x-5x^3-\ln(6-18x+15x^2-8x^3)+\ln(6+18x+15x^2+8x^3).$$

Since $g(0)=0$, it suffices to show that $g'(x)\ge 0$ for $x\in[0,1)$. From

$$\frac{1}{3}g'(x)=-2-5x^2+\frac{(6+8x^2)-10x}{6+15x^2-(18x+8x^3)}+\frac{(6+8x^2)+10x}{6+15x^2+(18x+8x^3)},$$

it follows that $g'(x)\ge 0$ is equivalent to

$$2(6+8x^2)(6+15x^2)-20x(18x+8x^3)\ge(2+5x^2)[(6+15x^2)^2-(18x+8x^3)^2].$$

Since

$$(6+15x^2)^2-(18x+8x^3)^2\le(6+15x^2)^2-324x^2-288x^4\le 4(9-36x^2),$$

it suffices to show that

$$(3+4x^2)(6+15x^2)-5x(18x+8x^3) \geq (2+5x^2)(9-36x^2).$$

This reduces to $6x^2 + 200x^4 \geq 0$, which is clearly true. The equality holds for $a = b = 1$.

Lemma. *If $t > -1$, then*

$$\ln(1+t) \leq t - \frac{t^2}{2} + \frac{t^3}{3}.$$

Proof. We need to prove that $f(t) \geq 0$, where

$$f(t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \ln(1+t).$$

Since

$$f'(t) = \frac{t^3}{t+1},$$

$f(t)$ is decreasing on $(-1, 0]$ and increasing on $[0, \infty)$. Therefore,

$$f(t) \geq f(0) = 0.$$

□

P 3.30. *If a, b are nonnegative real numbers such that $a + b = 2$, then*

$$a^{3b} + b^{3a} + \left(\frac{a-b}{2}\right)^4 \leq 2.$$

(Vasile Cîrtoaje, 2007)

Solution *(by M. Miyagi and Y. Nishizawa).* Using the substitution

$$a = 1 + x, \quad b = 1 - x, \qquad 0 \leq x \leq 1,$$

we can write the inequality as

$$(1+x)^{3(1-x)} + (1-x)^{3(1+x)} + x^4 \leq 2.$$

By Lemma below, we have

$$(1+x)^{1-x} \leq \frac{1}{4}(1+x)^2(2-x^2)(2-2x+x^2),$$

$$(1-x)^{1+x} \leq \frac{1}{4}(1-x)^2(2-x^2)(2+2x+x^2).$$

Therefore, it suffices to show that

$$(1+x)^6(2-x^2)^3(2-2x+x^2)^3+(1-x)^6(2-x^2)^3(2+2x+x^2)^3+64x^4 \le 128,$$

which is equivalent to

$$x^4(1-x^2)[x^6(x^6-8x^4+31x^2-34)-2(17x^6-38x^4+16x^2+8)] \le 0.$$

Thus, it suffices to show that

$$t^3-8t^2+31t-34<0$$

and

$$17t^3-38t^2+16t+8>0$$

for all $t \in [0,1]$. Indeed, we have

$$t^3-8t^2+31t-34 < t^3-8t^2+31t-24=(t-1)(t^2-7t+24) \le 0,$$

$$17t^3-38t^2+16t+8=17t(t-1)^2+(-4t^2-t+8)>0.$$

Lemma. *If* $-1 \le t \le 1$, *then*

$$(1+t)^{1-t} \le \frac{1}{4}(1+t)^2(2-t^2)(2-2t+t^2),$$

with equality for $t=-1$, $t=0$ *and* $t=1$.
Proof. It suffices to consider that

$$-1<t \le 1.$$

Rewrite the inequality as

$$(1+t)^{1+t}(2-t^2)(2-2t+t^2) \ge 4,$$

which is equivalent to $f(t) \ge 0$, where

$$f(t)=(1+t)\ln(1+t)+\ln(2-t^2)+\ln(2-2t+t^2)-\ln 4.$$

We have

$$f'(t)=1+\ln(1+t)-\frac{2t}{2-t^2}+\frac{2(t-1)}{2-2t+t^2},$$

$$f''(t)=\frac{t^2g(t)}{(1+t)(2-t^2)^2(2-2t+t^2)^2},$$

where

$$g(t)=t^6-8t^5+12t^4+8t^3-20t^2-16t+16.$$

Case 1: $0 \le t \le 1$. From

$$\begin{aligned} g'(t) &= 6t^5 - 40t^4 + 48t^3 + 24t^2 - 40t - 16 \\ &= 6t^5 - 8t - 16 - 8t(5t^3 - 6t^2 - 3t + 4) \\ &= (6t^5 - 8t - 16) - 8t(t-1)^2(5t+4) < 0, \end{aligned}$$

it follows that g is strictly decreasing on $[0,1]$. Since $g(0) = 16$ and $g(1) = -7$, there exists a number $c \in (0,1)$ such that $g(c) = 0$, $g(t) > 0$ for $0 < t < c$ and $g(t) < 0$ for $c < t \le 1$. Therefore, f' is strictly increasing on $[0,c]$ and strictly decreasing on $[c,1]$. From $f'(0) = 0$ and $f'(1) = \ln 2 - 1 < 0$, it follows that there exists a number $d \in (0,1)$ such that $f'(d) = 0$, $f'(t) > 0$ for $0 < t < d$ and $f'(t) < 0$ for $d < t \le 1$. As a consequence, f is strictly increasing on $[0,d]$ and strictly decreasing on $[d,1]$. Since $f(0) = 0$ and $f(1) = 0$, we have $f(t) \ge 0$ for $0 \le t \le 1$.

Case 2: $-1 < t \le 0$. From

$$g(t) = t^4(t-2)(t-6) + 4(t+1)(2t^2 - 7t + 3) + 4 > 0,$$

it follows that f' is strictly increasing on $(-1,0]$. Since $f'(0) = 0$, we have $f'(t) < 0$ for $-1 < t < 0$, hence f is strictly decreasing on $(-1,0]$. From $f(0) = 0$, it follows that $f(t) \ge 0$ for $-1 < t \le 0$.

Conjecture. *If a, b are nonnegative real numbers such that $a + b = 2$, then*

$$a^{3b} + b^{3a} + \left(\frac{a-b}{2}\right)^2 \ge 2.$$

□

P 3.31. *If a, b are positive real numbers such that $a + b = 2$, then*

$$a^{\frac{2}{a}} + b^{\frac{2}{b}} \le 2.$$

(Vasile Cîrtoaje, 2008)

Solution. Without loss of generality, assume that

$$0 < a \le 1 \le b < 2,$$

and write the inequality as

$$\frac{1}{\left(\frac{1}{a^2}\right)^{1/a}} + \frac{1}{\left(\frac{1}{b}\right)^{2/b}} \le 2.$$

By Bernoulli's inequality, we have

$$\left(\frac{1}{a^2}\right)^{1/a} \ge 1 + \frac{1}{a}\left(\frac{1}{a^2} - 1\right) = \frac{a^3 - a^2 + 1}{a^3},$$

$$\left(\frac{1}{b}\right)^{2/b} \ge 1+\frac{2}{b}\left(\frac{1}{b}-1\right)=\frac{b^2-2b+2}{b^2}.$$

Therefore, it suffices to show that

$$\frac{a^3}{a^3-a^2+1}+\frac{b^2}{b^2-2b+2} \le 2,$$

which is equivalent to

$$\frac{a^3}{a^3-a^2+1} \le \frac{(2-b)^2}{b^2-2b+2},$$

$$\frac{a^3}{a^3-a^2+1} \le \frac{a^2}{a^2-2a+2},$$

$$a^2(a-1)^2 \ge 0.$$

The equality happens for $a=b=1$. □

P 3.32. *If a,b are positive real numbers such that $a+b=2$, then*

$$a^{\frac{3}{a}}+b^{\frac{3}{b}} \ge 2.$$

(Vasile Cîrtoaje, 2008)

Solution. Assume that $a \le b$; that is,

$$0<a\le 1\le b<2.$$

There are two cases to consider: $0<a\le \dfrac{3}{5}$ and $\dfrac{3}{5}\le a\le 1$.

