This is Volume 4 of the five-volume book Mathematical Inequalities, which introduces and develops the main types of elementary inequalities. The first three volumes are a great opportunity to look into many old and new inequalities, as well as elementary procedures for solving them: Volume 1 - Symmetric Polynomial Inequalities, Volume 2 - Symmetric Rational and Nonrational Inequalities, Volume 3 - Cyclic and Noncyclic Inequalities. As a rule, the inequalities in these volumes are increasingly ordered according to the number of variables: two, three, four, ... , n-variables. The last two volumes (Volume 4 – Extensions and Refinements of Jensen’s Inequality, Volume 5 – Other Recent Methods for Creating and Solving Inequalities) present beautiful and original methods for solving inequalities, such as Half/Partial convex function method, Equal variables method, Arithmetic compensation method, Highest coefficient cancellation method, pqr method etc. The book is intended for a wide audience: advanced middle school students, high school students, college and university students, and teachers. Many problems and methods can be used as group projects for advanced high school students.

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978-613-9-91148-6

Mathematical Inequalities - Volume 4

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Extensions and Refinements of Jensen's Inequality

LAP LAMBERT Academic Publishing

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MATHEMATICAL INEQUALITIES

Volume 4

EXTENSIONS AND REFINEMENTS OF JENSEN'S INEQUALITY

LAP LAMBERT Academic Publishing

EDITURA UNIVERSITĂȚII PETROL-GAZE DIN PLOIEŞTI

2021

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Chapter 1

Half Convex Function Method

1.1 Theoretical Basis

Let $\mathbb{I}$ be a real interval, s an interior point of $\mathbb{I}$ and

$$\mathbb{I}_{\geq s} = \{u|u \in \mathbb{I}, u \geq s\}, \quad \mathbb{I}_{\leq s} = \{u|u \in \mathbb{I}, u \leq s\}.$$

The following statement is known as the Right Half Convex Function Theorem (RHCF-Theorem).

Right Half Convex Function Theorem (*Vasile Cîrtoaje*, 2004). *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\geq s}$, where $s \in int(\mathbb{I})$. If*

$$f(x) + (n-1)f(y) \geq nf(s)$$

for all $x, y \in \mathbb{I}$ so that $x \leq s \leq y$ and $x + (n-1)y = ns$, then the inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \tag{1}$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying $a_1 + a_2 + \cdots + a_n = ns$. In addition, the inequality (1) holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying $a_1 + a_2 + \cdots + a_n = ns_1$, where $s_1 \in int(\mathbb{I})$, $s_1 > s$.

Proof. Assume that

$$a_1 \leq a_2 \leq \cdots \leq a_n.$$

If $a_1 \geq s$, then the required inequality is just Jensen's inequality for convex functions. Otherwise, if $a_1 < s$, then there exists

$$k \in \{1, 2, \ldots, n-1\}$$

so that

$$a_1 \leq \cdots \leq a_k < s \leq a_{k+1} \leq \cdots \leq a_n.$$

Since f is convex on $\mathbb{I}_{\geq s}$, we may apply Jensen's inequality to get

$$f(a_{k+1})+\cdots+f(a_n)\geq (n-k)f(z),$$

where

$$z=\frac{a_{k+1}+\cdots+a_n}{n-k},\qquad z\in\mathbb{I}.$$

Thus, it suffices to show that

$$f(a_1)+\cdots+f(a_k)+(n-k)f(z)\geq nf(s). \tag{2}$$

Let $b_1,\ldots,b_k$ be defined by

$$a_i+(n-1)b_i=ns,\quad i=1,\ldots,k.$$

We claim that

$$z\geq b_1\geq\cdots\geq b_k>s,$$

which involves

$$b_1,\ldots,b_k\in\mathbb{I}_{\geq s}.$$

Indeed, we have

$$b_1\geq\cdots\geq b_k,$$

$$b_k-s=\frac{s-a_k}{n-1}>0,$$

and

$$z\geq b_1$$

because

$$\begin{aligned}(n-1)b_1=ns-a_1&=(a_2+\cdots+a_k)+a_{k+1}+\cdots+a_n\\&\leq(k-1)s+a_{k+1}+\cdots+a_n\\&=(k-1)s+(n-k)z\leq(n-1)z.\end{aligned}$$

Since $b_1,\ldots,b_k\in\mathbb{I}_{\geq s}$, by hypothesis we have

$$f(a_1)+(n-1)f(b_1)\geq nf(s),$$

$$\cdots$$

$$f(a_k)+(n-1)f(b_k)\geq nf(s),$$

hence

$$f(a_1)+\cdots+f(a_k)+(n-1)[f(b_1)+\cdots+f(b_k)]\geq knf(s),$$

$$f(a_1)+\cdots+f(a_k)\geq knf(s)-(n-1)[f(b_1)+\cdots+f(b_k)].$$

According to this result, the inequality (2) is true if

$$knf(s)-(n-1)[f(b_1)+\cdots+f(b_k)]+(n-k)f(z)\geq nf(s),$$

which is equivalent to

$$pf(z)+(k-p)f(s) \geq f(b_1)+\cdots+f(b_k), \qquad p=\frac{n-k}{n-1} \leq 1.$$

By Jensen's inequality, we have

$$pf(z)+(1-p)f(s) \geq f(w), \qquad w=pz+(1-p)s \geq s.$$

Thus, we only need to show that

$$f(w)+(k-1)f(s) \geq f(b_1)+\cdots+f(b_k).$$

Since the decreasingly ordered vector $\vec{A_k}=(w,s,\ldots,s)$ majorizes the decreasingly ordered vector $\vec{B_k}=(b_1,b_2,\ldots,b_k)$, this inequality follows from Karamata's inequality for convex functions.

According to this result, the inequality (1) holds for all $a_1,a_2,\ldots,a_n \in \mathbb{I}$ satisfying $a_1+a_2+\cdots+a_n=ns_1$ if $f(x_1)+(n-1)f(y_1) \geq nf(s_1)$ for all $x_1,y_1 \in \mathbb{I}$ so that $x_1 \leq s_1 \leq y_1$ and $x_1+(n-1)y_1=ns_1$. Thus, we need to show that if

$$f(x)+(n-1)f(y) \geq nf(s)$$

for all $x,y \in \mathbb{I}$ so that $x \leq s \leq y$ and $x+(n-1)y=ns$, then

$$f(x_1)+(n-1)f(y_1) \geq nf(s_1) \tag{3}$$

for all $x_1,y_1 \in \mathbb{I}$ so that $x_1 \leq s_1 \leq y_1$ and $x_1+(n-1)y_1=ns_1$. Since this is true for $x_1 \geq s$ (by Jensen's inequality), consider next $x_1<s$. By hypothesis, we have

$$f(x_1)+(n-1)f(y_2) \geq nf(s),$$

where $y_2 \in \mathbb{I}$ such that

$$x_1+(n-1)y_2=ns, \quad y_2>s.$$

Thus, (3) is true if

$$nf(s)-(n-1)f(y_2)+(n-1)f(y_1) \geq nf(s_1),$$

that is

$$(n-1)f(y_1)+nf(s) \geq (n-1)f(y_2)+nf(s_1).$$

Since

$$(n-1)y_1+ns=(n-1)y_2+ns_1$$

and the decreasingly ordered vector $\vec{C}_{2n-1}=(y_1,\ldots,y_1,s,\ldots,s)$ majorizes the vector $\vec{D}_{2n-1}=(y_2,\ldots,y_2,s_1,\ldots,s_1)$, this inequality follows from Karamata's inequality for convex functions.

Similarly, we can prove the Left Half Convex Function Theorem (LHCF-Theorem).

Left Half Convex Function Theorem. *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\le s}$, where $s \in int(\mathbb{I})$. If*

$$f(x)+(n-1)f(y) \ge nf(s)$$

for all $x, y \in \mathbb{I}$ so that $x \ge s \ge y$ and $x+(n-1)y = ns$, then the inequality

$$f(a_1)+f(a_2)+\cdots+f(a_n) \ge nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right) \tag{4}$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying $a_1+a_2+\cdots+a_n = ns$. In addition, the inequality (4) holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying $a_1+a_2+\cdots+a_n = ns_1$, where $s_1 \in int(\mathbb{I})$, $s_1 < s$.

From the RHCF-Theorem and the LHCF-Theorem, we find the HCF-Theorem (Half Convex Function Theorem).

Half Convex Function Theorem. *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\ge s}$ or $\mathbb{I}_{\le s}$, where $s \in int(\mathbb{I})$. The inequality*

$$f(a_1)+f(a_2)+\cdots+f(a_n) \ge nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying

$$a_1+a_2+\cdots+a_n = ns$$

if and only if

$$f(x)+(n-1)f(y) \ge nf(s)$$

for all $x, y \in \mathbb{I}$ so that $x+(n-1)y = ns$.

The following LCRCF-Theorem is also useful to prove some symmetric inequalities.

Left Convex-Right Concave Function Theorem (*Vasile Cîrtoaje*, 2004). *Let $a \le c$ be real numbers, let f be a continuous function defined on $\mathbb{I} = [a, \infty)$, strictly convex on $[a, c]$ and strictly concave on $[c, \infty)$, and let*

$$E(a_1, a_2, \ldots, a_n) = f(a_1)+f(a_2)+\cdots+f(a_n).$$

If $a_1, a_2, \ldots, a_n \in \mathbb{I}$ so that

$$a_1+a_2+\cdots+a_n = S = constant,$$

then

(a) *E is minimum for $a_1 = a_2 = \cdots = a_{n-1} \le a_n$;*

(b) *E is maximum for either $a_1 = a$* or *$a < a_1 \le a_2 = \cdots = a_n$.*

Proof. Without loss of generality, assume that $a_1 \le a_2 \le \cdots \le a_n$. Since the sum $E(a_1, a_2, \dots, a_n)$ is a continuous function on the compact set

$$\Lambda = \{(a_1, a_2, \dots, a_n) : \ a_1 + a_2 + \cdots + a_n = S, \ a_1, a_2, \dots, a_n \in \mathbb{I}\},$$

E attains its minimum and maximum values.

(a) For the sake of contradiction, suppose that E is minimum at $(b_1, b_2, \dots, b_n)$ with

$$b_1 \le b_2 \le \cdots \le b_n, \qquad b_1 < b_{n-1}.$$

For $b_{n-1} \le c$, by Jensen's inequality for strictly convex functions we have

$$f(b_1) + f(b_{n-1}) > 2f\left(\frac{b_1 + b_{n-1}}{2}\right),$$

while for $b_{n-1} > c$, by Karamata's inequality for strictly concave functions we have

$$f(b_{n-1}) + f(b_n) > f(c) + f(b_{n-1} + b_n - c).$$

The both results contradict the assumption that E is minimum at $(b_1, b_2, \dots, b_n)$.

(b) For the sake of contradiction, suppose that E is maximum at $(b_1, b_2, \dots, b_n)$ with

$$a < b_1 \le b_2 \le \cdots \le b_n, \qquad b_2 < b_n.$$

There are three cases to consider.

Case 1: $b_2 \ge c$. By Jensen's inequality for strictly concave functions, we have

$$f(b_2) + f(b_n) < 2f\left(\frac{b_2 + b_n}{2}\right).$$

Case 2: $b_2 < c$ and $b_1 + b_2 - a \le c$. By Karamata's inequality for strictly convex functions, we have

$$f(b_1) + f(b_2) < f(a) + f(b_1 + b_2 - a).$$

Case 3: $b_2 < c$ and $b_1 + b_2 - c \ge a$. By Karamata's inequality for strictly convex functions, we have

$$f(b_1) + f(b_2) < f(b_1 + b_2 - c) + f(c).$$

Clearly, all these results contradict the assumption that E is maximum at $(b_1, b_2, \dots, b_n)$.

Note 1. Let us denote

$$g(u) = \frac{f(u) - f(s)}{u - s}, \quad h(x, y) = \frac{g(x) - g(y)}{x - y}.$$

In many applications, it is useful to replace the hypothesis

$$f(x)+(n-1)f(y) \geq nf(s)$$

in the RHCF-Theorem, the LHCF-Theorem and the HCF-Theorem by the equivalent condition

$$h(x,y) \geq 0 \ \textit{ for all } \ x,y \in \mathbb{I} \ \textit{ so that } \ x+(n-1)y = ns.$$

This equivalence is true because

$$\begin{aligned} f(x)+(n-1)f(y)-nf(s) &= [f(x)-f(s)]+(n-1)[f(y)-f(s)] \\ &= (x-s)g(x)+(n-1)(y-s)g(y) \\ &= \frac{n-1}{n}(x-y)[g(x)-g(y)] \\ &= \frac{n-1}{n}(x-y)^2 h(x,y). \end{aligned}$$

Note 2. Assume that f is differentiable on $\mathbb{I}$, and let

$$H(x,y) = \frac{f'(x)-f'(y)}{x-y}.$$

The desired inequality of Jensen's type in the RHCF-Theorem, the LHCF-Theorem and the HCF-Theorem holds true by replacing the hypothesis

$$f(x)+(n-1)f(y) \geq nf(s)$$

with the more restrictive condition

$$H(x,y) \geq 0 \ \textit{ for all } \ x,y \in \mathbb{I} \ \textit{ so that } \ x+(n-1)y = ns.$$

To prove this, we will show that the new condition $H(x,y) \geq 0$ implies

$$f(x)+(n-1)f(y) \geq nf(s)$$

for all $x,y \in \mathbb{I}$ so that $x+(n-1)y = ns$. Write this inequality as

$$f_1(x) \geq nf(s),$$

where

$$f_1(x) = f(x)+(n-1)f(y) = f(x)+(n-1)f\left(\frac{ns-x}{n-1}\right).$$

From

$$\begin{aligned} f_1'(x) &= f'(x)-f'\left(\frac{ns-x}{n-1}\right) \\ &= f'(x)-f'(y) \\ &= \frac{n}{n-1}(x-s)H(x,y), \end{aligned}$$

it follows that f_1 is decreasing on $\mathbb{I}_{\le s}$ and increasing on $\mathbb{I}_{\ge s}$; therefore,

$$f_1(x) \ge f_1(s) = nf(s).$$

Note 3. From the proof of the RHCF-Theorem, it follows that the RHCF-Theorem, the LHCF-Theorem and the HCF-Theorem are also valid in the case when f is defined on $\mathbb{I} \setminus \{u_0\}$, where $u_0 \in \mathbb{I}_{<s}$ for the RHCF-Theorem, and $u_0 \in \mathbb{I}_{>s}$ for the LHCF-Theorem.

Note 4. The desired inequalities in the RHCF-Theorem, the LHCF-Theorem and the HCF-Theorem become equalities for

$$a_1 = a_2 = \cdots = a_n = s.$$

In addition, if there exist $x, y \in \mathbb{I}$ so that

$$x + (n-1)y = ns, \quad f(x) + (n-1)f(y) = nf(s), \quad x \ne y,$$

then the equality holds also for

$$a_1 = x, \quad a_2 = \cdots = a_n = y$$

(or any cyclic permutation). Notice that these equality conditions are equivalent to

$$x + (n-1)y = ns, \quad h(x, y) = 0$$

($x < y$ for the RHCF-Theorem, and $x > y$ for the LHCF-Theorem).

Note 5. The part (a) in LCRCF-Theorem is also true in the case where $\mathbb{I} = (a, \infty)$ and $f(a_+) = \infty$.

Note 6. Similarly, we can extend the *weighted* Jensen's inequality to right and left half convex functions establishing the WRHCF-Theorem, the WLHCF-Theorem and the WHCF-Theorem (*Vasile Cîrtoaje*, 2008).

WHCF-Theorem. *Let $p_1, p_2, \ldots, p_n$ be positive real numbers so that*

$$p_1 + p_2 + \cdots + p_n = 1, \qquad p = \min\{p_1, p_2, \ldots, p_n\},$$

and let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\ge s}$ or $\mathbb{I}_{\le s}$, where $s \in int(\mathbb{I})$. The inequality

$$p_1 f(a_1) + p_2 f(a_2) + \cdots + p_n f(a_n) \ge f(p_1 a_1 + p_2 a_2 + \cdots + p_n a_n)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ so that

$$p_1 a_1 + p_2 a_2 + \cdots + p_n a_n = s,$$

if and only if

$$pf(x) + (1-p)f(y) \ge f(s)$$

for all $x, y \in \mathbb{I}$ satisfying

$$px + (1-p)y = s.$$

1.2 Applications

1.1. If a, b, c are real numbers so that $a+b+c=3$, then

$$3(a^4+b^4+c^4)+a^2+b^2+c^2+6 \ge 6(a^3+b^3+c^3).$$

1.2. If $a_1, a_2, \ldots, a_n \ge \dfrac{1-2n}{n-2}$ so that $a_1+a_2+\cdots+a_n=n$, then

$$a_1^3+a_2^3+\cdots+a_n^3 \ge n.$$

1.3. If $a_1, a_2, \ldots, a_n \ge \dfrac{-n}{n-2}$ so that $a_1+a_2+\cdots+a_n=n$, then

$$a_1^3+a_2^3+\cdots+a_n^3 \ge a_1^2+a_2^2+\cdots+a_n^2.$$

1.4. If $a_1, a_2, \ldots, a_n$ are real numbers so that $a_1+a_2+\cdots+a_n=n$, then

$$(n^2-3n+3)(a_1^4+a_2^4+\cdots+a_n^4-n) \ge 2(n^2-n+1)(a_1^2+a_2^2+\cdots+a_n^2-n).$$

1.5. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that $a_1+a_2+\cdots+a_n=n$, then

$$(n^2+n+1)(a_1^3+a_2^3+\cdots+a_n^3-n) \ge (n+1)(a_1^4+a_2^4+\cdots+a_n^4-n).$$

1.6. If a, b, c are real numbers so that $a+b+c=3$, then

(a) $$a^4+b^4+c^4-3+2(7+3\sqrt{7})(a^3+b^3+c^3-3) \ge 0;$$

(b) $$a^4+b^4+c^4-3+2(7-3\sqrt{7})(a^3+b^3+c^3-3) \ge 0.$$

1.7. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1+a_2+\cdots+a_n=n$. If k is a positive integer satisfying $3 \le k \le n+1$, then

$$\frac{a_1^k+a_2^k+\cdots+a_n^k-n}{a_1^2+a_2^2+\cdots+a_n^2-n} \ge (n-1)\left[\left(\frac{n}{n-1}\right)^{k-1}-1\right].$$

1.8. Let $k \ge 3$ be an integer number. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then

$$\frac{a_1^k + a_2^k + \cdots + a_n^k - n}{a_1^2 + a_2^2 + \cdots + a_n^2 - n} \le \frac{n^{k-1} - 1}{n - 1}.$$

1.9. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then

$$n^2\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n\right) \ge 4(n-1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n).$$

1.10. If $a_1, a_2, \dots, a_8$ are positive real numbers so that $a_1 + a_2 + \cdots + a_8 = 8$, then

$$\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_8^2} \ge a_1^2 + a_2^2 + \cdots + a_8^2.$$

1.11. If $a_1, a_2, \dots, a_n$ are positive real numbers so that $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = n$, then

$$a_1^2 + a_2^2 + \cdots + a_n^2 - n \ge 2\left(1 + \frac{\sqrt{n-1}}{n}\right)(a_1 + a_2 + \cdots + a_n - n).$$

1.12. If a, b, c, d, e are positive real numbers so that $a^2 + b^2 + c^2 + d^2 + e^2 = 5$, then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} - 5 + \frac{4(1+\sqrt{5})}{5}(a + b + c + d + e - 5) \ge 0.$$

1.13. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{3a+b+c} + \frac{1}{3b+c+a} + \frac{1}{3c+a+b} \le \frac{2}{5}\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right).$$

1.14. If $a, b, c, d \ge 3 - \sqrt{7}$ so that $a + b + c + d = 4$, then

$$\frac{1}{2+a^2} + \frac{1}{2+b^2} + \frac{1}{2+c^2} + \frac{1}{2+d^2} \ge \frac{4}{3}.$$

1.15. If $a_1, a_2, \ldots, a_n \in [-\sqrt{n}, n-2]$ so that $a_1 + a_2 + \cdots + a_n = n$, then

$$\frac{1}{n+a_1^2} + \frac{1}{n+a_2^2} + \cdots + \frac{1}{n+a_n^2} \le \frac{n}{n+1}.$$

1.16. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$\frac{3-a}{9+a^2} + \frac{3-b}{9+b^2} + \frac{3-c}{9+c^2} \ge \frac{3}{5}.$$

1.17. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$\frac{1}{1-a+2a^2} + \frac{1}{1-b+2b^2} + \frac{1}{1-c+2c^2} \ge \frac{3}{2}.$$

1.18. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$\frac{1}{5+a+a^2} + \frac{1}{5+b+b^2} + \frac{1}{5+c+c^2} \ge \frac{3}{7}.$$

1.19. If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 4$, then

$$\frac{1}{10+a+a^2} + \frac{1}{10+b+b^2} + \frac{1}{10+c+c^2} + \frac{1}{10+d+d^2} \le \frac{1}{3}.$$

1.20. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If

$$k \ge 1 - \frac{1}{n},$$

then

$$\frac{1}{1+ka_1^2} + \frac{1}{1+ka_2^2} + \cdots + \frac{1}{1+ka_n^2} \ge \frac{n}{1+k}.$$

1.21. Let $a_1, a_2, \ldots, a_n$ be real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If

$$0 < k \le \frac{n-1}{n^2-n+1},$$

then

$$\frac{1}{1+ka_1^2} + \frac{1}{1+ka_2^2} + \cdots + \frac{1}{1+ka_n^2} \le \frac{n}{1+k}.$$

1.22. Let $a_1, a_2, \dots, a_n$ be nonnegative numbers so that $a_1 + a_2 + \cdots + a_n = n$. If

$$k \ge \frac{n^2}{4(n-1)},$$

then

$$\frac{a_1(a_1-1)}{a_1^2+k} + \frac{a_2(a_2-1)}{a_2^2+k} + \cdots + \frac{a_n(a_n-1)}{a_n^2+k} \ge 0.$$

1.23. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then

$$\frac{a_1-1}{(n-2a_1)^2} + \frac{a_2-1}{(n-2a_2)^2} + \cdots + \frac{a_n-1}{(n-2a_n)^2} \ge 0.$$

1.24. If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that

$$a_1 + a_2 + \cdots + a_n = n, \qquad a_1, a_2, \dots, a_n > -k, \qquad k \ge 1 + \frac{n}{\sqrt{n-1}},$$

then

$$\frac{a_1^2-1}{(a_1+k)^2} + \frac{a_2^2-1}{(a_2+k)^2} + \cdots + \frac{a_n^2-1}{(a_n+k)^2} \ge 0.$$

1.25. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If $0 < k \le 1 + \sqrt{\dfrac{2n-1}{n-1}}$, then

$$\frac{a_1^2-1}{(a_1+k)^2} + \frac{a_2^2-1}{(a_2+k)^2} + \cdots + \frac{a_n^2-1}{(a_n+k)^2} \le 0.$$

1.26. If $a_1, a_2, \dots, a_n \ge n - 1 - \sqrt{n^2-n+1}$ so that $a_1 + a_2 + \cdots + a_n = n$, then

$$\frac{a_1^2-1}{(a_1+2)^2} + \frac{a_2^2-1}{(a_2+2)^2} + \cdots + \frac{a_n^2-1}{(a_n+2)^2} \le 0.$$

1.27. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If $k \ge \dfrac{(n-1)(2n-1)}{n^2}$, then

$$\frac{1}{1+ka_1^3} + \frac{1}{1+ka_2^3} + \cdots + \frac{1}{1+ka_n^3} \ge \frac{n}{1+k}.$$

1.28. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If $0 < k \le \dfrac{n-1}{n^2-2n+2}$, then

$$\frac{1}{1+ka_1^3} + \frac{1}{1+ka_2^3} + \cdots + \frac{1}{1+ka_n^3} \le \frac{n}{1+k}.$$

1.29. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If $k \ge \dfrac{n^2}{n-1}$, then

$$\sqrt{\frac{a_1}{k-a_1}} + \sqrt{\frac{a_2}{k-a_2}} + \cdots + \sqrt{\frac{a_n}{k-a_n}} \le \frac{n}{\sqrt{k-1}}.$$

1.30. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then

$$n^{-a_1^2} + n^{-a_2^2} + \cdots + n^{-a_n^2} \ge 1.$$

1.31. If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 4$, then

$$(3a^2+1)(3b^2+1)(3c^2+1)(3d^2+1) \le 256.$$

1.32. If $a, b, c, d, e \ge -1$ so that $a + b + c + d + e = 5$, then

$$(a^2+1)(b^2+1)(c^2+1)(d^2+1)(e^2+1) \ge (a+1)(b+1)(c+1)(d+1)(e+1).$$

1.33. Let $a_1, a_2, \ldots, a_n$ be positive numbers so that $a_1 + a_2 + \cdots + a_n = n$. If

$$k \le \frac{2\sqrt{n-1}}{n} + 2\sqrt{\frac{2\sqrt{n-1}}{n}}, \qquad k \le 3,$$

then

$$k(\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n}) + \frac{1}{\sqrt{a_1}} + \frac{1}{\sqrt{a_2}} + \cdots + \frac{1}{\sqrt{a_n}} \ge (k+1)n.$$

1.34. If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are positive numbers so that $a_1 + a_2 + \cdots + a_n = 1$, then

$$\left(\frac{1}{\sqrt{a_1}} - \sqrt{a_1}\right)\left(\frac{1}{\sqrt{a_2}} - \sqrt{a_2}\right)\cdots\left(\frac{1}{\sqrt{a_n}} - \sqrt{a_n}\right) \ge \left(\sqrt{n} - \frac{1}{\sqrt{n}}\right)^n.$$

1.35. Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If

$$k \le \left(1 + \frac{2\sqrt{n-1}}{n}\right)^2,$$

then

$$\left(ka_1 + \frac{1}{a_1}\right)\left(ka_2 + \frac{1}{a_2}\right)\cdots\left(ka_n + \frac{1}{a_n}\right) \ge (k+1)^n.$$

1.36. If a, b, c, d are nonzero real numbers so that

$$a, b, c, d \ge \frac{-1}{2}, \quad a + b + c + d = 4,$$

then

$$3\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2}\right) + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \ge 16.$$

1.37. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that $a_1^2 + a_2^2 + \cdots + a_n^2 = n$, then

$$a_1^3 + a_2^3 + \cdots + a_n^3 - n + \sqrt{\frac{n}{n-1}}\,(a_1 + a_2 + \cdots + a_n - n) \ge 0.$$

1.38. If a, b, c, d, e are nonnegative real numbers so that $a^2 + b^2 + c^2 + d^2 + e^2 = 5$, then

$$\frac{1}{7-2a} + \frac{1}{7-2b} + \frac{1}{7-2c} + \frac{1}{7-2d} + \frac{1}{7-2e} \le 1.$$

1.39. Let $0 \le a_1, a_2, \ldots, a_n < k$ so that $a_1^2 + a_2^2 + \cdots + a_n^2 = n$. If

$$1 < k \le 1 + \sqrt{\frac{n}{n-1}},$$

then

$$\frac{1}{k-a_1} + \frac{1}{k-a_2} + \cdots + \frac{1}{k-a_n} \ge \frac{n}{k-1}.$$

1.40. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\sqrt{1 + \frac{48a}{b+c}} + \sqrt{1 + \frac{48b}{c+a}} + \sqrt{1 + \frac{48c}{a+b}} \ge 15.$$

1.41. If a, b, c are nonnegative real numbers, then

$$\sqrt{\frac{3a^2}{7a^2+5(b+c)^2}}+\sqrt{\frac{3b^2}{7b^2+5(c+a)^2}}+\sqrt{\frac{3c^2}{7c^2+5(a+b)^2}} \le 1.$$

1.42. If a, b, c are nonnegative real numbers, then

$$\sqrt{\frac{a^2}{a^2+2(b+c)^2}}+\sqrt{\frac{b^2}{b^2+2(c+a)^2}}+\sqrt{\frac{c^2}{c^2+2(a+b)^2}} \ge 1.$$

1.43. Let a, b, c be nonnegative real numbers, no two of which are zero. If

$$k \ge k_0, \quad k_0 = \frac{\ln 3}{\ln 2} - 1 \approx 0.585,$$

then

$$\left(\frac{2a}{b+c}\right)^k + \left(\frac{2b}{c+a}\right)^k + \left(\frac{2c}{a+b}\right)^k \ge 3.$$

1.44. If $a, b, c \in [1, 7+4\sqrt{3}]$, then

$$\sqrt{\frac{2a}{b+c}}+\sqrt{\frac{2b}{c+a}}+\sqrt{\frac{2c}{a+b}} \ge 3.$$

1.45. Let a, b, c be nonnegative real numbers so that $a+b+c=3$. If

$$0 < k \le k_0, \quad k_0 = \frac{\ln 2}{\ln 3 - \ln 2} \approx 1.71,$$

then

$$a^k(b+c)+b^k(c+a)+c^k(a+b) \le 6.$$

1.46. If a, b, c are nonnegative real numbers so that $a+b+c=3$, then

$$\sqrt{a}+\sqrt{b}+\sqrt{c}-3 \ge 13\left(\sqrt{\frac{a+b}{2}}+\sqrt{\frac{b+c}{2}}+\sqrt{\frac{c+a}{2}}-3\right).$$

1.47. Let a,b,c be nonnegative real numbers so that $a+b+c=3$. If $k>2$, then

$$a^k+b^k+c^k+3\ge 2\left(\frac{a+b}{2}\right)^k+2\left(\frac{b+c}{2}\right)^k+2\left(\frac{c+a}{2}\right)^k.$$

1.48. If $a_1,a_2,\dots,a_n$ are nonnegative real numbers so that $a_1+a_2+\cdots+a_n=n$, then

$$\sqrt{a_1}+\sqrt{a_2}+\cdots+\sqrt{a_n}+n(k-1)\le k\left(\sqrt{\frac{n-a_1}{n-1}}+\sqrt{\frac{n-a_2}{n-1}}+\cdots+\sqrt{\frac{n-a_n}{n-1}}\right),$$

where

$$k=(\sqrt{n}-1)(\sqrt{n}+\sqrt{n-1}).$$

1.49. If a,b,c are the lengths of the sides of a triangle so that $a+b+c=3$, then

$$\frac{1}{a+b-c}+\frac{1}{b+c-a}+\frac{1}{c+a-b}-3\ge 4(2+\sqrt{3})\left(\frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a}-3\right).$$

1.50. Let $a_1,a_2,\dots,a_5$ be nonnegative numbers so that $a_1+a_2+a_3+a_4+a_5\le 5$. If

$$k\ge k_0,\qquad k_0=\frac{29+\sqrt{761}}{10}\approx 5.66,$$

then

$$\sum\frac{1}{ka_1^2+a_2+a_3+a_4+a_5}\ge\frac{5}{k+4}.$$

1.51. Let $a_1,a_2,\dots,a_5$ be nonnegative numbers so that $a_1+a_2+a_3+a_4+a_5\le 5$. If

$$0<k\le k_0,\qquad k_0=\frac{11-\sqrt{101}}{10}\approx 0.095,$$

then

$$\sum\frac{1}{ka_1^2+a_2+a_3+a_4+a_5}\ge\frac{5}{k+4}.$$

1.52. Let $a_1,a_2,\dots,a_n$ be nonnegative real numbers so that $a_1+a_2+\cdots+a_n\le n$. If

$$0<k\le\frac{1}{n+1},$$

then

$$\frac{a_1}{ka_1^2+a_2+\cdots+a_n}+\frac{a_2}{a_1+ka_2^2+\cdots+a_n}+\cdots+\frac{a_n}{a_1+a_2+\cdots+ka_n^2}\ge\frac{n}{k+n-1}.$$

1.53. If $a_1, a_2, a_3, a_4, a_5 \le \frac{7}{2}$ so that $a_1 + a_2 + a_3 + a_4 + a_5 = 5$, then

$$\frac{a_1}{a_1^2 - a_1 + 5} + \frac{a_2}{a_2^2 - a_2 + 5} + \frac{a_3}{a_3^2 - a_3 + 5} + \frac{a_4}{a_4^2 - a_4 + 5} + \frac{a_5}{a_5^2 - a_5 + 5} \le 1.$$

1.54. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n \ge n$. If

$$0 < k \le \frac{1}{1 + \frac{1}{4(n-1)^2}},$$

then

$$\frac{a_1^2}{ka_1^2 + a_2 + \cdots + a_n} + \frac{a_2^2}{a_1 + ka_2^2 + \cdots + a_n} + \cdots + \frac{a_n^2}{a_1 + a_2 + \cdots + ka_n^2} \ge \frac{n}{k + n - 1}.$$

1.55. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n \le n$. If $k \ge n - 1$, then

$$\frac{a_1^2}{ka_1^2 + a_2 + \cdots + a_n} + \frac{a_2^2}{a_1 + ka_2^2 + \cdots + a_n} + \cdots + \frac{a_n^2}{a_1 + a_2 + \cdots + ka_n^2} \le \frac{n}{k + n - 1}.$$

1.56. Let $a_1, a_2, \ldots, a_n \in [0, n]$ so that $a_1 + a_2 + \cdots + a_n \ge n$. If $0 < k \le \frac{1}{n}$, then

$$\frac{a_1 - 1}{ka_1^2 + a_2 + \cdots + a_n} + \frac{a_2 - 1}{a_1 + ka_2^2 + \cdots + a_n} + \cdots + \frac{a_n - 1}{a_1 + a_2 + \cdots + ka_n^2} \ge 0.$$

1.57. If a, b, c are positive real numbers so that $abc = 1$, then

$$\sqrt{a^2 - a + 1} + \sqrt{b^2 - b + 1} + \sqrt{c^2 - c + 1} \ge a + b + c.$$

1.58. If $a, b, c, d \ge \frac{1}{1 + \sqrt{6}}$ so that $abcd = 1$, then

$$\frac{1}{a+2} + \frac{1}{b+2} + \frac{1}{c+2} + \frac{1}{d+2} \le \frac{4}{3}.$$

1.59. If a, b, c are positive real numbers so that $abc = 1$, then

$$a^2 + b^2 + c^2 - 3 \ge 2(ab + bc + ca - a - b - c).$$

1.60. If a,b,c are positive real numbers so that $abc=1$, then

$$a^2+b^2+c^2-3 \ge 18(a+b+c-ab-bc-ca).$$

1.61. If $a_1,a_2,\ldots,a_n$ are positive real numbers so that $a_1a_2\cdots a_n=1$, then

$$a_1^2+a_2^2+\cdots+a_n^2-n \ge 6\sqrt{3}\left(a_1+a_2+\cdots+a_n-\frac{1}{a_1}-\frac{1}{a_2}-\cdots-\frac{1}{a_n}\right).$$

1.62. If $a_1,a_2,\ldots,a_n$ $(n\ge 4)$ are positive real numbers so that $a_1a_2\cdots a_n=1$, then

$$(n-1)(a_1^2+a_2^2+\cdots+a_n^2)+n(n+3) \ge (2n+2)(a_1+a_2+\cdots+a_n).$$

1.63. Let $a_1,a_2,\ldots,a_n$ $(n\ge 3)$ be positive real numbers so that $a_1a_2\cdots a_n=1$. If p and q are nonnegative real numbers so that $p+q\ge n-1$, then

$$\frac{1}{1+pa_1+qa_1^2}+\frac{1}{1+pa_2+qa_2^2}+\cdots+\frac{1}{1+pa_n+qa_n^2} \ge \frac{n}{1+p+q}.$$

1.64. Let a,b,c,d be positive real numbers so that $abcd=1$. If p and q are nonnegative real numbers so that $p+q=3$, then

$$\frac{1}{1+pa+qa^3}+\frac{1}{1+pb+qb^3}+\frac{1}{1+pc+qc^3}+\frac{1}{1+pd+qd^3} \ge 1.$$

1.65. If $a_1,a_2,\ldots,a_n$ are positive real numbers so that $a_1a_2\cdots a_n=1$, then

$$\frac{1}{1+a_1+\cdots+a_1^{n-1}}+\frac{1}{1+a_2+\cdots+a_2^{n-1}}+\cdots+\frac{1}{1+a_n+\cdots+a_n^{n-1}} \ge 1.$$

1.66. Let $a_1,a_2,\ldots,a_n$ be positive real numbers so that $a_1a_2\cdots a_n=1$. If

$$k \ge n^2-1,$$

then

$$\frac{1}{\sqrt{1+ka_1}}+\frac{1}{\sqrt{1+ka_2}}+\cdots+\frac{1}{\sqrt{1+ka_n}} \ge \frac{n}{\sqrt{1+k}}.$$

1.67. Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that $a_1a_2\cdots a_n = 1$. If $p, q \ge 0$ so that $0 < p+q \le \frac{1}{n-1}$, then

$$\frac{1}{1+pa_1+qa_1^2}+\frac{1}{1+pa_2+qa_2^2}+\cdots+\frac{1}{1+pa_n+qa_n^2} \le \frac{n}{1+p+q}.$$

1.68. Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be positive real numbers so that $a_1a_2\cdots a_n = 1$. If

$$0 < k \le \frac{2n-1}{(n-1)^2},$$

then

$$\frac{1}{\sqrt{1+ka_1}}+\frac{1}{\sqrt{1+ka_2}}+\cdots+\frac{1}{\sqrt{1+ka_n}} \le \frac{n}{\sqrt{1+k}}.$$

1.69. If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1a_2\cdots a_n = 1$, then

$$\sqrt{a_1^4+\frac{2n-1}{(n-1)^2}}+\sqrt{a_2^4+\frac{2n-1}{(n-1)^2}}+\cdots+\sqrt{a_n^4+\frac{2n-1}{(n-1)^2}} \ge \frac{1}{n-1}(a_1+a_2+\cdots+a_n)^2.$$

1.70. If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1a_2\cdots a_n = 1$, then

$$a_1^{n-1}+a_2^{n-1}+\cdots+a_n^{n-1}+n(n-2) \ge (n-1)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right).$$

1.71. Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that $a_1a_2\cdots a_n = 1$. If $k \ge n$, then

$$a_1^k+a_2^k+\cdots+a_n^k+kn \ge (k+1)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right).$$

1.72. If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1a_2\cdots a_n = 1$, then

$$\left(1-\frac{1}{n}\right)^{a_1}+\left(1-\frac{1}{n}\right)^{a_2}+\cdots+\left(1-\frac{1}{n}\right)^{a_n} \le n-1.$$

1.73. If a, b, c are positive real numbers so that $abc = 1$, then

$$\frac{1}{1+\sqrt{1+3a}}+\frac{1}{1+\sqrt{1+3b}}+\frac{1}{1+\sqrt{1+3c}} \le 1.$$

1.74. If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1a_2\cdots a_n = 1$, then

$$\frac{1}{1+\sqrt{1+4n(n-1)a_1}}+\frac{1}{1+\sqrt{1+4n(n-1)a_2}}+\cdots+\frac{1}{1+\sqrt{1+4n(n-1)a_n}} \ge \frac{1}{2}.$$

1.75. If a, b, c are positive real numbers so that $abc = 1$, then

$$\frac{a^6}{1+2a^5}+\frac{b^6}{1+2b^5}+\frac{c^6}{1+2c^5} \ge 1.$$

1.76. If a, b, c are positive real numbers so that $abc = 1$, then

$$\sqrt{25a^2+144}+\sqrt{25b^2+144}+\sqrt{25c^2+144} \le 5(a+b+c)+24.$$

1.77. If a, b, c are positive real numbers so that $abc = 1$, then

$$\sqrt{16a^2+9}+\sqrt{16b^2+9}+\sqrt{16c^2+9} \ge 4(a+b+c)+3.$$

1.78. If ABC is a triangle, then

$$\sin A\left(2\sin\frac{A}{2}-1\right)+\sin B\left(2\sin\frac{B}{2}-1\right)+\sin C\left(2\sin\frac{C}{2}-1\right) \ge 0.$$

1.79. If ABC is an acute or right triangle, then

$$\sin 2A\left(1-2\sin\frac{A}{2}\right)+\sin 2B\left(1-2\sin\frac{B}{2}\right)+\sin 2C\left(1-2\sin\frac{C}{2}\right) \ge 0.$$

1.80. If a, b, c, d are real numbers so that $a+b+c+d = 4$, then

$$\frac{a}{a^2-a+4}+\frac{b}{b^2-b+4}+\frac{c}{c^2-c+4}+\frac{d}{d^2-d+4} \le 1.$$

1.81. Let a, b, c be nonnegative real numbers so that $a+b+c = 2$. If

$$k_0 \le k \le 3, \quad k_0 = \frac{\ln 2}{\ln 3 - \ln 2} \approx 1.71,$$

then

$$a^k(b+c)+b^k(c+a)+c^k(a+b) \le 2.$$

1.82. If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then

$$(n+1)^2\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \ge 4(n+2)(a_1^2 + a_2^2 + \cdots + a_n^2) + n(n^2 - 3n - 6).$$

1.83. If a, b, c, d, e are positive real numbers such that $a + b + c + d + e = 5$, then

$$27(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}) \ge 4(a^3 + b^3 + c^3 + d^3 + e^3) + 115.$$

1.84. If a, b, c are nonnegative real numbers so that $a + b + c = 12$, then

$$(a^2 + 10)(b^2 + 10)(c^2 + 10) \ge 13310.$$

1.85. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then

$$(a_1^2 + 1)(a_2^2 + 1)\cdots(a_n^2 + 1) \ge \frac{(n^2 - 2n + 2)^n}{(n-1)^{2n-2}}.$$

1.86. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$(a^2 + 2)(b^2 + 2)(c^2 + 2) \le 44.$$

1.87. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$(a^2 + 1)(b^2 + 1)(c^2 + 1) \le \frac{169}{16}.$$

1.88. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$(2a^2 + 1)(2b^2 + 1)(2c^2 + 1) \le \frac{121}{4}.$$

1.89. If a, b, c are nonnegative real numbers so that $a + b + c \ge k_0$, where

$$k_0 = \frac{3}{8}\sqrt{66 + 10\sqrt{105}} \approx 4.867,$$

then

$$\sqrt[3]{(a^2 + 1)(b^2 + 1)(c^2 + 1)} \le \left(\frac{a + b + c}{3}\right)^2 + 1.$$

1.90. If a,b,c,d are nonnegative real numbers so that $a+b+c+d=4$, then

$$(a^2+3)(b^2+3)(c^2+3)(d^2+3) \le 513.$$

1.91. If a,b,c,d are nonnegative real numbers so that $a+b+c+d=4$, then

$$(a^2+2)(b^2+2)(c^2+2)(d^2+2) \le 144.$$

1.92. If a,b,c,d are nonnegative real numbers such that

$$a+b+c+d=4,$$

then

$$\frac{a}{3a^3+2}+\frac{b}{3b^3+2}+\frac{c}{3c^3+2}+\frac{d}{3d^3+2} \le \frac{4}{5}.$$

1.93. If $a_1,a_2,\ldots,a_n$ are nonnegative real numbers such that $a_1+a_2+\cdots+a_n=1$, then

$$a_1^3+a_2^3+\cdots+a_n^3 \le \frac{1}{8}+a_1^4+a_2^4+\cdots+a_n^4.$$

1.3 Solutions

P 1.1. *If a, b, c are real numbers so that $a + b + c = 3$, then*

$$3(a^4 + b^4 + c^4) + a^2 + b^2 + c^2 + 6 \ge 6(a^3 + b^3 + c^3).$$

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Solution. Write the inequality as

$$f(a) + f(b) + f(c) \ge 3f(s), \quad s = \frac{a+b+c}{3} = 1,$$

where

$$f(u) = 3u^4 - 6u^3 + u^2, \quad u \in \mathbb{R}.$$

From

$$f''(u) = 2(18u^2 - 18u + 1),$$

it follows that $f''(u) > 0$ for $u \ge 1$, hence f is convex on $[s, \infty)$. By the RHCF-Theorem, it suffices to show that $f(x) + 2f(y) \ge 3f(1)$ for all real x, y so that $x + 2y = 3$. Let

$$E = f(x) + 2f(y) - 3f(1).$$

We have

$$\begin{aligned}
E &= [f(x) - f(1)] + 2[f(y) - f(1)] \\
&= (3x^4 - 6x^3 + x^2 + 2) + 2(3y^4 - 6y^3 + y^2 + 2) \\
&= (x-1)(3x^3 - 3x^2 - 2x - 2) + 2(y-1)(3y^3 - 3y^2 - 2y - 2) \\
&= (x-1)[(3x^3 - 3x^2 - 2x - 2) - (3y^3 - 3y^2 - 2y - 2)] \\
&= (x-1)[3(x^3 - y^3) - 3(x^2 - y^2) - 2(x-y)] \\
&= (x-1)(x-y)[3(x^2 + xy + y^2) - 3(x+y) - 2] \\
&= \frac{(x-1)^2[27(x^2 + xy + y^2) - 9(x+y)(x+2y) - 2(x+2y)^2]}{6} \\
&= \frac{(x-1)^2(4x-y)^2}{6} \ge 0.
\end{aligned}$$

The equality holds for $a = b = c = 1$, and also for $a = \frac{1}{3}$ and $b = c = \frac{4}{3}$ (or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

- *If $a_1, a_2, \ldots, a_n$ are real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then*

$$(a_1^2 - a_1)^2 + (a_2^2 - a_2)^2 + \cdots + (a_n^2 - a_n)^2 \ge \frac{n-1}{n^2 - 3n + 3}(a_1^2 + a_2^2 + \cdots + a_n^2 - n),$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$*, and also for*

$$a_1 = \frac{1}{n^2 - 3n + 3}, \qquad a_2 = a_3 = \cdots = a_n = 1 + \frac{n-2}{n^2 - 3n + 3}$$

(or any cyclic permutation).

□

P 1.2. *If* $a_1, a_2, \ldots, a_n \geq \dfrac{1-2n}{n-2}$ *so that* $a_1 + a_2 + \cdots + a_n = n$*, then*

$$a_1^3 + a_2^3 + \cdots + a_n^3 \geq n.$$

(Vasile C., 2000)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = u^3, \quad u \geq \frac{1-2n}{n-2}.$$

From $f''(u) = 6u$, it follows that f is convex on $[s, \infty)$. By the RHCF-Theorem and Note 1, it suffices to show that $h(x,y) \geq 0$ for all $x, y \geq \dfrac{1-2n}{n-2}$ so that $x + (n-1)y = n$. We have

$$g(u) = \frac{f(u) - f(1)}{u-1} = u^2 + u + 1,$$

$$h(x,y) = \frac{g(x) - g(y)}{x - y} = x + y + 1 = \frac{(n-2)x + 2n - 1}{n-1} \geq 0.$$

From $x + (n-1)y = n$ and $h(x,y) = 0$, we get

$$x = \frac{1-2n}{n-2}, \qquad y = \frac{n+1}{n-2}.$$

Therefore, according to Note 4, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = \frac{1-2n}{n-2}, \quad a_2 = a_3 = \cdots = a_n = \frac{n+1}{n-2}$$

(or any cyclic permutation).

□

P 1.3. *If* $a_1, a_2, \ldots, a_n \geq \dfrac{-n}{n-2}$ *so that* $a_1 + a_2 + \cdots + a_n = n$*, then*

$$a_1^3 + a_2^3 + \cdots + a_n^3 \geq a_1^2 + a_2^2 + \cdots + a_n^2.$$

Solution. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf(s), \quad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=u^3-u^2, \quad u\ge\frac{-n}{n-2}.$$

From $f''(u)=6u-2$, it follows that f is convex on $[s,\infty)$. According to the RHCF-Theorem and Note 1, it suffices to show that $h(x,y)\ge 0$ for $x,y\ge\dfrac{-n}{n-2}$ so that $x+(n-1)y=n$. We have

$$g(u)=\frac{f(u)-f(1)}{u-1}=u^2,$$

$$h(x,y)=\frac{g(x)-g(y)}{x-y}=x+y=\frac{(n-2)x+n}{n-1}\ge 0.$$

From $x+(n-1)y=n$ and $h(x,y)=0$, we get

$$x=\frac{-n}{n-2}, \quad y=\frac{n}{n-2}.$$

Therefore, in accordance with Note 4, the equality holds for $a_1=a_2=\cdots=a_n=1$, and also for

$$a_1=\frac{-n}{n-2}, \quad a_2=a_3=\cdots=a_n=\frac{n}{n-2}$$

(or any cyclic permutation).

□

P 1.4. *If $a_1,a_2,\ldots,a_n$ are real numbers so that $a_1+a_2+\cdots+a_n=n$, then*

$$(n^2-3n+3)(a_1^4+a_2^4+\cdots+a_n^4-n)\ge 2(n^2-n+1)(a_1^2+a_2^2+\cdots+a_n^2-n).$$

(Vasile C., 2009)

Solution. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf(s), \quad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=(n^2-3n+3)u^4-2(n^2-n+1)u^2, \quad u\in\mathbb{I}=\mathbb{R}.$$

For $u\ge s=1$, we have

$$\begin{aligned}\frac{1}{4}f''(u)&=3(n^2-3n+3)u^2-(n^2-n+1)\\&\ge 3(n^2-3n+3)-(n^2-n+1)=2(n-2)^2\ge 0;\end{aligned}$$

therefore, f is convex on $\mathbb{I}_{\ge s}$. By the RHCF-Theorem and Note 1, it suffices to show that $h(x,y) \ge 0$ for $x,y \in \mathbb{R}$ so that $x+(n-1)y = n$, where

$$h(x,y) = \frac{g(x)-g(y)}{x-y}, \quad g(u) = \frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u) = (n^2-3n+3)(u^3+u^2+u+1) - 2(n^2-n+1)(u+1)$$

and

$$\begin{aligned} h(x,y) &= (n^2-3n+3)(x^2+xy+y^2+x+y+1) - 2(n^2-n+1) \\ &= [(n^2-3n+3)y - n^2+n+1]^2 \ge 0. \end{aligned}$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = -1 + \frac{2}{n^2-3n+3}, \qquad a_2 = a_3 = \cdots = a_n = 1 + \frac{2n-4}{n^2-3n+3}$$

(or any cyclic permutation).

□

P 1.5. *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that $a_1+a_2+\cdots+a_n = n$, then*

$$(n^2+n+1)(a_1^3+a_2^3+\cdots+a_n^3-n) \ge (n+1)(a_1^4+a_2^4+\cdots+a_n^4-n).$$

(Vasile C., 2009)

Solution. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n) \ge nf(s), \quad s = \frac{a_1+a_2+\cdots+a_n}{n} = 1,$$

where

$$f(u) = (n^2+n+1)u^3 - (n+1)u^4, \quad u \in \mathbb{I} = [0,n].$$

The function f is convex on $\mathbb{I}_{\le s}$ because

$$\begin{aligned} f''(u) = 6u[n^2+n+1-2(n+1)u] &\ge 6u[n^2+n+1-2(n+1)] \\ &= 6(n^2-n-1)u \ge 0. \end{aligned}$$

By the LHCF-Theorem and Note 1, it suffices to show that $h(x,y) \ge 0$ for $x,y \ge 0$ so that $x+(n-1)y = n$, where

$$h(x,y) = \frac{g(x)-g(y)}{x-y}, \quad g(u) = \frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u) = (n^2+n+1)(u^2+u+1) - (n+1)(u^3+u^2+u+1)$$
$$= -(n+1)u^3 + n^2(u^2+u+1)$$

and

$$h(x,y) = -(n+1)(x^2+xy+y^2) + n^2(x+y+1)$$
$$= -(n+1)(x^2+xy+y^2) + n(x+y)[x+(n-1)y] + [x+(n-1)y]^2$$
$$= (n^2+n-3)xy + 2n(n-2)y^2 \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = n, \qquad a_2 = a_3 = \cdots = a_n = 0$$

(or any cyclic permutation).

□

P 1.6. *Let a,b,c be real numbers so that $a+b+c=3$. If*

$$-14-6\sqrt{7} \leq k \leq -14+6\sqrt{7},$$

then

$$a^4+b^4+c^4-3 \geq k(a^3+b^3+c^3-3).$$

(Vasile C., 2009)

Solution. Write the desired inequalities as

$$f(a)+f(b)+f(c) \geq 3f(s), \quad s = \frac{a+b+c}{3} = 1,$$

where

$$f(u) = u^4 - ku^3, \quad u \in \mathbb{R}.$$

From

$$f''(u) = 6u(2u^2-k),$$

it follows that $f''(u) > 0$ for $u \geq 1$, hence f is convex on $[s,\infty)$. By the RHCF-Theorem, it suffices to show that $f(x)+2f(y) \geq 3f(1)$ for all real x,y so that $x+2y=3$. Using Note 1, we only need to show that $h(x,y) \geq 0$, where

$$h(x,y) = \frac{g(x)-g(y)}{x-y}, \quad g(u) = \frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u) = u^3+u^2+u+1-k(u^2+u+1)+u+1 = u^3+(1-k)(u^2+u+1),$$

$$h(x,y)=x^2+xy+y^2+(1-k)(x+y+1)=3y^2-(10-k)y+13-4k$$
$$=3\left(y-\frac{10-k}{6}\right)^2+\frac{(6\sqrt{7}+14+k)(6\sqrt{7}-14-k)}{12}\geq 0.$$

The equality holds for $a=b=c=1$. If $k=-14-6\sqrt{7}$, then the equality holds also for

$$a=-5-2\sqrt{7}, \qquad b=c=4+\sqrt{7}$$

(or any cyclic permutation). If $k=-14+6\sqrt{7}$, then the equality holds also for

$$a=-5+2\sqrt{7}, \qquad b=c=4-\sqrt{7}$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

• *Let $a_1,a_2,\ldots,a_n$ be real numbers so that $a_1+a_2+\cdots+a_n=n$. If $k_1\leq k\leq k_2$, where*

$$k_1=\frac{-2(n^2-n+1)-2\sqrt{3(n^2-n+1)(n^2-3n+3)}}{(n-2)^2},$$

$$k_2=\frac{-2(n^2-n+1)+2\sqrt{3(n^2-n+1)(n^2-3n+3)}}{(n-2)^2},$$

then

$$a_1^4+a_2^4+\cdots+a_n^4-n\geq k(a_1^3+a_2^3+\cdots+a_n^3-n).$$

The equality holds for $a_1=a_2=\cdots=a_n=1$. If $k\in\{k_1,k_2\}$, then the equality holds also for

$$a_1=\frac{-2(n^2-3n+1)+(n-1)(n-2)k}{2(n^2-3n+3)},$$

$$a_2=a_3=\cdots=a_n=\frac{2(n^2-n-1)-(n-2)k}{2(n^2-3n+3)}$$

(or any cyclic permutation).

□

P 1.7. *Let $a_1,a_2,\ldots,a_n$ be nonnegative real numbers so that $a_1+a_2+\cdots+a_n=n$. If k is a positive integer satisfying $3\leq k\leq n+1$, then*

$$\frac{a_1^k+a_2^k+\cdots+a_n^k-n}{a_1^2+a_2^2+\cdots+a_n^2-n}\geq (n-1)\left[\left(\frac{n}{n-1}\right)^{k-1}-1\right].$$

(Vasile C., 2012)

Solution. Denote

$$m=(n-1)\left[\left(\frac{n}{n-1}\right)^{k-1}-1\right]=\left(\frac{n}{n-1}\right)^{k-2}+\left(\frac{n}{n-1}\right)^{k-3}+\cdots+1,$$

and write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf(s),\quad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=u^k-mu^2,\quad u\in[0,n].$$

We will show that f is convex on $[1,n]$. Since

$$f''(u)=k(k-1)u^{k-2}-2m\ge k(k-1)-2m,$$

we need to show that

$$\frac{k(k-1)}{2}\ge\left(\frac{n}{n-1}\right)^{k-2}+\left(\frac{n}{n-1}\right)^{k-3}+\cdots+1.$$

Since $n\ge k-1$, this inequality is true if

$$\frac{k(k-1)}{2}\ge\left(\frac{k-1}{k-2}\right)^{k-2}+\left(\frac{k-1}{k-2}\right)^{k-3}+\cdots+1.$$

By Bernoulli's inequality, we have

$$\left(\frac{k-1}{k-2}\right)^{j}=\frac{1}{\left(1-\frac{1}{k-1}\right)^{j}}\le\frac{1}{1-\frac{j}{k-1}}=\frac{k-1}{k-j-1},\qquad j=0,1,\ldots,k-2.$$

Therefore, it suffices to show that

$$\frac{k(k-1)}{2}\ge(k-1)\left(1+\frac{1}{2}+\cdots+\frac{1}{k-1}\right).$$

This is true if

$$\frac{k}{2}\ge1+\frac{1}{2}+\cdots+\frac{1}{k-1},$$

which can be easily proved by induction. According to the RHCF-Theorem and Note 1, we only need to show that $h(x,y)\ge0$ for $x,y\ge0$ so that $x+(n-1)y=n$, where

$$h(x,y)=\frac{g(x)-g(y)}{x-y},\quad g(u)=\frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u)=\frac{(u^k-1)-m(u^2-1)}{u-1}=(u^{k-1}+u^{k-2}+\cdots+1)-m(u+1),$$

$$h(x,y)=\left(\frac{x^{k-1}-y^{k-1}}{x-y}+\frac{x^{k-2}-y^{k-1}}{x-y}+\cdots+1\right)-m$$
$$=\sum_{j=1}^{k-2}\left[\frac{x^{j+1}-y^{j+1}}{x-y}-\left(\frac{n}{n-1}\right)^{j}\right].$$

It suffices to show that $f_j(y)\ge 0$ for $y\in\left[0,\frac{n}{n-1}\right]$ and $j=1,2,\ldots,k-2$, where

$$f_j(y)=x^j+x^{j-1}y+\cdots+xy^{j-1}+y^j-\left(\frac{n}{n-1}\right)^j,\quad x=n-(n-1)y.$$

For $j=1$, we have

$$f_1(y)=x+y-\frac{n}{n-1}=\frac{(n-2)x}{n-1}\ge 0.$$

For $j\ge 2$, from $x'=-(n-1)$ and $n-1\ge k-2\ge j$, we get

$$\begin{aligned}f_j'(y)&=-(n-1)[jx^{j-1}+(j-1)x^{j-2}y+\cdots+y^{j-1}]+x^{j-1}+2x^{j-2}y+\cdots+jy^{j-1}\\&\le -j[jx^{j-1}+(j-1)x^{j-2}y+\cdots+y^{j-1}]+x^{j-1}+2x^{j-2}y+\cdots+jy^{j-1}\\&=-(j\cdot j-1)x^{j-1}-[j\cdot(j-1)-2]x^{j-2}y-\cdots-(j\cdot 2-j+1)xy^{j-2}\le 0.\end{aligned}$$

As a consequence, f_j is decreasing, hence it is minimum for $y=\frac{n}{n-1}$ (when $x=0$):

$$f_j(y)\ge f_j\left(\frac{n}{n-1}\right)=0.$$

From $x+(n-1)y=n$ and $h(x,y)=0$, we get

$$x=0,\qquad y=\frac{n}{n-1}.$$

Therefore, the equality holds for

$$a_1=0,\qquad a_2=a_3=\cdots=a_n=\frac{n}{n-1}$$

(or any cyclic permutation).

Remark. For $k=3$ and $k=4$, we get the following statements (*Vasile C.* , 2002):

- *If $a_1,a_2,\ldots,a_n$ are nonnegative real numbers so that $a_1+a_2+\cdots+a_n=n$, then*

$$(n-1)(a_1^3+a_2^3+\cdots+a_n^3-n)\ge(2n-1)(a_1^2+a_2^2+\cdots+a_n^2-n),$$

which is equivalent to

$$\frac{3}{n-2}\sum_{1\le i<j<k\le n}a_ia_ja_k+n^2\ge\frac{3n-1}{n-1}\sum_{1\le i<j\le n}a_ia_j,$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = 0, \qquad a_2 = a_3 = \cdots = a_n = \frac{n}{n-1}$$

(or any cyclic permutation).

- *If $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ are nonnegative real numbers so that*

$$a_1 + a_2 + \cdots + a_n = n,$$

then

$$(n-1)^2(a_1^4 + a_2^4 + \cdots + a_n^4 - n) \geq (3n^2 - 3n + 1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n),$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = 0, \qquad a_2 = a_3 = \cdots = a_n = \frac{n}{n-1}$$

(or any cyclic permutation).

□

P 1.8. *Let $k \geq 3$ be an integer number. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then*

$$\frac{a_1^k + a_2^k + \cdots + a_n^k - n}{a_1^2 + a_2^2 + \cdots + a_n^2 - n} \leq \frac{n^{k-1} - 1}{n-1}.$$

(Vasile C., 2012)

Solution. Denote

$$m = \frac{n^{k-1} - 1}{n-1} = n^{k-2} + n^{k-3} + \cdots + 1,$$

and write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \qquad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = mu^2 - u^k, \qquad u \in [0, n].$$

We will show that f is convex on $[0, 1]$. Since

$$f''(u) = 2m - k(k-1)u^{k-2} \geq 2m - k(k-1),$$

we need to show that

$$n^{k-2} + n^{k-3} + \cdots + 1 \geq \frac{k(k-1)}{2}.$$

This is true if

$$2^{k-2}+2^{k-3}+\cdots+1\geq\frac{k(k-1)}{2},$$

which is equivalent to

$$2^{k-1}-1\geq\frac{k(k-1)}{2},$$

$$2^k\geq k^2-k+2.$$

Since

$$\begin{aligned}2^k=(1+1)^k&\geq 1+\binom{k}{1}+\binom{k}{2}+\binom{k}{3}\\&=1+k+\frac{k(k-1)}{2}+\frac{k(k-1)(k-2)}{6},\end{aligned}$$

it suffices to show that

$$1+k+\frac{k(k-1)}{2}+\frac{k(k-1)(k-2)}{6}\geq k^2-k+2,$$

which reduces to

$$(k-1)(k-2)(k-3)\geq 0.$$

According to the LHCF-Theorem and Note 1, we only need to show that $h(x,y)\geq 0$ for $x,y\geq 0$ so that $x+(n-1)y=n$, where

$$h(x,y)=\frac{g(x)-g(y)}{x-y},\quad g(u)=\frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u)=\frac{m(u^2-1)-(u^k-1)}{u-1}=m(u+1)-(u^{k-1}+u^{k-2}+\cdots+1)$$

and

$$\begin{aligned}h(x,y)&=m-\frac{x^{k-1}-y^{k-1}}{x-y}-\frac{x^{k-2}-y^{k-1}}{x-y}-\cdots-1\\&=\left(n^{k-2}-\frac{x^{k-1}-y^{k-1}}{x-y}\right)+\left(n^{k-3}-\frac{x^{k-2}-y^{k-2}}{x-y}\right)+\cdots+\left(n-\frac{x^2-y^2}{x-y}\right).\end{aligned}$$

It suffices to show that

$$n^j\geq\frac{x^{j+1}-y^{j+1}}{x-y},\qquad j=1,2,\ldots,k-2.$$

We will show that

$$n^j\geq(x+y)^j\geq\frac{x^{j+1}-y^{j+1}}{x-y}.$$

The left inequality is true since

$$n-(x+y)=x+(n-1)y-(x+y)=(n-2)y\geq 0.$$

The right inequality is also true since

$$(x+y)^j=x^j+\binom{j}{1}x^{j-1}y+\cdots+\binom{j}{j-1}xy^{j-1}+y^j$$

and

$$\frac{x^{j+1}-y^{j+1}}{x-y}=x^j+x^{j-1}y+\cdots+xy^{j-1}+y^j.$$

The equality holds for $a_1=n$ and $a_2=a_3=\cdots=a_n=0$ (or any cyclic permutation).

Remark. For $k=3$ and $k=4$, we get the following statements (*Vasile C.* , 2002):

- *If $a_1,a_2,\ldots,a_n$ are nonnegative real numbers so that $a_1+a_2+\cdots+a_n=n$, then*

$$a_1^3+a_2^3+\cdots+a_n^3-n\leq(n+1)(a_1^2+a_2^2+\cdots+a_n^2-n),$$

with equality for $a_1=a_2=\cdots=a_n=1$, and also for

$$a_1=n,\qquad a_2=a_3=\cdots=a_n=0$$

(or any cyclic permutation).

- *If $a_1,a_2,\ldots,a_n$ are nonnegative real numbers so that $a_1+a_2+\cdots+a_n=n$, then*

$$a_1^4+a_2^4+\cdots+a_n^4-n\leq(n^2+n+1)(a_1^2+a_2^2+\cdots+a_n^2-n),$$

with equality for $a_1=a_2=\cdots=a_n=1$, and also for

$$a_1=n,\qquad a_2=a_3=\cdots=a_n=0$$

(or any cyclic permutation).

□

P 1.9. *If $a_1,a_2,\ldots,a_n$ are positive real numbers so that $a_1+a_2+\cdots+a_n=n$, then*

$$n^2\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}-n\right)\geq 4(n-1)(a_1^2+a_2^2+\cdots+a_n^2-n).$$

(Vasile C., 2004)

Solution. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf(s), \quad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=\frac{n^2}{u}-4(n-1)u^2, \quad u\in\mathbb{I}=(0,n).$$

For $u\in(0,1]$, we have

$$f''(u)=\frac{2n^2}{u^3}-8(n-1)\ge 2n^2-8(n-1)=2(n-2)^2\ge 0.$$

Thus, f is convex on $\mathbb{I}_{\le s}$. By the LHCF-Theorem and Note 1, it suffices to show that $h(x,y)\ge 0$ for $x,y>0$ so that $x+(n-1)y=n$, where

$$h(x,y)=\frac{g(x)-g(y)}{x-y}, \quad g(u)=\frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u)=\frac{-n^2}{u}-4(n-1)(u+1)$$

and

$$h(x,y)=\frac{n^2}{xy}-4(n-1)=\frac{[x+(n-1)y]^2}{xy}-4(n-1=\frac{[x-(n-1)y]^2}{xy}.$$

In accordance with Note 4, the equality holds for $a_1=a_2=\cdots=a_n=1$, and also for

$$a_1=\frac{n}{2}, \qquad a_2=a_3=\cdots=a_n=\frac{n}{2n-2}$$

(or any cyclic permutation).

□

P 1.10. *If $a_1,a_2,\ldots,a_8$ are positive real numbers so that $a_1+a_2+\cdots+a_8=8$, then*

$$\frac{1}{a_1^2}+\frac{1}{a_2^2}+\cdots+\frac{1}{a_8^2}\ge a_1^2+a_2^2+\cdots+a_8^2.$$

(Vasile C., 2007)

Solution. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_8)\ge 8f(s), \quad s=\frac{a_1+a_2+\cdots+a_8}{8}=1,$$

where

$$f(u)=\frac{1}{u^2}-u^2, \quad u\in(0,8).$$

For $u \in (0,1]$, we have

$$f''(u) = \frac{6}{u^4} - 2 \geq 6 - 2 > 0.$$

Thus, f is convex on $(0,s]$. By the LHCF-Theorem and Note 1, it suffices to show that $h(x,y) \geq 0$ for $x,y > 0$ so that $x + 7y = 8$, where

$$h(x,y) = \frac{g(x)-g(y)}{x-y}, \quad g(u) = \frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u) = -u - 1 - \frac{1}{u} - \frac{1}{u^2}$$

and

$$h(x,y) = -1 + \frac{1}{xy} + \frac{x+y}{x^2y^2}.$$

From $8 = x + 7y \geq 2\sqrt{7xy}$, we get $xy \leq 16/7$. Therefore,

$$\begin{aligned} h(x,y) &\geq -1 + \frac{1}{xy} + \frac{7(x+y)}{16xy} = \frac{112y^2 - 170y + 72}{16xy} \\ &> \frac{112y^2 - 176y + 72}{16xy} = \frac{14y^2 - 22y + 9}{2xy} > 0. \end{aligned}$$

The equality holds for $a_1 = a_2 = \cdots = a_8 = 1$.

Remark. In the same manner, we can prove the following generalization:

- *If $a_1, a_2, \ldots, a_n$ ($n \geq 4$) are positive real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then*

$$\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2} + 8 - n \geq \frac{8}{n}\left(a_1^2 + a_2^2 + \cdots + a_n^2\right).$$

□

P 1.11. *If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = n$, then*

$$a_1^2 + a_2^2 + \cdots + a_n^2 - n \geq 2\left(1 + \frac{\sqrt{n-1}}{n}\right)(a_1 + a_2 + \cdots + a_n - n).$$

(Vasile C., 2006)

Solution. Replacing each a_i by $1/a_i$, we need to prove that

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where
$$f(u)=\frac{1}{u^2}-\frac{2k}{u}, \quad k=1+\frac{\sqrt{n-1}}{n}, \quad u\in(0,n).$$
For $u\in(0,1]$, we have
$$f''(u)=\frac{6-4ku}{u^4}\geq\frac{6-4k}{u^4}=\frac{2(\sqrt{n-1}-1)^2}{nu^4}\geq 0.$$
Thus, f is convex on $(0,s]$. By the LHCF-Theorem and Note 1, it suffices to show that $h(x,y)\geq 0$ for $x,y>0$ so that $x+(n-1)y=n$, where
$$h(x,y)=\frac{g(x)-g(y)}{x-y}, \quad g(u)=\frac{f(u)-f(1)}{u-1}.$$
We have
$$g(u)=\frac{-1}{u^2}+\frac{2k-1}{u}$$
and
$$h(x,y)=\frac{1}{xy}\left(\frac{1}{x}+\frac{1}{y}+1-2k\right).$$
We only need to show that
$$\frac{1}{x}+\frac{1}{y}\geq 2k-1.$$
Indeed, using the Cauchy-Schwarz inequality, we get
$$\frac{1}{x}+\frac{1}{y}\geq\frac{(1+\sqrt{n-1})^2}{x+(n-1)y}=\frac{(1+\sqrt{n-1})^2}{n}=2k-1,$$
with equality for $x=\sqrt{n-1}y$. From $x+(n-1)y=n$ and $h(x,y)=0$, we get
$$x=\frac{n}{1+\sqrt{n-1}}, \qquad y=\frac{n}{n-1+\sqrt{n-1}}.$$
In accordance with Note 4, the original equality holds for $a_1=a_2=\cdots=a_n=1$, and also for
$$a_1=\frac{1+\sqrt{n-1}}{n}, \qquad a_2=a_3=\cdots=a_n=\frac{n-1+\sqrt{n-1}}{n}$$
(or any cyclic permutation).

□

P 1.12. *If a,b,c,d,e are positive real numbers so that $a^2+b^2+c^2+d^2+e^2=5$, then*
$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}-5+\frac{4(1+\sqrt{5})}{5}(a+b+c+d+e-5)\geq 0.$$

(Vasile C., 2006)

Solution. Replacing a,b,c,d,e by $\sqrt{a},\sqrt{b},\sqrt{c},\sqrt{d},\sqrt{e}$, respectively, we need to prove that

$$f(a)+f(b)+f(c)+f(d)+f(e)\ge 5f(s),\qquad s=\frac{a+b+c+d+e}{5}=1,$$

where

$$f(u)=\frac{1}{\sqrt{u}}+k\sqrt{u},\qquad k=\frac{4(1+\sqrt{5})}{5}\approx 2.59,\qquad u\in(0,5).$$

For $u\in(0,1]$, we have

$$f''(u)=\frac{3-ku}{4u^2\sqrt{u}}>0;$$

therefore, f is convex on $(0,s]$. By the LHCF-Theorem and Note 1, it suffices to show that $h(x,y)\ge 0$ for $x,y>0$ so that $x+4y=5$. We have

$$g(u)=\frac{f(u)-f(1)}{u-1}=\frac{k\sqrt{u}-1}{u+\sqrt{u}}$$

and

$$h(x,y)=\frac{g(x)-g(y)}{x-y}=\frac{\sqrt{x}+\sqrt{y}+1-k\sqrt{xy}}{\sqrt{xy}(\sqrt{x}+\sqrt{y})(\sqrt{x}+1)(\sqrt{y}+1)}.$$

Thus, we only need to show that

$$\sqrt{x}+\sqrt{y}+1-k\sqrt{xy}\ge 0,$$

which is true if

$$2\sqrt[4]{xy}+1-k\sqrt{xy}\ge 0.$$

Let

$$t=\sqrt[4]{xy}.$$

From

$$5=x+4y\ge 4\sqrt{xy}=4t^2,$$

we get

$$t\le\frac{\sqrt{5}}{2}.$$

Thus,

$$\begin{aligned}2\sqrt[4]{xy}+1-k\sqrt{xy}&=2t+1-kt^2\\&=\left(1-\frac{2}{\sqrt{5}}t\right)\left[1+2\left(1+\frac{1}{\sqrt{5}}\right)t\right]\ge 0.\end{aligned}$$

The equality holds for $a=b=c=d=e=1$.