Case 1: $0<a\le \dfrac{3}{5}$. From $a+b=2$, we get $\dfrac{7}{5}\le b<2$. Let

$$f(x)=x^{\frac{3}{x}}, \quad 0<x<2.$$

Since

$$f'(x)=3x^{\frac{3}{x}-2}(1-\ln x)>0,$$

$f(x)$ is increasing on $(0,2)$, hence $f(b)\ge f\left(\dfrac{7}{5}\right)$; that is,

$$b^{\frac{3}{b}} \ge \left(\frac{7}{5}\right)^{15/7}.$$

Using Bernoulli's inequality gives

$$\left(\frac{7}{5}\right)^{15/7}=\frac{7}{5}\left(1+\frac{2}{5}\right)^{8/7}>\frac{7}{5}\left(1+\frac{16}{35}\right)=\frac{51}{25}>2,$$

therefore

$$a^{\frac{3}{a}} + b^{\frac{3}{b}} > 2.$$

Case 2: $\frac{3}{5} \le a \le 1$. From $a + b = 2$, we get $1 \le b \le \frac{7}{5}$. By Lemma below, we have

$$2a^{\frac{3}{a}} \ge 3 - 15a + 21a^2 - 7a^3$$

and

$$2b^{\frac{3}{b}} \ge 3 - 15b + 21b^2 - 7b^3.$$

Summing these inequalities, we get

$$\begin{aligned} 2\left(a^{\frac{3}{a}} + b^{\frac{3}{b}}\right) &\ge 6 - 15(a+b) + 21(a^2+b^2) - 7(a^3+b^3) \\ &= 6 - 15(a+b) + 21(a+b)^2 - 7(a+b)^3 = 4. \end{aligned}$$

This completes the proof. The equality holds for $a = b = 1$.

Lemma. *If* $\frac{3}{5} \le x \le 2$, *then*

$$2x^{\frac{3}{x}} \ge 3 - 15x + 21x^2 - 7x^3,$$

with equality for $x = 1$.

Proof. First, we show that $h(x) > 0$, where

$$h(x) = 3 - 15x + 21x^2 - 7x^3.$$

From

$$h'(x) = 3(-5 + 14x - 7x^2),$$

it follows that $h(x)$ is increasing on $\left[1 - \sqrt{\frac{2}{7}}, 1 + \sqrt{\frac{2}{7}}\right]$, and decreasing on $\left[1 + \sqrt{\frac{2}{7}}, \infty\right)$. Then, it suffices to show that $f\left(\frac{3}{5}\right) \ge 0$ and $f(2) \ge 0$. Indeed

$$f\left(\frac{3}{5}\right) = \frac{6}{125}, \quad f(2) = 1.$$

Write now the desired inequality as $f(x) \ge 0$, where

$$f(x) = \ln 2 + \frac{3}{x}\ln x - \ln(3 - 15x + 21x^2 - 7x^3), \quad \frac{3}{5} \le x \le 2.$$

We have

$$\frac{x^2}{3}f'(x) = g(x), \quad g(x) = 1 - \ln x + \frac{x^2(7x^2 - 14x + 5)}{3 - 15x + 21x^2 - 7x^3},$$

$$g'(x) = \frac{g_1(x)}{x(3-15x+21x^2-7x^3)^2},$$

where

$$g_1(x) = -49x^7 + 245x^6 - 280x^5 - 147x^4 + 471x^3 - 321x^2 + 90x - 9.$$

In addition,

$$g_1(x = (x-1)^2 g_2(x), \quad g_2(x) = -49x^5 + 147x^4 + 63x^3 - 168x^2 + 72x - 9,$$

$$g_2(x) = 11x^5 + 3g_3(x), \quad g_3(x) = -20x^5 + 49x^4 + 21x^3 - 56x^2 + 24x - 3,$$

$$g_3(x) = (4x-1)g_4(x), \quad g_4(x) = -5x^4 + 11x^3 + 8x^2 - 12x + 3,$$

$$g_4(x) = x^5 + g_5(x), \quad g_5(x) = -6x^4 + 11x^3 + 8x^2 - 12x + 3,$$

$$g_5(x) = (2x-1)g_6(x), \quad g_6(x) = -3x^3 + 4x^2 + 6x - 3,$$

$$g_6(x) = 1 + (2-x)(3x^2 + 2x - 2).$$

Therefore, we get in succession $g_6(x) > 0$, $g_5(x) > 0$, $g_4(x) > 0$, $g_3(x) > 0$, $g_2(x) > 0$, $g_1(x) \geq 0$, $g'(x) \geq 0$, $g(x)$ is increasing. Since $g(1) = 0$, we have $g(x) < 0$ on $\left[\frac{3}{5}, 1\right)$ and $g(x) > 0$ on $(1,2]$. Then, $f(x)$ is decreasing on $\left[\frac{3}{5}, 1\right]$ and increasing on $[1,2]$, hence $f(x) \geq f(1) = 0$.

□

P 3.33. *If a, b are positive real numbers such that $a + b = 2$, then*

$$a^{5b^2} + b^{5a^2} \leq 2.$$

(Vasile Cîrtoaje, 2010)

Solution. Assume that $a \geq b$. For $a = 2$ and $b = 0$, the inequality is obvious. Otherwise, using the substitution $a = 1 + x$ and $b = 1 - x$, $0 \leq x < 1$, we can write the desired inequality as

$$e^{5(1-x)^2 \ln(1+x)} + e^{5(1+x)^2 \ln(1-x)} \leq 2.$$

According to Lemma below, it suffices to show that $f(x) \leq 2$, where

$$f(x) = e^{5(u-v)} + e^{-5(u+v)},$$

$$u = x + \frac{7}{3}x^3 + \frac{31}{30}x^5, \quad v = \frac{5}{2}x^2 + \frac{17}{12}x^4 + \frac{9}{20}x^6.$$

If $f'(x) \leq 0$, then $f(x)$ is decreasing, hence

$$f(x) \leq f(0) = 2.$$

Since

$$f'(x) = 5(u' - v')e^{5(u-v)} - 5(u' + v')e^{-5(u+v)},$$

$$u' = 1 + 7x^2 + \frac{31}{6}x^4, \quad v' = 5x + \frac{17}{3}x^3 + \frac{27}{10}x^5,$$

the inequality $f'(x) \le 0$ becomes

$$e^{-10u}(u' + v') \ge u' - v'$$

For the nontrivial case $u' - v' > 0$, we rewrite this inequality as $g(x) \ge 0$, where

$$g(x) = -10u + \ln(u' + v') - \ln(u' - v').$$

If $g'(x) \ge 0$, then $g(x)$ is increasing, hence

$$g(x) \ge f(0) = 0.$$

We have

$$g'(x) = -10u' + \frac{u'' + v''}{u' + v'} - \frac{u'' - v''}{u' - v'},$$

where

$$u'' = 14x + \frac{62}{3}x^3, \quad v'' = 5 + 17x^2 + \frac{27}{2}x^4.$$

Write the inequality $g'(x) \ge 0$ as

$$u'v'' - v'u'' \ge 5u'(u' + v')(u' - v'),$$

$$a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + a_7 t^7 \ge 0,$$

where $t = x^2$, $0 \le t < 1$, and

$$a_1 = 2, \quad a_2 = 321.5, \quad a_3 \approx 152.1, \quad a_4 \approx -498.2,$$

$$a_5 \approx -168.5, \quad a_6 \approx 356.0, \quad a_7 \approx 188.3.$$

This inequality is true if

$$300t^2 + 150t^3 - 500t^4 - 200t^5 + 250t^6 \ge 0.$$

Since the last inequality is equivalent to the obvious inequality

$$50t^2(1 - t)(6 + 9t - t^2 - 5t^3) \ge 0,$$

the proof is completed. The equality holds for $a = b = 1$.

Lemma. *If* $-1 < t < 1$, *then*

$$(1 - t)^2 \ln(1 + t) \le t - \frac{5}{2}t^2 + \frac{7}{3}t^3 - \frac{17}{12}t^4 + \frac{31}{30}t^5 - \frac{9}{20}t^6.$$

Proof. We show that

$$(1-t)^2\ln(1+t)\le(1-t)^2\left(t-\frac{1}{2}t^2+\frac{1}{3}t^3-\frac{1}{4}t^4+\frac{1}{5}t^5\right)$$
$$\le t-\frac{5}{2}t^2+\frac{7}{3}t^3-\frac{17}{12}t^4+\frac{31}{30}t^5-\frac{9}{20}t^6.$$

The left inequality is equivalent to $f(t)\ge 0$, where

$$f(t)=t-\frac{1}{2}t^2+\frac{1}{3}t^3-\frac{1}{4}t^4+\frac{1}{5}t^5-\ln(1+t).$$

Since

$$f'(t)=\frac{t^5}{1+t},$$

$f(t)$ is decreasing on $(-1,0]$ and increasing on $[0,1)$; therefore, $f(t)\ge f(0)=0$. The right inequality is equivalent to $t^6(t-1)\le 0$, which is clearly true.

□

P 3.34. *If a,b are positive real numbers such that $a+b=2$, then*

$$a^{2\sqrt{b}}+b^{2\sqrt{a}}\le 2.$$

(Vasile Cîrtoaje, 2010)

Solution. Assume that $a\ge b$. For $a=2$ and $b=0$, the inequality is obvious. Otherwise, using the substitution $a=1+x$ and $b=1-x$, $0\le x<1$, we can write the desired inequality as $f(x)\le 2$, where

$$f(x)=(1+x)^{2\sqrt{1-x}}+(1-x)^{2\sqrt{1+x}}=e^{2\sqrt{1-x}\ln(1+x)}+e^{2\sqrt{1+x}\ln(1-x)}.$$

There are two cases to consider.

Case 1: $13/20\le x<1$. If f is decreasing on $[13/20,1)$, then

$$f(x)\le f\left(\frac{13}{20}\right)=\left(\frac{33}{20}\right)^{\sqrt{7/5}}+\left(\frac{7}{20}\right)^{\sqrt{33/5}}<\left(\frac{5}{3}\right)^{5/4}+\left(\frac{1}{4}\right)^2<2.$$

Since the function $(1-x)^{2\sqrt{1+x}}$ is decreasing, it suffices to show that

$$g(x)=(1+x)^{2\sqrt{1-x}}$$

is decreasing. This is true if $g'(x)\le 0$ for $x\in[13/20,1)$, that is equivalent to $h(x)\le 0$, where

$$h(x)=\frac{2(1-x)}{1+x}-\ln(1+x).$$

Clearly, h is decreasing, hence

$$h(x) \le h\left(\frac{13}{20}\right) = \frac{14}{33} - \ln\frac{33}{20} < 0.$$

Case 2: $0 \le x \le 13/20$. By Lemma below, it suffices to show that $g(x) \le 2$, where

$$g(x) = e^{2x-2x^2+\frac{11}{12}x^3-\frac{1}{2}x^4} + e^{-(2x+2x^2+\frac{11}{12}x^3+\frac{1}{2}x^4)}.$$

If $g'(x) \le 0$ for $x \in [0, 13/20]$, then g is decreasing, hence $g(x) \le g(0) = 2$. Since

$$\begin{aligned} g'(x) =&(2 - 4x + \frac{11}{4}x^2 - 2x^3)e^{2x-2x^2+\frac{11}{12}x^3-\frac{1}{2}x^4} \\ &- (2 + 4x + \frac{11}{4}x^2 + 2x^3)e^{-(2x+2x^2+\frac{11}{12}x^3+\frac{1}{2}x^4)}, \end{aligned}$$

the inequality $g'(x) \le 0$ is equivalent to

$$e^{-4x-\frac{11}{6}x^3} \ge \frac{8 - 16x + 11x^2 - 8x^3}{8 + 16x + 11x^2 + 8x^3}.$$

For the nontrivial case $8-16x+11x^2-8x^3 > 0$, rewrite this inequality as $h(x) \ge 0$, where

$$h(x) = -4x - \frac{11}{6}x^3 - \ln(8 - 16x + 11x^2 - 8x^3) + \ln(8 + 16x + 11x^2 + 8x^3).$$

If $h' \ge 0$, then h is increasing, hence $h(x) \ge h(0) = 0$. From

$$h'(x) = -4 - \frac{11}{2}x^2 + \frac{(16 + 24x^2) - 22x}{8 + 11x^2 - (16x + 8x^3)} + \frac{(16 + 24x^2) + 22x}{8 + 11x^2 + (16x + 8x^3)},$$

it follows that $h'(x) \ge 0$ is equivalent to

$$(16+24x^2)(8+11x^2)-22x(16x+8x^3) \ge \frac{1}{4}(8+11x^2)[(8+11x^2)^2-(16x+8x^3)^2].$$

Since

$$(8 + 11x^2)^2 - (16x + 8x^3)^2 \le (8 + 11x^2)^2 - 256x^2 - 256x^4 \le 16(4 - 5x^2),$$

it suffices to show that

$$(4 + 6x^2)(8 + 11x^2) - 11x(8x + 4x^3) \ge (8 + 11x^2)(4 - 5x^2).$$

This inequality reduces to $77x^4 \ge 0$. The proof is completed. The equality holds for $a = b = 1$.

Lemma. *If* $-1 < t \le \dfrac{13}{20}$, *then*

$$\sqrt{1-t}\,\ln(1+t) \le t - t^2 + \frac{11}{24}t^3 - \frac{1}{4}t^4.$$

Proof. Consider two cases.