□

P 1.13. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

$$\frac{1}{3a+b+c}+\frac{1}{3b+c+a}+\frac{1}{3c+a+b} \le \frac{2}{5}\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right).$$

(Vasile C., 2006)

Solution. Due to homogeneity, we may assume that $a+b+c=3$. So, we need to show that

$$f(a)+f(b)+f(c) \ge 3f(s), \quad s=\frac{a+b+c}{3}=1,$$

where

$$f(u)=\frac{2}{3-u}-\frac{5}{2u+3}, \quad u \in [0,3).$$

For $u \in [1,3)$, we have

$$f''(u)=\frac{4}{(3-u)^3}-\frac{40}{(2u+3)^3}=\frac{36[2u^3+3u^2+9(u-1)(3-u)]}{(3-u)^3(2u+3)^3}>0;$$

therefore, f is convex on $[s,3)$. By the RHCF-Theorem and Note 1, it suffices to show that $h(x,y) \ge 0$ for $x, y \ge 0$ so that $x+2y=3$, where

$$h(x,y)=\frac{g(x)-g(y)}{x-y}, \quad g(u)=\frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u)=\frac{1}{3-u}+\frac{2}{2u+3}$$

and

$$\begin{aligned} h(x,y) &= \frac{1}{(3-x)(3-y)}-\frac{4}{(2x+3)(2y+3)} \\ &= \frac{9(2x+2y-3)}{(3-x)(3-y)(2x+3)(2y+3)} \\ &= \frac{9x}{(3-x)(3-y)(2x+3)(2y+3)} \ge 0. \end{aligned}$$

The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.14. *If $a, b, c, d \ge 3-\sqrt{7}$ so that $a+b+c+d=4$, then*

$$\frac{1}{2+a^2}+\frac{1}{2+b^2}+\frac{1}{2+c^2}+\frac{1}{2+d^2} \ge \frac{4}{3}.$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d) \geq 4f(s), \qquad s=\frac{a+b+c+d}{4}=1,$$

where

$$f(u)=\frac{1}{2+u^2}, \quad u \geq 3-\sqrt{7}.$$

For $u \geq s = 1$, $f(u)$ is convex because

$$f''(u)=\frac{3(3u^2-2)}{(2+u^2)^3}>0.$$

By the RHCF-Theorem and Note 1, it suffices to show that $h(x,y) \geq 0$ for $x,y \geq 3-\sqrt{7}$ so that $x+3y=4$. We have

$$g(u)=\frac{f(u)-f(1)}{u-1}=\frac{-1-u}{3(2+u^2)}$$

and

$$h(x,y)=\frac{g(x)-g(y)}{x-y}=\frac{xy+x+y-2}{3(2+x^2)(2+y^2)},$$

where

$$\begin{aligned} xy+x+y-2 &= \frac{-x^2+6x-2}{3}=\frac{(3+\sqrt{7}-x)(x-3+\sqrt{7})}{3} \\ &= \frac{(-1+\sqrt{7}+3y)(x-3+\sqrt{7})}{3} \geq 0. \end{aligned}$$

In accordance with Note 4, the equality holds for $a=b=c=d=1$, and also for

$$a=3-\sqrt{7}, \qquad b=c=d=\frac{1+\sqrt{7}}{3}$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- *If $a_1,a_2,\ldots,a_n \geq n-1-\sqrt{n^2-3n+3}$ so that $a_1+a_2+\cdots+a_n=n$, then*

$$\frac{1}{2+a_1^2}+\frac{1}{2+a_2^2}+\cdots+\frac{1}{2+a_n^2} \geq \frac{n}{3},$$

with equality for $a_1=a_2=\cdots=a_n=1$, and also for

$$a_1=n-1-\sqrt{n^2-3n+3}, \qquad a_2=a_3=\cdots=a_n=\frac{1+\sqrt{n^2-3n+3}}{n-1}$$

(or any cyclic permutation).

□

P 1.15. *If $a_1, a_2, \dots, a_n \in [-\sqrt{n}, n-2]$ so that $a_1 + a_2 + \cdots + a_n = n$, then*

$$\frac{1}{n+a_1^2} + \frac{1}{n+a_2^2} + \cdots + \frac{1}{n+a_n^2} \le \frac{n}{n+1}.$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \qquad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{-1}{n+u^2}, \quad n \ge 3, \quad u \in [-\sqrt{n}, n-2].$$

For $u \in [-\sqrt{n}, 1]$, we have

$$f''(u) = \frac{2(n-u^2)}{(n+u^2)^3} \ge 0,$$

hence f is convex on $[-\sqrt{n}, s]$. By the LHCF-Theorem and Note 1, it suffices to show that $h(x,y) \ge 0$ for $x, y \in [-\sqrt{n}, n-2]$ so that $x + (n-1)y = n$. We have

$$g(u) = \frac{f(u) - f(1)}{u-1} = \frac{u+1}{(n+1)(n+u^2)}$$

and

$$\begin{aligned} h(x,y) = \frac{g(x) - g(y)}{x-y} &= \frac{n - x - y - xy}{(n+1)(n+x^2)(n+y^2)} \\ &= \frac{(n-x)(n-2-x)}{(n^2-1)(n+x^2)(n+y^2)} \ge 0. \end{aligned}$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = n-2, \qquad a_2 = a_3 = \cdots = a_n = \frac{2}{n-1}$$

(or any cyclic permutation).

□

P 1.16. *If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then*

$$\frac{3-a}{9+a^2} + \frac{3-b}{9+b^2} + \frac{3-c}{9+c^2} \ge \frac{3}{5}.$$

(Vasile C., 2013)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)\ge 3f(s), \qquad s=\frac{a+b+c}{3}=1,$$

where

$$f(u)=\frac{3-u}{9+u^2}, \qquad u\in[0,3].$$

For $u\in[1,3]$, we have

$$\frac{1}{2}f''(u)=\frac{u^2(9-u)+27(u-1)}{(9+u^2)^3}>0.$$

Thus, f is convex on $[s,3]$. By the RHCF-Theorem and Note 1, it suffices to show that $h(x,y)\ge 0$ for $x,y\ge 0$ so that $x+2y=3$, where

$$h(x,y)=\frac{g(x)-g(y)}{x-y}, \quad g(u)=\frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u)=\frac{-(6+u)}{5(9+u^2)}$$

and

$$h(x,y)=\frac{xy+6x+6y-9}{5(9+x^2)(9+y^2)}=\frac{x(9-x)}{10(9+x^2)(9+y^2)}\ge 0.$$

The equality holds for $a=b=c=1$, and also for $a=0$ and $b=c=\frac{3}{2}$ (or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

- *If $a_1,a_2,\ldots,a_n$ are nonnegative real numbers so that $a_1+a_2+\cdots+a_n=n$, then*

$$\frac{n-a_1}{n^2+(n^2-3n+1)a_1^2}+\frac{n-a_2}{n^2+(n^2-3n+1)a_2^2}+\cdots+\frac{n-a_n}{n^2+(n^2-3n+1)a_n^2}\ge\frac{n}{2n-1},$$

with equality for $a_1=a_2=\cdots=a_n=1$, and also for

$$a_1=0, \qquad a_2=a_3=\cdots=a_n=\frac{n}{n-1}$$

(or any cyclic permutation).

□

P 1.17. *If a,b,c are nonnegative real numbers so that $a+b+c=3$, then*

$$\frac{1}{1-a+2a^2}+\frac{1}{1-b+2b^2}+\frac{1}{1-c+2c^2}\ge\frac{3}{2}.$$

(Vasile C., 2012)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)\geq 3f(s), \qquad s=\frac{a+b+c}{3}=1,$$

where

$$f(u)=\frac{1}{1-u+2u^2}, \qquad u\in[0,3].$$

For $u\in[1,3]$, we have

$$\frac{1}{2}f''(u)=\frac{12u^2-6u-1}{(1-u+2u^2)^3}>0.$$

Thus, f is convex on $[s,3]$. By the RHCF-Theorem and Note 1, it suffices to show that $h(x,y)\geq 0$ for $x,y\geq 0$ so that $x+2y=3$, where

$$h(x,y)=\frac{g(x)-g(y)}{x-y}, \quad g(u)=\frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u)=\frac{-(1+2u)}{2(1-u+2u^2)}$$

and

$$h(x,y)=\frac{4xy+2x+2y-3}{2(1-x+2x^2)(1-y+2y^2)}=\frac{x(1+4y)}{2(1-x+2x^2)(1-y+2y^2)}\geq 0.$$

The equality holds for $a=b=c=1$, and also for $a=0$ and $b=c=\dfrac{3}{2}$ (or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

- *Let $a_1,a_2,\ldots,a_n$ be nonnegative real numbers so that $a_1+a_2+\cdots+a_n=n$. If*

$$k\geq k_1, \qquad k_1=\frac{3n-2+\sqrt{5n^2-8n+4}}{2n},$$

then

$$\frac{1}{1-a_1+ka_1^2}+\frac{1}{1-a_2+ka_2^2}+\cdots+\frac{1}{1-a_n+ka_n^2}\geq\frac{n}{k},$$

with equality for $a_1=a_2=\cdots=a_n=1$. If $k=k_1$, then the equality holds also for

$$a_1=0, \qquad a_2=a_3=\cdots=a_n=\frac{n}{n-1}$$

(or any cyclic permutation).

□

P 1.18. *If a, b, c are nonnegative real numbers so that $a+b+c=3$, then*

$$\frac{1}{5+a+a^2}+\frac{1}{5+b+b^2}+\frac{1}{5+c+c^2} \ge \frac{3}{7}.$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a)+f(b)+f(c) \ge 3f(s), \qquad s=\frac{a+b+c}{3}=1,$$

where

$$f(u)=\frac{1}{5+u+u^2}, \qquad u \in [0,3].$$

For $u \ge 1$, from

$$f''(u)=\frac{2(3u^2+3u-4)}{(5+u+u^2)^3}>0,$$

it follows that f is convex on [s,3]. By the RHCF-Theorem and Note 1, it suffices to show that $h(x,y) \ge 0$ for $x, y \ge 0$ so that $x+2y=3$. We have

$$g(u)=\frac{f(u)-f(1)}{u-1}=\frac{-2-u}{7(5+u+u^2)}$$

and

$$\begin{aligned} h(x,y)=\frac{g(x)-g(y)}{x-y} &= \frac{xy+2(x+y)-3}{7(5+x+x^2)(5+y+y^2)} \\ &= \frac{x(5-x)}{14(5+x+x^2)(5+y+y^2)} \ge 0. \end{aligned}$$

According to Note 4, the equality holds for $a=b=c=1$, and also for $a=0$ and $b=c=\frac{3}{2}$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1+a_2+\cdots+a_n=n$. If*

$$0<k \le k_1, \qquad k_1=\frac{2(2n-1)}{n-1},$$

then

$$\frac{1}{k+a_1+a_1^2}+\frac{1}{k+a_2+a_2^2}+\cdots+\frac{1}{k+a_n+a_n^2} \ge \frac{n}{k+2},$$

with equality for $a_1=a_2=\cdots=a_n=1$. If $k=k_1$, then the equality holds also for

$$a_1=0, \qquad a_2=a_3=\cdots=a_n=\frac{n}{n-1}$$

(or any cyclic permutation).

□

P 1.19. *If a,b,c,d are nonnegative real numbers so that $a+b+c+d=4$, then*

$$\frac{1}{10+a+a^2}+\frac{1}{10+b+b^2}+\frac{1}{10+c+c^2}+\frac{1}{10+d+d^2}\le\frac{1}{3}.$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d)\ge 4f(s),\qquad s=\frac{a+b+c+d}{4}=1,$$

where

$$f(u)=\frac{-1}{10+u+u^2},\quad u\in[0,4].$$

For $u\in[0,1]$, we have

$$f''(u)=\frac{6(3-u-u^2)}{(10+u+u^2)^3}>0.$$

Thus, f is convex on [0,s]. By the LHCF-Theorem and Note 1, it suffices to show that $h(x,y)\ge 0$ for $x,y\ge 0$ so that $x+3y=4$. We have

$$g(u)=\frac{f(u)-f(1)}{u-1}=\frac{2+u}{12(10+u+u^2)}$$

and

$$h(x,y)=\frac{g(x)-g(y)}{x-y}=\frac{8-2(x+y)-xy}{12(10+x+x^2)(10+y+y^2)}$$
$$=\frac{3y^2}{12(10+x+x^2)(10+y+y^2)}\ge 0.$$

The equality holds for $a=b=c=d=1$, and also for $a=4$ and $b=c=d=0$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- *Let $a_1,a_2,\dots,a_n$ ($n\ge 4$) be nonnegative real numbers so that*

$$a_1+a_2+\cdots+a_n=n.$$

If $k\ge 2n+2$, then

$$\frac{1}{k+a_1+a_1^2}+\frac{1}{k+a_2+a_2^2}+\cdots+\frac{1}{k+a_n+a_n^2}\le\frac{n}{k+2},$$

with equality for $a_1=a_2=\cdots=a_n=1$. If $k=2n+2$, then the equality holds also for

$$a_1=n,\qquad a_2=a_3=\cdots=a_n=0$$

(or any cyclic permutation).

□

P 1.20. *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If*

$$k \geq 1 - \frac{1}{n},$$

then

$$\frac{1}{1+ka_1^2} + \frac{1}{1+ka_2^2} + \cdots + \frac{1}{1+ka_n^2} \geq \frac{n}{1+k}.$$

(Vasile C., 2005)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \qquad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{1}{1+ku^2}, \qquad u \in [0, n].$$

For $u \in [1, n]$, we have

$$f''(u) = \frac{2k(3ku^2 - 1)}{(1+ku^2)^3} \geq \frac{2k(3k-1)}{(1+ku^2)^3} > 0.$$

Thus, f is convex on $[s, n]$. By the RHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y \geq 0$ so that $x + (n-1)y = n$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-k(u+1)}{(1+k)(1+ku^2)}$$

and

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{k^2(x + y + xy) - k}{(1+k)(1+kx^2)(1+ky^2)}.$$

We need to show that

$$k(x + y + xy) - 1 \geq 0.$$

Indeed, we have

$$k(x + y + xy) - 1 \geq \left(1 - \frac{1}{n}\right)(x + y + xy) - 1 = \frac{x(2n - 2 - x)}{n} \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $k = 1 - \frac{1}{n}$, then the equality holds also for

$$a_1 = 0, \qquad a_2 = a_3 = \cdots = a_n = \frac{n}{n-1}$$

(or any cyclic permutation).

□

P 1.21. *Let $a_1, a_2, \ldots, a_n$ be real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If*

$$0 < k \le \frac{n-1}{n^2-n+1},$$

then

$$\frac{1}{1+ka_1^2} + \frac{1}{1+ka_2^2} + \cdots + \frac{1}{1+ka_n^2} \le \frac{n}{1+k}.$$

(Vasile C., 2005)

Solution. Replacing all negative numbers a_i by $-a_i$, we need to show the same inequality for

$$a_1, a_2, \ldots, a_n \ge 0, \qquad a_1 + a_2 + \cdots + a_n \ge n.$$

Since the left side of the desired inequality is decreasing with respect to each a_i, is sufficient to consider that $a_1 + a_2 + \cdots + a_n = n$. Write this inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \qquad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{-1}{1+ku^2}, \qquad u \in [0, n].$$

For $u \in [0, 1]$, we have

$$f''(u) = \frac{2k(1-3ku^2)}{(1+ku^2)^3} \ge 0,$$

since

$$1 - 3ku^2 \ge 1 - 3k \ge 1 - \frac{3(n-1)}{n^2-n+1} = \frac{(n-2)^2}{n^2-n+1} \ge 0.$$

Thus, f is convex on $[0, s]$. By the LHCF-Theorem and Note 1, it suffices to show that $h(x, y) \ge 0$ for $x, y \ge 0$ so that $x + (n-1)y = n$. We have

$$g(u) = \frac{f(u) - f(1)}{u-1} = \frac{k(u+1)}{(1+k)(1+ku^2)}$$

and

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{k - k^2(x + y + xy)}{(1+k)(1+kx^2)(1+ky^2)}.$$

It suffices to show that

$$1 - k(x + y + xy) \ge 0.$$

Indeed, we have

$$1 - k(x + y + xy) \ge 1 - \frac{n-1}{n^2-n+1}(x + y + xy) = \frac{(x-n+1)^2}{n^2-n+1} \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $k = \dfrac{n-1}{n^2-n+1}$, then the equality holds also for

$$a_1 = n-1, \qquad a_2 = a_3 = \cdots = a_n = \frac{1}{n-1}$$

(or any cyclic permutation).

□

P 1.22. *Let $a_1, a_2, \ldots, a_n$ be nonnegative numbers so that $a_1 + a_2 + \cdots + a_n = n$. If $k \ge \dfrac{n^2}{4(n-1)}$, then*

$$\frac{a_1(a_1-1)}{a_1^2+k} + \frac{a_2(a_2-1)}{a_2^2+k} + \cdots + \frac{a_n(a_n-1)}{a_n^2+k} \ge 0.$$

(Vasile C., 2012)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \qquad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{u(u-1)}{u^2+k}, \quad u \in [0, n].$$

From

$$f'(u) = \frac{u^2+2ku-k}{(u^2+k)^2}, \quad f''(u) = \frac{2(k^2-u^3)+6ku(1-u)}{(u^2+k)^3},$$

it follows that f is convex on $[0,1]$. By the LHCF-Theorem and Note 1, it suffices to show that $h(x,y) \ge 0$ for $x, y \ge 0$ so that $x + (n-1)y = n$, where

$$h(x,y) = \frac{g(x)-g(y)}{x-y}, \quad g(u) = \frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u) = \frac{u}{u^2+k}$$

and

$$\begin{aligned} h(x,y) &= \frac{k-xy}{(x^2+k)(y^2+k)} \ge \frac{n^2-4(n-1)xy}{4(n-1)(x^2+k)(y^2+k)} \\ &= \frac{[x+(n-1)y]^2-4(n-1)xy}{4(n-1)(x^2+k)(y^2+k)} = \frac{[x-(n-1)y]^2}{4(n-1)(x^2+k)(y^2+k)} \ge 0. \end{aligned}$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = n/2, \qquad a_2 = a_3 = \cdots = a_n = n/(2n-2)$$

(or any cyclic permutation).

□

P 1.23. *If $a_1,a_2,\ldots,a_n$ are nonnegative real numbers so that $a_1+a_2+\cdots+a_n=n$, then*

$$\frac{a_1-1}{(n-2a_1)^2}+\frac{a_2-1}{(n-2a_2)^2}+\cdots+\frac{a_n-1}{(n-2a_n)^2}\ge 0.$$

(Vasile C., 2012)

Solution. For $n=2$, the inequality is an identity. Consider further $n\ge 3$ and write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf(s), \qquad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=\frac{u-1}{(n-2u)^2}, \qquad u\in\mathbb{I}=[0,n]\setminus\{n/2\}.$$

From

$$f'(u)=\frac{2u+n-4}{(n-2u)^3}, \qquad f''(u)=\frac{8(u+n-3)}{(n-2u)^4},$$

it follows that f is convex on $\mathbb{I}_{\le s}$. By the LHCF-Theorem, Note 1 and Note 3, it suffices to show that $h(x,y)\ge 0$ for $x,y\in\mathbb{I}$ so that $x+(n-1)y=n$. We have

$$g(u)=\frac{f(u)-f(1)}{u-1}=\frac{1}{(n-2u)^2}$$

and

$$h(x,y)=\frac{g(x)-g(y)}{x-y}=\frac{4(n-x-y)}{(n-2x)^2(n-2y)^2}=\frac{4(n-2)y}{(n-2x)^2(n-2y)^2}\ge 0.$$

In accordance with Note 4, the equality holds for $a_1=a_2=\cdots=a_n=1$, and also for

$$a_1=n, \qquad a_2=a_3=\cdots=a_n=0$$

(or any cyclic permutation).

□

P 1.24. *If $a_1,a_2,\ldots,a_n$ are nonnegative real numbers so that*

$$a_1+a_2+\cdots+a_n=n, \qquad a_1,a_2,\ldots,a_n>-k, \qquad k\ge 1+\frac{n}{\sqrt{n-1}},$$

then

$$\frac{a_1^2-1}{(a_1+k)^2}+\frac{a_2^2-1}{(a_2+k)^2}+\cdots+\frac{a_n^2-1}{(a_n+k)^2}\ge 0.$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf(s),\qquad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=\frac{u^2-1}{(u+k)^2},\qquad u>-k.$$

For $u\in(-k,1]$, we have

$$f''(u)=\frac{2(k^2-3-2ku)}{(u+k)^4}\ge\frac{2(k^2-2k-3)}{(u+k)^4}=\frac{2(k+1)(k-3)}{(u+k)^4}\ge 0.$$

Thus, f is convex on $(-k,s]$. By the LHCF-Theorem and Note 1, it suffices to show that $h(x,y)\ge 0$ for $x,y>-k$ so that $x+(n-1)y=n$. We have

$$g(u)=\frac{f(u)-f(1)}{u-1}=\frac{u+1}{(u+k)^2}$$

and

$$h(x,y)=\frac{g(x)-g(y)}{x-y}=\frac{(k-1)^2-(1+x)(1+y)}{(x+k)^2(y+k)^2}.$$

Since

$$(k-1)^2\ge\frac{n^2}{n-1},$$

we need to show that

$$n^2\ge(n-1)(1+x)(1+y).$$

Indeed,

$$n^2-(n-1)(1+x)(1+y)=n^2-(1+x)(2n-1-x)=(x-n+1)^2\ge 0.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$. If $k=1+\dfrac{n}{\sqrt{n-1}}$, then the equality holds also for

$$a_1=n-1,\qquad a_2=a_3=\cdots=a_n=\frac{1}{n-1}$$

(or any cyclic permutation).

□

P 1.25. *Let $a_1,a_2,\ldots,a_n$ be nonnegative real numbers so that $a_1+a_2+\cdots+a_n=n$. If $0<k\le 1+\sqrt{\dfrac{2n-1}{n-1}}$, then*

$$\frac{a_1^2-1}{(a_1+k)^2}+\frac{a_2^2-1}{(a_2+k)^2}+\cdots+\frac{a_n^2-1}{(a_n+k)^2}\le 0.$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf(s), \qquad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=\frac{1-u^2}{(u+k)^2}, \qquad u\in[0,n].$$

For $u\ge 1$, we have

$$f''(u)=\frac{2(2ku-k^2+3)}{(u+k)^4}\ge\frac{2(2k-k^2+3)}{(u+k)^4}=\frac{2(1+k)(3-k)}{(u+k)^4}>0.$$

Thus, f is convex on $[s,n]$. By the RHCF-Theorem and Note 1, it suffices to show that $h(x,y)\ge 0$ for $x,y\ge 0$ so that $x+(n-1)y=n$. We have

$$g(u)=\frac{f(u)-f(1)}{u-1}=\frac{-u-1}{(u+k)^2}$$

and

$$h(x,y)=\frac{g(x)-g(y)}{x-y}=\frac{2k-k^2+x+y+xy}{(x+k)^2(y+k)^2}\ge\frac{2k-k^2+x+y}{(x+k)^2(y+k)^2}.$$

Since

$$x+y\ge\frac{x+(n-1)y}{n-1}=\frac{n}{n-1},$$

we get

$$2k-k^2+x+y\ge 2k-k^2+\frac{n}{n-1}=-(k-1)^2+\frac{2n-1}{n-1}\ge 0,$$

hence $h(x,y)\ge 0$.

The equality holds for $a_1=a_2=\cdots=a_n=1$. If $k=1+\sqrt{\dfrac{2n-1}{n-1}}$, then the equality holds also for

$$a_1=0, \qquad a_2=a_3=\cdots=a_n=\frac{n}{n-1}$$

(or any cyclic permutation).

□

P 1.26. *If $a_1,a_2,\dots,a_n\ge n-1-\sqrt{n^2-n+1}$ so that $a_1+a_2+\cdots+a_n=n$, then*

$$\frac{a_1^2-1}{(a_1+2)^2}+\frac{a_2^2-1}{(a_2+2)^2}+\cdots+\frac{a_n^2-1}{(a_n+2)^2}\le 0.$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf(s), \quad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=\frac{1-u^2}{(u+2)^2}, \quad u\ge n-1-\sqrt{n^2-n+1}.$$

For $u\ge 1$, we have

$$f''(u)=\frac{2(4u-1)}{(u+2)^4}>0.$$

Thus, $f(u)$ is convex for $u\ge s$. By the RHCF-Theorem and Note 1, it suffices to show that $h(x,y)\ge 0$ for

$$n-1-\sqrt{n^2-n+1}\le x\le 1\le y, \qquad x+(n-1)y=n.$$

Since

$$g(u)=\frac{f(u)-f(1)}{u-1}=\frac{-u-1}{(u+2)^2},$$

$$h(x,y)=\frac{g(x)-g(y)}{x-y}=\frac{x+y+xy}{(x+2)^2(y+2)^2}=\frac{-x^2+2(n-1)x+n}{(n-1)(x+2)^2(y+2)^2},$$

we need to show that

$$n-1-\sqrt{n^2-n+1}\le x\le n-1+\sqrt{n^2-n+1}.$$

This is true because

$$n-1-\sqrt{n^2-n+1}\le x\le 1<n-1+\sqrt{n^2-n+1}.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$, and also for

$$a_1=n-1-\sqrt{n^2-n+1}, \qquad a_2=a_3=\cdots=a_n=\frac{1+\sqrt{n^2-n+1}}{n-1}$$

(or any cyclic permutation).

□

P 1.27. *Let $a_1,a_2,\ldots,a_n$ be nonnegative real numbers so that $a_1+a_2+\cdots+a_n=n$. If $k\ge\dfrac{(n-1)(2n-1)}{n^2}$, then*

$$\frac{1}{1+ka_1^3}+\frac{1}{1+ka_2^3}+\cdots+\frac{1}{1+ka_n^3}\ge\frac{n}{1+k}.$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\geq nf(s), \qquad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=\frac{1}{1+ku^3}, \quad u\in[0,n].$$

For $u\in[1,n]$, we have

$$f''(u)=\frac{6ku(2ku^3-1)}{(1+ku^3)^3}\geq\frac{6ku(2k-1)}{(1+ku^3)^3}>0.$$

Thus, f is convex on $[s,n]$. By the RHCF-Theorem and Note 1, it suffices to show that $h(x,y)\geq 0$ for $x,y\geq 0$ so that $x+(n-1)y=n$, where

$$h(x,y)=\frac{g(x)-g(y)}{x-y}, \quad g(u)=\frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u)=\frac{-k(u^2+u+1)}{(1+k)(1+ku^3)}$$

and

$$\frac{h(x,y)}{k^2}=\frac{x^2y^2+xy(x+y-1)+(x+y)^2-(x+y+1)/k}{(1+k)(1+kx^3)(1+ky^3)}.$$

Since

$$x+y\geq\frac{x+(n-1)y}{n-1}=\frac{n}{n-1}>1,$$

it suffices to show that

$$(x+y)^2\geq\frac{x+y+1}{k}.$$

From $x+y\geq\dfrac{n}{n-1}$, we get

$$k(x+y)\geq\frac{2n-1}{n},$$

hence

$$k(x+y)^2-x-y=(x+y)[k(x+y)-1]\geq\frac{n}{n-1}\left(\frac{2n-1}{n}-1\right)=1.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$. If $k=\dfrac{(n-1)(2n-1)}{n^2}$, then the equality holds also for

$$a_1=0, \qquad a_2=a_3=\cdots=a_n=\frac{n}{n-1}$$

(or any cyclic permutation).

$\square$

P 1.28. *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If $0 < k \le \dfrac{n-1}{n^2-2n+2}$, then*

$$\frac{1}{1+ka_1^3} + \frac{1}{1+ka_2^3} + \cdots + \frac{1}{1+ka_n^3} \le \frac{n}{1+k}.$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \qquad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{-1}{1+ku^3}, \quad u \in [0, n].$$

For $u \in [0,1]$, we have

$$f''(u) = \frac{6ku(1-2ku^3)}{(1+ku^3)^3} \ge \frac{6ku(1-2k)}{(1+ku^3)^3} \ge 0.$$

Thus, f is convex on $[0,s]$. By the LHCF-Theorem and Note 1, it suffices to show that $h(x,y) \ge 0$ for $x, y \ge 0$ so that $x + (n-1)y = n$, where

$$h(x,y) = \frac{g(x)-g(y)}{x-y}, \quad g(u) = \frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u) = \frac{k(u^2+u+1)}{(1+k)(1+ku^3)}$$

and

$$\frac{h(x,y)}{k^2} = \frac{(x+y+1)/k - x^2y^2 - xy(x+y-1) - (x+y)^2}{(1+k)(1+kx^3)(1+ky^3)}.$$

It suffices to show that

$$\frac{(n^2-2n+2)(x+y+1)}{n-1} - x^2y^2 - xy(x+y-1) - (x+y)^2 \ge 0,$$

which is equivalent to

$$[2 + ny - (n-1)y^2][1-(n-1)y]^2 \ge 0.$$

This is true because

$$2 + ny - (n-1)y^2 = 2 + y[n-(n-1)y] = 2 + xy > 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $k = \dfrac{n-1}{n^2-2n+2}$, then the equality holds also for

$$a_1 = n-1, \qquad a_2 = a_3 = \cdots = a_n = \frac{1}{n-1}$$

(or any cyclic permutation).

□

P 1.29. *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If $k \ge \dfrac{n^2}{n-1}$, then*

$$\sqrt{\frac{a_1}{k-a_1}} + \sqrt{\frac{a_2}{k-a_2}} + \cdots + \sqrt{\frac{a_n}{k-a_n}} \le \frac{n}{\sqrt{k-1}}.$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \qquad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = -\sqrt{\frac{u}{k-u}}, \quad u \in [0, n].$$

For $u \in [0, 1]$, we have

$$f''(u) = \frac{k(k-4u)}{4u^{3/2}(k-u)^{5/2}} \ge \frac{k(k-4)}{4u^{3/2}(k-u)^{5/2}} \ge 0.$$

Thus, f is convex on $[0, s]$. By the LHCF-Theorem, it suffices to prove that

$$f(x) + (n-1)f(y) \ge nf(1)$$

for $x \ge 1 \ge y \ge 0$ so that $x + (n-1)y = n$. We write the inequality as

$$\sqrt{\frac{(k-1)x}{k-x}} + (n-1)\sqrt{\frac{(k-1)y}{k-y}} \le n,$$

$$\sqrt{1 + \frac{(n-1)k(1-y)}{(n-1)y + k - n}} \le 1 + (n-1)\left[1 - \sqrt{\frac{(k-1)y}{k-y}}\,\right].$$

Let

$$z = \sqrt{\frac{(k-1)y}{k-y}}, \quad z \le 1,$$

which yields

$$y = \frac{kz^2}{z^2 + k - 1},$$

$$1 - y = \frac{(k-1)(1-z^2)}{z^2 + k - 1}, \quad (n-1)y + k - n = \frac{(k-1)(nz^2 + k - n)}{z^2 + k - 1}.$$

Since

$$\begin{aligned}\frac{k(1-y)}{(n-1)y + k - n} &= \frac{k(1-z^2)}{k - n(1-z^2)} = \frac{1-z^2}{1 - n(1-z^2)/k} \\ &\le \frac{1-z^2}{1 - (1-z^2)(n-1)/n} = \frac{n(1-z^2)}{(n-1)z^2 + 1},\end{aligned}$$

it suffices to show that

$$\sqrt{1+\frac{n(n-1)(1-z^2)}{(n-1)z^2+1}} \le 1+(n-1)(1-z).$$

By squaring, we get the obvious inequality

$$(z-1)^2[(n-1)z-1]^2 \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $k = \dfrac{n^2}{n-1}$, then the equality holds also for

$$a_1 = \frac{n(n-1)^2}{n^2-2n+2}, \qquad a_2 = a_3 = \cdots = a_n = \frac{n}{(n-1)(n^2-2n+2)}$$

(or any cyclic permutation).

□

P 1.30. *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then*

$$n^{-a_1^2} + n^{-a_2^2} + \cdots + n^{-a_n^2} \ge 1.$$

(Vasile C., 2006)

Solution. Let $k = \ln n$. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \qquad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = n^{-u^2}, \quad u \in [0, n].$$

For $u \ge 1$, we have

$$f''(u) = 2kn^{-u^2}(2ku^2 - 1) \ge 2kn^{-u^2}(2k-1) \ge 2kn^{-u^2}(2\ln 2 - 1) > 0;$$

therefore, f is convex on $[s, n]$. By the RHCF-Theorem, it suffices to show that

$$f(x) + (n-1)f(y) \ge nf(1)$$

for $0 \le x \le 1 \le y$ and $x + (n-1)y = n$. The desired inequality is equivalent to $g(x) \ge 0$, where

$$g(x) = n^{-x^2} + (n-1)n^{-y^2} - 1, \quad y = \frac{n-x}{n-1}, \quad 0 \le x \le 1.$$

Since $y' = -1/(n-1)$, we get

$$g'(x) = -2xkn^{-x^2} - 2(n-1)kyy'n^{-y^2} = 2k(yn^{-y^2} - xn^{-x^2}).$$

The derivative $g'(x)$ has the same sign as $g_1(x)$, where

$$g_1(x) = \ln(yn^{-y^2}) - \ln(xn^{-x^2}) = \ln y - \ln x + k(x^2 - y^2),$$

$$g_1'(x) = \frac{y'}{y} - \frac{1}{x} + 2k(x - yy') = n\left[\frac{-1}{x(n-x)} + \frac{2k(1+nx-2x)}{(n-1)^2}\right].$$

For $0 < x \le 1$, $g_1'(x)$ has the same sign as

$$h(x) = \frac{-(n-1)^2}{2k} + x(n-x)(1+nx-2x).$$

Since

$$\begin{aligned} h'(x) &= n + 2(n^2-2n-1)x - 3(n-2)x^2 \\ &\ge nx + 2(n^2-2n-1)x - 3(n-2)x \\ &= 2(n-1)(n-2)x \ge 0, \end{aligned}$$

h is strictly increasing on $[0,1]$. From

$$h(0) = \frac{-(n-1)^2}{2k} < 0, \quad h(1) = (n-1)^2\left(1 - \frac{1}{2k}\right) > 0,$$

it follows that there is $x_1 \in (0,1)$ so that $h(x_1) = 0$, $h(x) < 0$ for $x \in [0, x_1)$ and $h(x) > 0$ for $x \in (x_1, 1]$. Therefore, g_1 is strictly decreasing on $(0, x_1]$ and strictly increasing on $[x_1, 1]$. Since $g_1(0_+) = \infty$ and $g_1(1) = 0$, there is $x_2 \in (0, x_1)$ so that $g_1(x_2) = 0$, $g_1(x) > 0$ for $x \in (0, x_2)$ and $g_1(x) < 0$ for $x \in (x_2, 1)$. Consequently, g is strictly increasing on $[0, x_2]$ and strictly decreasing on $[x_2, 1]$. Because $g(0) > 0$ and $g(1) = 0$, it follows that $g(x) \ge 0$ for $x \in [0,1]$. The proof is completed.

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.31. *If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 4$, then*

$$(3a^2+1)(3b^2+1)(3c^2+1)(3d^2+1) \le 256.$$

(Vasile C., 2006)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) \ge 4f(s), \quad s = \frac{a+b+c+d}{4} = 1,$$

where

$$f(u) = -\ln(3u^2+1), \quad u \in [0,4].$$

For $u \in [1,4]$, we have
$$f''(u) = \frac{6(3u^2-1)}{(3u^2+1)^2} > 0.$$
Therefore, f is convex on $[s,4]$. By the RHCF-Theorem, we only need to show that
$$f(x)+3f(y) \geq 4f(1)$$
for $0 \leq x \leq 1 \leq y$ so that $x+3y=4$; that is, to show that $g(x) \geq 0$ for $x \in [0,1]$, where
$$g(x) = f(x)+3f(y)-4f(1), \qquad y = \frac{4-x}{3}.$$
Since $y'(x) = -1/3$, we have
$$\begin{aligned} g'(x) &= f'(x)+3y'f'(y) = \frac{-6x}{3x^2+1} + \frac{6y}{3y^2+1} \\ &= \frac{6(x-y)(3xy-1)}{(3x^2+1)(3y^2+1)} = \frac{8(1-x)(x^2-4x+1)}{(3x^2+1)(3y^2+1)} \geq 0. \end{aligned}$$
Since g is increasing on $[0, 2-\sqrt{3}]$ and decreasing on $[2-\sqrt{3}, 1]$, it suffices to show that $g(0) \geq 0$ and $g(1) \geq 0$. The inequality $g(0) \geq 0$ is true if the original inequality holds for $a=0$ and $b=c=d=4/3$. This reduces to $19^3 \leq 27 \cdot 256$, which is true because $27 \cdot 256 - 19^3 = 53 > 0$. The inequality $g(1) \geq 0$ is also true because $g(1)=0$.