Case 1: $0 \le t \le \dfrac{13}{20}$. We can prove the desired inequality by multiplying the following inequalities

$$\sqrt{1-t} \le 1 - \frac{1}{2}t - \frac{1}{8}t^2 - \frac{1}{16}t^3,$$

$$\ln(1+t) \le t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + \frac{1}{5}t^5,$$

$$\left(1 - \frac{1}{2}t - \frac{1}{8}t^2 - \frac{1}{16}t^3\right)\left(t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + \frac{1}{5}t^5\right) \le t - t^2 + \frac{11}{24}t^3 - \frac{1}{4}t^4.$$

The first inequality is equivalent to $f(t) \ge 0$, where

$$f(t) = \ln\left(1 - \frac{1}{2}t - \frac{1}{8}t^2 - \frac{1}{16}t^3\right) - \frac{1}{2}\ln(1-t).$$

Since

$$f'(t) = \frac{1}{2(1-t)} - \frac{8+4t+3t^2}{16-8t-2t^2-t^3} = \frac{5t^3}{2(1-t)(16-8t-2t^2-t^3)} \ge 0,$$

f is increasing, hence $f(t) \ge f(0) = 0$.
The second inequality is equivalent to $f(t) \ge 0$, where

$$f(t) = t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + \frac{1}{5}t^5 - \ln(1+t).$$

Since

$$f'(t) = 1 - t + t^2 - t^3 + t^4 - \frac{1}{1+t} = \frac{t^5}{1+t} \ge 0,$$

$f(t)$ is increasing, hence $f(t) \ge f(0) = 0$.
The third inequality is equivalent to

$$t^4(160 - 302t + 86t^2 + 9t^3 + 12t^4) \ge 0.$$

This is true since

$$160 - 302t + 86t^2 + 9t^3 + 12t^4 \ge 2(80 - 151t + 43t^2) > 0.$$

Case 2: $-1 < t \le 0$. Write the desired inequality as

$$-\sqrt{1-t}\,\ln(1+t) \ge -t + t^2 - \frac{11}{24}t^3 + \frac{1}{4}t^4.$$

This is true if

$$\sqrt{1-t} \ge 1 - \frac{1}{2}t - \frac{1}{8}t^2,$$

$$-\ln(1+t) \geq -t + t^2 - \frac{1}{3}t^3 + \frac{1}{4}t^4,$$

$$\left(1 - \frac{1}{2}t - \frac{1}{8}t^2\right)\left(-t + t^2 - \frac{1}{3}t^3 + \frac{1}{4}t^4\right) \geq -t + t^2 - \frac{11}{24}t^3 + \frac{1}{4}t^4.$$

The first inequality is equivalent to $f(t) \geq 0$, where

$$f(t) = \frac{1}{2}\ln(1-t) - \ln\left(1 - \frac{1}{2}t - \frac{1}{8}t^2\right).$$

Since

$$f'(t) = \frac{-1}{2(1-t)} + \frac{2(2+t)}{8-4t-t^2} = \frac{-3t^2}{2(1-t)(8-4t-t^2)} \leq 0,$$

f is decreasing, hence $f(t) \geq f(0) = 0$.
The second inequality is equivalent to $f(t) \geq 0$, where

$$f(t) = t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 - \ln(1+t).$$

Since

$$f'(t) = 1 - t + t^2 - t^3 - \frac{1}{1+t} = \frac{-t^4}{1+t} \leq 0,$$

f is decreasing, hence $f(t) \geq f(0) = 0$.
The third inequality reduces to the obvious inequality

$$t^4(10 - 8t - 3t^2) \geq 0.$$

□

P 3.35. *If a, b are nonnegative real numbers such that $a + b = 2$, then*

$$\frac{ab(1-ab)^2}{2} \leq a^{b+1} + b^{a+1} - 2 \leq \frac{ab(1-ab)^2}{3}.$$

(Vasile Cîrtoaje, 2010)

Solution. Assume that $a \geq b$, which yields $1 \leq a \leq 2$ and $0 \leq b \leq 1$.

(a) To prove the left inequality we apply Lemma 1 below. For $x = a$ and $k = b$, we have

$$a^{b+1} \geq 1 + (1+b)(a-1) + \frac{b(1+b)}{2}(a-1)^2 - \frac{b(1+b)(1-b)}{6}(a-1)^3,$$

$$a^{b+1} \geq a - b + ab + \frac{b(1+b)}{2}(a-1)^2 - \frac{b(1+b)}{6}(a-1)^4. \qquad (*)$$

Also, for $x = b$ and $k = a-1$, we have

$$b^a \ge 1 + a(b-1) + \frac{a(a-1)}{2}(b-1)^2 - \frac{a(a-1)(2-a)}{6}(b-1)^3,$$

$$b^a \ge 1 - a + ab + \frac{a}{2}(a-1)^3 + \frac{ab}{6}(a-1)^4,$$

$$b^{a+1} \ge b - ab + ab^2 + \frac{ab}{2}(a-1)^3 + \frac{ab^2}{6}(a-1)^4. \tag{**}$$

Summing up (*) and (**) gives

$$\begin{aligned} a^{b+1} + b^{a+1} - 2 &\ge -b(a-1)^2 + \frac{b(3-ab)}{2}(a-1)^2 - \frac{b(1+b-ab)}{6}(a-1)^4 \\ &= \frac{b}{2}(a-1)^4 - \frac{b(1+b-ab)}{6}(a-1)^4 \\ &= \frac{ab(1+b)}{6}(a-1)^4 \ge \frac{ab}{6}(a-1)^4 = \frac{ab(1-ab)^2}{6}. \end{aligned}$$

The equality holds for $a = b = 1$, for $a = 2$ and $b = 0$, and for $a = 0$ and $b = 2$.

(b) To prove the right inequality we apply Lemma 2 below. For $x = a$ and $k = b$, we have

$$\begin{aligned} a^{b+1} \le 1 + (b+1)(a-1) + \frac{(b+1)b}{2}(a-1)^2 + \frac{(b+1)b(b-1)}{6}(a-1)^3 \\ + \frac{(b+1)b(b-1)(b-2)}{24}(a-1)^4, \end{aligned}$$

$$a^{b+1} \le 1+(b+1)(a-1)+\frac{b(b+1)}{2}(a-1)^2 - \frac{b(b+1)}{6}(a-1)^4 + \frac{ab(b+1)}{24}(a-1)^5.$$

Also, for $x = b$ and $k = a$, we have

$$b^{a+1} \le 1+(a+1)(b-1)+\frac{a(a+1)}{2}(b-1)^2 - \frac{a(a+1)}{6}(b-1)^4 + \frac{ab(a+1)}{24}(b-1)^5.$$

Summing up these inequalities and having in view that

$$(b+1)(a-1)^5 + (a+1)(b-1)^5 = -2(a-1)^5 \le 0$$

give

$$\begin{aligned} a^{b+1} + b^{a+1} - 2 &\le -2(a-1)^2 + \frac{a^2+b^2+2}{2}(a-1)^2 - \frac{a^2+b^2+2}{6}(a-1)^4 \\ &\le \frac{a^2+b^2-2}{2}(a-1)^2 - \frac{a^2+b^2+2}{6}(a-1)^4 \\ &= (a-1)^4 - \frac{a^2+b^2+2}{6}(a-1)^4 \\ &= \frac{ab}{3}(a-1)^4 = \frac{ab(1-ab)^2}{3}. \end{aligned}$$

The equality holds for $a=b=1$, for $a=2$ and $b=0$, and for $a=0$ and $b=2$.