The equality holds for $a=b=c=d=1$.

□

P 1.32. *If $a,b,c,d,e \geq -1$ so that $a+b+c+d+e=5$, then*
$$(a^2+1)(b^2+1)(c^2+1)(d^2+1)(e^2+1) \geq (a+1)(b+1)(c+1)(d+1)(e+1).$$

(Vasile C., 2007)

Solution. Consider the nontrivial case $a,b,c,d,e > -1$, and write the inequality as
$$f(a)+f(b)+f(c)+f(d)+f(e) \geq nf(s), \qquad s = \frac{a+b+c+d+e}{5} = 1,$$
where
$$f(u) = \ln(u^2+1) - \ln(u+1), \quad u > -1.$$
For $u \in (-1,1]$, we have
$$f''(u) = \frac{2(1-u^2)}{(u^2+1)^2} + \frac{1}{(u+1)^2} > 0.$$

Therefore, f is convex on $(-1,s]$. By the LHCF-Theorem and Note 2, it suffices to show that $H(x,y) \ge 0$ for $x,y > -1$ so that $x+4y=5$, where

$$H(x,y) = \frac{f'(x)-f'(y)}{x-y} = \frac{2(1-xy)}{(x^2+1)(y^2+1)} + \frac{1}{(x+1)(y+1)};$$

thus, we need to show that

$$2(1-xy) + \frac{(x^2+1)(y^2+1)}{(x+1)(y+1)} \ge 0.$$

Since

$$\frac{x^2+1}{x+1} \ge \frac{x+1}{2}, \qquad \frac{y^2+1}{y+1} \ge \frac{y+1}{2},$$

it suffices to prove that

$$2(1-xy) + \frac{(x+1)(y+1)}{4} \ge 0,$$

which is equivalent to

$$x+y+9-7xy \ge 0,$$

$$28x^2-38x+14 \ge 0,$$

$$(28x-19)^2+31 \ge 0.$$

The equality holds for $a=b=c=d=e=1$.

□

P 1.33. *Let $a_1,a_2,\dots,a_n$ be positive numbers so that $a_1+a_2+\cdots+a_n=n$. If*

$$k \le \frac{2\sqrt{n-1}}{n} + 2\sqrt{\frac{2\sqrt{n-1}}{n}}, \qquad k \le 3,$$

then

$$k(\sqrt{a_1}+\sqrt{a_2}+\cdots+\sqrt{a_n}) + \frac{1}{\sqrt{a_1}} + \frac{1}{\sqrt{a_2}} + \cdots + \frac{1}{\sqrt{a_n}} \ge (k+1)n.$$

(Vasile C., 2006)

Solution. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n) \ge nf(s), \qquad s = \frac{a_1+a_2+\cdots+a_n}{n} = 1,$$

where

$$f(u) = \frac{k}{\sqrt{u}} + \sqrt{u}, \quad u \in (0,n).$$

From

$$f''(u) = \frac{3-ku}{4u^{5/2}},$$

it follows that f is convex on $(0,1]$. Thus, according to the LHCF-Theorem and Note 1, it suffices to show that $h(x,y) \geq 0$ for $x \geq 1 \geq y > 0$ such that $x+(n-1)y = n$, where

$$h(x,y) = \frac{g(x)-g(y)}{x-y}, \quad g(u) = \frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u) = \frac{k}{\sqrt{u}+1} - \frac{1}{u+\sqrt{u}}$$

and

$$(\sqrt{x}+\sqrt{y})(\sqrt{x}+1)(\sqrt{y}+1)h(x,y) = -k + \frac{\sqrt{x}+\sqrt{y}+1}{\sqrt{xy}}.$$

So, we need to show that

$$\frac{\sqrt{x}+\sqrt{y}+1}{\sqrt{xy}} \geq k.$$

Since

$$\sqrt{x}+\sqrt{y} \geq 2\sqrt[4]{xy},$$

it suffices to show that

$$\frac{2\sqrt[4]{xy}+1}{\sqrt{xy}} \geq k,$$

which is equivalent to

$$\frac{1}{\sqrt{xy}} + \frac{2}{\sqrt[4]{xy}} \geq k.$$

From

$$n = x+(n-1)y \geq 2\sqrt{(n-1)xy},$$

we get

$$\frac{1}{\sqrt{xy}} \geq \frac{2\sqrt{n-1}}{n},$$

hence

$$\frac{1}{\sqrt{xy}} + \frac{2}{\sqrt[4]{xy}} \geq \frac{2\sqrt{n-1}}{n} + 2\sqrt{\frac{2\sqrt{n-1}}{n}} \geq k.$$

The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark. Since

$$1 < \frac{2\sqrt{n-1}}{n} + 2\sqrt{\frac{2\sqrt{n-1}}{n}}$$

for $n \leq 134$, the following inequality holds for $a_1, a_2, \ldots, a_{134} > 0$ such that $a_1 + a_2 + \cdots + a_{134} = 134$:

$$\sqrt{a_1}+\sqrt{a_2}+\cdots+\sqrt{a_{134}} + \frac{1}{\sqrt{a_1}} + \frac{1}{\sqrt{a_2}} + \cdots + \frac{1}{\sqrt{a_{134}}} \geq 268.$$

Since

$$2 < \frac{2\sqrt{n-1}}{n} + 2\sqrt{\frac{2\sqrt{n-1}}{n}}$$

for $n \le 12$, the following inequality holds for $a_1, a_2, \dots, a_{12} > 0$ such that $a_1 + a_2 + \cdots + a_{12} = 12$:

$$2(\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_{12}}) + \frac{1}{\sqrt{a_1}} + \frac{1}{\sqrt{a_2}} + \cdots + \frac{1}{\sqrt{a_{12}}} \ge 36.$$

$\square$

P 1.34. *If $a_1, a_2, \dots, a_n$ $(n \ge 3)$ are positive numbers so that $a_1 + a_2 + \cdots + a_n = 1$, then*

$$\left(\frac{1}{\sqrt{a_1}} - \sqrt{a_1}\right)\left(\frac{1}{\sqrt{a_2}} - \sqrt{a_2}\right)\cdots\left(\frac{1}{\sqrt{a_n}} - \sqrt{a_n}\right) \ge \left(\sqrt{n} - \frac{1}{\sqrt{n}}\right)^n.$$

(Vasile C., 2006)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \qquad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{1}{n},$$

where

$$f(u) = \ln\left(\frac{1}{\sqrt{u}} - \sqrt{u}\right) = \ln(1-u) - \frac{1}{2}\ln u, \qquad u \in (0,1).$$

From

$$f'(u) = \frac{-1}{1-u} - \frac{1}{2u}, \qquad f''(u) = \frac{1-2u-u^2}{2u^2(1-u)^2},$$

it follows that $f''(u) \ge 0$ for $u \in (0, \sqrt{2}-1]$. Since

$$s = \frac{1}{n} \le \frac{1}{3} < \sqrt{2} - 1,$$

f is convex on $(0, s]$. Thus, we can apply the LHCF-Theorem.

First Solution. By the LHCF-Theorem, it suffices to show that

$$f(x) + (n-1)f(y) \ge nf\left(\frac{1}{n}\right)$$

for all $x, y > 0$ so that $x + (n-1)y = 1$; that is, to show that

$$\left(\frac{1}{\sqrt{x}} - \sqrt{x}\right)\left(\frac{1}{\sqrt{y}} - \sqrt{y}\right)^{n-1} \ge \left(\sqrt{n} - \frac{1}{\sqrt{n}}\right)^n.$$

Write this inequality as

$$n^{n/2}(1-y)^{n-1} \ge (n-1)^{n-1}x^{1/2}y^{(n-3)/2}.$$

By squaring, this inequality becomes as follows:

$$n^n(1-y)^{2n-2} \ge (n-1)^{2n-2}xy^{n-3},$$

$$(2-2y)^{2n-2} \ge \frac{(2n-2)^{2n-2}}{n^n}xy^{n-3},$$

$$\left[n\cdot\frac{1}{n}+x+(n-3)y\right]^{2n-2} \ge [n+1+(n-3)]^{n+1+(n-3)}\cdot\frac{1}{n^n}\cdot x\cdot y^{n-3}.$$

The last inequality follows from the AM-GM inequality. The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1/n$.

Second Solution. By the LHCF-Theorem and Note 2, it suffices to prove that $H(x,y) \ge 0$ for $x,y>0$ so that $x+(n-1)y=1$, where

$$H(x,y)=\frac{f'(x)-f'(y)}{x-y}.$$

We have

$$H(x,y)=\frac{1-x-y-xy}{2xy(1-x)(1-y)}=\frac{n(y+1)-y-3}{2x(1-x)(1-y)}$$
$$\ge\frac{3(y+1)-y-3}{2x(1-x)(1-y)}=\frac{y}{x(1-x)(1-y)}>0.$$

Remark 1. We may write the inequality in P 1.34 in the form

$$\prod_{i=1}^{n}\left(\frac{1}{\sqrt{a_i}}-1\right)\cdot\prod_{i=1}^{n}(1+\sqrt{a_i}\,)\ge\left(\sqrt{n}-\frac{1}{\sqrt{n}}\right)^n.$$

On the other hand, by the AM-GM inequality and the Cauchy-Schwarz inequality, we have

$$\prod_{i=1}^{n}(1+\sqrt{a_i})\le\left(1+\frac{1}{n}\sum_{i=1}^{n}\sqrt{a_i}\right)^n\le\left(1+\sqrt{\frac{1}{n}\sum_{i=1}^{n}a_i}\right)^n=\left(1+\frac{1}{\sqrt{n}}\right)^n.$$

Thus, the following statement follows:

- *If $a_1,a_2,\ldots,a_n$ $(n\ge 3)$ are positive real numbers so that $a_1+a_2+\cdots+a_n=1$, then*

$$\left(\frac{1}{\sqrt{a_1}}-1\right)\left(\frac{1}{\sqrt{a_2}}-1\right)\cdots\left(\frac{1}{\sqrt{a_n}}-1\right)\ge(\sqrt{n}-1)^n,$$

with equality for $a_1=a_2=\cdots=a_n=1/n$.

Remark 2. By squaring, the inequality in P 1.34 becomes

$$\prod_{i=1}^{n}\frac{(1-a_i)^2}{a_i}\ge\frac{(n-1)^{2n}}{n^n}.$$

On the other hand, since the function $f(x)=\ln\dfrac{1+x}{1-x}$ is convex on $(0,1)$, by Jensen's inequality we have

$$\prod_{i=1}^{n}\left(\frac{1+a_i}{1-a_i}\right)\ge\left(\frac{1+\dfrac{a_1+a_2+\cdots+a_n}{n}}{1-\dfrac{a_1+a_2+\cdots+a_n}{n}}\right)^n=\left(\frac{n+1}{n-1}\right)^n.$$

Multiplying these inequalities yields the following result (*Kee-Wai Lau*, 2000):

- *If $a_1,a_2,\ldots,a_n$ $(n\ge 3)$ are positive real numbers so that $a_1+a_2+\cdots+a_n=1$, then*

$$\left(\frac{1}{a_1}-a_1\right)\left(\frac{1}{a_2}-a_2\right)\cdots\left(\frac{1}{a_n}-a_n\right)\ge\left(n-\frac{1}{n}\right)^n,$$

with equality for $a_1=a_2=\cdots=a_n=1/n$.

□

P 1.35. *Let $a_1,a_2,\ldots,a_n$ be positive real numbers so that $a_1+a_2+\cdots+a_n=n$. If*

$$0<k\le\left(1+\frac{2\sqrt{n-1}}{n}\right)^2,$$

then

$$\left(ka_1+\frac{1}{a_1}\right)\left(ka_2+\frac{1}{a_2}\right)\cdots\left(ka_n+\frac{1}{a_n}\right)\ge(k+1)^n.$$

(Vasile C., 2006)

Solution. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf(s),\quad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=\ln\left(ku+\frac{1}{u}\right),\quad u\in(0,n).$$

We have

$$f'(u)=\frac{ku^2-1}{u(ku^2+1)},\quad f''(u)=\frac{1+4ku^2-k^2u^4}{u^2(ku^2+1)^2}.$$

For $u\in(0,1]$, we get $f''(u)>0$ since

$$1+4ku^2-k^2u^4>ku^2(4-ku^2)\ge ku^2(4-k)\ge 0.$$

Therefore, f is convex on $(0,s]$. By the LHCF-Theorem and Note 2, it suffices to prove that $H(x,y)\ge 0$ for $x,y>0$ so that $x+(n-1)y=n$, where

$$H(x,y)=\frac{f'(x)-f'(y)}{x-y}.$$

Since

$$H(x,y)=\frac{1+k(x+y)^2-k^2x^2y^2}{xy(kx^2+1)(ky^2+1)}>\frac{k[(x+y)^2-kx^2y^2]}{xy(kx^2+1)(ky^2+1)},$$

it suffices to show that

$$x+y\ge\sqrt{k}\,xy.$$

Indeed, by the Cauchy-Schwarz inequality, we have

$$(x+y)[(n-1)y+x]\ge(\sqrt{n-1}+1)^2xy,$$

hence

$$x+y\ge\frac{1}{n}(\sqrt{n-1}+1)^2xy=\left(1+\frac{2\sqrt{n-1}}{n}\right)xy\ge\sqrt{k}\,xy.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$.

$\square$

P 1.36. *If a,b,c,d are nonzero real numbers so that*

$$a,b,c,d\ge\frac{-1}{2},\quad a+b+c+d=4,$$

then

$$3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right)+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\ge 16.$$

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d)\ge 4f(s),\qquad s=\frac{a+b+c+d}{4}=1,$$

where

$$f(u)=\frac{3}{u^2}+\frac{1}{u},\quad u\in\mathbb{I}=\left[\frac{-1}{2},\frac{11}{2}\right]\setminus\{0\},$$

is convex on $\mathbb{I}_{\ge s}$ (because $3/u^2$ and $1/u$ are convex). By the RHCF-Theorem, Note 1 and Note 3, it suffices to prove that $h(x,y)\ge 0$ for $x,y\in\mathbb{I}$ so that

$$x+3y=4,$$

where

$$h(x,y)=\frac{g(x)-g(y)}{x-y},\qquad g(u)=\frac{f(u)-f(1)}{u-1}.$$

Indeed, we have

$$g(u) = -\frac{4}{u} - \frac{3}{u^2},$$

$$h(x,y) = \frac{4xy + 3x + 3y}{x^2y^2} = \frac{2(1+2x)(6-x)}{3x^2y^2} \ge 0.$$

In accordance with Note 4, the equality holds for $a = b = c = d = 1$, and also for

$$a = \frac{-1}{2}, \qquad b = c = d = \frac{3}{2}$$

(or any cyclic permutation).

□

P 1.37. *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that $a_1^2 + a_2^2 + \cdots + a_n^2 = n$, then*

$$a_1^3 + a_2^3 + \cdots + a_n^3 - n + \sqrt{\frac{n}{n-1}}\,(a_1 + a_2 + \cdots + a_n - n) \ge 0.$$

(Vasile C., 2007)

Solution. Replacing each a_i by $\sqrt{a_i}$, we have to prove that

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s),$$

where

$$s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1$$

and

$$f(u) = u\sqrt{u} + k\sqrt{u}, \quad k = \sqrt{\frac{n}{n-1}}, \quad u \in [0,n].$$

For $u \ge 1$, we have

$$f''(u) = \frac{3u-k}{4u\sqrt{u}} \ge \frac{3-k}{4u\sqrt{u}} > 0.$$

Therefore, f is convex on $[s,n]$. According to the RHCF-Theorem and Note 1, it suffices to show that $h(x,y) \ge 0$ for $x, y \ge 0$ so that $x + (n-1)y = n$. Since

$$g(u) = \frac{f(u) - f(1)}{u-1} = 1 + \frac{u+k}{\sqrt{u}+1}$$

and

$$h(x,y) = \frac{g(x) - g(y)}{x-y} = \frac{\sqrt{x} + \sqrt{y} + \sqrt{xy} - k}{(\sqrt{x}+\sqrt{y})(\sqrt{x}+1)(\sqrt{y}+1)},$$

we need to show that

$$\sqrt{x} + \sqrt{y} + \sqrt{xy} \ge k.$$

Since

$$\sqrt{x}+\sqrt{y}+\sqrt{xy} \ge \sqrt{x}+\sqrt{y} \ge \sqrt{x+y},$$

it suffices to show that

$$x+y \ge k^2.$$

Indeed, we have

$$x+y \ge \frac{x}{n-1}+y = \frac{n}{n-1} = k^2.$$

In accordance with Note 4, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = 0, \qquad a_2 = \cdots = a_n = \sqrt{\frac{n}{n-1}}$$

(or any cyclic permutation).

□

P 1.38. *If a,b,c,d,e are nonnegative real numbers so that $a^2+b^2+c^2+d^2+e^2=5$, then*

$$\frac{1}{7-2a}+\frac{1}{7-2b}+\frac{1}{7-2c}+\frac{1}{7-2d}+\frac{1}{7-2e} \le 1.$$

(Vasile C., 2010)

Solution. Replacing a,b,c,d,e by $\sqrt{a},\sqrt{b},\sqrt{c},\sqrt{d},\sqrt{e}$, we have to prove that

$$f(a)+f(b)+f(c)+f(d)+f(e) \ge 5f(s),$$

where

$$s = \frac{a+b+c+d+e}{5} = 1$$

and

$$f(u) = \frac{1}{2\sqrt{u}-7}, \qquad u \in [0,5].$$

For $u \in [0,1]$, we have

$$f''(u) = \frac{7-6\sqrt{u}}{2u\sqrt{u}(7-2\sqrt{u})^3} > 0.$$

Therefore, f is convex on $[0,s]$. According to the LHCF-Theorem and Note 1, it suffices to show that $h(x,y) \ge 0$ for $x,y \ge 0$ so that $x+4y=5$. Since

$$g(u) = \frac{f(u)-f(1)}{u-1} = \frac{-2}{5(7-2\sqrt{u})(1+\sqrt{u})}$$

and

$$h(x,y) = \frac{g(x)-g(y)}{x-y} = \frac{2(5-2\sqrt{x}-2\sqrt{y})}{(\sqrt{x}+\sqrt{y})(1+\sqrt{x})(1+\sqrt{y})(7-2\sqrt{x})(7-2\sqrt{y})},$$

we need to show that

$$\sqrt{x}+\sqrt{y}\le \frac{5}{2}.$$

Indeed, by the Cauchy-Schwarz inequality, we have

$$(\sqrt{x}+\sqrt{y})^2\le \left(1+\frac{1}{4}\right)(x+4y)=\frac{25}{4}.$$

The proof is completed. The equality holds for $a=b=c=d=e=1$, and also for

$$a=2,\qquad b=c=d=e=\frac{1}{2}$$

(or any cyclic permutation).

Remark In the same manner, we can prove the following generalization:

- *Let $a_1,a_2,\ldots,a_n$ be nonnegative real numbers so that $a_1^2+a_2^2+\cdots+a_n^2=n$. If $k\ge 1+\frac{n}{\sqrt{n-1}}$, then*

$$\frac{1}{k-a_1}+\frac{1}{k-a_2}+\cdots+\frac{1}{k-a_n}\le \frac{n}{k-1},$$

with equality for $a_1=a_2=\cdots=a_n=1$. If $k=1+\frac{n}{\sqrt{n-1}}$, then the equality holds also for

$$a_1=\sqrt{n-1},\qquad a_2=\cdots=a_n=\frac{1}{\sqrt{n-1}}$$

(or any cyclic permutation).

□

P 1.39. *Let $0\le a_1,a_2,\ldots,a_n<k$ so that $a_1^2+a_2^2+\cdots+a_n^2=n$. If*

$$1<k\le 1+\sqrt{\frac{n}{n-1}},$$

then

$$\frac{1}{k-a_1}+\frac{1}{k-a_2}+\cdots+\frac{1}{k-a_n}\ge \frac{n}{k-1}.$$

(Vasile C., 2010)

Solution. Replacing $a_1,a_2,\ldots,a_n$ by $\sqrt{a_1},\sqrt{a_2},\ldots,\sqrt{a_n}$, we have to prove that

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf(s),$$

where

$$s=\frac{a_1+a_2+\cdots+a_n}{n}=1$$

and

$$f(u)=\frac{1}{k-\sqrt{u}}, \quad u\in[0,k^2).$$

From

$$f''(u)=\frac{3\sqrt{u}-k}{4u\sqrt{u}(k-\sqrt{u})^3},$$

it follows that f is convex on $[s,k^2)$. According to the RHCF-Theorem and Note 1, it suffices to show that $h(x,y)\geq 0$ for all $x,y\in[0,k^2)$ so that $x+(n-1)y=n$. Since

$$g(u)=\frac{f(u)-f(1)}{u-1}=\frac{1}{(k-1)(k-\sqrt{u})(1+\sqrt{u})}$$

and

$$h(x,y)=\frac{g(x)-g(y)}{x-y}=\frac{\sqrt{x}+\sqrt{y}+1-k}{(k-1)(\sqrt{x}+\sqrt{y})(1+\sqrt{x})(1+\sqrt{y})(k-\sqrt{x})(k-\sqrt{y})},$$

we need to show that

$$\sqrt{x}+\sqrt{y}\geq k-1.$$

Indeed,

$$\sqrt{x}+\sqrt{y}\geq\sqrt{x+y}\geq\sqrt{\frac{x}{n-1}+y}=\sqrt{\frac{n}{n-1}}\geq k-1.$$

The proof is completed. The equality holds for $a_1=a_2=\cdots=a_n=1$, and also for

$$a_1=0, \quad a_2=\cdots=a_n=\sqrt{\frac{n}{n-1}}$$

(or any cyclic permutation).

□

P 1.40. *If a,b,c are nonnegative real numbers, no two of which are zero, then*

$$\sqrt{1+\frac{48a}{b+c}}+\sqrt{1+\frac{48b}{c+a}}+\sqrt{1+\frac{48c}{a+b}}\geq 15.$$

(Vasile C., 2005)

Solution. Due to homogeneity, we may assume that $a+b+c=1$. Thus, we need to show that

$$f(a)+f(b)+f(c)\geq 3f(s),$$

where

$$s=\frac{a+b+c}{3}=\frac{1}{3}$$

and

$$f(u)=\sqrt{\frac{1+47u}{1-u}},\quad u\in[0,1).$$

From

$$f''(u)=\frac{48(47u-11)}{\sqrt{(1-u)^5(1+47u)^3}},$$

it follows that f is convex on $[s,1)$. By the RHCF-Theorem, it suffices to show that

$$f(x)+2f(y)\geq 3f\left(\frac{1}{3}\right)$$

for $x,y\geq 0$ so that $x+2y=1$; that is,

$$\sqrt{\frac{1+47x}{1-x}}+2\sqrt{\frac{49-47x}{1+x}}\geq 15.$$

Setting

$$t=\sqrt{\frac{49-47x}{1+x}},\quad 1<t\leq 7,$$

the inequality turns into

$$\sqrt{\frac{1175-23t^2}{t^2-1}}\geq 15-2t.$$

By squaring, this inequality becomes

$$350-15t-61t^2+15t^3-t^4\geq 0,$$

$$(5-t)^2(2+t)(7-t)\geq 0.$$

The original inequality is an equality for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.41. *If a,b,c are nonnegative real numbers, then*

$$\sqrt{\frac{3a^2}{7a^2+5(b+c)^2}}+\sqrt{\frac{3b^2}{7b^2+5(c+a)^2}}+\sqrt{\frac{3c^2}{7c^2+5(a+b)^2}}\leq 1.$$

(Vasile C., 2008)

Solution. Due to homogeneity, we may assume that $a+b+c=3$. Thus, we need to show that

$$f(a)+f(b)+f(c) \geq 3f(s),$$

where

$$s=\frac{a+b+c}{3}=1$$

and

$$f(u)=-\sqrt{\frac{3u^2}{7u^2+5(3-u)^2}}=\frac{-u}{\sqrt{4u^2-10u+15}}, \quad u \in [0,3].$$

From

$$f''(u)=\frac{5(-8u^2+41u-30)}{(4u^2-10u+15)^{5/2}} \geq \frac{5(-8u^2+38u-30)}{(4u^2-10u+15)^{5/2}}=\frac{10(u-1)(15-4u)}{(4u^2-10u+15)^{5/2}},$$

it follows that f is convex on $[s,3]$. By the RHCF-Theorem, it suffices to prove the original homogeneous inequality for $b=c=0$ and $b=c=1$. For the nontrivial case $b=c=1$, we need to show that

$$\sqrt{\frac{3a^2}{7a^2+20}}+2\sqrt{\frac{3}{5a^2+10a+12}} \leq 1.$$

By squaring two times, the inequality becomes

$$a(5a^3+10a^2+16a+50) \geq 3a\sqrt{(7a^2+20)(5a^2+10a+12)},$$

$$a^2(5a^6+20a^5-11a^4+38a^3-80a^2-40a+68) \geq 0,$$

$$a^2(a-1)^2(5a^4+30a^3+44a^2+96a+68) \geq 0.$$

The last inequality is clearly true.

The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.42. *If a,b,c are nonnegative real numbers, then*

$$\sqrt{\frac{a^2}{a^2+2(b+c)^2}}+\sqrt{\frac{b^2}{b^2+2(c+a)^2}}+\sqrt{\frac{c^2}{c^2+2(a+b)^2}} \geq 1.$$

(Vasile C., 2008)

Solution. Due to homogeneity, we may assume that $a+b+c=3$. Thus, we need to show that

$$f(a)+f(b)+f(c) \geq 3f(s),$$

where

$$s=\frac{a+b+c}{3}=1$$

and

$$f(u)=\sqrt{\frac{3u^2}{u^2+2(3-u)^2}}=\frac{u}{\sqrt{u^2-4u+6}},\quad u\in[0,3].$$

From

$$f''(u)=\frac{2(2u^2-11u+12)}{(u^2-4u+6)^{5/2}}\geq\frac{2(-11u+12)}{(u^2-4u+6)^{5/2}},$$

it follows that f is convex on $[0,s]$. By the LHCF-Theorem, it suffices to prove the original homogeneous inequality for $b=c=0$ and $b=c=1$. For the nontrivial case $b=c=1$, the inequality has the form

$$\frac{a}{\sqrt{a^2+8}}+\frac{2}{\sqrt{2a^2+4a+3}}\geq 1.$$

By squaring, the inequality becomes

$$a\sqrt{(a^2+8)(2a^2+4a+3)}\geq 3a^2+8a-2.$$

For the nontrivial case $3a^2+8a-2>0$, by squaring both sides we get

$$a^6+2a^5+5a^4-8a^3-14a^2+16a-2\geq 0,$$

$$(a-1)^2[a^4+4a^3+9a^2+4a+(3a^2+8a-2)]\geq 0.$$

The equality holds for $a=b=c$, and also for $b=c=0$ (or any cyclic permutation).

□

P 1.43. *Let a,b,c be nonnegative real numbers, no two of which are zero. If*

$$k\geq k_0,\quad k_0=\frac{\ln 3}{\ln 2}-1\approx 0.585,$$

then

$$\left(\frac{2a}{b+c}\right)^k+\left(\frac{2b}{c+a}\right)^k+\left(\frac{2c}{a+b}\right)^k\geq 3.$$

(Vasile C., 2005)

Solution. For $k=1$, the inequality is just the well known Nesbitt's inequality

$$\frac{2a}{b+c}+\frac{2b}{c+a}+\frac{2c}{a+b}\geq 3.$$

For $k \ge 1$, the inequality follows from Nesbitt's inequality and Jensens's inequality applied to the convex function $f(u) = u^k$:

$$\left(\frac{2a}{b+c}\right)^k + \left(\frac{2b}{c+a}\right)^k + \left(\frac{2c}{a+b}\right)^k \ge 3\left(\frac{\frac{2a}{b+c} + \frac{2b}{c+a} + \frac{2c}{a+b}}{3}\right)^k \ge 3.$$

Consider now that

$$k_0 \le k < 1.$$

Due to homogeneity, we may assume that $a + b + c = 1$. Thus, we need to show that

$$f(a) + f(b) + f(c) \ge 3f(s),$$

where

$$s = \frac{a+b+c}{3} = \frac{1}{3}$$

and

$$f(u) = \left(\frac{2u}{1-u}\right)^k, \quad u \in [0,1).$$

From

$$f''(u) = \frac{4k}{(1-u)^4}\left(\frac{2u}{1-u}\right)^{k-2}(2u+k-1),$$

it follows that f is convex on $[s,1)$ (because $u \ge s = 1/3$ involves $2u + k - 1 \ge 2/3 + k - 1 = k - 1/3 > 0$). By the RHCF-Theorem, it suffices to prove the original homogeneous inequality for $b = c = 1$ and $a \in [0,1]$; that is, to show that $h(a) \ge 3$, where

$$h(a) = a^k + 2\left(\frac{2}{a+1}\right)^k, \quad a \in [0,1].$$

For $a \in (0,1]$, the derivative

$$h'(a) = ka^{k-1} - k\left(\frac{2}{a+1}\right)^{k+1}$$

has the same sign as

$$g(a) = (k-1)\ln a - (k+1)\ln\frac{2}{a+1}.$$

From

$$g'(a) = \frac{2ka + k - 1}{a(a+1)},$$

it follows that $g'(a_0) = 0$ for $a_0 = (1-k)/(2k) < 1$, $g'(a) < 0$ for $a \in (0, a_0)$ and $g'(a) > 0$ for $a \in (a_0, 1]$. Consequently, g is strictly decreasing on $(0, a_0]$ and strictly increasing on $(a_0, 1]$. Since $g(0_+) = \infty$ and $g(1) = 0$, there exists $a_1 \in (0, a_0)$ so that $g(a_1)=0$, $g(a)>0$ for $a\in(0,a_1)$ and $g(a)<0$ for $a\in(a_1,1)$; therefore, $h(a)$ is strictly increasing on $[0,a_1]$ and strictly decreasing on $[a_1,1]$. As a result,

$$h(a)\ge \min\{h(0),h(1)\}.$$

Since $h(0)=2^{k+1}\ge 3$ and $h(1)=3$, we get $h(a)\ge 3$. The proof is completed. The equality holds for $a=b=c$. If $k=k_0$, then the equality holds also for $a=0$ and $b=c$ (or any cyclic permutation).

Remark. For $k=2/3$, we can give the following solution (based on the AM-GM inequality):

$$\begin{aligned}\sum\left(\frac{2a}{b+c}\right)^{2/3} &= \sum\frac{2a}{\sqrt[3]{2a\cdot(b+c)\cdot(b+c)}}\\ &\ge \sum\frac{6a}{2a+(b+c)+(b+c)}=3.\end{aligned}$$

□

P 1.44. *If $a,b,c\in[1,7+4\sqrt{3}]$, then*

$$\sqrt{\frac{2a}{b+c}}+\sqrt{\frac{2b}{c+a}}+\sqrt{\frac{2c}{a+b}}\ge 3.$$

(Vasile C., 2007)

Solution. Denoting

$$s=\frac{a+b+c}{3},\qquad 1\le s\le 7+4\sqrt{3},$$

we need to show that

$$f(a)+f(b)+f(c)\ge 3f(s),$$

where

$$f(u)=\sqrt{\frac{2u}{3s-u}},\quad 1\le u<3s.$$

For $u\ge s$, we have

$$f''(u)=3s\left(\frac{3s-u}{2u}\right)^{3/2}\frac{4u-3s}{(3s-u)^4}>0.$$

Therefore, $f(u)$ is convex for $u\ge s$. By the RHCF-Theorem, it suffices to prove the original inequality for $b=c$; that is,

$$\sqrt{\frac{a}{b}}+2\sqrt{\frac{2b}{a+b}}\ge 3.$$

Putting $t = \sqrt{\frac{b}{a}}$, the condition $a, b \in [1, 7+4\sqrt{3}]$ involves

$$2-\sqrt{3} \le t \le 2+\sqrt{3}.$$

We need to show that

$$2\sqrt{\frac{2t^2}{t^2+1}} \ge 3 - \frac{1}{t}.$$

This is true if

$$\frac{8t^2}{t^2+1} \ge \left(3-\frac{1}{t}\right)^2,$$

which is equivalent to the obvious inequality

$$(t-1)^2(t-2+\sqrt{3}\,)(t-2-\sqrt{3}\,) \le 0.$$

The equality holds for $a = b = c$, and also for $a = 1$, and $b = c = 7+4\sqrt{3}$ (or any cyclic permutation).

□

P 1.45. *Let a, b, c be nonnegative real numbers so that $a+b+c = 3$. If*

$$0 < k \le k_0, \quad k_0 = \frac{\ln 2}{\ln 3 - \ln 2} \approx 1.71,$$

then

$$a^k(b+c) + b^k(c+a) + c^k(a+b) \le 6.$$

Solution. For $0 < k \le 1$, the inequality follows from Jensens's inequality applied to the convex function $f(u) = -u^k$:

$$(b+c)a^k + (c+a)b^k + (a+b)c^k \le 2(a+b+c)\left[\frac{(b+c)a+(c+a)b+(a+b)c}{2(a+b+c)}\right]^k$$
$$= 6\left(\frac{ab+bc+ca}{3}\right)^k \le 6\left(\frac{a+b+c}{3}\right)^{2k} = 6.$$

Consider now that

$$1 < k \le k_0,$$

and write the inequality as

$$f(a) + f(b) + f(c) \ge 3f(s),$$

where

$$s = \frac{a+b+c}{3} = 1$$

and

$$f(u) = u^k(u-3), \quad u \in [0,3].$$

For $u \geq 1$, we have

$$f''(u) = ku^{k-2}[(k+1)u - 3k + 3] \geq ku^{k-2}[(k+1) - 3k + 3] = 2k(2-k)u^{k-2} > 0;$$

therefore, f is convex on $[1,s]$. By the RHCF-Theorem, it suffices to consider the case $a \leq b = c$. So, we only need to prove the homogeneous inequality

$$a^k(b+c) + b^k(c+a) + c^k(a+b) \leq 6\left(\frac{a+b+c}{3}\right)^{k+1}$$

for $b = c = 1$ and $a \in [0,1]$; that is, to show that $g(a) \geq 0$ for $a \geq 0$, where

$$g(a) = 3\left(\frac{a+2}{3}\right)^{k+1} - a^k - a - 1.$$

We have

$$g'(a) = (k+1)\left(\frac{a+2}{3}\right)^k - ka^{k-1} - 1, \quad \frac{1}{k}g''(a) = \frac{k+1}{3}\left(\frac{a+2}{3}\right)^{k-1} - \frac{k-1}{a^{2-k}}.$$

Since g'' is strictly increasing, $g''(0_+) = -\infty$ and $g''(1) = 2k(2-k)/3 > 0$, there exists $a_1 \in (0,1)$ so that $g''(a_1) = 0$, $g''(a) < 0$ for $a \in (0, a_1)$, $g''(a) > 0$ for $a \in (a_1, 1]$. Therefore, g' is strictly decreasing on $[0, a_1]$ and strictly increasing on $[a_1, 1]$. Since

$$g'(0) = (k+1)(2/3)^k - 1 \geq (k+1)(2/3)^{k_0} - 1 = \frac{k+1}{2} - 1 = \frac{k-1}{2} > 0,$$

$$g'(1) = 0,$$

there exists $a_2 \in (0, a_1)$ so that $g'(a_2) = 0$, $g'(a) > 0$ for $a \in [0, a_2)$, $g'(a) < 0$ for $a \in (a_2, 1]$. Thus, g is strictly increasing on $[0, a_2]$ and strictly decreasing on $[a_2, 1]$; consequently,

$$g(a) \geq \min\{g(0), g(1)\}.$$

From

$$g(0) = 3(2/3)^{k+1} - 1 \geq 3(2/3)^{k_0+1} - 1 = 1 - 1 = 0, \quad g(1) = 0,$$

we get $g(a) \geq 0$. This completes the proof. The equality holds for $a = b = c = 1$. If $k = k_0$, then the equality holds also for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

Remark 1. Using the Cauchy-Schwarz inequality and the inequality in P 1.45, we get

$$\sum \frac{a}{b^k + c^k} \geq \frac{(a+b+c)^2}{\sum a(b^k + c^k)} = \frac{9}{\sum a^k(b+c)} \geq \frac{3}{2}.$$

Thus, the following statement holds.

- *Let a, b, c be nonnegative real numbers so that $a+b+c=3$. If*

$$0 < k \le k_0, \quad k_0 = \frac{\ln 2}{\ln 3 - \ln 2} \approx 1.71,$$

then

$$\frac{a}{b^k+c^k} + \frac{b}{c^k+a^k} + \frac{c}{a^k+b^k} \ge \frac{3}{2},$$

with equality for $a=b=c=1$. If $k=k_0$, then the equality holds also for $a=0$ and $b=c=3/2$ (or any cyclic permutation).

Remark 2. Also, the following statement holds:

- *Let a, b, c be nonnegative real numbers so that $a+b+c=3$. If*

$$k \ge k_1, \quad k_1 = \frac{\ln 9 - \ln 8}{\ln 3 - \ln 2} \approx 0.2905,$$

then

$$\frac{a^k}{b+c} + \frac{b^k}{c+a} + \frac{c^k}{a+b} \ge \frac{3}{2},$$

with equality for $a=b=c=1$. If $k=k_1$, then the equality holds also for $a=0$ and $b=c=3/2$ (or any cyclic permutation).

For $k_1 \le k \le 2$, the inequality can be proved using the Cauchy-Schwarz inequality and the inequality in P 1.45, as follows:

$$\sum \frac{a^k}{b+c} \ge \frac{(a+b+c)^2}{\sum a^{2-k}(b+c)} = \frac{9}{\sum a^{2-k}(b+c)} \ge \frac{3}{2}.$$

For $k \ge 2$, the inequality can be deduced from the Cauchy-Schwarz inequality and Bernoulli's inequality, as follows:

$$\sum \frac{a^k}{b+c} \ge \frac{\left(\sum a^{k/2}\right)^2}{\sum(b+c)} = \frac{\left(\sum a^{k/2}\right)^2}{6},$$

$$\sum a^{k/2} \ge \sum \left[1 + \frac{k}{2}(a-1)\right] = 3.$$

□

P 1.46. *If a,b,c are nonnegative real numbers so that $a+b+c=3$, then*

$$\sqrt{a}+\sqrt{b}+\sqrt{c}-3 \ge 13\left(\sqrt{\frac{a+b}{2}}+\sqrt{\frac{b+c}{2}}+\sqrt{\frac{c+a}{2}}-3\right).$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)\ge 3f(s), \quad s=\frac{a+b+c}{3}=1,$$

where

$$f(u)=\sqrt{u}-13\sqrt{\frac{3-u}{2}}, \quad u\in[0,3].$$

For $u\in[1,3)$, we have

$$4f''(u)=-u^{-3/2}+\frac{13}{4}\left(\frac{3-u}{2}\right)^{-3/2}\ge -1+\frac{13}{4}>0.$$

Therefore, f is convex on $[s,3]$. By the RHCF-Theorem, it suffices to consider only the case $a\le b=c$. Write the original inequality in the homogeneous form

$$\sqrt{a}+\sqrt{b}+\sqrt{c}-3\sqrt{\frac{a+b+c}{3}}\ge 13\left(\sqrt{\frac{a+b}{2}}+\sqrt{\frac{b+c}{2}}+\sqrt{\frac{c+a}{2}}-3\sqrt{\frac{a+b+c}{3}}\right).$$

Due to homogeneity, we may assume that $b=c=1$. Moreover, it is convenient to use the notation $\sqrt{a}=x$. Thus, we need to show that $g(x)\ge 0$ for $x\in[0,1]$, where

$$g(x)=x-11+36\sqrt{\frac{x^2+2}{3}}-26\sqrt{\frac{x^2+1}{2}}.$$

We have

$$g'(x)=1+12x\sqrt{\frac{3}{x^2+2}}-13x\sqrt{\frac{2}{x^2+1}},$$

$$g''(x)=\frac{13}{2}\left(\frac{2}{x^2+1}\right)^{3/2}\left[\left(m\cdot\frac{x^2+1}{x^2+2}\right)^{3/2}-1\right],$$

where

$$m=\frac{6\sqrt[3]{52}}{13}\approx 1.72.$$

Clearly, $g''(x)$ has the same sign as $h(x)$, where

$$h(x)=m\cdot\frac{x^2+1}{x^2+2}-1.$$

Since h is strictly increasing,

$$h(0)=\frac{m}{2}-1<0, \quad h(1)=\frac{2m}{3}-1>0,$$

there is $x_1 \in (0,1)$ so that $h(x_1)=0$, $h(x)<0$ for $x \in [0,x_1)$ and $h(x)>0$ for $x \in (x_1,1]$. Therefore, g' is strictly decreasing on $[0,x_1]$ and strictly increasing on $[x_1,1]$. Since $g'(0)=1$ and $g'(1)=0$, there is $x_2 \in (0,x_1)$ so that $g'(x_2)=0$, $g'(x)>0$ for $x \in (0,x_2)$ and $g'(x)<0$ for $x \in (x_2,1)$. Thus, $g(x)$ is strictly increasing on $[0,x_2]$ and strictly decreasing on $[x_2,1]$. From

$$g(0)=-11+12\sqrt{6}-13\sqrt{2}>0$$

and $g(1)=0$, it follows that $g(x)\ge 0$ for $x\in[0,1]$. This completes the proof. The equality holds for $a=b=c=1$.

Remark. Similarly, we can prove the following generalizations:

- *Let a,b,c be nonnegative real numbers so that $a+b+c=3$. If $k \ge k_0$, where*

$$k_0=\frac{\sqrt{6}-2}{\sqrt{6}-\sqrt{2}-1}=(2+\sqrt{2})(2+\sqrt{3})\approx 12.74\,,$$

then

$$\sqrt{a}+\sqrt{b}+\sqrt{c}-3 \ge k\left(\sqrt{\frac{a+b}{2}}+\sqrt{\frac{b+c}{2}}+\sqrt{\frac{c+a}{2}}-3\right),$$

with equality for $a=b=c=1$. If $k=k_0$, then the equality holds also for $a=0$ and $b=c=3/2$ (or any cyclic permutation).

- *Let $a_1,a_2,\ldots,a_n$ be nonnegative real numbers so that $a_1+a_2+\cdots+a_n=n$. If $k \ge k_0$, where*

$$k_0=\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n}-\sqrt{n-2}-\frac{1}{\sqrt{n-1}}}\,,$$

then

$$\sqrt{a_1}+\sqrt{a_2}+\cdots+\sqrt{a_n}-n \ge k\left(\sqrt{\frac{n-a_1}{n-1}}+\sqrt{\frac{n-a_2}{n-1}}+\cdots+\sqrt{\frac{n-a_n}{n-1}}-n\right),$$

with equality for $a_1=a_2=\cdots=a_n=1$. If $k=k_0$, then the equality holds also for $a_1=0$ and $a_2=a_3=\cdots=a_n=\dfrac{n}{n-1}$ (or any cyclic permutation).

□

P 1.47. *Let a,b,c be nonnegative real numbers so that $a+b+c=3$. If $k>2$, then*

$$a^k+b^k+c^k+3 \ge 2\left(\frac{a+b}{2}\right)^k+2\left(\frac{b+c}{2}\right)^k+2\left(\frac{c+a}{2}\right)^k.$$

Solution. Write the inequality as

$$f(a)+f(b)+f(c)\geq 3f(s), \quad s=\frac{a+b+c}{3}=1,$$

where

$$f(u)=u^k-2\left(\frac{3-u}{2}\right)^k, \quad u\in[0,3].$$

For $u\geq 1$, we have

$$\frac{f''(u)}{k(k-1)}=u^{k-2}-\frac{1}{2}\left(\frac{3-u}{2}\right)^{k-2}\geq 1-\frac{1}{2}>0.$$

Therefore, f is convex on $[s,3]$. By the RHCF-Theorem, it suffices to consider only the case $a\leq b=c$. Write the original inequality in the homogeneous form

$$a^k+b^k+c^k+3\left(\frac{a+b+c}{3}\right)^k\geq 2\left(\frac{a+b}{2}\right)^k+2\left(\frac{b+c}{2}\right)^k+2\left(\frac{c+a}{2}\right)^k.$$

Due to homogeneity, we may assume that $b=c=1$. Thus, we need to prove that

$$a^k+3\left(\frac{a+2}{3}\right)^k\geq 4\left(\frac{a+1}{2}\right)^k$$

for $a\in[0,1]$. Substituting

$$a^k=t, \qquad t\in[0,1],$$

we need to show that $g(t)\geq 0$, where

$$g(t)=t+3\left(\frac{t^{1/k}+2}{3}\right)^k-4\left(\frac{t^{1/k}+1}{2}\right)^k.$$

We have

$$g'(t)=1+t^{1/k-1}\left(\frac{t^{1/k}+2}{3}\right)^{k-1}-2t^{1/k-1}\left(\frac{t^{1/k}+1}{2}\right)^{k-1},$$

$$\frac{kt^{2-1/k}}{k-1}g''(t)=\left(\frac{t^{1/k}+1}{2}\right)^{k-2}-\frac{2}{3}\left(\frac{t^{1/k}+2}{3}\right)^{k-2}.$$

Setting

$$m=\left(\frac{2}{3}\right)^{\frac{1}{k-2}}, \qquad 0<m<1,$$

we see that $g''(t)$ has the same sign as $h(t)$, where

$$h(t)=6\left(\frac{t^{1/k}+1}{2}-m\frac{t^{1/k}+2}{3}\right)=(3-2m)t^{1/k}+3-4m$$

is strictly increasing. There are two cases to consider: $0 < m \le 3/4$ and $3/4 < m < 1$.

Case 1: $0 < m \le 3/4$. Since $h(0) = 3 - 4m \ge 0$, we have $h(t) > 0$ for $t \in (0,1]$, hence g' is strictly increasing on $(0,1]$. From $g'(1) = 0$, it follows that $g'(t) < 0$ for $t \in (0,1)$, hence g is strictly decreasing on $[0,1]$. Since $g(1) = 0$, we get $g(t) > 0$ for $t \in [0,1)$.

Case 2: $3/4 < m < 1$. From $m > 3/4$, we get

$$2^{2k-3} > 3^{k-1}.$$

Since $h(0) = 3 - 4m < 0$ and $h(1) = 3(1-m) > 0$, there is $t_1 \in (0,1)$ so that $h(t_1) = 0$, $h(t) < 0$ for $t \in [0, t_1)$ and $h(t) > 0$ for $t \in (t_1, 1]$. Thus, $g'(t)$ is strictly decreasing on $(0, t1]$ and strictly increasing on $[t_1, 1]$. Since $g'(0_+) = +\infty$ and $g'(1) = 0$, there exists $t_2 \in (0, t_1)$ so that $g'(t_2) = 0$, $g'(t) > 0$ for $t \in (0, t_2)$ and $g'(t) < 0$ for $t \in (t_2, 1)$. Therefore, $g(t)$ is strictly increasing on $[0, t_2]$ and strictly decreasing on $[t_2, 1]$. Since

$$g(0) = \frac{2^{2k-2} - 3^{k-1}}{2^k 3^{k-1}} > 0$$

and $g(1) = 0$, we have $g(t) \ge 0$ for $t \in [0,1]$.

The equality holds for $a = b = c = 1$.

Remark 1. The inequality in P 1.47 is Popoviciu's inequality

$$f(a) + f(b) + f(c) + 3f\left(\frac{a+b+c}{3}\right) \ge 2f\left(\frac{a+b}{2}\right) + 2f\left(\frac{b+c}{2}\right) + 2f\left(\frac{c+a}{2}\right)$$

applied to the convex function $f(x) = x^k$ defined on $[0, \infty)$.

Remark 2. In the same manner, we can prove the following refinements (*Vasile C.*, 2008):

- *Let a, b, c be nonnegative real numbers so that $a + b + c = 3$. If $k > 2$ and $m \le m_0$, where*

$$m_0 = \frac{2^k(3^{k-1} - 2^{k-1})}{6^{k-1} + 3^{k-1} - 2^{2k-1}} > 2,$$

then

$$a^k + b^k + c^k - 3 \ge m\left[\left(\frac{a+b}{2}\right)^k + \left(\frac{b+c}{2}\right)^k + \left(\frac{c+a}{2}\right)^k - 3\right],$$

with equality for $a = b = c = 1$. If $m = m_0$, then the equality holds also for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

- *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If $k > 2$ and $m \le m_1$, where*

$$m_1 = \frac{\frac{1}{(n-1)^{k-1}} - \frac{1}{n^{k-1}}}{\frac{1}{(n-1)^k} + \frac{(n-2)^k}{(n-1)^{2k-1}} - \frac{1}{n^{k-1}}} > n-1,$$

then

$$a_1^k + a_2^k + \cdots + a_n^k - n \geq m\left[\left(\frac{n-a_1}{n-1}\right)^k + \left(\frac{n-a_2}{n-1}\right)^k + \cdots + \left(\frac{n-a_n}{n-1}\right)^k - n\right],$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$*. If* $m = m_1$*, then the equality holds also for* $a_1 = 0$ *and* $a_2 = a_3 = \cdots = a_n = \dfrac{n}{n-1}$ *(or any cyclic permutation).*

□

P 1.48. *If* $a_1, a_2, \ldots, a_n$ *are nonnegative real numbers so that* $a_1 + a_2 + \cdots + a_n = n$*, then*

$$\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n} + n(k-1) \leq k\left(\sqrt{\frac{n-a_1}{n-1}} + \sqrt{\frac{n-a_2}{n-1}} + \cdots + \sqrt{\frac{n-a_n}{n-1}}\right),$$

where

$$k = (\sqrt{n}-1)(\sqrt{n}+\sqrt{n-1}).$$

(Vasile C., 2008)

Solution. For $n = 2$, the inequality is an identity. Consider further that $n \geq 3$. We will show first that

$$n-1 < k < 2(n-1).$$

The left inequality reduces to

$$(\sqrt{n}-1)(\sqrt{n-1}-1) > 0,$$

while the right inequality is equivalent to

$$(\sqrt{n}-1)(\sqrt{n}-\sqrt{n-1}+2) > 0.$$

Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = -\sqrt{u} + k\sqrt{\frac{n-u}{n-1}}, \quad u \in [0, n].$$

For $u \leq 1$, we have

$$\begin{aligned} 4f''(u) &= u^{-3/2} - \frac{k}{\sqrt{n-1}}(n-u)^{-3/2} \geq 1 - \frac{k}{\sqrt{n-1}}(n-1)^{-3/2} \\ &= 1 - \frac{k}{(n-1)^2} \geq 1 - \frac{k}{2(n-1)} > 0. \end{aligned}$$

Therefore, f is convex on $[0,s]$. By the LHCF-Theorem, it suffices to consider the case

$$a_1 \ge a_2 = \cdots = a_n.$$

Write the original inequality in the homogeneous form

$$\sum \sqrt{a_1} + n(k-1)\sqrt{\frac{a_1+a_2+\cdots+a_n}{n}} \le k\sum\sqrt{\frac{a_2+\cdots+a_n}{n-1}}.$$

Do to homogeneity, we need to prove this inequality for $a_2 = \cdots = a_n = 1$ and $\sqrt{a_1} = x \ge 1$; that is, to show that $g(x) \le 0$ for $x \ge 1$, where

$$g(x) = x + n - 1 - k + (k-1)\sqrt{n(x^2+n-1)} - k\sqrt{(n-1)(x^2+n-2)}.$$

We have

$$g'(x) = 1 + (k-1)\sqrt{\frac{nx^2}{x^2+n-1}} - k\sqrt{\frac{(n-1)x^2}{x^2+n-2}},$$

$$g''(x) = \frac{k(n-2)\sqrt{n-1}}{(x^2+n-2)^{3/2}}\left[\left(m\cdot\frac{x^2+n-2}{x^2+n-1}\right)^{3/2} - 1\right],$$

where

$$m = \sqrt[3]{\frac{(k-1)^2 n(n-1)}{k^2(n-2)^2}}.$$

Clearly, $g''(x)$ has the same sign as $h(x)$, where

$$h(x) = \frac{m(x^2+n-2)}{x^2+n-1} - 1 = m\left(1 - \frac{1}{x^2+n-1}\right) - 1.$$

We have

$$h(1) = \frac{m(n-1)}{n} - 1, \qquad \lim_{x\to\infty} h(x) = m - 1.$$

We will show that $h(1) < 0$ and $\lim_{x\to\infty} h(x) > 0$; that is, to show that

$$1 < m < \frac{n}{n-1}.$$

The inequality $m > 1$ is equivalent to

$$1 - \frac{1}{k} > \frac{n-2}{\sqrt{n(n-1)}},$$

which is true since

$$1 - \frac{1}{k} > 1 - \frac{1}{n-1} = \frac{n-2}{n-1} > \frac{n-2}{\sqrt{n(n-1)}}.$$

The inequality $m < \dfrac{n}{n-1}$ is equivalent to

$$1-\frac{1}{k} < \frac{n(n-2)}{(n-1)^2},$$

which is also true because

$$1-\frac{1}{k} < 1-\frac{1}{2(n-1)} = \frac{2n-3}{2(n-1)} \le \frac{n(n-2)}{(n-1)^2}.$$

Since h is strictly increasing on $[1,\infty)$, $h(1) < 0$ and $\lim_{x\to\infty} h(x) > 0$, there is $x_1 \in (1,\infty)$ so that $h(x_1) = 0$, $h(x) < 0$ for $x \in [1,x_1)$ and $h(x) > 0$ for $x \in (x_1,\infty)$. Therefore, g' is strictly decreasing on $[1,x_1]$ and strictly increasing on $[x_1,\infty)$. Since $g'(1) = 0$ and $\lim_{x\to\infty} g'(x) = 0$, it follows that $g'(x) < 0$ for $x \in (1,\infty)$. Thus, $g(x)$ is strictly decreasing on $[1,\infty)$, hence $g(x) \le g(1) = 0$.

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = n, \qquad a_2 = a_3 = \cdots = a_n = 0$$

(or any cyclic permutation).

Remark. Since $k > n-1$ for $n \ge 3$, the inequality in P 1.48 is sharper than Popoviciu's inequality applied to the convex function $f(x) = -\sqrt{x}$, $x \ge 0$:

$$\sqrt{a_1}+\sqrt{a_2}+\cdots+\sqrt{a_n}+n(n-2) \le (n-1)\left(\sqrt{\frac{n-a_1}{n-1}}+\sqrt{\frac{n-a_2}{n-1}}+\cdots+\sqrt{\frac{n-a_n}{n-1}}\right).$$

□

P 1.49. *If a,b,c are the lengths of the sides of a triangle so that $a+b+c=3$, then*

$$\frac{1}{a+b-c}+\frac{1}{b+c-a}+\frac{1}{c+a-b}-3 \ge 4(2+\sqrt{3})\left(\frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a}-3\right).$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a)+f(b)+f(c) \ge 3f(s), \quad s = \frac{a+b+c}{3} = 1,$$

where

$$f(u) = \frac{1}{3-2u} - \frac{4k}{3-u}, \quad k = 2(2+\sqrt{3}) \approx 7.464, \qquad u \in [0,3/2).$$

For $u \ge 1$, we have

$$f''(u) = \frac{8}{(3-2u)^3} - \frac{8k}{(3-u)^3} > 8\left[\left(\frac{1}{3-2u}\right)^3 - \left(\frac{2}{3-u}\right)^3\right].$$

Since

$$\frac{1}{3-2u} \ge \frac{2}{3-u}, \qquad u \in [1,3/2),$$

it follows that f is convex on $[s,3/2)$. By the RHCF-Theorem and Note 1, it suffices to show that $h(x,y) \ge 0$ for $x,y \in [0,3/2)$ so that $x+2y=3$. We have

$$g(u) = \frac{f(u)-f(1)}{u-1} = \frac{2}{3-2u} - \frac{2k}{3-u}$$

and

$$\begin{aligned} h(x,y) = \frac{g(x)-g(y)}{x-y} &= \frac{2}{(3-2x)(3-2y)} - \frac{k}{(3-x)(3-y)} \\ &= \frac{2}{(2y-x)x} - \frac{k}{2y(x+y)} \\ &= \frac{kx^2-2(k-2)xy+4y^2}{2xy(x+y)(2y-x)} \\ &= \frac{[(\sqrt{3}+1)x-2y]^2}{2xy(x+y)(2y-x)} \ge 0. \end{aligned}$$

According to Note 4, the equality holds for $a=b=c=1$, and also for

$$a = 3(2-\sqrt{3}), \qquad b = c = \frac{3(\sqrt{3}-1)}{2}$$

(or anu cyclic permutation).

□

P 1.50. *Let* $a_1,a_2,\dots,a_5$ *be nonnegative numbers so that* $a_1+a_2+a_3+a_4+a_5 \le 5$. *If*

$$k \ge k_0, \qquad k_0 = \frac{29+\sqrt{761}}{10} \approx 5.66,$$

then

$$\sum \frac{1}{ka_1^2+a_2+a_3+a_4+a_5} \ge \frac{5}{k+4}.$$

(Vasile C., 2006)

Solution. Since each term of the left hand side of the inequality decreases by increasing any number a_i, it suffices to consider the case

$$a_1+a_2+a_3+a_4+a_5 = 5,$$

when the desired inequality can be written as

$$f(a_1)+f(a_2)+f(a_3)+f(a_4)+f(a_4) \ge 5f(s),$$

where
$$s=\frac{a_1+a_2+a_3+a_4+a_5}{5}=1$$
and
$$f(u)=\frac{1}{ku^2-u+5},\quad u\in[0,5].$$
For $u\ge 1$, we have
$$\begin{aligned}f''(u)&=\frac{2[3ku(ku-1)-5k+1]}{(ku^2-u+5)^3}\\&\ge\frac{2[3k(k-1)-5k+1]}{(ku^2-u+5)^3}\\&=\frac{2[k(3k-8)+1]}{(ku^2-u+5)^3}>0;\end{aligned}$$
therefore, f is convex on $[s,5]$. By the RHCF-Theorem, it suffices to show that
$$\frac{1}{kx^2-x+5}+\frac{4}{ky^2-y+5}\ge\frac{5}{k+4}$$
for
$$0\le x\le 1\le y,\qquad x+4y=5.$$
Write this inequality as follows:
$$\frac{1}{kx^2-x+5}-\frac{1}{k+4}+4\left[\frac{1}{ky^2-y+5}-\frac{1}{k+4}\right]\ge 0,$$
$$\frac{(x-1)(1-k-kx)}{kx^2-x+5}+\frac{4(y-1)(1-k-ky)}{ky^2-y+5}\ge 0.$$
Since
$$4(y-1)=1-x,$$
the inequality is equivalent to
$$(x-1)\left(\frac{1-k-kx}{kx^2-x+5}-\frac{1-k-ky}{ky^2-y+5}\right)\ge 0,$$
$$\frac{5(x-1)^2g(x,y,k)}{4(kx^2-x+5)(ky^2-y+5)}\ge 0,$$
where
$$g(x,y,k)=k^2xy+k(k-1)(x+y)-6k+1.$$
For fixed x and y, let $h(k)=g(x,y,k)$. Since
$$\begin{aligned}h'(k)&=2kxy+(2k-1)(x+y)-6\ge(2k-1)(x+y)-6\\&\ge(2k-1)\left(x+\frac{y}{4}\right)-6=\frac{10k-29}{4}>0,\end{aligned}$$

it suffices to show that $g(x,y,k_0) \ge 0$. We have

$$\begin{aligned} g(x,y,k_0) &= k_0^2 xy + k_0(k_0-1)(x+y) - 6k_0 + 1 \\ &= -4k_0^2 y^2 + k_0(2k_0+3)y + 5k_0^2 - 11k_0 + 1. \end{aligned}$$

Since

$$5k_0^2 - 29k_0 + 4 = 0,$$

we get

$$g(x,y,k_0) = (5-4y)\left(k_0^2 y + k_0^2 - \frac{11k_0-1}{5}\right) = x\left(k_0^2 y + k_0^2 - \frac{11k_0-1}{5}\right).$$

It suffices to show that

$$k_0^2 - \frac{11k_0-1}{5} \ge 0.$$

Indeed,

$$k_0^2 - \frac{11k_0-1}{5} = \frac{k_0(5k_0-11)+1}{5} > 0.$$

The equality holds for $a_1 = a_2 = a_3 = a_4 = a_5 = 1$. If $k = k_0$, then the equality holds also for

$$a_1 = 0, \qquad a_2 = a_3 = a_4 = a_5 = \frac{5}{4}$$

(or any cyclic permutation).

Remark. In the same manner, we can prove the following statement:

- *Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n \le n$. If*

$$k \ge k_0, \qquad k_0 = \frac{n^2+n-1+\sqrt{n^4+2n^3-5n^2+2n+1}}{2n},$$

then

$$\sum \frac{1}{ka_1^2 + a_2 + \cdots + a_n} \ge \frac{n}{k+n-1},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$. If $k = k_0$, then the equality holds also for

$$a_1 = 0, \qquad a_2 = \cdots = a_n = \frac{n}{n-1}$$

(or any cyclic permutation).

□

P 1.51. *Let $a_1, a_2, \dots, a_5$ be nonnegative numbers so that $a_1 + a_2 + a_3 + a_4 + a_5 \le 5$. If*

$$0 < k \le k_0, \qquad k_0 = \frac{11-\sqrt{101}}{10} \approx 0.095,$$

then

$$\sum \frac{1}{ka_1^2 + a_2 + a_3 + a_4 + a_5} \ge \frac{5}{k+4}.$$

(Vasile C., 2006)

Solution. As shown at the preceding P 1.50, it suffices to consider the case

$$a_1 + a_2 + a_3 + a_4 + a_5 = 5,$$

when the desired inequality can be written as

$$f(a_1) + f(a_2) + f(a_3) + f(a_4) + f(a_4) \ge 5f(s),$$

where

$$s = \frac{a_1 + a_2 + a_3 + a_4 + a_5}{5} = 1,$$

and

$$f(u) = \frac{1}{ku^2 - u + 5}, \quad u \in [0, 5].$$

For $u \in [0, 1]$, we have

$$u(ku - 1) - (k - 1) = (1 - u)(1 - ku) \ge 0,$$

hence

$$\begin{aligned} f''(u) &= \frac{2[3ku(ku-1) - 5k + 1]}{(ku^2 - u + 5)^3} \\ &\ge \frac{2[3k(k-1) - 5k + 1]}{(ku^2 - u + 5)^3} \\ &= \frac{2[(1-8k) + 3k^2]}{(ku^2 - u + 5)^3} > 0; \end{aligned}$$

therefore, f is convex on $[0, s]$. By the LHCF-Theorem, it suffices to show that

$$\frac{1}{kx^2 - x + 5} + \frac{4}{ky^2 - y + 5} \ge \frac{5}{k+4}$$

for

$$x \ge 1 \ge y \ge 0, \qquad x + 4y = 5.$$

Write this inequality as follows:

$$\frac{1}{kx^2 - x + 5} - \frac{1}{k+4} + 4\left[\frac{1}{ky^2 - y + 5} - \frac{1}{k+4}\right] \ge 0,$$

$$\frac{(x-1)(1-k-kx)}{kx^2 - x + 5} + \frac{4(y-1)(1-k-ky)}{ky^2 - y + 5} \ge 0.$$

Since

$$4(y - 1) = 1 - x,$$

the inequality is equivalent to

$$(x-1)\left(\frac{1-k-kx}{kx^2 - x + 5} - \frac{1-k-ky}{ky^2 - y + 5}\right) \ge 0,$$

$$\frac{5(x-1)^2 g(x,y,k)}{4(kx^2-x+5)(ky^2-y+5)} \geq 0,$$

where

$$g(x,y,k) = k^2xy - k(1-k)(x+y) - 6k + 1.$$

For fixed x and y, let $h(k) = g(x,y,k)$. Since

$$\begin{aligned} h'(k) &= 2kxy - (1-2k)(x+y) - 6 \leq 2kxy - 6 \\ &\leq \frac{k(x+4y)^2}{8} - 6 = \frac{25k}{8} - 6 < 0, \end{aligned}$$

it suffices to show that $g(x,y,k_0) \geq 0$. We have

$$\begin{aligned} g(x,y,k_0) &= k_0^2xy + k_0(k_0-1)(x+y) - 6k + 1 \\ &= -4k_0^2y^2 + k_0(2k_0+3)y + 5k_0^2 - 11k_0 + 1. \end{aligned}$$

Since

$$5k_0^2 - 11k_0 + 1 = 0,$$

we get

$$g(x,y,k_0) = k_0y(-4k_0y + 2k_0 + 3) \geq k_0y(-4k_0 + 2k_0 + 3) = k_0(3-2k_0)y \geq 0.$$

The equality holds for $a_1 = a_2 = a_3 = a_4 = a_5 = 1$. If $k = k_0$, then the equality holds also for

$$a_1 = 5, \qquad a_2 = a_3 = a_4 = a_5 = 0$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following statement:

- *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n \leq n$. If*

$$0 \leq k \leq k_0, \qquad k_0 = \frac{2n+1-\sqrt{4n^2+1}}{2n},$$

then

$$\sum \frac{1}{ka_1^2 + a_2 + \cdots + a_n} \geq \frac{n}{k+n-1},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$. If $k = k_0$, then the equality holds also for

$$a_1 = n, \qquad a_2 = \cdots = a_n = 0$$

(or any cyclic permutation).

□

P 1.52. *Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n \le n$. If*

$$0 < k \le \frac{1}{n+1},$$

then

$$\frac{a_1}{ka_1^2 + a_2 + \cdots + a_n} + \frac{a_2}{a_1 + ka_2^2 + \cdots + a_n} + \cdots + \frac{a_n}{a_1 + a_2 + \cdots + ka_n^2} \ge \frac{n}{k+n-1}.$$

(Vasile C., 2006)

Solution. Using the notation

$$x_1 = \frac{a_1}{s},\ x_2 = \frac{a_2}{s},\ \dots,\ x_n = \frac{a_n}{s},$$

where

$$s = \frac{a_1 + a_2 + \cdots + a_n}{n} \le 1,$$

we need to show that $x_1 + x_2 + \cdots + x_n = n$ involves

$$\frac{x_1}{ksx_1^2 + x_2 + \cdots + x_n} + \cdots + \frac{x_n}{x_1 + x_2 + \cdots + ksx_n^2} \ge \frac{n}{k+n-1}.$$

Since $s \le 1$, it suffices to prove the inequality for $s = 1$; that is, to show that

$$\frac{a_1}{ka_1^2 - a_1 + n} + \frac{a_2}{ka_2^2 - a_2 + n} + \cdots + \frac{a_n}{ka_n^2 - a_n + n} \ge \frac{n}{k+n-1}$$

for

$$a_1 + a_2 + \cdots + a_n = n.$$

Write the desired inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s),$$

where

$$s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1$$

and

$$f(u) = \frac{u}{u^2 - u + n}, \qquad u \in [0, n].$$

We have

$$f'(u) = \frac{n - ku^2}{(ku^2 - u + n)^2}, \qquad f''(u) = \frac{f_1(u)}{(u^2 - u + n)^3},$$

where

$$f_1(u) = k^2u^3 - 3knu + n.$$

For $u \in [0,1]$, we have

$$\begin{aligned} f_1(u) &\geq -3knu + n \geq -3kn + n \\ &\geq -\frac{3n}{n+1} + n = \frac{n(n-2)}{n+1} \geq 0. \end{aligned}$$

Since $f''(u) > 0$, it follows that f is convex on $[0,s]$. By the LHCF-Theorem, we only need to show that

$$\frac{x}{kx^2 - x + n} + \frac{(n-1)y}{ky^2 - y + n} \geq \frac{n}{k+n-1}$$

for all nonnegative x, y which satisfy $x + (n-1)y = n$. Write this inequality as follows:

$$\frac{x}{kx^2 - x + n} - \frac{1}{k+n-1} + (n-1)\left[\frac{y}{ky^2 - y + n} - \frac{1}{k+n-1}\right] \geq 0,$$

$$(x-1)\left(\frac{n-kx}{kx^2 - x + n} - \frac{n-ky}{ky^2 - y + n}\right) \geq 0,$$

$$\frac{(x-1)^2 h(x,y)}{(kx^2 - x + n)(ky^2 - y + n)} \geq 0,$$

where

$$h(x,y) = k^2xy - kn(x+y) + n - nk.$$

We need to show that $h(x,y) \geq 0$. Indeed,

$$\begin{aligned} h(x,y) &= ky[n(k+n-2) - k(n-1)y] + n[1 - k(n+1)] \\ &= ky[n(n-2) + kx] + n[1 - k(n+1)] \geq 0. \end{aligned}$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $k = \dfrac{1}{n+1}$, then the equality holds also for

$$a_1 = n, \qquad a_2 = a_3 = \cdots = a_n = 0$$

(or any cyclic permutation).

□

P 1.53. *If* $a_1, a_2, a_3, a_4, a_5 \leq \dfrac{7}{2}$ *so that* $a_1 + a_2 + a_3 + a_4 + a_5 = 5$*, then*

$$\frac{a_1}{a_1^2 - a_1 + 5} + \frac{a_2}{a_2^2 - a_2 + 5} + \frac{a_3}{a_3^2 - a_3 + 5} + \frac{a_4}{a_4^2 - a_4 + 5} + \frac{a_5}{a_5^2 - a_5 + 5} \leq 1.$$

(Vasile C., 2006)

Solution. Write the desired inequality as

$$f(a_1)+f(a_2)+f(a_3)+f(a_4)+f(a_5)\ge 5f(s),$$

where

$$s=\frac{a_1+a_2+a_3+a_4+a_5}{5}=1$$

and

$$f(u)=\frac{-u}{u^2-u+5},\qquad u\le\frac{7}{2}.$$

For $u\in\left[1,\frac{7}{2}\right]$, we have

$$\begin{aligned}f''(u)&=\frac{-u^3+15u-5}{(u^2-u+5)^3}\\&=\frac{(2u+9)(u-1)(7-2u)+43-7u}{4(u^2-u+5)^3}>0.\end{aligned}$$

Thus, f is convex on $\left[s,\frac{7}{2}\right]$. By the RHCF-Theorem, it suffices to show that

$$\frac{x}{x^2-x+5}+\frac{4y}{y^2-y+5}\le 1$$

for all nonnegative $x,y\le\frac{7}{2}$ which satisfy $x+4y=5$. Write this inequality as follows:

$$\frac{x}{x^2-x+5}-\frac{1}{5}+4\left(\frac{y}{y^2-y+5}-\frac{1}{5}\right)]\le 0,$$

$$(x-1)\left(\frac{5-x}{x^2-x+5}-\frac{5-y}{y^2-y+5}\right)\le 0,$$

$$\frac{(x-1)^2[5(x+y)-xy]}{(x^2-x+5)(y^2-y+5)}\ge 0,$$

$$\frac{(x-1)^2[(x+4y)(x+y)-xy]}{(x^2-x+5)(y^2-y+5)}\ge 0,$$

$$\frac{(x-1)^2(x+2y)^2}{(x^2-x+5)(y^2-y+5)}\ge 0.$$

The equality holds for $a_1=a_2=a_3=a_4=a_5=1$, and also for

$$a_1=-5,\qquad a_2=a_3=a_4=a_5=\frac{5}{2}$$

(or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

- *Let* $a_1, a_2, \ldots, a_n \le \sqrt{3}$ *so that* $a_1 + a_2 + \cdots + a_n \le n$. *If*

$$k = \frac{n^2 + 2n - 2 - 2\sqrt{(n-1)(2n^2-1)}}{n},$$

then

$$\frac{a_1}{ka_1^2 - a_1 + n} + \frac{a_2}{ka_2^2 - a_2 + n} + \cdots + \frac{a_n}{ka_n^2 - a_n + n} \le \frac{n}{k-1+n},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, *and also for*

$$a_1 = \frac{n(k-n+2)}{2k}, \qquad a_2 = \cdots = a_n = \frac{n(k+n-2)}{2k(n-1)}$$

(or any cyclic permutation).