Lemma 1. *If $x \ge 0$ and $0 \le k \le 1$, then*

$$x^{k+1} \ge 1+(1+k)(x-1)+\frac{k(1+k)}{2}(x-1)^2-\frac{k(1+k)(1-k)}{6}(x-1)^3,$$

with equality for $x=1$, for $k=0$ and for $k=1$.

Proof. For $k=0$ and $k=1$, the inequality is an identity. For fixed k, $0<k<1$, let us define

$$f(x)=x^{k+1}-1-(1+k)(x-1)-\frac{k(1+k)}{2}(x-1)^2+\frac{k(1+k)(1-k)}{6}(x-1)^3.$$

We need to show that $f(x) \ge 0$. We have

$$\frac{1}{1+k}f'(x)=x^k-1-k(x-1)+\frac{k(1-k)}{2}(x-1)^2,$$

$$\frac{1}{k(1+k)}f''(x)=x^{k-1}-1+(1-k)(x-1),$$

$$\frac{1}{k(1+k)(1-k)}f'''(x)=-x^{k-2}+1.$$

Case 1: $0 \le x \le 1$. Since $f''' \le 0$, f'' is decreasing, $f''(x) \ge f''(1)=0$, f' is increasing, $f'(x) \le f'(1)=0$, f is decreasing, hence $f(x) \ge f(1)=0$.

Case 2: $x \ge 1$. Since $f''' \ge 0$, f'' is increasing, $f''(x) \ge f''(1)=0$, f' is increasing, $f'(x) \ge f'(1)=0$, f is increasing, hence $f(x) \ge f(1)=0$.

Lemma 2. *If either $x \ge 1$ and $0 \le k \le 1$, or $0 \le x \le 1$ and $1 \le k \le 2$, then*

$$x^{k+1} \le 1+(k+1)(x-1)+\frac{(k+1)k}{2}(x-1)^2+\frac{(k+1)k(k-1)}{6}(x-1)^3$$
$$+\frac{(k+1)k(k-1)(k-2)}{24}(x-1)^4,$$

with equality for $x=1$, for $k=0$, for $k=1$ and for $k=2$.

Proof. For $k=0$, $k=1$ and $k=2$, the inequality is an identity. For fixed k, $k \in (0,1)\cup(1,2)$, let us define

$$f(x)=x^{k+1}-1-(k+1)(x-1)-\frac{(k+1)k}{2}(x-1)^2-\frac{(k+1)k(k-1)}{6}(x-1)^3$$
$$-\frac{(k+1)k(k-1)(k-2)}{24}(x-1)^4.$$

We need to show that $f(x) \le 0$. We have

$$\frac{1}{k+1}f'(x) = x^k - 1 - k(x-1) - \frac{k(k-1)}{2}(x-1)^2 - \frac{k(k-1)(k-2)}{6}(x-1)^3,$$

$$\frac{1}{k(k+1)}f''(x) = x^{k-1} - 1 - (k-1)(x-1) - \frac{(k-1)(k-2)}{2}(x-1)^2,$$

$$\frac{1}{k(k+1)(k-1)}f'''(x) = x^{k-2} - 1 - (k-2)(x-1),$$

$$\frac{1}{k(k+1)(k-1)(k-2)}f^{(4)}(x) = x^{k-3} - 1.$$

Case 1: $x \ge 1$, $0 < k < 1$. Since $f^{(4)}(x) \le 0$, $f'''(x)$ is decreasing, $f'''(x) \le f'''(1) = 0$, f'' is decreasing, $f''(x) \le f''(1) = 0$, f' is decreasing, $f'(x) \le f'(1) = 0$, f is decreasing, hence $f(x) \le f(1) = 0$.

Case 2: $0 \le x \le 1$, $1 < k < 2$. Since $f^{(4)} \le 0$, f''' is decreasing, $f'''(x) \ge f'''(1) = 0$, f'' is increasing, $f''(x) \le f''(1) = 0$, f' is decreasing, $f'(x) \ge f'(1) = 0$, f is increasing, hence $f(x) \le f(1) = 0$.

□

P 3.36. *If a, b are nonnegative real numbers such that $a + b = 1$, then*

$$a^{2b} + b^{2a} \le 1.$$

(Vasile Cîrtoaje, 2007)

Solution. Without loss of generality, assume that

$$0 \le b \le \frac{1}{2} \le a \le 1.$$

Applying Lemma 1 below for $c = 2b$, $0 \le c \le 1$, we get

$$a^{2b} \le (1-2b)^2 + 4ab(1-b) - 2ab(1-2b)\ln a,$$

which is equivalent to

$$a^{2b} \le 1 - 4ab^2 - 2ab(a-b)\ln a.$$

Similarly, applying Lemma 2 below for $d = 2a - 1$, $d \ge 0$, we get

$$b^{2a-1} \le 4a(1-a) + 2a(2a-1)\ln(2a+b-1),$$

which is equivalent to

$$b^{2a} \le 4ab^2 + 2ab(a-b)\ln a.$$

Adding up these inequalities, the desired inequality follows. The equality holds for $a = b = 1/2$, for $a = 0$ and $b = 1$, and for $a = 1$ and $b = 0$.

Lemma 1. *If* $0 < a \le 1$ *and* $c \ge 0$*, then*

$$a^c \le (1-c)^2 + ac(2-c) - ac(1-c)\ln a,$$

with equality for $a = 1$*, for* $c = 0$ *and for* $c = 1$*.*

Proof. Making the substitution

$$a = e^{-x}, \qquad x \ge 0,$$

we need to prove that $f(x) \ge 0$, where

$$f(x) = (1-c)^2 e^x + c(2-c) + c(1-c)x - e^{(1-c)x},$$

$$f'(x) = (1-c)[(1-c)e^x + c - e^{(1-c)x}].$$

If $f' \ge 0$ on $[0, \infty)$, then f is increasing, and hence $f(x) \ge f(0) = 0$. In order to prove that $f' \ge 0$, we consider two cases.

Case 1: $0 \le c \le 1$. By the weighted AM-GM inequality, we have

$$(1-c)e^x + c \ge e^{(1-c)x},$$

hence $f'(x) \ge 0$.

Case 2: $c \ge 1$. By the weighted AM-GM inequality, we have

$$(c-1)e^x + e^{(1-c)x} \ge c,$$

which yields

$$f'(x) = (c-1)[(c-1)e^x + e^{(1-c)x} - c] \ge 0.$$

Lemma 2. *If* $0 \le b \le 1$ *and* $d \ge 0$*, then*

$$b^d \le 1 - d^2 + d(1+d)\ln(b+d),$$

with equality for $b = 0$ *and for* $d = 0$*.*

Proof. Consider $0 < b \le 1$ and $d > 0$, and write the inequality as

$$(1+d)[1 - d + d\ln(b+d)] \ge b^d.$$

Since

$$1 - d + d\ln(b+d) > 1 - d + d\ln d \ge 0,$$

we can rewrite the inequality in the form

$$\ln(1+d) + \ln[1 - d + d\ln(b+d)] \ge d\ln b.$$

Using the substitution

$$b = e^{-x} - d, \qquad -\ln(1+d) \le x < -\ln d,$$

we need to prove that $f(x) \ge 0$, where

$$f(x) = \ln(1+d) + \ln(1-d-dx) + dx - d\ln(1-de^x).$$

Since

$$f'(x) = \frac{d^2(e^x - 1 - x)}{(1-d-dx)(1-de^x)} \ge 0,$$

f is increasing, hence

$$f(x) \ge f(-\ln(1+d)) = \ln[1 - d^2 + d(1+d)\ln(1+d)].$$

To complete the proof, we only need to show that $-d^2 + d(1+d)\ln(1+d) \ge 0$; that is,

$$(1+d)\ln(1+d) \ge d.$$

This inequality follows from $e^x \ge 1+x$, where $x = \dfrac{-d}{1+d}$.