□

P 1.54. *Let* $a_1, a_2, \ldots, a_n$ *be nonnegative real numbers so that* $a_1 + a_2 + \cdots + a_n \ge n$. *If*

$$0 < k \le \frac{1}{1 + \frac{1}{4(n-1)^2}},$$

then

$$\frac{a_1^2}{ka_1^2 + a_2 + \cdots + a_n} + \frac{a_2^2}{a_1 + ka_2^2 + \cdots + a_n} + \cdots + \frac{a_n^2}{a_1 + a_2 + \cdots + ka_n^2} \ge \frac{n}{k+n-1}.$$

(Vasile C., 2006)

Solution. Using the substitution

$$x_1 = \frac{a_1}{s},\ x_2 = \frac{a_2}{s},\ \ldots,\ x_n = \frac{a_n}{s},$$

where

$$s = \frac{a_1 + a_2 + \cdots + a_n}{n} \ge 1,$$

we need to show that $x_1 + x_2 + \cdots + x_n = n$ involves

$$\frac{x_1^2}{kx_1^2 + (x_2 + \cdots + x_n)/s} + \cdots + \frac{x_n^2}{(x_1 + \cdots + x_{n-1})/s + kx_n^2} \ge \frac{n}{k+n-1}.$$

Since $s \ge 1$, it suffices to prove the inequality for $s = 1$; that is, to show that

$$\frac{a_1^2}{ka_1^2 - a_1 + n} + \frac{a_2^2}{ka_2^2 - a_2 + n} + \cdots + \frac{a_n^2}{ka_n^2 - a_n + n} \ge \frac{n}{k+n-1}$$

for

$$a_1 + a_2 + \cdots + a_n = n.$$

Write the desired inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf(s),$$

where

$$s=\frac{a_1+a_2+\cdots+a_n}{n}=1$$

and

$$f(u)=\frac{u^2}{u^2-u+n}, \qquad u\in[0,n].$$

We have

$$f'(u)=\frac{u(2n-u)}{(ku^2-u+n)^2}, \qquad f''(u)=\frac{2f_1(u)}{(u^2-u+n)^3},$$

where

$$f_1(u)=ku^3-3knu^2+n^2.$$

For $u\in[0,1]$ and $n\ge 3$, we have

$$f_1(u)\ge -3knu^2+n^2\ge -3kn+n^2>-3n+n^2\ge 0.$$

Also, for $u\in[0,1]$ and $n=2$, we have

$$\begin{aligned}f_1(u)&=4-ku^2(6-u)\ge 4-\frac{4}{5}u^2(6-u)\\&\ge 4-\frac{4}{5}u(6-u)=\frac{4(1-u)(5-u)}{5}\ge 0.\end{aligned}$$

Since $f''(u)\ge 0$ for $u\in[0,1]$, it follows that f is convex on $[0,s]$. By the LHCF-Theorem, we need to show that

$$\frac{x^2}{kx^2-x+n}+\frac{(n-1)y^2}{ky^2-y+n}\ge\frac{n}{k+n-1}$$

for all nonnegative x,y which satisfy $x+(n-1)y=n$. Write this inequality as follows:

$$\frac{x^2}{kx^2-x+n}-\frac{1}{k+n-1}+(n-1)\left[\frac{y^2}{ky^2-y+n}-\frac{1}{k+n-1}\right]\ge 0,$$

$$\frac{(x-1)(nx-x+n)}{kx^2-x+5}+\frac{4(y-1)(ny-y+n)}{ky^2-y+5}\ge 0,$$

$$(x-1)\left(\frac{nx-x+n}{kx^2-x+n}-\frac{ny-y+n}{ky^2-y+n}\right)\ge 0,$$

$$\frac{(x-1)^2h(x,y)}{(kx^2-x+n)(ky^2-y+n)}\ge 0,$$

where

$$h(x,y)=n^2-kn(x+y)-k(n-1)xy.$$

Since

$$0 < k \le k_0, \qquad k_0 = \frac{1}{1+\frac{1}{4(n-1)^2}},$$

we have

$$\begin{aligned} h(x,y) &\ge n^2 - k_0 n(x+y) - k_0(n-1)xy \\ &= (n-1)^2 k_0 y^2 - nk_0 y + n^2(1-k_0) \\ &= k_0\left[(n-1)y - \frac{n}{2(n-1)}\right]^2 \ge 0. \end{aligned}$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $k = k_0$, then the equality holds also for

$$a_1 = \frac{n(2n-3)}{2(n-1)}, \qquad a_2 = a_3 = \cdots = a_n = \frac{n}{2(n-1)^2}$$

(or any cyclic permutation).

□

P 1.55. *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n \le n$. If $k \ge n-1$, then*

$$\frac{a_1^2}{ka_1^2 + a_2 + \cdots + a_n} + \frac{a_2^2}{a_1 + ka_2^2 + \cdots + a_n} + \cdots + \frac{a_n^2}{a_1 + a_2 + \cdots + ka_n^2} \le \frac{n}{k+n-1}.$$

(Vasile C., 2006)

Solution. Using the notation

$$x_1 = \frac{a_1}{s},\ x_2 = \frac{a_2}{s},\ \ldots,\ x_n = \frac{a_n}{s},$$

where

$$s = \frac{a_1 + a_2 + \cdots + a_n}{n} \le 1,$$

we need to show that $x_1 + x_2 + \cdots + x_n = n$ involves

$$\frac{x_1^2}{kx_1^2 + (x_2 + \cdots + x_n)/s} + \cdots + \frac{x_n^2}{(x_1 + \cdots + x_{n-1})/s + kx_n^2} \le \frac{n}{k+n-1}.$$

Since $s \le 1$, it suffices to prove the inequality for $s = 1$; that is, to show that

$$\frac{a_1^2}{ka_1^2 - a_1 + n} + \frac{a_2^2}{ka_2^2 - a_2 + n} + \cdots + \frac{a_n^2}{ka_n^2 - a_n + n} \le \frac{n}{k+n-1}$$

for

$$a_1 + a_2 + \cdots + a_n = n.$$

Write the desired inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf(s), \quad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=\frac{-u^2}{u^2-u+n}, \quad u\in[0,n].$$

We have

$$f'(u)=\frac{u(u-2n)}{(ku^2-u+n)^2}, \qquad f''(u)=\frac{2f_1(u)}{(u^2-u+n)^3},$$

where

$$f_1(u)=-ku^3+3knu^2-n^2.$$

For $u\in[1,n]$, we have

$$\begin{aligned}f_1(u)&\ge -knu^2+3knu^2-n^2=2knu^2-n^2\\&\ge 2kn-n^2\ge 2(n-1)n-n^2=n(n-2)\ge 0.\end{aligned}$$

Since $f''(u)\ge 0$ for $u\in[1,n]$, it follows that f is convex on $[s,n]$. By the RHCF-Theorem, it suffices to show that

$$\frac{x^2}{kx^2-x+n}+\frac{(n-1)y^2}{ky^2-y+n}\le\frac{n}{k+n-1}$$

for all nonnegative x,y which satisfy $x+(n-1)y=n$. As shown in the proof of the preceding P 1.54, we only need to show that $h(x,y)\ge 0$, where

$$h(x,y)=kn(x+y)+k(n-1)xy-n^2.$$

Since $k\ge n-1$, we have

$$\begin{aligned}h(x,y)&\ge n(n-1)(x+y)+(n-1)^2xy-n^2\\&=-(n-1)^3y^2+n(n-1)y+n^2(n-2)\\&=[n-(n-1)y][n(n-2)+(n-1)^2y]\\&=x[n(n-2)+(n-1)^2y]\ge 0.\end{aligned}$$

The equality holds for $a_1=a_2=\cdots=a_n=1$. If $k=n-1$, then the equality holds also for

$$a_1=0, \qquad a_2=a_3=\cdots=a_n=\frac{n}{n-1}$$

(or any cyclic permutation).

□

P 1.56. *Let $a_1, a_2, \ldots, a_n \in [0, n]$ so that $a_1 + a_2 + \cdots + a_n \geq n$. If $0 < k \leq \dfrac{1}{n}$, then*

$$\frac{a_1 - 1}{ka_1^2 + a_2 + \cdots + a_n} + \frac{a_2 - 1}{a_1 + ka_2^2 + \cdots + a_n} + \cdots + \frac{a_n - 1}{a_1 + a_2 + \cdots + ka_n^2} \geq 0.$$

(Vasile C., 2006)

Solution. Let

$$s = \frac{a_1 + a_2 + \cdots + a_n}{n}, \qquad s \geq 1.$$

Case 1: $s > 1$ Without loss of generality, assume that

$$a_1 \geq \cdots \geq a_j > 1 \geq a_{j+1} \cdots \geq a_n, \qquad j \in \{1, 2, \ldots, n\}.$$

Clearly, there are $b_1, b_2, \ldots, b_n$ so that $b_1 + b_2 + \cdots + b_n = n$ and

$$a_1 \geq b_1 \geq 1, \ \ldots, \ a_j \geq b_j \geq 1, \ b_{j+1} = a_{j+1}, \ \ldots, \ b_n = a_n.$$

Write the desired inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq 0,$$

where

$$f(u) = \frac{u - 1}{ku^2 - u + ns}, \qquad u \in [0, n],$$

$$f'(u)\frac{f_1(u)}{(ku^2 - u + ns)^2}, \qquad f_1(u) = k(-u^2 + 2u) + ns - 1.$$

For $u \in [1, n)$, we have

$$\begin{aligned} f_1(u) &\geq k(-nu + 2u) + ns - 1 = -k(n-2)u + ns - 1 \\ &\geq -k(n-2)n + ns - 1 \geq -(n-2) + ns - 1 = n(s-1) + 1 > 0. \end{aligned}$$

Consequently, f is strictly increasing on $[1, n]$ and

$$f(b_1) \leq f(a_1), \ \ldots, \ f(b_j) \leq f(a_j), \ f(b_{j+1}) = f(a_{j+1}), \ \ldots, \ f(b_n) = f(a_n).$$

Since

$$f(b_1) + f(b_2) + \cdots + f(b_n) \leq f(a_1) + f(a_2) + \cdots + f(a_n),$$

it suffices to show that $f(b_1) + f(b_2) + \cdots + f(b_n) \geq 0$ for $b_1 + b_2 + \cdots + b_n = n$. This inequality is proved at Case 2.

Case 2: $s = 1$. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \qquad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u)=\frac{u-1}{ku^2-u+n},\quad u\in[0,n],$$

$$f''(u)=\frac{2g(u)}{(ku^2-u+n)^3},\quad g(u)=k^2u^3-3k^2u^2-3k(n-1)u+kn+n-1.$$

We will show that $f''(u)\ge 0$ for $u\in[0,1]$. From

$$g'(u)=3k^2u(u-2)-3k(n-1),$$

it follows that $g'(u)<0$, g is decreasing, hence

$$\begin{aligned} g(u)\ge g(1)&=-2k^2-(2n-3)k+n-1\\ &\ge\frac{-2}{n^2}-\frac{2n-3}{n}+n-1\\ &=\frac{(n-1)^3-1}{n^2}\ge 0. \end{aligned}$$

Thus, f is convex on $[0,s]$. By the LHCF-Theorem, it suffices to show that

$$\frac{x-1}{kx^2-x+n}+\frac{(n-1)(y-1)}{ky^2-y+n}\ge 0$$

for all nonnegative real x,y so that $x+(n-1)y=n$. Since $(n-1)(y-1)=1-x$, we have

$$\begin{aligned} \frac{x-1}{kx^2-x+n}+\frac{(n-1)(y-1)}{ky^2-y+n}&=(x-1)\left(\frac{1}{kx^2-x+n}-\frac{1}{ky^2-y+n}\right)\\ &=\frac{(x-1)(x-y)(1-kx-ky)}{(kx^2-x+n)(ky^2-y+n)}\\ &=\frac{n(x-1)^2(1-kx-ky)}{(n-1)(kx^2-x+n)(ky^2-y+n)}\\ &\ge\frac{n(x-1)^2(1-\frac{x+y}{n})}{(n-1)(kx^2-x+n)(ky^2-y+n)}\\ &=\frac{(n-2)y(x-1)^2}{(n-1)(kx^2-x+n)(ky^2-y+n)}\ge 0. \end{aligned}$$

The proof is completed. The equality holds for $a_1=a_2=\cdots=a_n=1$. If $k=\dfrac{1}{n}$, then the equality holds also for

$$a_1=n,\qquad a_2=a_3=\cdots=a_n=0.$$

□

P 1.57. *If a, b, c are positive real numbers so that $abc = 1$, then*

$$\sqrt{a^2-a+1}+\sqrt{b^2-b+1}+\sqrt{c^2-c+1} \ge a+b+c.$$

Solution. Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z,$$

we need to show that

$$f(x)+f(y)+f(z) \ge 3f(s), \quad s = \frac{x+y+z}{3} = 0,$$

where

$$f(u) = \sqrt{e^{2u}-e^u+1}-e^u, \quad u \in \mathbb{I} = \mathbb{R}.$$

We claim that f is convex on $\mathbb{I}_{\ge s}$. Since

$$e^{-u}f''(u) = \frac{4e^{3u}-6e^{2u}+9e^u-2}{4(e^{2u}-e^u+1)^{3/2}} - 1,$$

we need to show that $4x^3-6x^2+9x-2 > 0$ and

$$(4x^3-6x^2+9x-2)^2 \ge 16(x^2-x+1)^3,$$

where $x = e^u \ge 1$. Indeed,

$$4x^3-6x^2+9x-2 = x(x-3)^2+(3x^3-2) > 0$$

and

$$(4x^3-6x^2+9x-2)^2-16(x^2-x+1)^3 = 12x^3(x-1)+9x^2+12(x-1) > 0.$$

By the RHCF-Theorem, it suffices to prove the original inequality for

$$b = c := t, \quad a = 1/t^2, \quad t > 0;$$

that is,

$$\frac{\sqrt{t^4-t^2+1}}{t^2}+2\sqrt{t^2-t+1} \ge \frac{1}{t^2}+2t,$$

$$\frac{t^2-1}{\sqrt{t^4-t^2+1}+1}+\frac{2(1-t)}{\sqrt{t^2-t+1}+t} \ge 0.$$

Since

$$\frac{t^2-1}{\sqrt{t^4-t^2+1}} \ge \frac{t^2-1}{t^2+1},$$

it suffices to show that

$$\frac{t^2-1}{t^2+1}+\frac{2(1-t)}{\sqrt{t^2-t+1}+t} \ge 0,$$

which is equivalent to

$$(t-1)\left[\frac{t+1}{t^2+1}-\frac{2}{\sqrt{t^2-t+1}+t}\right]\ge 0,$$

$$(t-1)\left[(t+1)\sqrt{t^2-t+1}-t^2+t-2\right]\ge 0,$$

$$\frac{(t-1)^2(3t^2-2t+3)}{(t+1)\sqrt{t^2-t+1}+t^2-t+2}\ge 0.$$

The equality holds for $a=b=c=1$.

□

P 1.58. *If* $a,b,c,d\ge \dfrac{1}{1+\sqrt{6}}$ *so that* $abcd=1$*, then*

$$\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}+\frac{1}{d+2}\le\frac{4}{3}.$$

(Vasile C., 2005)

Solution. Using the notation

$$a=e^x,\quad b=e^y,\quad c=e^z,\quad d=e^w,$$

we need to show that

$$f(x)+f(y)+f(z)+f(w)\ge 4f(s),\quad s=\frac{x+y+z+w}{4}=0,$$

where

$$f(u)=\frac{-1}{e^u+2},\quad u\in\mathbb{I}=\mathbb{R}.$$

For $u\le 0$, we have

$$f''(u)=\frac{e^u(2-e^u)}{(e^u+2)^3}>0,$$

hence f is convex on $\mathbb{I}_{\le s}$. By the LHCF-Theorem, it suffices to prove the original inequality for

$$b=c=d:=t,\quad a=1/t^3,\quad t\ge\frac{1}{1+\sqrt{6}};$$

that is,

$$\frac{t^3}{2t^3+1}+\frac{3}{t+2}\le\frac{4}{3},$$

which is equivalent to the obvious inequality

$$(t-1)^2(5t^2+2t-1)\ge 0.$$

According to Note 4, the equality holds for $a = b = c = d = 1$, and also for

$$a = 19 + 9\sqrt{6}, \qquad b = c = d = \frac{1}{1+\sqrt{6}}$$

(or any cyclic permutation).

□

P 1.59. *If a, b, c are positive real numbers so that $abc = 1$, then*

$$a^2 + b^2 + c^2 - 3 \geq 2(ab + bc + ca - a - b - c).$$

Solution. Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z,$$

we need to show that

$$f(x) + f(y) + f(z) \geq 3f(s), \quad s = \frac{x+y+z}{3} = 0,$$

where

$$f(u) = e^{2u} - 1 + 2(e^u - e^{-u}), \quad u \in \mathbb{R} = \mathbb{R}.$$

For $u \geq 0$, we have

$$f''(u) = 4e^{2u} + 2(e^u - e^{-u}) > 0,$$

hence f is convex on $\mathbb{I}_{\geq s}$. By the RHCF-Theorem, it suffices to prove the original inequality for $b = c := t$ and $a = 1/t^2$, where $t > 0$; that is, to show that

$$4t^5 - 3t^4 - 4t^3 + 2t^2 + 1 \geq 0,$$

which is equivalent to

$$(t-1)^2(4t^3 + 5t^2 + 2t + 1) \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 1.60. *If a, b, c are positive real numbers so that $abc = 1$, then*

$$a^2 + b^2 + c^2 - 3 \geq 18(a + b + c - ab - bc - ca).$$

Solution. Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z,$$

we need to show that

$$f(x) + f(y) + f(z) \ge 3f(s), \quad s = \frac{x+y+z}{3} = 0,$$

where

$$f(u) = e^{2u} - 1 - 18(e^u - e^{-u}), \quad u \in \mathbb{R}.$$

For $u \le 0$, we have

$$f''(u) = 4e^{2u} + 18(e^{-u} - e^u) > 0,$$

hence f is convex on $\mathbb{I}_{\le s}$. By the LHCF-Theorem, it suffices to prove the original inequality for $b = c := t$ and $a = 1/t^2$, where $t > 0$. Since

$$a^2 + b^2 + c^2 - 3 = \frac{1}{t^4} + 2t^2 - 3 = \frac{(t^2-1)^2(2t^2+1)}{t^4}$$

and

$$a + b + c - ab - bc - ca = \frac{-(t^4 - 2t^3 + 2t - 1)}{t^2} = \frac{-(t-1)^3(t+1)}{t^2},$$

we get

$$a^2+b^2+c^2-3-18(a+b+c-ab-bc-ca) = \frac{(t-1)^2(2t-1)^2(t+1)(5t+1)}{t^4} \ge 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 4$ and $b = c = 1/2$ (or any cyclic permutation).

□

P 1.61. *If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1 a_2 \cdots a_n = 1$, then*

$$a_1^2 + a_2^2 + \cdots + a_n^2 - n \ge 6\sqrt{3}\left(a_1 + a_2 + \cdots + a_n - \frac{1}{a_1} - \frac{1}{a_2} - \cdots - \frac{1}{a_n}\right).$$

Solution. Using the notation $a_i = e^{x_i}$ for $i = 1, 2, \ldots, n$, we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge nf(s), \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

where

$$f(u) = e^{2u} - 1 - 6\sqrt{3}\,(e^u - e^{-u}), \quad u \in \mathbb{I} = \mathbb{R}.$$

For $u \le 0$, we have

$$f''(u) = 4e^{2u} + 6\sqrt{3}(e^{-u} - e^{u}) > 0,$$

hence f is convex on $\mathbb{I}_{\le s}$. By the LHCF-Theorem and Note 2, it suffices to show that $H(x,y) \ge 0$ for $x, y \in \mathbb{R}$ so that $x + (n-1)y = 0$, where

$$H(x,y) = \frac{f'(x) - f'(y)}{x - y}.$$

From

$$f'(u) = 2e^{2u} - 6\sqrt{3}\,(e^{u} + e^{-u}),$$

we get

$$H(x,y) = \frac{2(e^{x} - e^{y})}{x - y}\left(e^{x} + e^{y} - 3\sqrt{3} + 3\sqrt{3}\,e^{-x-y}\right).$$

Since $(e^{x} - e^{y})/(x - y) > 0$, we need to prove that

$$e^{x} + e^{y} + 3\sqrt{3}\,e^{-x-y} \ge 3\sqrt{3}.$$

Indeed, by the AM-GM inequality, we have

$$e^{x} + e^{y} + 3\sqrt{3}\,e^{-x-y} \ge 3\sqrt[3]{e^{x} \cdot e^{y} \cdot 3\sqrt{3}\,e^{-x-y}} = 3\sqrt{3}.$$

The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.62. *If $a_1, a_2, \ldots, a_n$ $(n \ge 4)$ are positive real numbers so that $a_1 a_2 \cdots a_n = 1$, then*

$$(n-1)(a_1^2 + a_2^2 + \cdots + a_n^2) + n(n+3) \ge (2n+2)(a_1 + a_2 + \cdots + a_n).$$

Solution. Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \ldots, n$, we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge nf(s), \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

where

$$f(u) = (n-1)e^{2u} - (2n+2)e^{u}, \quad u \in \mathbb{I} = \mathbb{R}.$$

For $u \ge 0$, we have

$$\begin{aligned} f''(u) &= 4(n-1)e^{2u} - (2n+2)e^{u} \\ &= 2e^{u}[2(n-1)e^{u} - n - 1] \\ &\ge 2e^{u}[2(n-1) - n - 1] = 2(n-3)e^{u} > 0. \end{aligned}$$

Therefore, f is convex on $\mathbb{I}_{\geq s}$. By the RHCF-Theorem and Note 2, it suffices to show that $H(x,y)\geq 0$ for $x,y\in\mathbb{R}$ so that $x+(n-1)y=0$, where

$$H(x,y)=\frac{f'(x)-f'(y)}{x-y}.$$

From

$$f'(u)=2(n-1)e^{2u}-(2n+2)e^{u},$$

we get

$$H(x,y)=\frac{2(e^x-e^y)}{x-y}\left[(n-1)(e^x+e^y)-(n+1)\right].$$

Since $(e^x-e^y)/(x-y)>0$, we need to prove that $(n-1)(e^x+e^y)\geq n+1$. Using the AM-GM inequality, we have

$$\begin{aligned}(n-1)(e^x+e^y)&=(n-1)e^x+e^y+e^y+\cdots+e^y\\&\geq n\sqrt[n]{(n-1)e^x\cdot e^y\cdot e^y\cdots e^y}\\&=n\sqrt[n]{(n-1)e^{x+(n-1)y}}=n\sqrt[n]{n-1}.\end{aligned}$$

Thus, it suffices to show that

$$n\sqrt[n]{n-1}\geq n+1,$$

which is equivalent to

$$n-1\geq\left(1+\frac{1}{n}\right)^n.$$

This is true for $n\geq 4$, since

$$n-1\geq 3>\left(1+\frac{1}{n}\right)^n.$$

The proof is completed. The equality holds for $a_1=a_2=\cdots=a_n=1$.

Remark. From the proof above, the following sharper inequality follows (*Gabriel Dospinescu* and *Calin Popa*):

- *If $a_1,a_2,\ldots,a_n$ are positive real numbers so that $a_1a_2\cdots a_n=1$, then*

$$a_1^2+a_2^2+\cdots+a_n^2-n\geq\frac{2n\sqrt[n]{n-1}}{n-1}(a_1+a_2+\cdots+a_n-n).$$

□

P 1.63. *Let $a_1,a_2,\ldots,a_n$ $(n\geq 3)$ be positive real numbers so that $a_1a_2\cdots a_n=1$. If $p,q\geq 0$ so that $p+q\geq n-1$, then*

$$\frac{1}{1+pa_1+qa_1^2}+\frac{1}{1+pa_2+qa_2^2}+\cdots+\frac{1}{1+pa_n+qa_n^2}\geq\frac{n}{1+p+q}.$$

(Vasile C., 2007)

Solution. Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \ldots, n$, we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s), \qquad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

where

$$f(u) = \frac{1}{1 + pe^u + qe^{2u}}, \qquad u \in \mathbb{I} = \mathbb{R}.$$

For $u \geq 0$, we have

$$\begin{aligned} f''(u) &= \frac{e^u[4q^2e^{3u} + 3pqe^{2u} + (p^2 - 4q)e^u - p]}{(1 + pe^u + qe^{2u})^3} \\ &\geq \frac{e^{2u}[4q^2 + 3pq + (p^2 - 4q) - p]}{(1 + pe^u + qe^{2u})^3} \\ &= \frac{e^{2u}[(p + 2q)(p + q - 2) + 2q^2 + p]}{(1 + pe^u + qe^{2u})^3} > 0, \end{aligned}$$

therefore f is convex on $\mathbb{I}_{\geq s}$. By the RHCF-Theorem, it suffices to prove the original inequality for

$$a_1 = 1/t^{n-1}, \quad a_2 = \cdots = a_n = t, \qquad t > 0.$$

Write this inequality as

$$\frac{t^{2n-2}}{t^{2n-2} + pt^{n-1} + q} + \frac{n-1}{1 + pt + qt^2} \geq \frac{n}{1 + p + q}.$$

Applying the Cauchy-Schwarz inequality, it suffices to prove that

$$\frac{(t^{n-1} + n - 1)^2}{(t^{2n-2} + pt^{n-1} + q) + (n-1)(1 + pt + qt^2)} \geq \frac{n}{1 + p + q},$$

which is equivalent to

$$pB + qC \geq A,$$

where

$$A = (n-1)(t^{n-1} - 1)^2 \geq 0,$$

$$B = (t^{n-1} - 1)^2 + nE = \frac{A}{n-1} + nE, \qquad E = t^{n-1} + n - 2 - (n-1)t,$$

$$C = (t^{n-1} - 1)^2 + nF = \frac{A}{n-1} + nF, \qquad F = 2t^{n-1} + n - 3 - (n-1)t^2.$$

By the AM-GM inequality applied to $n-1$ positive numbers, we have $E \geq 0$ and $F \geq 0$ for $n \geq 3$. Since $A \geq 0$ and $p + q \geq n - 1$, we have

$$pB + qC - A \geq pB + qC - \frac{(p+q)A}{n-1} = n(pE + qF) \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark 1. For $p = 2k$ and $q = k^2$, we get the following result:

- *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be positive real numbers so that $a_1 a_2 \cdots a_n = 1$. If $k \ge \sqrt{n} - 1$, then*

$$\frac{1}{(1+ka_1)^2} + \frac{1}{(1+ka_2)^2} + \cdots + \frac{1}{(1+ka_n)^2} \ge \frac{n}{(1+k)^2},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$.

In addition, for $n = 4$ and $k = 1$, we get the known inequality (*Vasile C.*, 1999):

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \ge 1,$$

where $a, b, c, d > 0$ so that $abcd = 1$.

Remark 2. For $p + q = n - 1$ $(n \ge 3)$, we get the beautiful inequality

$$\frac{1}{1+pa_1+qa_1^2} + \frac{1}{1+pa_2+qa_2^2} + \cdots + \frac{1}{1+pa_n+qa_n^2} \ge 1,$$

which is a generalization of the following inequalities:

$$\frac{1}{1+(n-1)a_1} + \frac{1}{1+(n-1)a_2} + \cdots + \frac{1}{1+(n-1)a_n} \ge 1,$$

$$\frac{1}{[1+(\sqrt{n}-1)a_1]^2} + \frac{1}{[1+(\sqrt{n}-1)a_1]^2} + \cdots + \frac{1}{[1+(\sqrt{n}-1)a_1]^2} \ge 1,$$

$$\frac{1}{2+(n-1)(a_1+a_1^2)} + \frac{1}{2+(n-1)(a_2+a_2^2)} + \cdots + \frac{1}{2+(n-1)(a_n+a_n^2)} \ge \frac{1}{2}.$$

□

P 1.64. *Let a, b, c, d be positive real numbers so that $abcd = 1$. If p and q are nonnegative real numbers so that $p + q = 3$, then*

$$\frac{1}{1+pa+qa^3} + \frac{1}{1+pb+qb^3} + \frac{1}{1+pc+qc^3} + \frac{1}{1+pd+qd^3} \ge 1.$$

(Vasile C., 2007)

Solution. Using the notation

$$a = e^x, \quad b = e^y, \quad c = e^z, \quad d = e^w,$$

we need to show that

$$f(x) + f(y) + f(z) + f(w) \geq 4f(s), \qquad s = \frac{x+y+z+w}{4} = 0,$$

where

$$f(u) = \frac{1}{1 + pe^u + qe^{3u}}, \qquad u \in \mathbb{I} = \mathbb{R}.$$

We will show that $f''(u) > 0$ for $u \geq 0$, hence f is convex on $\mathbb{I}_{\geq s}$. Since

$$f''(u) = \frac{th(t)}{(1 + pt + qt^3)^3},$$

where

$$h(t) = 9q^2t^5 + 2pqt^3 - 9qt^2 + p^2t - p, \qquad t = e^u,$$

we need to show that $h(t) \geq 0$ for $t \geq 1$. Indeed, we have

$$h(t) \geq 9q^2t^3 + 2pqt^3 - 9qt^2 + p^2t - pt = tg(t),$$

where

$$\begin{aligned}
g(t) &= (9q^2 + 2pq)t^2 - 9qt + p^2 - p \\
&\geq (9q^2 + 2pq)(2t - 1) - 9qt + p^2 - p \\
&= q(18q + 4p - 9)t - 9q^2 - 2pq + p^2 - p \\
&\geq q(18q + 4p - 9) - 9q^2 - 2pq + p^2 - p \\
&= p^2 + 2pq + 9q^2 - p - 9q \\
&= p^2 + 2pq + 9q^2 - \frac{(p + 9q)(p + q)}{3} \\
&= \frac{2(p - q)^2 + 16q^2}{3} \geq 0.
\end{aligned}$$

By the RHCF-Theorem, it suffices to prove the original inequality for

$$b = c = d = t, \quad a = 1/t^3, \qquad t > 0;$$

that is,

$$\frac{t^9}{t^9 + pt^6 + q} + \frac{3}{1 + pt + qt^3} \geq 1,$$

$$\frac{3}{1 + pt + qt^3} \geq \frac{pt^6 + q}{t^9 + pt^6 + q},$$

$$(3 - pq)t^9 - p^2t^7 + 2pt^6 - q^2t^3 - pqt + 2q \geq 0,$$

$$[(p+q)^2-3pq]t^9-3p^2t^7+2p(p+q)t^6-3q^2t^3-3pqt+2q(p+q)\geq 0,$$

$$Ap^2+Bq^2\geq Cpq,$$

where

$$A=t^9-3t^7+2t^6=t^6(t-1)^2(t+2)\geq 0,$$

$$B=t^9-3t^3+2=(t^3-1)^2(t^3+2)\geq 0,$$

$$C=t^9-2t^6+3t-2.$$

Since $A\geq 0$ and $B\geq 0$, it suffices to consider the case $C\geq 0$. Since

$$Ap^2+Bq^2\geq 2\sqrt{AB}pq,$$

we only need to show that $4AB\geq C^2$. From

$$t^3-3t+2=(t-1)^2(t+2)\geq 0,$$

we get $3t-2\leq t^3$. Therefore

$$C\leq t^9-2t^6+t^3=t^3(t^3-1)^2,$$

hence

$$\begin{aligned}4AB-C^2&\geq 4AB-t^6(t^3-1)^4\\&=t^6(t-1)^2(t^3-1)^2[4(t+2)(t^3+2)-(t^2+t+1)^2]\\&=t^6(t-1)^2(t^3-1)^2(3t^4+6t^3-3t^2+6t+15)\geq 0.\end{aligned}$$

The proof is completed. The inequality holds for $a=b=c=d=1$.

Remark 1. For $p=1$ and $p=2$, we get the following nice inequalities:

$$\frac{1}{1+a+2a^3}+\frac{1}{1+b+2b^3}+\frac{1}{1+c+2c^3}+\frac{1}{1+d+2d^3}\geq 1,$$

$$\frac{1}{1+2a+a^3}+\frac{1}{1+2b+b^3}+\frac{1}{1+2c+c^3}+\frac{1}{1+2d+d^3}\geq 1.$$

Remark 2. Similarly, we can prove the following generalizations:

- *Let a,b,c,d be positive real numbers so that $abcd=1$. If p and q are nonnegative real numbers so that $p+q\geq 3$, then*

$$\frac{1}{1+pa+qa^3}+\frac{1}{1+pb+qb^3}+\frac{1}{1+pc+qc^3}+\frac{1}{1+pd+qd^3}\geq\frac{4}{1+p+q}.$$

• *Let $a_1,a_2,\ldots,a_n$ $(n\ge 4)$ be positive real numbers so that $a_1a_2\cdots a_n=1$. If $p,q,r\ge 0$ so that $p+q+r\ge n-1$, then*

$$\sum_{i=1}^{n}\frac{1}{1+pa_i+qa_i^2+ra_i^3}\ge\frac{n}{1+p+q+r}.$$

For $n=4$ and $p+q+r=3$, we get the beautiful inequality

$$\sum_{i=1}^{4}\frac{1}{1+pa_i+qa_i^2+ra_i^3}\ge 1.$$

Since

$$a_i^2\le\frac{a_i+a_i^3}{2},$$

the best inequality with respect to q if for $q=0$:

$$\sum_{i=1}^{4}\frac{1}{1+pa_i+ra_i^3}\ge 1,\quad p+r=3.$$

□

P 1.65. *If $a_1,a_2,\ldots,a_n$ are positive real numbers so that $a_1a_2\cdots a_n=1$, then*

$$\frac{1}{1+a_1+\cdots+a_1^{n-1}}+\frac{1}{1+a_2+\cdots+a_2^{n-1}}+\cdots+\frac{1}{1+a_n+\cdots+a_n^{n-1}}\ge 1.$$

(Vasile C., 2007)

Solution. Using the substitutions $a_i=e^{x_i}$ for $i=1,2,\ldots,n$, we need to show that

$$f(x_1)+f(x_2)+\cdots+f(x_n)\ge nf(s),\qquad s=\frac{x_1+x_2+\cdots+x_n}{n}=0,$$

where

$$f(u)=\frac{1}{1+e^u+\cdots+e^{(n-1)u}},\quad u\in\mathbb{I}=\mathbb{R}.$$

We will show by induction on n that f is convex on $\mathbb{I}_{\ge s}$. Setting $t=e^u$, the condition $f''(u)\ge 0$ for $u\ge 0$ $(t\ge 1)$ is equivalent to

$$2A^2\ge B(1+C),$$

where

$$A=t+2t^2+\cdots+(n-1)t^{n-1},$$

$$B=t+4t^2+\cdots+(n-1)^2t^{n-1},$$

$$C=t+t^2+\cdots+t^{n-1}.$$

For $n=2$, the inequality becomes $t(t-1)\ge 0$. Assume now that the inequality is true for n and prove it for $n+1$, $n\ge 2$. So, we need to show that $2A^2\ge B(1+C)$ involves

$$2(A+nt^n)^2\ge (B+n^2t^n)(1+C+t^n),$$

which is equivalent to

$$2A^2-B(1+C)+t^n[n^2(t^n-1)+D]\ge 0,$$

where

$$D=4nA-B-n^2C=\sum_{i=1}^{n-1}b_it^i,\quad b_i=3n^2-(2n-i)^2.$$

Since $2A^2-B(1+C)\ge 0$ (by the induction hypothesis), it suffices to show that $D\ge 0$. Since

$$b_1<b_2<\cdots<b_{n-1},\quad t\le t^2\le\cdots\le t^{n-1},$$

we may apply Chebyshev's inequality to get

$$D\ge\frac{1}{n}(b_1+b_2+\cdots+b_{n-1})(t+t^2+\cdots+t^{n-1}).$$

Thus, it suffices to show that $b_1+b_2+\cdots+b_{n-1}\ge 0$. Indeed,

$$b_1+b_2+\cdots+b_{n-1}=\sum_{i=1}^{n-1}[3n^2-(2n-i)^2]=\frac{n(n-1)(4n+1)}{6}>0.$$

By the RHCF-Theorem, it suffices to prove the original inequality for

$$a_1=1/t^{n-1},\quad a_2=\cdots=a_n=t,\quad t\ge 1,$$

Setting $k=n-1$ $(k\ge 1)$, we need to show that

$$\frac{t^{k^2}}{1+t^k+\cdots+t^{k^2}}+\frac{k}{1+t+\cdots+t^k}\ge 1.$$

For the nontrivial case $t>1$, this inequality is equivalent to each of the following inequalities:

$$\frac{k}{1+t+\cdots+t^k}\ge\frac{1+t^k+\cdots+t^{(k-1)k}}{1+t^k+\cdots+t^{k^2}},$$

$$\frac{k(t-1)}{t^{k+1}-1}\ge\frac{t^{k^2}-1}{t^k-1}\cdot\frac{t^k-1}{t^{(k+1)k}-1},$$

$$\frac{k(t-1)}{t^{k+1}-1}\ge\frac{t^{k^2}-1}{t^{(k+1)k}-1},$$

$$k\frac{t^{(k+1)k}-1}{t^{k+1}-1}\ge\frac{t^{k^2}-1}{t-1},$$

$$k\left[1+t^{k+1}+t^{2(k+1)}+\cdots+t^{(k-1)(k+1)}\right]\geq 1+t+t^2+\cdots+t^{(k-1)(k+1)},$$

$$k\left[1\cdot 1+t\cdot t^k+\cdots+t^{k-1}\cdot t^{(k-1)k}\right]\geq\left(1+t+\cdots+t^{k-1}\right)\left[1+t^k+\cdots+t^{(k-1)k}\right].$$

Since $1<t<\cdots<t^{k-1}$ and $1<t^k<\cdots<t^{(k-1)k}$, the last inequality follows from Chebyshev's inequality.