Conjecture. *If a, b are nonnegative real numbers such that $1 \le a+b \le 15$, then*

$$a^{2b} + b^{2a} \le a^{a+b} + b^{a+b}.$$

□

P 3.37. *If a, b are positive real numbers such that $a+b=1$, then*

$$2a^a b^b \ge a^{2b} + b^{2a}.$$

Solution. Taking into account the inequality $a^{2b} + b^{2a} \le 1$ from the preceding P 3.36, it suffices to show that

$$2a^a b^b \ge 1.$$

Write this inequality as

$$2a^a b^b \ge a^{a+b} + b^{a+b},$$

$$2 \ge \left(\frac{a}{b}\right)^b + \left(\frac{b}{a}\right)^a.$$

Since $a < 1$ and $b < 1$, we apply Bernoulli's inequality as follows:

$$\left(\frac{a}{b}\right)^b + \left(\frac{b}{a}\right)^a \le 1 + b\left(\frac{a}{b} - 1\right) + 1 + a\left(\frac{b}{a} - 1\right) = 2.$$

Thus, the proof is completed. The equality holds for $a = b = 1/2$.

□

P 3.38. *If a, b are positive real numbers such that $a+b=1$, then*

$$a^{-2a}+b^{-2b} \le 4.$$

Solution. Applying Lemma below, we have

$$a^{-2a} \le 4-2\ln 2-4(1-\ln 2)a,$$

$$b^{-2b} \le 4-2\ln 2-4(1-\ln 2)b.$$

Adding these inequalities, the desired inequality follows. The equality holds for $a=b=1/2$.

Lemma. *If $x \in (0,1]$, then*

$$x^{-2x} \le 4-2\ln 2-4(1-\ln 2)x,$$

with equality for $x=1/2$.

Proof. Write the inequality as

$$\frac{1}{4}x^{-2x} \le 1-c-(1-2c)x, \quad c=\frac{1}{2}\ln 2 \approx 0.346.$$

This is true if $f(x) \le 0$, where

$$f(x)=-2\ln 2-2x\ln x-\ln[1-c-(1-2c)x].$$

We have

$$f'(x)=-2-2\ln x+\frac{1-2c}{1-c-(1-2c)x},$$

$$f''(x)=-\frac{2}{x}+\frac{(1-2c)^2}{[1-c-(1-2c)x]^2}=\frac{g(x)}{x[1-c-(1-2c)x]^2},$$

where

$$g(x)=2(1-2c)^2x^2-(1-2c)(5-6c)x+2(1-c)^2.$$

Since

$$\begin{aligned} g'(x) &= (1-2c)[4(1-2c)x-5+6c] \le (1-2c)[4(1-2c)-5+6c] \\ &= (1-2c)(-1-2c)<0, \end{aligned}$$

g is decreasing on $(0,1]$, hence $g(x) \ge g(1)=-2c^2+4c-1>0$, $f''(x)>0$ for $x \in (0,1]$, f' is increasing. Since $f'(1/2)=0$, we have $f'(x) \le 0$ for $x \in (0,1/2]$ and $f'(x) \ge 0$ for $x \in [1/2,1]$. Therefore, f is decreasing on $(0,1/2]$ and increasing on $[1/2,1]$, hence $f(x) \ge f(1/2)=0$.

Remark. According to the inequalities in P 3.36 and P 3.38, the following inequality holds for all positive numbers a, b such that $a+b=1$:

$$\left(a^{2b}+b^{2a}\right)\left(\frac{1}{a^{2a}}+\frac{1}{b^{2b}}\right) \le 4.$$

Actually, this inequality holds for all $a, b \in (0, 1]$. In this case, it is sharper than the inequality in P 3.19.

□

P 3.39. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 a_2 \cdots a_n = 1$, then*

$$\left(1-\frac{1}{n}\right)^{a_1} + \left(1-\frac{1}{n}\right)^{a_2} + \cdots + \left(1-\frac{1}{n}\right)^{a_n} \le n-1.$$

(Vasile Cîrtoaje, 2004)

Solution. We will prove the more general inequality

$$\left(1-\frac{1}{n}\right)^{a_1} + \left(1-\frac{1}{n}\right)^{a_2} + \cdots + \left(1-\frac{1}{n}\right)^{a_n} \le n\left(1-\frac{1}{n}\right)^{a}, \qquad (*)$$

where $a = \sqrt[n]{a_1 a_2 \cdots a_n} \le 1$. Using the substitution

$$x_i = a_i \ln \frac{n}{n-1}, \qquad i = 1, 2, \ldots, n,$$

the inequality becomes as follows:

$$e^{-x_1} + e^{-x_2} + \cdots + e^{-x_n} \le n e^{-r}, \qquad (**)$$

where

$$r = \sqrt[n]{x_1 x_2 \cdots x_n} \le \ln \frac{n}{n-1}.$$

To prove this inequality, we use the induction technique. For $n = 1$, (**) is an equality. Consider now that (**) holds for $n-1$ numbers, $n \ge 2$, and show that it also holds for n numbers. Assume that

$$x_1 \le x_2 \le \cdots \le x_n,$$

and denote

$$x = \sqrt[n-1]{x_1 x_2 \cdots x_{n-1}}.$$

Because

$$x \le r \le \ln \frac{n}{n-1} < \ln \frac{n-1}{(n-1)-1},$$

the induction hypothesis yields

$$e^{-x_1} + e^{-x_2} + \cdots + e^{-x_{n-1}} \le (n-1)e^{-x}.$$

Thus, we only need to show that

$$e^{-x_n} + (n-1)e^{-x} \le n e^{-r},$$

which is equivalent to

$$f(x) \le ne^{-r}$$

for

$$0 < x \le r \le \ln \frac{n}{n-1} < 1,$$

where

$$f(x) = e^{-r^n/x^{n-1}} + (n-1)e^{-x}.$$

We have

$$\frac{x^n e^{r^n/x^{n-1}}}{n-1} f'(x) = g(x), \qquad g(x) = r^n - x^n e^{r^n/x^{n-1}-x},$$

$$e^{x-r^n/x^{n-1}} g'(x) = h(x), \qquad h(x) = x^n - nx^{n-1} + (n-1)r^n,$$

$$h'(x) = nx^{n-2}(x-n+1).$$

Since $h'(x) < 0$, h is strictly decreasing, and from

$$h(0) = (n-1)r^n > 0, \qquad h(r) = nr^{n-1}(r-1) < 0,$$

it follows that there exists $x_1 \in (0, r)$ such that $h(x_1) = 0$, $h(x) > 0$ for $x \in (0, x_1)$, $h(x) < 0$ for $x \in (x_1, r]$. Therefore, g is strictly increasing on $(0, x_1]$ and strictly decreasing on $[x_1, r]$. Since $g(0_+) = -\infty$ and $g(r) = 0$, there exists $x_2 \in (0, x_1)$ such that $g(x_2) = 0$, $g(x) < 0$ for $x \in (0, x_2)$, $g(x) > 0$ for $x \in (x_2, r]$. Consequently, f is strictly decreasing on $(0, x_2]$ and strictly increasing on $[x_2, r]$, hence

$$f(x) \le \max\{f(0_+), f(r)\} = \max\{n-1, ne^{-r}\} = ne^{-r}.$$

Thus, the proof is completed. The inequality (**) is an equality for

$$x_1 = x_2 = \cdots = x_n \le \ln \frac{n}{n-1},$$

the inequality (*) for

$$a_1 = a_2 = \cdots = a_n \le 1,$$

and the original inequality for

$$a_1 = a_2 = \cdots = a_n = 1.$$

Appendix A

Glosar

1. AM-GM (ARITHMETIC MEAN-GEOMETRIC MEAN) INEQUALITY

If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers, then

$$a_1 + a_2 + \cdots + a_n \geq n\sqrt[n]{a_1 a_2 \cdots a_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