The equality holds for $a_1=a_2=\cdots=a_n=1$.

Remark. Actually, the following generalization holds:

• *Let $a_1,a_2,\ldots,a_n$ be positive numbers so that $a_1a_2\cdots a_n=1$, and let $k_1,k_2,\ldots,k_m\geq 0$ so that $k_1+k_2+\cdots+k_m\geq n-1$. If $m\leq n-1$, then*

$$\sum_{i=1}^{n}\frac{1}{1+k_1a_i+k_2a_i^2+\cdots+k_ma_i^m}\geq\frac{n}{1+k_1+k_2+\cdots+k_m}.$$

In addition, since

$$a_i^k\leq\frac{(m-k)a_i+(k-1)a_i^m}{m-1},\qquad k=2,3,\ldots,m-1$$

(by the AM-GM inequality applied to $m-1$ positive numbers), the best inequality with respect to $k_2,\ldots,k_{m-1}$ is for $k_2=0,\ldots,k_{m-1}=0$; that is,

$$\sum_{i=1}^{n}\frac{1}{1+k_1a_i+k_ma_i^m}\geq\frac{n}{1+k_1+k_m},\qquad k_1+k_m\geq n-1,\quad 1\leq m\leq n-1.$$

If $k_1+k_m=n-1$, then

$$\sum_{i=1}^{n}\frac{1}{1+k_1a_i+k_ma_i^m}\geq 1,\qquad 1\leq m\leq n-1,$$

therefore

$$\sum_{i=1}^{n}\frac{1}{1+k_1a_i+k_{n-1}a_i^{n-1}}\geq 1,\qquad k_1+k_{n-1}=n-1.$$

For $k_1=1$ and $k_1=n-2$, we get the following strong inequalities:

$$\sum_{i=1}^{n}\frac{1}{1+a_i+(n-2)a_i^{n-1}}\geq 1,$$

$$\sum_{i=1}^{n}\frac{1}{1+(n-2)a_i+a_i^{n-1}}\geq 1.$$

□

P 1.66. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that $a_1a_2\cdots a_n = 1$. If*

$$k \geq n^2 - 1,$$

then

$$\frac{1}{\sqrt{1+ka_1}} + \frac{1}{\sqrt{1+ka_2}} + \cdots + \frac{1}{\sqrt{1+ka_n}} \geq \frac{n}{\sqrt{1+k}}.$$

Solution. Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \ldots, n$, we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s), \qquad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

where

$$f(u) = \frac{1}{\sqrt{1+ke^u}}, \qquad u \in \mathbb{I} = \mathbb{R}.$$

For $u \geq 0$, we have

$$f''(u) = \frac{ke^u(ke^u - 2)}{4(1+ke^u)^{5/2}} \geq \frac{ke^u(k-2)}{4(1+ke^u)^{5/2}} > 0.$$

Therefore, f is convex on $\mathbb{I}_{\geq s}$. By the RHCF-Theorem, it suffices to prove the original inequality for

$$a_1 = 1/t^{n-1}, \qquad a_2 = \cdots = a_n = t, \qquad t \geq 1.$$

Write this inequality as $h(t) \geq 0$, where

$$h(t) = \sqrt{\frac{t^{n-1}}{t^{n-1}+k}} + \frac{n-1}{\sqrt{1+kt}} - \frac{n}{\sqrt{1+k}}.$$

The derivative

$$h'(t) = \frac{(n-1)kt^{(n-3)/2}}{2(t^{n-1}+k)^{3/2}} - \frac{(n-1)k}{2(kt+1)^{3/2}}$$

has the same sign as

$$h_1(t) = t^{n/3-1}(kt+1) - t^{n-1} - k.$$

Denoting $m = n/3$ $(m \geq 2/3)$, we see that

$$h_1(t) = kt^m + t^{m-1} - t^{3m-1} - k = k(t^m - 1) - t^{m-1}(t^{2m} - 1) = (t^m - 1)h_2(t),$$

where

$$h_2(t) = k - t^{m-1} - t^{2m-1}.$$

For $t > 1$, we have

$$\begin{aligned} h_2'(t) &= t^{m-2}[-m + 1 - (2m-1)t^m] < t^{m-2}[-m + 1 - (2m-1)] \\ &= -(3m-2)t^{m-2} \leq 0, \end{aligned}$$

hence $h_2(t)$ is strictly decreasing for $t \ge 1$. Since

$$h_2(1) = k - 2 > 0, \qquad \lim_{t\to\infty} h_2(t) = -\infty,$$

there exists $t_1 > 1$ so that $h_2(t_1) = 0$, $h_2(t) > 0$ for $t \in [1, t_1)$, $h_2(t) < 0$ for $t \in (t_1, \infty)$. Since $h_2(t)$, $h_1(t)$ and $h'(t)$ has the same sign for $t > 1$, $h(t)$ is strictly increasing for $t \in [1, t_1]$ and strictly decreasing for $t \in [t_1, \infty)$; this yields

$$h(t) \ge \min\{h(1), h(\infty)\}.$$

From $h(1) = 0$ and $h(\infty) = 1 - \dfrac{n}{\sqrt{1+k}} \ge 0$, it follows that $h(t) \ge 0$ for all $t \ge 1$. The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark. The following generalization holds (*Vasile C.*, 2005):

- *Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that $a_1 a_2 \cdots a_n = 1$. If k and m are positive numbers so that*

$$m \le n - 1, \qquad k \ge n^{1/m} - 1,$$

then

$$\frac{1}{(1+ka_1)^m} + \frac{1}{(1+ka_2)^m} + \cdots + \frac{1}{(1+ka_n)^m} \ge \frac{n}{(1+k)^m},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$.

For $0 < m \le n - 1$ and $k = n^{1/m} - 1$, we get the beautiful inequality

$$\frac{1}{(1+ka_1)^m} + \frac{1}{(1+ka_2)^m} + \cdots + \frac{1}{(1+ka_n)^m} \ge 1.$$

□

P 1.67. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that $a_1 a_2 \cdots a_n = 1$. If $p, q \ge 0$ so that $0 < p + q \le \dfrac{1}{n-1}$, then*

$$\frac{1}{1+pa_1+qa_1^2} + \frac{1}{1+pa_2+qa_2^2} + \cdots + \frac{1}{1+pa_n+qa_n^2} \le \frac{n}{1+p+q}.$$

(Vasile C., 2007)

Solution. Using the notation $a_i = e^{x_i}$ for $i = 1, 2, \ldots, n$, we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge nf(s), \qquad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

where

$$f(u) = \frac{-1}{1 + pe^u + qe^{2u}}, \qquad u \in \mathbb{I} = \mathbb{R}.$$

For $u \le 0$, we have

$$f''(u) = \frac{e^u[-4q^2e^{3u} - 3pqe^{2u} + (4q - p^2)e^u + p]}{(1 + pe^u + qe^{2u})^3}$$
$$= \frac{e^{2u}[-4q^2e^{2u} - 3pqe^u + (4q - p^2) + pe^{-u}]}{(1 + pe^u + qe^{2u})^3}$$
$$\ge \frac{e^{2u}[-4q^2 - 3pq + (4q - p^2) + p]}{(1 + pe^u + qe^{2u})^3}$$
$$= \frac{e^{2u}[(p + 4q)(1 - p - q) + 2pq]}{(1 + pe^u + qe^{2u})^3} \ge 0,$$

therefore f is convex on $\mathbb{I}_{\le s}$. By the LHCF-Theorem, it suffices to prove the original inequality for

$$a_1 = 1/t^{n-1}, \quad a_2 = \cdots = a_n = t, \quad t > 0.$$

Write this inequality as

$$\frac{t^{2n-2}}{t^{2n-2} + pt^{n-1} + q} + \frac{n-1}{1 + pt + qt^2} \le \frac{n}{1 + p + q},$$

$$p^2A + q^2B + pqC \le pD + qE,$$

where

$$A = t^{n-1}(t^n - nt + n - 1), \quad B = t^{2n} - nt^2 + n - 1,$$
$$C = t^{2n-1} + t^{2n} - nt^{n+1} + (n-1)t^{n-1} - nt + n - 1,$$
$$D = t^{n-1}[(n-1)t^n + 1 - nt^{n-1}], \quad E = (n-1)t^{2n} + 1 - nt^{2n-2}.$$

Applying the AM-GM inequality to n positive numbers yields $D \ge 0$ and $E \ge 0$. Since $(n-1)(p+q) \le 1$ involves $pD + qE \ge (n-1)(p+q)(pD + qE)$, it suffices to show that

$$p^2A + q^2B + pqC \le (n-1)(p+q)(pD + qE).$$

Write this inequality as

$$p^2A_1 + q^2B_1 + pqC_1 \ge 0,$$

where

$$A_1 = (n-1)D - A = nt^n[(n-2)t^{n-1} + 1 - (n-1)t^{n-2}],$$
$$B_1 = (n-1)E - B = nt^2[(n-2)t^{2n-2} + 1 - (n-1)t^{2n-4}],$$
$$C_1 = (n-1)(D+E) - C = nt[(n-2)(t^{2n-1} + t^{2n-2}) - 2(n-1)t^{2n-3} + t^n + 1].$$

Applying the AM-GM inequality to $n-1$ nonnegative numbers yields $A_1 \ge 0$ and $B_1 \ge 0$. So, it suffices to show that $C_1 \ge 0$. Indeed, we have

$$(n-2)(t^{2n-1} + t^{2n-2}) - 2(n-1)t^{2n-3} + t^n + 1 = A_2 + B_2 + C_2,$$

where

$$A_2 = (n-2)t^{2n-1} + t - (n-1)t^{2n-3} \ge 0,$$

$$B_2 = (n-2)t^{2n-2} + t^{n-1} - (n-1)t^{2n-3} \geq 0,$$
$$C_2 = t^n - t^{n-1} - t + 1 = (t-1)(t^{n-1} - 1) \geq 0.$$

The inequalities $A_2 \geq 0$ and $B_2 \geq 0$ follow by applying the AM-GM inequality to $n-1$ nonnegative numbers.

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark 1. For $p + q = \dfrac{1}{n-1}$, we get the inequality

$$\frac{1}{1+pa_1+qa_1^2} + \frac{1}{1+pa_2+qa_2^2} + \cdots + \frac{1}{1+pa_n+qa_n^2} \leq n-1,$$

which is a generalization of the following inequalities:

$$\frac{1}{n-1+a_1} + \frac{1}{n-1+a_2} + \cdots + \frac{1}{n-1+a_n} \leq 1,$$

$$\frac{1}{2n-2+a_1+a_1^2} + \frac{1}{2n-2+a_2+a_2^2} + \cdots + \frac{1}{2n-2+a_n+a_n^2} \leq \frac{1}{2}.$$

Remark 2. For

$$p = \frac{4n-3}{2(n-1)(2n-1)}, \qquad q = \frac{1}{2(n-1)(2n-1)},$$

we get the inequality

$$\frac{1}{(a_1+2n-2)(a_1+2n-1)} + \cdots + \frac{1}{(a_n+2n-2)(a_n+2n-1)} \leq \frac{1}{4n-2},$$

which is equivalent to

$$\frac{1}{a_1+2n-2} + \cdots + \frac{1}{a_n+2n-2} \leq \frac{1}{4n-2} + \frac{1}{a_1+2n-1} + \cdots + \frac{1}{a_n+2n-1}.$$

Remark 3. For $p = 2k$ and $q = k^2$, we get the following statement:

- *Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that $a_1 a_2 \cdots a_n = 1$. If*

$$0 < k \leq \sqrt{\frac{n}{n-1}} - 1,$$

then

$$\frac{1}{(1+ka_1)^2} + \frac{1}{(1+ka_2)^2} + \cdots + \frac{1}{(1+ka_n)^2} \leq \frac{n}{(1+k)^2},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.68. *Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be positive real numbers so that $a_1 a_2 \cdots a_n = 1$. If*

$$0 < k \leq \frac{2n-1}{(n-1)^2},$$

then

$$\frac{1}{\sqrt{1+ka_1}} + \frac{1}{\sqrt{1+ka_2}} + \cdots + \frac{1}{\sqrt{1+ka_n}} \leq \frac{n}{\sqrt{1+k}}.$$

Solution. Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \ldots, n$, we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s), \qquad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

where

$$f(u) = \frac{-1}{\sqrt{1+ke^u}}, \qquad u \in \mathbb{I} = \mathbb{R}.$$

For $u \leq 0$, we have

$$f''(u) = \frac{ke^u(2-ke^u)}{4(1+ke^u)^{5/2}} \geq \frac{ke^u(2-k)}{4(1+ke^u)^{5/2}} > 0.$$

Therefore, f is convex on $\mathbb{I}_{\leq s}$. By the LHCF-Theorem, it suffices to prove the original inequality for

$$a_1 = 1/t^{n-1}, \qquad a_2 = \cdots = a_n = t. \qquad 0 < t \leq 1.$$

Write this inequality as $h(t) \leq 0$, where

$$h(t) = \sqrt{\frac{t^{n-1}}{t^{n-1}+k}} + \frac{n-1}{\sqrt{1+kt}} - \frac{n}{\sqrt{1+k}}.$$

The derivative

$$h'(t) = \frac{(n-1)kt^{(n-3)/2}}{2(t^{n-1}+k)^{3/2}} - \frac{(n-1)k}{2(kt+1)^{3/2}}$$

has the same sign as

$$h_1(t) = t^{n/3-1}(kt+1) - t^{n-1} - k.$$

Denoting $m = n/3$, $m \geq 1$, we see that

$$h_1(t) = kt^m + t^{m-1} - t^{3m-1} - k = -k(1-t^m) + t^{m-1}(1-t^{2m}) = (1-t^m)h_2(t),$$

where

$$h_2(t) = t^{m-1} + t^{2m-1} - k$$

is strictly increasing for $t \in [0,1]$. There are two possible cases: $h_2(0) \geq 0$ and $h_2(0) < 0$.

Case 1: $h_2(0) \ge 0$. This case is possible only for $m = 1$ and $k \le 1$, when $h_2(t) = t + 1 - k > 0$ for $t \in (0,1]$. Also, we have $h_1(t) > 0$ and $h'(t) > 0$ for $t \in (0,1)$. Therefore, h is strictly increasing on $[0,1]$, hence $h(t) \le h(1) = 0$.

Case 2: $h_2(0) < 0$. This case is possible for either $m = 1$ ($n = 3$) and $1 < k \le 5/4$, or $m > 1$ ($n \ge 4$). Since $h_2(1) = 2-k > 0$, there exists $t_1 \in (0,1)$ so that $h_2(t_1) = 0$, $h_2(t) < 0$ for $t \in (0,t_1)$, and $h_2(t) > 0$ for $t \in (t_1,1)$. Since h' has the same sign as h_2 on $(0,1)$, it follows that h is strictly decreasing on $[0,t_1]$ and strictly increasing on $[t_1,1]$. Therefore, $h(t) \le \max\{h(0),h(1)\}$. Since $h(0) = n-1-\dfrac{n}{\sqrt{1+k}} \le 0$ and $h(1) = 0$, we have $h(t) \le 0$ for all $t \in (0,1]$.

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark. The following generalization holds (*Vasile C.*, 2005):

- *Let $a_1, a_2, \ldots, a_n$ ($n \ge 3$) be positive real numbers so that $a_1a_2\cdots a_n = 1$. If k and m are positive numbers so that*

$$m \ge \frac{1}{n-1}, \qquad k \le \left(\frac{n}{n-1}\right)^{1/m} - 1,$$

then

$$\frac{1}{(1+ka_1)^m} + \frac{1}{(1+ka_2)^m} + \cdots + \frac{1}{(1+ka_n)^m} \le \frac{n}{(1+k)^m},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$.

For $n \ge 3$, $m \ge \dfrac{1}{n-1}$ and $k = \left(\dfrac{n}{n-1}\right)^{1/m} - 1$, we get the beautiful inequality

$$\frac{1}{(1+ka_1)^m} + \frac{1}{(1+ka_2)^m} + \cdots + \frac{1}{(1+ka_n)^m} \le n-1.$$

□

P 1.69. *If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1a_2\cdots a_n = 1$, then*

$$\sqrt{a_1^4 + \frac{2n-1}{(n-1)^2}} + \sqrt{a_2^4 + \frac{2n-1}{(n-1)^2}} + \cdots + \sqrt{a_n^4 + \frac{2n-1}{(n-1)^2}} \ge \frac{1}{n-1}(a_1+a_2+\cdots+a_n)^2.$$

(Vasile C., 2006)

Solution. According to the preceding P 1.68, the following inequality holds

$$\sum \frac{1}{\sqrt{1 + \frac{2n-1}{(n-1)^2}a_1^{-4}}} \le n-1.$$

On the other hand, by the Cauchy-Schwarz inequality

$$\left(\sum \frac{1}{\sqrt{1+\frac{2n-1}{(n-1)^2}a_1^{-4}}}\right)\left(\sum a_1^2\sqrt{1+\frac{2n-1}{(n-1)^2}a_1^{-4}}\ \right) \ge \left(\sum a_1\right)^2.$$

From these inequalities, we get

$$(n-1)\left(\sum a_1^2\sqrt{1+\frac{2n-1}{(n-1)^2}a_1^{-4}}\right) \ge \left(\sum a_1\right)^2,$$

which is the desired inequality.

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.70. *If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1a_2\cdots a_n = 1$, then*

$$a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} + n(n-2) \ge (n-1)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

Solution. Using the notation $a_i = e^{x_i}$ for $i = 1, 2, \ldots, n$, we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge nf(s), \qquad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

where

$$f(u) = e^{(n-1)u} - (n-1)e^{-u}, \quad u \in \mathbb{I} = \mathbb{R}.$$

For $u \ge 0$, we have

$$f''(u) = (n-1)^2e^{(n-1)u} - (n-1)e^{-u} = (n-1)e^{-u}[(n-1)e^{nu} - 1] \ge 0;$$

therefore, f is convex on $\mathbb{I}_{\ge s}$. By the RHCF-Theorem and Note 2, it suffices to show that $H(x, y) \ge 0$ for $x, y \in \mathbb{R}$ so that $x + (n-1)y = 0$, where

$$H(x,y) = \frac{f'(x) - f'(y)}{x - y}.$$

From

$$f'(u) = (n-1)[e^{(n-1)u} + e^{-u}],$$

we get

$$\begin{aligned} H(x,y) &= \frac{(n-1)(e^x - e^y)}{x-y}\left[e^{(n-2)x} + e^{(n-3)x+y} + \cdots + e^{x+(n-3)y} + e^{(n-2)y} - e^{-x-y}\right] \\ &= \frac{(n-1)(e^x - e^y)}{x-y}\left[e^{(n-2)x} + e^{(n-3)x+y} + \cdots + e^{x+(n-3)y})\right]. \end{aligned}$$

Since $(e^x - e^y)/(x - y) > 0$, we have $H(x, y) > 0$.

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.71. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that $a_1 a_2 \cdots a_n = 1$. If $k \ge n$, then*
$$a_1^k + a_2^k + \cdots + a_n^k + kn \ge (k+1)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

(Vasile C., 2006)

Solution. Using the notations $a_i = e^{x_i}$ for $i = 1, 2, \ldots, n$, we need to show that
$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge nf(s), \qquad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$
where
$$f(u) = e^{ku} - (k+1)e^{-u}, \qquad u \in \mathbb{I} = \mathbb{R}.$$
For $u \ge 0$, we have
$$f''(u) = k^2 e^{ku} - (k+1)e^{-u} = e^{-u}\left[k^2 e^{(k+1)u} - k - 1\right] \ge e^{-u}(k^2 - k - 1) > 0;$$
therefore, f is convex on $\mathbb{I}_{\ge s}$. By the RHCF-Theorem, it suffices to to prove the original inequality for $a_1 \le 1 \le a_2 = \cdots = a_n$; that is, to show that
$$a^k + (n-1)b^k - \frac{k+1}{a} - \frac{(k+1)(n-1)}{b} + kn \ge 0$$
for
$$ab^{n-1} = 1, \qquad 0 < a \le 1 \le b.$$
By the weighted AM-GM inequality, we have
$$a^k + (kn - k - 1) \ge [1 + (kn - k - 1)]a^{\frac{k}{1+(kn-k-1)}} = \frac{k(n-1)}{b}.$$
Thus, we still have to show that
$$(n-1)\left(b^k - \frac{1}{b}\right) - (k+1)\left(\frac{1}{a} - 1\right) \ge 0,$$
which is equivalent to $h(b) \ge 0$ for $b \ge 1$, where
$$h(b) = (n-1)(b^{k+1} - 1) - (k+1)(b^n - b).$$
Since
$$\begin{aligned}
\frac{h'(b)}{k+1} &= (n-1)b^k - nb^{n-1} + 1 \ge (n-1)b^n - nb^{n-1} + 1 \\
&= nb^{n-1}(b-1) - (b^n - 1) \\
&= (b-1)\left[(b^{n-1} - b^{n-2}) + (b^{n-1} - b^{n-3}) + \cdots + (b^{n-1} - 1)\right] \ge 0,
\end{aligned}$$
h is increasing on $[1, \infty)$, hence $h(b) \ge h(1) = 0$. The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.72. *If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1 a_2 \cdots a_n = 1$, then*

$$\left(1-\frac{1}{n}\right)^{a_1}+\left(1-\frac{1}{n}\right)^{a_2}+\cdots+\left(1-\frac{1}{n}\right)^{a_n} \le n-1.$$

(Vasile C., 2006)

Solution. Let

$$k=\frac{n}{n-1}, \quad k>1,$$

and

$$m=\ln k, \quad 0<m\le \ln 2<1.$$

Using the substitutions $a_i=e^{x_i}$ for $i=1,2,\ldots,n$, we need to show that

$$f(x_1)+f(x_2)+\cdots+f(x_n)\ge nf(s), \qquad s=\frac{x_1+x_2+\cdots+x_n}{n}=0,$$

where

$$f(u)=-k^{-e^u}, \quad u\in\mathbb{I}=\mathbb{R}.$$

From

$$f''(u)=me^u k^{-e^u}(1-me^u),$$

it follows that $f''(u)>0$ for $u\le 0$, since

$$1-me^u\ge 1-m\ge 1-\ln 2>0.$$

Therefore, f is convex on $\mathbb{I}_{\le s}$. By the LHCF-Theorem and Note 5, it suffices to prove the original inequality for

$$a_2=\cdots=a_n:=t, \qquad a_1=t^{-n+1}, \qquad 0<t\le 1.$$

Write this inequality as

$$h(t)\le n-1,$$

where

$$h(t)=k^{-t^{-n+1}}+(n-1)k^{-t}, \quad t\in(0,1].$$

We have

$$h'(t)=(n-1)mt^{-n}k^{-t^{-n+1}}h_1(t), \qquad h_1(t)=1-t^n k^{t^{-n+1}-t},$$

$$h_1'(t)=k^{t^{-n+1}-t}h_2(t), \qquad h_2(t)=m(n-1+t^n)-nt^{n-1}.$$

Since

$$h_2'(t)=nt^{n-2}(mt-n+1)\le nt^{n-2}(m-n+1)\le nt^{n-2}(m-1)<0,$$

h_2 is strictly decreasing on $[0,1]$. From

$$h_2(0)=(n-1)m>0, \qquad h_2(1)=n(m-1)<0,$$

it follows that there is $t_1 \in (0,1)$ so that $h_2(t_1) = 0$, $h_2(t) > 0$ for $t \in [0,t_1)$ and $h_2(t) < 0$ for $t \in (t_1,1]$. Therefore, h_1 is strictly increasing on $(0,t_1]$ and strictly decreasing on $[t_1,1]$. Since $h_1(0_+) = -\infty$ and $h_1(1) = 0$, there is $t_2 \in (0,t_1)$ so that $h_1(t_2) = 0$, $h_1(t) < 0$ for $t \in (0,t_2)$, $h_1(t) > 0$ for $t \in (t_2,1)$. Thus, h is strictly decreasing on $(0,t_2]$ and strictly increasing on $[t_2,1]$. Since $h(0_+) = n-1$ and $h(1) = n-1$, we have $h(t) \le n-1$ for all $t \in (0,1]$. This completes the proof. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.73. *If a,b,c are positive real numbers so that $abc = 1$, then*

$$\frac{1}{1+\sqrt{1+3a}} + \frac{1}{1+\sqrt{1+3b}} + \frac{1}{1+\sqrt{1+3c}} \le 1.$$

(Vasile C., 2008)

Solution. Write the inequality as

$$\frac{\sqrt{1+3a}-1}{3a} + \frac{\sqrt{1+3b}-1}{3b} + \frac{\sqrt{1+3c}-1}{3c} \le 1,$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+3 \ge \sqrt{\frac{1}{a^2}+\frac{3}{a}} + \sqrt{\frac{1}{b^2}+\frac{3}{b}} + \sqrt{\frac{1}{c^2}+\frac{3}{c}}.$$

Replacing a,b,c by $1/a, 1/b, 1/c$, respectively, we need to prove that $abc = 1$ involves

$$a+b+c+3 \ge \sqrt{a^2+3a} + \sqrt{b^2+3b} + \sqrt{c^2+3c}. \tag{*}$$

Using the notation

$$a = e^x, \quad b = e^y, \quad c = e^z,$$

we need to show that

$$f(x)+f(y)+f(z) \ge 3f(s), \qquad s = \frac{x+y+z}{3} = 0,$$

where

$$f(u) = e^u - \sqrt{e^{2u}+3e^u}, \quad u \in \mathbb{I} = \mathbb{R}.$$

We have

$$f''(u) = t\left[1 - \frac{4t^2+18t+9}{4(t+3)\sqrt{t(t+3)}}\right], \qquad t = e^u \ge 1.$$

For $u \ge 0$, which involves $t \ge 1$, from

$$16t(t+3)^3 - (4t^2+18t+9)^2 = 9(4t^2+12t-9) > 0,$$

it follows that $f'' > 0$, hence f is convex on $\mathbb{I}_{\geq s}$. By the RHCF-Theorem, it suffices to prove the inequality (*) for $b = c$. Thus, we need to show that

$$a - \sqrt{a^2 + 3a} + 2(b - \sqrt{b^2 + 3b}\ ) + 3 \geq 0$$

for $ab^2 = 1$. Write this inequality as

$$2b^3 + 3b^2 + 1 \geq \sqrt{3b^2 + 1} + 2b^2\sqrt{b^2 + 3b}.$$

Squaring and dividing by b^2, the inequality becomes

$$9b^2 + 4b + 3 \geq 4\sqrt{(b^2 + 3b)(3b^2 + 1)}.$$

Since

$$2\sqrt{(b^2 + 3b)(3b^2 + 1)} \leq (b^2 + 3b) + (3b^2 + 1) = 4b^2 + 3b + 1,$$

it suffices to show that

$$9b^2 + 4b + 3 \geq 2(4b^2 + 3b + 1),$$

which is equivalent to $(b - 1)^2 \geq 0$. The equality holds for $a = b = c = 1$.

Remark. In the same manner, we can prove the following generalization:

- *Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that $a_1 a_2 \cdots a_n = 1$. If*

$$0 < k \leq \frac{4n}{(n-1)^2},$$

then

$$\frac{1}{1 + \sqrt{1 + ka_1}} + \frac{1}{1 + \sqrt{1 + ka_2}} + \cdots + \frac{1}{1 + \sqrt{1 + ka_n}} \leq \frac{n}{1 + \sqrt{1 + k}}.$$

□

P 1.74. *If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1 a_2 \cdots a_n = 1$, then*

$$\frac{1}{1 + \sqrt{1 + 4n(n-1)a_1}} + \frac{1}{1 + \sqrt{1 + 4n(n-1)a_2}} + \cdots + \frac{1}{1 + \sqrt{1 + 4n(n-1)a_n}} \geq \frac{1}{2}.$$

(Vasile C., 2008)

Solution. Denote

$$k = 4n(n-1), \qquad k \geq 8,$$

and write the inequality as follows:

$$\frac{\sqrt{1+ka_1}-1}{ka_1} + \frac{\sqrt{1+ka_2}-1}{ka_2} + \cdots + \frac{\sqrt{1+ka_n}-1}{ka_n} \geq \frac{1}{2},$$

$$\sqrt{\frac{1}{a_1^2}+\frac{k}{a_1}} + \sqrt{\frac{1}{a_2^2}+\frac{k}{a_2}} + \cdots + \sqrt{\frac{1}{a_1^2}+\frac{k}{a_1}} \geq \frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}+\frac{k}{2}.$$

Replacing $a_1, a_2, \ldots, a_n$ by $1/a_1, 1/a_2, \ldots, 1/a_n$, we need to prove that $a_1a_2\cdots a_n = 1$ implies

$$\sqrt{a_1^2+ka_1} + \sqrt{a_2^2+ka_2} + \cdots + \sqrt{a_n^2+ka_n} \geq a_1+a_2+\cdots+a_n+\frac{k}{2}. \qquad (*)$$

Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \ldots, n$, we need to show that

$$f(x_1)+f(x_2)+\cdots+f(x_n) \geq nf(s), \qquad s = \frac{x_1+x_2+\cdots+x_n}{n} = 0,$$

where

$$f(u) = \sqrt{e^{2u}+ke^u} - e^u, \qquad u \in \mathbb{I} = \mathbb{R}.$$

We will show that $f''(u) > 0$ for $u \leq 0$. Indeed, denoting $t = e^u$, $t \in (0,1]$, we have

$$f''(u) = t\left[\frac{4t^2+6kt+k^2}{4(t+k)\sqrt{t(t+k)}} - 1\right] > 0$$

because

$$(4t^2+6kt+k^2)^2 - 16t(t+k)^3 = k^2(k^2-4kt-4t^2) \geq k^2(k^2-4k-4) > 0.$$

Thus, f is convex on $\mathbb{I}_{\leq s}$. By the LHCF-Theorem, it suffices to prove the inequality (*) for $a_2 = a_3 = \cdots = a_n$; that is, to show that

$$\sqrt{a^2+ka} - a + (n-1)\left(\sqrt{b^2+kb} - b\right) \geq n\left(\sqrt{1+k}-1\right),$$

for all positive a, b satisfying $ab^{n-1} = 1$. Write this inequality as

$$\sqrt{kb^{n-1}+1} + (n-1)\sqrt{kb^{2n-1}+b^{2n}} \geq (n-1)b^n + 2n(n-1)b^{n-1} + 1.$$

By Minkowski's inequality, we have

$$\sqrt{kb^{n-1}+1} + (n-1)\sqrt{kb^{2n-1}+b^{2n}} \geq$$

$$\geq \sqrt{kb^{n-1}[1+(n-1)b^{n/2}]^2 + [1+(n-1)b^n]^2}.$$

Thus, it suffices to show that

$$kb^{n-1}[1+(n-1)b^{n/2}]^2+[1+(n-1)b^n]^2 \ge [(n-1)b^n+2n(n-1)b^{n-1}+1]^2,$$

which is equivalent to

$$4n(n-1)^2 b^{\frac{3n-2}{2}}\left[2+(n-2)b^{\frac{n}{2}}-nb^{\frac{n-2}{2}}\right] \ge 0.$$

This inequality follows immediately by the AM-GM inequality applied to n positive numbers.

The equality holds for $a_1=a_2=\cdots=a_n=1$.

□

P 1.75. *If a,b,c are positive real numbers so that $abc=1$, then*

$$\frac{a^6}{1+2a^5}+\frac{b^6}{1+2b^5}+\frac{c^6}{1+2c^5} \ge 1.$$

(Vasile C., 2008)

Solution. Using the substitution

$$a=e^x, \quad b=e^y, \quad c=e^z,$$

we need to show that

$$f(x)+f(y)+f(z) \ge 3f(s), \qquad s=\frac{x+y+z}{3}=0,$$

where

$$f(u)=\frac{e^{6u}}{1+2e^{5u}}, \qquad u \in \mathbb{I}=\mathbb{R}.$$

For $u \le 0$, which involves $w=e^u \in (0,1]$, we have

$$f''(u)=\frac{2w^6(2-w^5)(9-2w^5)}{(1+2w^5)^3}>0.$$

Therefore, f is convex on $\mathbb{I}_{\le s}$. By the LHCF-Theorem, it suffices to prove the original inequality for $b=c$ and $ab^2=1$; that is,

$$\frac{1}{b^2(b^{10}+2)}+\frac{2b^6}{1+2b^5} \ge 1.$$

Since

$$1+2b^5 \le 1+b^4+b^6,$$

it suffices to show that

$$\frac{1}{x(x^5+2)}+\frac{2x^3}{1+x^2+x^3}\geq 1,\quad x=\sqrt{b}.$$

This inequality can be written as follows:

$$x^3(x^6-x^5-x^3+2x-1)+(x-1)^2\geq 0,$$

$$x^3(x-1)^2(x^4+x^3+x^2-1)+(x-1)^2\geq 0,$$

$$(x-1)^2[x^7+x^5+(x^6-x^3+1)]\geq 0.$$

The equality holds for $a=b=c=1$.

□

P 1.76. *If a,b,c are positive real numbers so that $abc=1$, then*

$$\sqrt{25a^2+144}+\sqrt{25b^2+144}+\sqrt{25c^2+144}\leq 5(a+b+c)+24.$$

(Vasile C., 2008)

Solution. Using the notation

$$a=e^x,\quad b=e^y,\quad c=e^z,$$

we need to show that

$$f(x)+f(y)+f(z)\geq 3f(s),\quad s=\frac{x+y+z}{3}=0,$$

where

$$f(u)=5e^u-\sqrt{25e^{2u}+144},\quad u\in\mathbb{R}.$$

We will show that $f(u)$ is convex for $u\leq 0$. From

$$f''(u)=5w\left[1-\frac{5w(25w^2+288)}{(25w^2+144)^{3/2}}\right],\quad w=e^u\in(0,1],$$

we need to show that

$$(25w^2+144)^3\geq 25w^2(25w^2+288)^2.$$

Setting $25w^2=144z$, we have $z\in\left(0,\frac{25}{144}\right]$ and

$$\begin{aligned}(25w^2+144)^3-25w^2(25w^2+288)^2&=144^3(z+1)^3-144^3z(z+2)^2\\&=144^3(1-z-z^2)>0.\end{aligned}$$

By the LHCF-Theorem, it suffices to prove the original inequality for

$$a = t^2, \qquad b = c = 1/t, \qquad t > 0;$$

that is,

$$5t^3 + 24t + 10 \geq \sqrt{25t^6 + 144t^2} + 2\sqrt{25 + 144t^2}.$$

Squaring and dividing by $4t$ give

$$60t^3 + 25t^2 - 36t + 120 \geq \sqrt{(25t^4 + 144)(144t^2 + 25)}.$$

Squaring again and dividing by 120, the inequality becomes

$$25t^5 - 36t^4 + 105t^3 - 112t^2 - 72t + 90 \geq 0,$$

$$(t-1)^2(25t^3 + 14t^2 + 108t + 90) \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 1.77. *If a, b, c are positive real numbers so that $abc = 1$, then*

$$\sqrt{16a^2 + 9} + \sqrt{16b^2 + 9} + \sqrt{16c^2 + 9} \geq 4(a + b + c) + 3.$$

(Vasile C., 2008)

Solution. Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z,$$

we need to show that

$$f(x) + f(y) + f(z) \geq 3f(s), \qquad s = \frac{x + y + z}{3} = 0,$$

where

$$f(u) = \sqrt{16e^{2u} + 9} - 4e^u, \quad u \in \mathbb{R}.$$

We will show that $f(u)$ is convex for $u \geq 0$. From

$$f''(u) = 4w\left[\frac{4w(16w^2 + 18)}{(16w^2 + 9)^{3/2}} - 1\right], \quad w = e^u \geq 1,$$

we need to show that

$$16w^2(16w^2 + 18)^2 \geq (16w^2 + 9)^3.$$

Setting $16w^2 = 9z$, we have $z \geq \dfrac{16}{9}$ and

$$\begin{aligned}16w^2(16w^2+18)^2-(16w^2+9)^3 &= 729z(z+2)^2-729(z+1)^3\\ &= 729(z^2+z-1) > 0.\end{aligned}$$

By the RHCF-Theorem, it suffices to prove the original inequality for

$$a = t^2, \qquad b = c = 1/t, \qquad t > 0;$$

that is,

$$\sqrt{16t^6+9t^2}+2\sqrt{16+9t^2} \geq 4t^3+3t+8.$$

Squaring and dividing by $4t$ give

$$\sqrt{(16t^4+9)(9t^2+16)} \geq 6t^3+16t^2-9t+12.$$

Squaring again and dividing by $12t$, the inequality becomes

$$9t^5-16t^4+9t^3+12t^2-32t+18 \geq 0,$$

$$(t-1)^2(9t^3+2t^2+4t+18) \geq 0.$$

The equality holds for $a = b = c = 1$.