2. WEIGHTED AM-GM INEQUALITY

Let $p_1, p_2, \ldots, p_n$ be positive real numbers satisfying

$$p_1 + p_2 + \cdots + p_n = 1.$$

If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers, then

$$p_1 a_1 + p_2 a_2 + \cdots + p_n a_n \geq a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

3. AM-HM (ARITHMETIC MEAN-HARMONIC MEAN) INEQUALITY

If $a_1, a_2, \ldots, a_n$ are positive real numbers, then

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2,$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

4. POWER MEAN INEQUALITY

The power mean of order k of positive real numbers $a_1, a_2, \ldots, a_n$, that is

$$M_k = \begin{cases} \left(\frac{a_1^k + a_2^k + \cdots + a_n^k}{n}\right)^{\frac{1}{k}}, & k \neq 0 \\ \sqrt[n]{a_1 a_2 \cdots a_n}, & k = 0 \end{cases},$$

is an increasing function with respect to $k \in \mathbb{R}$. For instant, $M_2 \geq M_1 \geq M_0 \geq M_{-1}$ is equivalent to

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}.$$

5. BERNOULLI'S INEQUALITY

For any real number $x \geq -1$, we have

a) $(1+x)^r \geq 1 + rx$ for $r \geq 1$ and $r \leq 0$;

b) $(1+x)^r \leq 1 + rx$ for $0 \leq r \leq 1$.

If $a_1, a_2, \ldots, a_n$ are real numbers such that either $a_1, a_2, \ldots, a_n \geq 0$ or

$$-1 \leq a_1, a_2, \ldots, a_n \leq 0,$$

then

$$(1+a_1)(1+a_2)\cdots(1+a_n) \geq 1 + a_1 + a_2 + \cdots + a_n.$$

6. SCHUR'S INEQUALITY

For any nonnegative real numbers a, b, c and any positive number k, the inequality holds

$$a^k(a-b)(a-c) + b^k(b-c)(b-a) + c^k(c-a)(c-b) \geq 0,$$

with equality for $a = b = c$, and for $a = 0$ and $b = c$ (or any cyclic permutation). For $k = 1$, we get the third degree Schur's inequality, which can be rewritten as follows

$$a^3 + b^3 + c^3 + 3abc \geq ab(a+b) + bc(b+c) + ca(c+a),$$

$$(a+b+c)^3 + 9abc \geq 4(a+b+c)(ab+bc+ca),$$

$$a^2 + b^2 + c^2 + \frac{9abc}{a+b+c} \geq 2(ab+bc+ca),$$

$$(b-c)^2(b+c-a) + (c-a)^2(c+a-b) + (a-b)^2(a+b-c) \geq 0.$$

For $k = 2$, we get the fourth degree Schur's inequality, which holds for any real numbers a, b, c, and can be rewritten as follows

$$a^4 + b^4 + c^4 + abc(a + b + c) \geq ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2),$$

$$a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2 \geq (ab + bc + ca)(a^2 + b^2 + c^2 - ab - bc - ca),$$

$$(b-c)^2(b+c-a)^2 + (c-a)^2(c+a-b)^2 + (a-b)^2(a+b-c)^2 \geq 0,$$

$$6abcp \geq (p^2 - q)(4q - p^2), \qquad p = a + b + c, \ \ q = ab + bc + ca.$$

A generalization of the fourth degree Schur's inequality, which holds for any real numbers a, b, c and any real number m, is the following (*Vasile Cirtoaje*, 2004)

$$\sum (a - mb)(a - mc)(a - b)(a - c) \geq 0,$$

where the equality holds for $a = b = c$, and for $a/m = b = c$ (or any cyclic permutation). This inequality is equivalent to

$$\sum a^4 + m(m+2)\sum a^2b^2 + (1 - m^2)abc\sum a \geq (m+1)\sum ab(a^2 + b^2),$$

$$\sum (b-c)^2(b + c - a - ma)^2 \geq 0.$$

A more general result is given by the following theorem (*Vasile Cirtoaje*, 2008).

Theorem. *Let*

$$f_4(a, b, c) = \sum a^4 + \alpha \sum a^2b^2 + \beta abc \sum a - \gamma \sum ab(a^2 + b^2),$$

where α, β, γ *are real constants such that* $1 + \alpha + \beta = 2\gamma$. *Then,*

(a) $f_4(a, b, c) \geq 0$ *for all* $a, b, c \in \mathbb{R}$ *if and only if*

$$1 + \alpha \geq \gamma^2;$$

(b) $f_4(a, b, c) \geq 0$ *for all* $a, b, c \geq 0$ *if and only if*

$$\alpha \geq (\gamma - 1)\max\{2, \gamma + 1\}.$$

7. CAUCHY-SCHWARZ INEQUALITY

If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers, then

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2,$$

with equality for

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}.$$

Notice that the equality conditions are also valid for $a_i = b_i = 0$, where $1 \leq i \leq n$.

8. HÖLDER'S INEQUALITY

If x_{ij} $(i = 1, 2, \cdots, m; j = 1, 2, \cdots n)$ are nonnegative real numbers, then

$$\prod_{i=1}^{m}\left(\sum_{j=1}^{n} x_{ij}\right) \ge \left(\sum_{j=1}^{n} \sqrt[m]{\prod_{i=1}^{m} x_{ij}}\right)^{m}.$$

9. CHEBYSHEV'S INEQUALITY

Let $a_1 \ge a_2 \ge \cdots \ge a_n$ be real numbers.

a) If $b_1 \ge b_2 \ge \cdots b_n$, then

$$n\sum_{i=1}^{n} a_i b_i \ge \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right);$$

b) If $b_1 \le b_2 \le \cdots \le b_n$, then

$$n\sum_{i=1}^{n} a_i b_i \le \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right).$$

10. CONVEX FUNCTIONS

A function f defined on a real interval $\mathbb{I}$ is said to be *convex* if

$$f(\alpha x + \beta y) \le \alpha f(x) + \beta f(y)$$

for all x, $y \in \mathbb{I}$ and any α, $\beta \ge 0$ with $\alpha + \beta = 1$. If the inequality is reversed, then f is said to be concave.

If f is differentiable on $\mathbb{I}$, then f is (strictly) convex if and only if the derivative f' is (strictly) increasing. If $f'' \ge 0$ on $\mathbb{I}$, then f is convex on $\mathbb{I}$.

Jensen's inequality. *Let $p_1, p_2, \ldots, p_n$ be positive real numbers. If f is a convex function on a real interval $\mathbb{I}$, then for any a_1, a_2, $\ldots$, $a_n \in \mathbb{I}$, the inequality holds*

$$\frac{p_1 f(a_1) + p_2 f(a_2) + \cdots + p_n f(a_n)}{p_1 + p_2 + \cdots + p_n} \ge f\left(\frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_1 + p_2 + \cdots + p_n}\right).$$

For $p_1 = p_2 = \cdots = p_n$, Jensen's inequality becomes

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge n f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right).$$

11. KARAMATA'S MAJORIZATION INEQUALITY

Let f be a convex function on a real interval $\mathbb{I}$. If a decreasingly ordered sequence

$$A = (a_1, a_2, \ldots, a_n), \quad a_i \in \mathbb{I},$$

majorizes a decreasingly ordered sequence

$$B = (b_1, b_2, \ldots, b_n), \quad b_i \in \mathbb{I},$$

then

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge f(b_1) + f(b_2) + \cdots + f(b_n).$$

We say that a sequence $A = (a_1, a_2, \ldots, a_n)$ with $a_1 \ge a_2 \ge \cdots \ge a_n$ majorizes a sequence $B = (b_1,\ b_2,\ \ldots,\ b_n)$ with $b_1 \ge b_2 \ge \cdots \ge b_n$, and write it as

$$A \succ B,$$

if

$$\begin{gathered} a_1 \ge b_1, \\ a_1 + a_2 \ge b_1 + b_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_1 + a_2 + \cdots + a_{n-1} \ge b_1 + b_2 + \cdots + b_{n-1}, \\ a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n. \end{gathered}$$

12. SYMMETRIC INEQUALITIES OF DEGREE THREE, FOUR OR FIVE

Theorem (*Vasile Cirtoaje*, 2010) *Let $f_n(a, b, c)$ be a symmetric homogeneous polynomial of degree n.*

(a) *The inequality $f_4(a, b, c) \ge 0$ holds for all real numbers a, b, c if and only if $f_4(a, 1, 1) \ge 0$ for all real a;*

(b) *For $n \in \{3, 4, 5\}$, the inequality $f_n(a, b, c) \ge 0$ holds for all $a, b, c \ge 0$ if and only if $f_n(a, 1, 1) \ge 0$ and $f_n(0, b, c) \ge 0$ for all $a, b, c \ge 0$.*

13. SYMMETRIC HOMOGENEOUS INEQUALITIES OF DEGREE SIX

Any sixth degree symmetric homogeneous polynomial $f_6(a, b, c)$ can be written in the form

$$f_6(a, b, c) = Ar^2 + B(p, q)r + C(p, q),$$

where A is called *the highest coefficient* of f_6, and

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

Theorem (*Vasile Cirtoaje*, 2010). *Let $f_6(a, b, c)$ be a sixth degree symmetric homogeneous polynomial having the highest coefficient $A \le 0$.*

(a) *The inequality* $f_6(a,b,c) \ge 0$ *holds for all real numbers* a,b,c *if and only if* $f_6(a,1,1) \ge 0$ *for all real* a;

(b) *The inequality* $f_6(a,b,c) \ge 0$ *holds for all* $a,b,c \ge 0$ *if and only if* $f_6(a,1,1) \ge 0$ *and* $f_6(0,b,c) \ge 0$ *for all* $a,b,c \ge 0$.

This theorem is also valid for the case where $B(p,q)$ and $C(p,q)$ are homogeneous rational functions.

For $A > 0$, we can use the *highest coefficient cancellation method* (*Vasile Cirtoaje*, 2010). This method consists in finding some suitable real numbers B, C and D such that the following sharper inequality holds

$$f_6(a,b,c) \ge A\left(r + Bp^3 + Cpq + D\frac{q^2}{p}\right)^2.$$

Because the function g_6 defined by

$$g_6(a,b,c) = f_6(a,b,c) - A\left(r + Bp^3 + Cpq + D\frac{q^2}{p}\right)^2$$

has the highest coefficient $A_1 = 0$, we can prove the inequality $g_6(a,b,c) \ge 0$ using Theorem above.

Notice that sometimes it is useful to break the problem into two parts, $p^2 \le \xi q$ and $p^2 > \xi q$, where ξ is a suitable real number.

A symmetric homogeneous polynomial of degree six in three variables has the form

$$f_6(a,b,c) = A_1\sum a^6 + A_2\sum ab(a^4+b^4) + A_3\sum a^2b^2(a^2+b^2)$$

$$+A_4\sum a^3b^3 + A_5 abc\sum a^3 + A_6 abc\sum ab(a+b) + 3A_7a^2b^2c^2,$$

where $A_1,\dots,A_7$ are real constants. In order to write this polynomial as a function of p, q and r, the following relations are useful:

$$\sum a^3 = 3r + p^3 - 3pq,$$

$$\sum ab(a+b) = -3r + pq,$$

$$\sum a^3b^3 = 3r^2 - 3pqr + q^3,$$

$$\sum a^2b^2(a^2+b^2) = -3r^2 - 2(p^3 - 2pq)r + p^2q^2 - 2q^3,$$

$$\sum ab(a^4+b^4) = -3r^2 - 2(p^3 - 7pq)r + p^4q - 4p^2q^2 + 2q^3,$$

$$\sum a^6 = 3r^2 + 6(p^3 - 2pq)r + p^6 - 6p^4q + 9p^2q^2 - 2q^3,$$

$$(a-b)^2(b-c)^2(c-a)^2 = -27r^2 + 2(9pq - 2p^3)r + p^2q^2 - 4q^3.$$

According to these relations, the highest coefficient A of the polynomial $f_6(a,b,c)$ is

$$A = 3(A_1 - A_2 - A_3 + A_4 + A_5 - A_6 + A_7).$$

The polynomials

$$P_1(a,b,c) = \sum (A_1a^2 + A_2bc)(B_1a^2 + B_2bc)(C_1a^2 + C_2bc),$$

$$P_2(a,b,c) = \sum (A_1a^2 + A_2bc)(B_1b^2 + B_2ca)(C_1c^2 + C_2ab)$$

and

$$P_3(a,b,c) = (A_1a^2 + A_2bc)(A_1b^2 + A_2ca)(A_1c^2 + A_2ab)$$

has the highest coefficients

$$P_1(1,1,1), \quad P_2(1,1,1), \quad P_3(1,1,1),$$

respectively. The polynomial

$$P_4(a,b,c) = (a^2 + mab + b^2)(b^2 + mbc + c^2)(c^2 + mca + a^2)$$

has the highest coefficient

$$A = (m-1)^3.$$

14. VASC'S POWER EXPONENTIAL INEQUALITIES

Theorem. *Let* $0 < k \le e$.

(a) *If* $a, b > 0$, *then (Vasile Cîrtoaje, 2006)*

$$a^{ka} + b^{kb} \ge a^{kb} + b^{ka};$$

(b) *If* $a, b \in (0, 1]$, *then (Vasile Cîrtoaje, 2010)*

$$2\sqrt{a^{ka}b^{kb}} \ge a^{kb} + b^{ka}.$$

Appendix B

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