$\square$

P 1.78. *If ABC is a triangle, then*

$$\sin A\left(2\sin\frac{A}{2}-1\right)+\sin B\left(2\sin\frac{B}{2}-1\right)+\sin C\left(2\sin\frac{C}{2}-1\right) \geq 0.$$

(Lorian Saceanu, 2015)

Solution. Write the inequality as

$$f(A)+f(B)+f(C) \geq 3f(s), \qquad s = \frac{A+B+C}{3} = \frac{\pi}{3},$$

where

$$f(u) = \sin u\left(2\sin\frac{u}{2}-1\right) = \cos\frac{u}{2}-\cos\frac{3u}{2}-\sin u, \quad u \in \mathbb{I} = [0,\pi].$$

We will show that f is convex on $\mathbb{I}_{\leq s}$. Indeed, for $u \in [0,\pi/3]$, we have

$$\begin{aligned}f''(u) &= \cos\frac{u}{2}\left(2+2\sin\frac{u}{2}-9\sin^2\frac{u}{2}\right) \geq \cos\frac{u}{2}\left(2+2\sin\frac{u}{2}-12\sin^2\frac{u}{2}\right)\\ &= 2\cos\frac{u}{2}\left(1+3\sin\frac{u}{2}\right)\left(1-2\sin\frac{u}{2}\right) \geq 0.\end{aligned}$$

By the LHCF-Theorem, it suffices to prove the original inequality for $B = C$, when it transforms into

$$\sin 2B(2\cos B - 1) + 2\sin B\left(2\sin\frac{B}{2} - 1\right) \geq 0,$$

$$\sin B \sin\frac{B}{2}\left(\sin\frac{B}{2} + 1\right)\left(2\sin\frac{B}{2} - 1\right)^2 \geq 0.$$

The equality occurs for an equilateral triangle, and for a degenerate triangle with $A = \pi$ and $B = C = 0$ (or any cyclic permutation).

Remark. Based on this inequality, we can prove the following statement:

- *If ABC is a triangle, then*

$$\sin 2A(2\cos A - 1) + \sin 2B(2\cos B - 1) + \sin 2C(2\cos C - 1) \geq 0,$$

with equality for an equilateral triangle, for a degenerate triangle with $A = 0$ and $B = C = \pi/2$ (or any cyclic permutation), and for a degenerate triangle with $A = \pi$ and $B = C = 0$ (or any cyclic permutation).

If ABC is an acute or right triangle, then this inequality follows by replacing A, B and C with $\pi - 2A$, $\pi - 2B$ and $\pi - 2C$ in the inequality from P 1.78. Consider now that

$$A > \frac{\pi}{2} > B \geq C \geq 0.$$

The inequality is true for $B \leq \pi/3$, because

$$\sin 2A(2\cos A - 1) \geq 0, \quad \sin 2B(2\cos B - 1) \geq 0, \quad \sin 2C(2\cos C - 1) \geq 0.$$

Consider further that

$$\frac{2\pi}{3} > A > \frac{\pi}{2} > B > \frac{\pi}{3} > C \geq 0.$$

From

$$1 - 2\cos A > 1 - 2\cos B,$$

it follows that

$$(-\sin 2A)(1 - 2\cos A) > (-\sin 2A)(1 - 2\cos B).$$

Therefore it suffices to

$$(-\sin 2A)(1 - 2\cos B) + \sin 2B(2\cos B - 1) + \sin 2C(2\cos C - 1) \geq 0,$$

which is equivalent to

$$(\sin 2A + \sin 2B)(2\cos B - 1) + \sin 2C(2\cos C - 1) \geq 0,$$

$$2\sin C\cos(A-B)(2\cos B-1)+2\sin C\cos C(2\cos C-1)\geq 0.$$

This inequality is true if

$$\cos(A-B)(2\cos B-1)+\cos C(2\cos C-1)\geq 0,$$

which can be written as

$$\cos C(2\cos C-1)\geq \cos(A-B)(1-2\cos B).$$

Since

$$C<A-B<\frac{2\pi}{3}-\frac{\pi}{3}=\frac{\pi}{3},$$

we have $\cos C>\cos(A-B)$. Therefore, it suffices to show that

$$2\cos C-1\geq 1-2\cos B,$$

which is equivalent to

$$\cos B+\cos C\geq 1.$$

From $B+C<\pi/2$, we get $\cos B>\cos(\pi/2-C)=\sin C$, hence

$$\cos B+\cos C>\sin C+\cos C=\sqrt{1+\sin 2C}\geq 1.$$

□

P 1.79. *If ABC is an acute or right triangle, then*

$$\sin 2A\left(1-2\sin\frac{A}{2}\right)+\sin 2B\left(1-2\sin\frac{B}{2}\right)+\sin 2C\left(1-2\sin\frac{C}{2}\right)\geq 0.$$

(Vasile C., 2015)

Solution. Write the inequality as

$$f(A)+f(B)+f(C)\geq 3f(s),\qquad s=\frac{A+B+C}{3}=\frac{\pi}{3},$$

where

$$f(u)=\sin 2u\left(1-2\sin\frac{u}{2}\right)=\sin 2u-\cos\frac{3u}{2}+\cos\frac{5u}{2},\quad u\in\mathbb{I}=[0,\pi/2].$$

We will show that f is convex on $[s,\pi/2]$. From

$$f''(u)=-4\sin 2u+\frac{9}{4}\cos\frac{3u}{2}-\frac{25}{4}\cos\frac{5u}{2}$$

and

$$\cos\frac{3u}{2}-\cos\frac{5u}{2}=2\sin\frac{u}{2}\sin 2u\geq 0,$$

we get

$$\begin{aligned} f''(u) &\geq -4\sin 2u + \frac{9}{4}\cos\frac{5u}{2} - \frac{25}{4}\cos\frac{5u}{2} \\ &= -4\left[\sin 2u + \sin\frac{\pi - 5u}{2}\right] = 8\sin\frac{\pi - u}{4}\cos\frac{5\pi - 9u}{4}. \end{aligned}$$

For $\pi/3 \leq u \leq \pi/2$, we have

$$\frac{\pi}{8} \leq \frac{5\pi - 9u}{4} \leq \frac{\pi}{2},$$

hence $f''(u) \geq 0$. By the RHCF-Theorem, it suffices to prove the original inequality for $B = C$, $0 \leq B \leq \pi/2$, when it becomes

$$-\sin 4B(1 - 2\cos B) + 2\sin 2B\left(1 - 2\sin\frac{B}{2}\right) \geq 0,$$

$$2\sin 2B\left[\cos 2B(2\cos B - 1) + 1 - \sin\frac{B}{2}\right] \geq 0.$$

We need to show that

$$\cos 2B(2\cos B - 1) + 1 - \sin\frac{B}{2} \geq 0,$$

which is equivalent to $g(t) \geq 0$, where

$$g(t) = (1 - 8t^2 + 8t^4)(1 - 4t^2) + 1 - 2t, \quad t = \sin\frac{B}{2}, \quad 0 \leq t \leq \frac{1}{\sqrt{2}}.$$

Indeed, we have

$$g(t) = 2(1-t)^2(1 + 3t + 2t^2 - 4t^3 - 4t^4) \geq 0$$

because

$$1 + 3t + 2t^2 - 4t^3 - 4t^4 \geq 1 + 3t + 2t^2 - 2t - 2t^2 = 1 + t > 0.$$

The equality occurs for an equilateral triangle, for a degenerate triangle with $A = 0$ and and $B = C = \pi/2$ (or any cyclic permutation), and for a degenerate triangle with $A = \pi$ and $B = C = 0$ (or any cyclic permutation).

Remark 1. Actually, the inequality holds also for an obtuse triangle ABC. To prove this, consider that

$$A > \frac{\pi}{2} > B \geq C \geq 0.$$

The inequality is true for $B \leq \pi/3$, because

$$\sin 2A\left(1 - 2\sin\frac{A}{2}\right) \geq 0, \quad \sin 2B\left(1 - 2\sin\frac{B}{2}\right) \geq 0, \quad \sin 2C\left(1 - 2\sin\frac{C}{2}\right) \geq 0.$$

Consider further that

$$\frac{2\pi}{3} > A > \frac{\pi}{2} > B > \frac{\pi}{3} > C \geq 0.$$

From

$$2\sin\frac{A}{2} - 1 > 2\sin\frac{B}{2} - 1,$$

it follows that

$$(-\sin 2A)\left(2\sin\frac{A}{2} - 1\right) > (-\sin 2A)\left(2\sin\frac{B}{2} - 1\right).$$

Therefore it suffices to

$$(-\sin 2A)\left(2\sin\frac{B}{2} - 1\right) + \sin 2B\left(1 - 2\sin\frac{B}{2}\right) + \sin 2C\left(1 - 2\sin\frac{C}{2}\right) \geq 0,$$

which is equivalent to

$$(\sin 2A + \sin 2B)\left(1 - 2\sin\frac{B}{2}\right) + \sin 2C\left(1 - 2\sin\frac{C}{2}\right) \geq 0,$$

$$2\sin C\cos(A-B)\left(1 - 2\sin\frac{B}{2}\right) + 2\sin C\cos C\left(1 - 2\sin\frac{C}{2}\right) \geq 0.$$

This inequality is true if

$$\cos(A-B)\left(1 - 2\sin\frac{B}{2}\right) + \cos C\left(1 - 2\sin\frac{C}{2}\right) \geq 0,$$

which can be written as

$$\cos C\left(1 - 2\sin\frac{C}{2}\right) \geq \cos(A-B)\left(2\sin\frac{B}{2} - 1\right).$$

Since

$$C < A - B < \frac{2\pi}{3} - \frac{\pi}{3} = \frac{\pi}{3},$$

we have $\cos C > \cos(A-B)$. Therefore, it suffices to show that

$$1 - 2\sin\frac{C}{2} \geq 2\sin\frac{B}{2} - 1,$$

which is equivalent to

$$\sin\frac{B}{2} + \sin\frac{C}{2} \leq 1,$$

$$2\sin\frac{B+C}{4}\cos\frac{B-C}{4} \leq 1.$$

This is true since

$$2\sin\frac{B+C}{4} < 2\sin\frac{\pi}{8} < 1, \quad \cos\frac{B-C}{4} < 1.$$

Remark 2. Replacing A, B and C in P 1.79 by $\pi-2A$, $\pi-2B$ and $\pi-2C$, respectively, we get the following inequality for an acute or right triangle ABC:

$$\sin 4A(2\cos A-1)+\sin 4B(2\cos B-1)+\sin 4C(2\cos C-1)\geq 0,$$

with equality for an equilateral triangle, for a triangle with $A=\pi/2$ and $B=C=\pi/4$ (or any cyclic permutation), and for a degenerate triangle with $A=0$ and and $B=C=\pi/2$ (or any cyclic permutation).

□

P 1.80. *If a,b,c,d are real numbers so that $a+b+c+d=4$, then*

$$\frac{a}{a^2-a+4}+\frac{b}{b^2-b+4}+\frac{c}{c^2-c+4}+\frac{d}{d^2-d+4}\leq 1.$$

(Sqing, 2015)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d)\geq 4f(s), \qquad s=\frac{a+b+c+d}{4}=1,$$

where

$$f(u)=\frac{-u}{u^2-u+4}, \qquad u\in\mathbb{R}.$$

We see that

$$f(u)-f(2)=\frac{(u-2)^2}{3(u^2-u+4}\geq 0.$$

From

$$f''(u)=\frac{2(-u^3+12u-4)}{(u^2-u+4)^3},$$

it follows that f is convex on $[1,2]$. Define the function

$$f_0(u)=\begin{cases} f(u), & u\leq 2\\ f(2), & u>2 \end{cases}.$$

Since $f_0(u)\leq f(u)$ for $u\in\mathbb{R}$ and $f_0(1)=f(1)$, it suffices to show that

$$f_0(a)+f_0(b)+f_0(c)+f_0(d)\geq 4f_0(s).$$

The function f_0 is convex on $[1,\infty)$ because it is differentiable on $[1,\infty)$ and its derivative

$$f_0'(u)=\begin{cases} f'(u), & u\leq 2\\ 0, & u>2 \end{cases}$$

is continuous and increasing on $[1,\infty)$. Therefore, by the RHCF-Theorem, we only need to show that $f_0(x)+3f_0(y)\geq 4f_0(1)$ for all $x,y\in\mathbb{R}$ so that $x\leq 1\leq y$ and $x+3y=4$. There are two cases to consider: $y\leq 2$ and $y>2$.

Case 1: $y\leq 2$. The inequality $f_0(x)+3f_0(y)\geq 4f_0(1)$ is equivalent to $f(x)+3f(y)\geq 4f(1)$. According to Note 1, this is true if $h(x,y)\geq 0$ for $x+3y=4$. We have

$$g(u)=\frac{f(u)-f(1)}{u-1}=\frac{u-4}{4(u^2-u+4)},$$

$$\begin{aligned}h(x,y)=\frac{g(x)-g(y)}{x-y}&=\frac{4(x+y)-xy}{4(x^2-x+4)(y^2-y+4)}\\&=\frac{3(y-2)^2+4}{4(x^2-x+4)(y^2-y+4)}>0.\end{aligned}$$

Case 2: $y>2$. From $y>2$ and $x+3y=4$, we get $x<-2$ and

$$f_0(x)+3f_0(y)-4f_0(1)=f(x)+3f(2)-4f(1)=\frac{-x}{x^2-x+4}>0.$$

The equality holds for $a=b=c=d=1$.

□

P 1.81. *Let a,b,c be nonnegative real numbers so that $a+b+c=2$. If*

$$k_0\leq k\leq 3,\quad k_0=\frac{\ln 2}{\ln 3-\ln 2}\approx 1.71,$$

then

$$a^k(b+c)+b^k(c+a)+c^k(a+b)\leq 2.$$

Solution. Write the inequality as

$$f(a)+f(b)+f(c)\leq 2,$$

where

$$f(u)=u^k(2-u),\quad u\in[0,\infty).$$

From

$$f''(u)=ku^{k-2}[2k-2-(k+1)u],$$

it follows that f is convex on $\left[0,\dfrac{2k-2}{k+1}\right]$ and concave on $\left[\dfrac{2k-2}{k+1},2\right]$. According to LCRCF-Theorem, the sum $f(a)+f(b)+f(c)$ is maximum when either $a=0$ or $0<a\leq b=c$.

Case 1: $a=0$. We need to show that

$$bc(b^{k-1}+c^{k-1})\le 2$$

for $b+c=2$. Since $0<(k-1)/2\le 1$, Bernoulli's inequality gives

$$\begin{aligned}b^{k-1}+c^{k-1}&=(b^2)^{(k-1)/2}+(c^2)^{(k-1)/2}\le 1+\frac{k-1}{2}(b^2-1)+1+\frac{k-1}{2}(c^2-1)\\&=3-k+\frac{k-1}{2}(b^2+c^2).\end{aligned}$$

Thus, it suffices to show that

$$(3-k)bc+\frac{k-1}{2}bc(b^2+c^2)\le 2.$$

Since

$$bc\le\left(\frac{b+c}{2}\right)^2=1,$$

we only need to show that

$$3-k+\frac{k-1}{2}bc(b^2+c^2)\le 2,$$

which is equivalent to

$$bc(b^2+c^2)\le 2.$$

Indeed, we have

$$8[2-bc(b^2+c^2)]=(b+c)^4-8bc(b^2+c^2)=(b-c)^4\ge 0.$$

Case 2: $0<a\le b=c$. We only need to prove the homogeneous inequality

$$a^k(b+c)+b^k(c+a)+c^k(a+b)\le 2\left(\frac{a+b+c}{2}\right)^{k+1}$$

for $b=c=1$ and $0<a\le 1$; that is,

$$\left(1+\frac{a}{2}\right)^{k+1}-a^k-a-1\ge 0.$$

Since $\left(1+\frac{a}{2}\right)^{k+1}$ is increasing and a^k is decreasing with respect to k, it suffices consider the case $k=k_0$; that is, to prove that $g(a)\ge 0$, where

$$g(a)=\left(1+\frac{a}{2}\right)^{k_0+1}-a^{k_0}-a-1,\qquad 0<a\le 1.$$

We have

$$g'(a)=\frac{k_0+1}{2}\left(1+\frac{a}{2}\right)^{k_0}-k_0a^{k_0-1}-1,$$

$$\frac{1}{k_0}g''(a) = \frac{k_0+1}{4}\left(1+\frac{a}{2}\right)^{k_0-1} - \frac{k_0-1}{a^{2-k_0}}.$$

Since g'' is increasing on $(0,1]$, $g''(0_+) = -\infty$ and

$$\frac{1}{k_0}g''(1) = \frac{k_0+1}{4}\left(\frac{3}{2}\right)^{k_0-1} - k_0 + 1 = \frac{k_0+1}{3} - k_0 + 1 = \frac{2(2-k_0)}{3} > 0,$$

there exists $a_1 \in (0,1)$ so that $g''(a_1) = 0$, $g''(a) < 0$ for $a \in (0,a_1)$, $g''(a) > 0$ for $a \in (a_1,1]$. Therefore, g' is strictly decreasing on $[0,a_1]$ and strictly increasing on $[a_1,1]$. Since

$$g'(0) = \frac{k_0-1}{2} > 0, \qquad g'(1) = \frac{k_0+1}{2}\left[(3/2)^{k_0} - 2\right] = 0,$$

there exists $a_2 \in (0,a_1)$ so that $g'(a_2) = 0$, $g'(a) > 0$ for $a \in [0,a_2)$, $g'(a) < 0$ for $a \in (a_2,1)$. Thus, g is strictly increasing on $[0,a_2]$ and strictly decreasing on $[a_2,1]$. Consequently,

$$g(a) \ge \min\{g(0), g(1)\},$$

and from

$$g(0) = 0, \quad g(1) = (3/2)^{k_0+1} - 3 = 0,$$

we get $g(a) \ge 0$.

The equality holds for $a = 0$ and $b = c$ (or any cyclic permutation). If $k = k_0$, then the equality holds also for $a = b = c$.

□

P 1.82. *If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then*

$$(n+1)^2\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \ge 4(n+2)(a_1^2 + a_2^2 + \cdots + a_n^2) + n(n^2 - 3n - 6).$$

(Vasile C., 2006)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge n(n^2 - 3n - 6),$$

where

$$f(u) = \frac{(n+1)^2}{u} - 4(n+2)u^2, \quad u \in (0,\infty).$$

From

$$f''(u) = \frac{2(n+1)^2}{u^3} - 8(n+2),$$

it follows that f is strictly convex on $(0,c]$ and strictly concave on $[c,\infty)$, where

$$c = \sqrt[3]{\frac{(n+1)^2}{4(n+2)}}.$$

According to LCRCF-Theorem and Note 5, it suffices to consider the case

$$a_1 = a_2 = \cdots = a_{n-1} = x, \quad a_n = n-(n-1)x, \quad 0 < x \le 1,$$

when the inequality becomes as follows:

$$(n+1)^2\left(\frac{n-1}{x}+\frac{1}{a_n}\right) \ge 4(n+2)[(n-1)x^2+a_n^2)+n(n^2-3n-6),$$

$$n(n-1)(2x-1)^2[(n+2)(n-1)x^2-(n+2)(2n-1)x+(n+1)^2] \ge 0.$$

The last inequality is true since

$$(n-1)x^2-(2n-1)x+\frac{(n+1)^2}{n+2} = (n-1)\left(x-\frac{2n-1}{2n-2}\right)^2+\frac{3(n-2)}{4(n-1)(n+2)} \ge 0.$$

The equality holds for

$$a_1 = a_2 = \cdots = a_{n-1} = \frac{1}{2}, \quad a_n = \frac{n+1}{2}$$

(or any cyclic permutation).

□

P 1.83. *If a,b,c,d,e are positive real numbers such that $a+b+c+d+e=5$, then*

$$27\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right) \ge 4(a^3+b^3+c^3+d^3+e^3)+115.$$

(Vasile Cîrtoaje)

Proof. Write the inequality as

$$f(a)+f(b)+f(c)+f(d)+f(e) \ge 5f(s), \quad s = \frac{a+b+c+d+e}{5} = 1,$$

where

$$f(u) = \frac{27}{u} - 4u^3, \quad 0 < u < 5.$$

From

$$f''(u) = \frac{6(9-4u^4)}{u^3},$$

it follows that f is convex on $(0,1]$. According to LHCF-Theorem, it suffices to prove that

$$f(x)+4f(y) \ge 5f(1)$$

for $x \ge 1 \ge y > 0$ and $x+4y=5$. This occurs if $h(x,y) \ge 0$, where

$$h(x,y) = \frac{g(x)-g(y)}{x-y}, \quad g(u) = \frac{f(u)-f(1)}{u-1}.$$

Since

$$g(u)=-\frac{27}{u}-4(u^2+u+1),$$

$$h(x,y)=\frac{A(x,y)}{xy}, \quad A(x,y)=27-4xy(x+y+1),$$

we need show that $A(x,y)\geq 0$. Indeed,

$$\frac{1}{3}A(x,y)=9-4y(4y-5)(y-2)=9-40y+52y^2-16y^3$$

$$=(1-2y)^2(9-4y)\geq 0.$$

The equality holds for $a=b=c=d=e=1$, and for $a=3$ and $b=c=d=e=1/2$ (or any cyclic permutation).

Generalization. If $a_1,a_2,...,a_n$ are positive real numbers such that

$$a_1+a_2+\cdots+a_n=n,$$

then

$$(n+1)^2(2n-1)(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}-n)\geq 27(n-1)^2(a_1^3+a_2^3+\cdots+a_n^3-n),$$

with equality for $a_1=a_2=\cdots=a_n=1$, and for

$$a_1=\frac{2n-1}{3}, \ \ a_2=\cdots=a_n=\frac{n+1}{3(n-1)}$$

(or any cyclic permutation).

□

P 1.84. *If a,b,c are nonnegative real numbers so that $a+b+c=12$, then*

$$(a^2+10)(b^2+10)(c^2+10)\geq 13310.$$

(Vasile C., 2006)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)\geq 2\ln 11+\ln 110,$$

where

$$f(u)=\ln(u^2+10), \quad u\in[0,12].$$

From

$$f''(u)=\frac{2(10-u^2)}{(u^2+10)^2},$$

it follows that f is convex on $[0,\sqrt{10}]$ and concave on $[\sqrt{10},12]$. According to LCRCF-Theorem, the sum $f(a)+f(b)+f(c)$ is minimum when $a=b\le c$. Therefore, it suffices to prove that $g(a)\ge 0$, where

$$g(a)=2f(a)+f(c)-2\ln 11-\ln 110, \qquad c=12-2a, \quad a\in[0,4].$$

Since $c'(a)=-2$, we have

$$\begin{aligned} g'(a)&=2f'(a)-2f'(c)=4\left(\frac{a}{a^2+10}-\frac{c}{c^2+10}\right)\\ &=\frac{4(a-c)(10-ac)}{(a^2+10)(c^2+10)}=\frac{24(4-a)(5-a)(a-1)}{(a^2+10)(c^2+10)}. \end{aligned}$$

Therefore, $g'(a)<0$ for $a\in[0,1)$ and $g'(a)>0$ for $a\in(1,4)$, hence g is strictly decreasing on $[0,1]$ and strictly increasing on $[1,4]$. Thus, we have

$$g(a)\ge g(1)=0.$$

The equality holds for $a=b=1$ and $c=10$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- *Let $a_1,a_2,\ldots,a_n$ be nonnegative real numbers so that $a_1+a_2+\cdots+a_n=2n(n-1)$. If $k=(n-1)(2n-1)$, then*

$$(a_1^2+k)(a_2^2+k)\cdots(a_n^2+k)\ge k(k+1)^n,$$

with equality for $a_1=k$ and $a_2=\cdots=a_n=1$ (or any cyclic permutation).

□

P 1.85. *If $a_1,a_2,\ldots,a_n$ are nonnegative real numbers so that $a_1+a_2+\cdots+a_n=n$, then*

$$(a_1^2+1)(a_2^2+1)\cdots(a_n^2+1)\ge\frac{(n^2-2n+2)^n}{(n-1)^{2n-2}}.$$

(Vasile C., 2006)

Solution. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge\ln k, \qquad k=\frac{(n^2-2n+2)^n}{(n-1)^{2n-2}},$$

where

$$f(u)=\ln(u^2+1), \quad u\in[0,n].$$

From

$$f''(u)=\frac{2(1-u^2)}{(u^2+1)^2},$$

it follows that f is strictly convex on $[0,1]$ and strictly concave on $[1,n]$. According to LCRCF-Theorem, it suffices to consider the case $a_1 = a_2 = \cdots = a_{n-1} \le a_n$; that is, to show that $g(x) \ge 0$, where

$$g(x) = (n-1)f(x) + f(y) - \ln k, \qquad y = n-(n-1)x, \quad x \in [0,1].$$

Since $y'(x) = -(n-1)$, we get

$$\begin{aligned} g'(x) &= (n-1)f'(x) - (n-1)f'(y) = (n-1)[f'(x) - f'(y)] \\ &= 2(n-1)\left(\frac{x}{x^2+1} - \frac{y}{y^2+1}\right) = \frac{2(n-1)(x-y)(1-xy)}{(x^2+1)(y^2+1)} \\ &= \frac{2n(n-1)(x-1)^2[(n-1)x-1]}{(x^2+1)(y^2+1)}. \end{aligned}$$

Therefore, $g'(x) \le 0$ for $x \in \left[0, \frac{1}{n-1}\right]$ and $g'(x) \ge 0$ for $x \in \left[\frac{1}{n-1}, n\right]$, hence g is decreasing on $\left[0, \frac{1}{n-1}\right]$ and increasing on $\left[\frac{1}{n-1}, 1\right]$. Since $g\left(\frac{1}{n-1}\right) = 0$, the conclusion follows.

The equality holds for $a_1 = a_2 = \cdots = a_{n-1} = \dfrac{1}{n-1}$ and $a_n = n-1$ (or any cyclic permutation).

□

P 1.86. *If a, b, c are nonnegative real numbers so that $a+b+c = 3$, then*

$$(a^2+2)(b^2+2)(c^2+2) \le 44.$$

(Vasile C., 2006)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) \le \ln 44,$$

where

$$f(u) = \ln(u^2+2), \quad u \in [0,3].$$

From

$$f''(u) = \frac{2(2-u^2)}{(u^2+2)^2},$$

it follows that f is strictly convex on $[0, \sqrt{2}]$ and strictly concave on $[\sqrt{2}, 3]$. According to LCRCF-Theorem, the sum $f(a)+f(b)+f(c)$ is maximum for either $a = 0$ or $0 < a \le b = c$.

Case 1: $a = 0$. We need to show that $b + c = 3$ involves

$$(b^2+2)(c^2+2) \le 22,$$

which is equivalent to

$$bc(bc-4) \le 0.$$

This is true because

$$bc \le \left(\frac{b+c}{2}\right)^2 = \frac{9}{4} < 4.$$

Case 2: $0 < a \le b = c$. We need to show that $a+2b=3$ $(0 < a \le 1)$ involves

$$(a^2+2)(b^2+2)^2 \le 44,$$

which is equivalent to $g(a) \le 0$, where

$$g(a) = \ln(a^2+2) + 2\ln(b^2+2) - \ln 44, \quad b = \frac{3-a}{2}, \quad a \in (0,1].$$

Since $b'(a) = -1/2$, we have

$$\begin{aligned} g'(a) &= \frac{2a}{a^2+2} - \frac{2b}{b^2+2} = \frac{2(a-b)(2-ab)}{(a^2+2)(b^2+2)} \\ &= \frac{3(a-1)(a^2-3a+4)}{2(a^2+2)(b^2+2)}. \end{aligned}$$

Because

$$a^2 - 3a + 4 = (a-2)^2 + a > 0,$$

we have $g'(a) < 0$ for $a \in (0,1)$, g is strictly decreasing on $[0,1]$, hence it suffices to show that $g(0) \le 0$. This reduces to $16 \cdot 22 \ge 17^2$, which is true because

$$16 \cdot 22 - 17^2 = 63 > 0.$$

The equality holds for $a = b = 0$ and $c = 3$ (or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

- *Let a, b, c be nonnegative real numbers so that $a+b+c=3$. If $k \ge \dfrac{9}{8}$, then*

$$(a^2+k)(b^2+k)(c^2+k) \le k^2(k+9),$$

with equality for $a = b = 0$ and $c = 3$ (or any cyclic permutation). If $k = 9/8$, then the equality holds also for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

□

P 1.87. *If a, b, c are nonnegative real numbers so that $a+b+c=3$, then*

$$(a^2+1)(b^2+1)(c^2+1) \le \frac{169}{16}.$$

(Vasile C., 2006)

Solution. Write the inequality as

$$f(a)+f(b)+f(c) \le \ln 169 - \ln 16,$$

where

$$f(u) = \ln(u^2+1), \quad u \in [0,3].$$

From

$$f''(u) = \frac{2(1-u^2)}{(u^2+1)^2},$$

it follows that f is strictly convex on $[0,1]$ and strictly concave on $[1,3]$. According to LCRCF-Theorem, it suffices to consider the cases $a=0$ and $0<a \le b=c$.

Case 1: $a=0$. We need to show that $b+c=3$ involves

$$(b^2+1)(c^2+1) \le \frac{169}{16},$$

which is equivalent to

$$(4bc+1)(4bc-9) \le 0.$$

This is true because

$$4bc \le (b+c)^2 = 9.$$

Case 2: $0<a \le b=c$. We need to show that $a+2b=3$ $(0<a \le 1)$ involves

$$(a^2+1)(b^2+1)^2 \le \frac{169}{16},$$

which is equivalent to $g(a) \le 0$, where

$$g(a) = \ln(a^2+1) + 2\ln(b^2+1) - \ln 169 + \ln 16, \quad b = \frac{3-a}{2}, \quad a \in (0,1].$$

Since $b'(a) = -1/2$, we have

$$\begin{aligned} g'(a) &= \frac{2a}{a^2+1} - \frac{2b}{b^2+1} = \frac{2(a-b)(1-ab)}{(a^2+1)(b^2+1)} \\ &= \frac{3(a-1)^2(a-2)}{2(a^2+1)(b^2+1)} \le 0, \end{aligned}$$

hence g is strictly decreasing. Consequently, we have

$$g(a) < g(0) = 0.$$

The equality holds for $a=0$ and $b=c=3/2$ (or any cyclic permutation).

$\square$

P 1.88. *If a,b,c are nonnegative real numbers so that $a+b+c=3$, then*

$$(2a^2+1)(2b^2+1)(2c^2+1) \le \frac{121}{4}.$$

(Vasile C., 2006)

Solution. Write the inequality as

$$f(a)+f(b)+f(c) \le \ln 121 - \ln 4,$$

where

$$f(u) = \ln(2u^2+1), \quad u \in [0,3].$$

From

$$f''(u) = \frac{4(1-2u^2)}{(2u^2+1)^2},$$

it follows that f is strictly convex on $[0,1/\sqrt{2}]$ and strictly concave on $[1/\sqrt{2},3]$. By LCRCF-Theorem, it suffices to consider the cases $a=0$ and $0<a\le b=c$.

Case 1: $a=0$. We need to show that $b+c=3$ involves

$$(2b^2+1)(2c^2+1) \le \frac{121}{4},$$

which is equivalent to

$$(4bc+5)(4bc-9) \le 0.$$

This is true because

$$4bc \le (b+c)^2 = 9.$$

Case 2: $0<a\le b=c$. We need to show that $a+2b=3$ $(0<a\le 1)$ involves

$$(2a^2+1)(2b^2+1)^2 \le \frac{121}{4},$$

which is equivalent to $g(a)\le 0$, where

$$g(a) = \ln(2a^2+1) + 2\ln(2b^2+1) - \ln 121 + \ln 4, \quad b = \frac{3-a}{2}, \quad a \in (0,1].$$

Since $b'(a) = -1/2$, we have

$$\begin{aligned} g'(a) &= \frac{4a}{2a^2+1} - \frac{4b}{2b^2+1} = \frac{4(a-b)(1-2ab)}{(2a^2+1)(2b^2+1)} \\ &= \frac{6(a-1)(a^2-3a+1)}{(2a^2+1)(2b^2+1)} \\ &= \frac{3(1-a)(3+\sqrt{5}-2a)(2a-3+\sqrt{5})}{2(2a^2+1)(2b^2+1)}, \end{aligned}$$

hence $g'\left(\frac{3-\sqrt{5}}{2}\right)=0$, $g'(a)<0$ for $a\in\left[0,\frac{3-\sqrt{5}}{2}\right)$, $g'(a)>0$ for $a\in\left(\frac{3-\sqrt{5}}{2},1\right)$. Therefore, g is strictly decreasing on $\left[0,\frac{3-\sqrt{5}}{2}\right]$ and strictly increasing on $\left[\frac{3-\sqrt{5}}{2},1\right]$. Since $g(0)=0$, it suffices to show that $g(1)\le 0$, which reduces to $27\cdot 4\le 121$.

The equality holds for $a=0$ and $b=c=3/2$ (or any cyclic permutation).

□

P 1.89. *If a,b,c are nonnegative real numbers so that $a+b+c\ge k_0$, where*

$$k_0=\frac{3}{8}\sqrt{66+10\sqrt{105}}\approx 4.867,$$

then

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)}\le\left(\frac{a+b+c}{3}\right)^2+1.$$

(Vasile C., 2018)

Solution. Consider first the case $a+b+c=k_0$, and write the inequality as

$$f(a)+f(b)+f(c)\ge 3f(s),\qquad s=\frac{a+b+c}{3}=\frac{k_0}{3},$$

where

$$f(u)=-\ln(u^2+1),\qquad u\in[0,k_0].$$

For $u\in[s,k_0]$, $f(u)$ is convex because

$$f''(u)=\frac{6(3u^2-1)}{(3u^2+1)^2}>0.$$

By the RHCF-Theorem, we only need to show that

$$f(x)+2f(y)\ge 3f(s)$$

for $0\le x\le s\le y$ so that $x+2y=3s$; that is, to show that $g(x)\ge 0$ for $x\in[0,s]$, where

$$g(x)=f(x)+2f(y)-3f(s),\qquad y=\frac{k_0-x}{2}.$$

Since $y'(x)=-1/3$, we have

$$g'(x)=f'(x)+2y'f'(y)=\frac{-2x}{x^2+1}+\frac{2y}{y^2+1}$$
$$=\frac{2(x-y)(xy-1)}{(x^2+1)(y^2+1)}=\frac{3(s-x)(x^2-k_0x+2)}{2(x^2+1)(y^2+1)}.$$

Since g is increasing on $[0,s_1]$ and decreasing on $[s_1,s]$, where $s_1=\dfrac{k_0-\sqrt{k_0^2-8}}{2}$, it suffices to show that $g(0)\geq 0$ and $g(s)\geq 0$. These inequalities are true because $g(0)=0$ and $g(s)=0$. The equality $g(0)=0$ is equivalent to

$$\sqrt[3]{(y^2+1)^2}=\left(\frac{2y}{3}\right)^2+1,$$

where $y=\dfrac{k_0}{2}$.

According to RHCF-Theorem, if the inequality

$$f(a)+f(b)+f(c)\geq 3f\left(\frac{a+b+c}{3}\right)$$

holds for $a+b+c=k_0$, then it holds for $a+b+c>k_0$, too.

The equality holds for $a=b=c$. In addition, for $a+b+c=k_0$, the equality occurs again for $a=0$ and $b=c=k_0/2$ (or any cyclic permutation).

□

P 1.90. *If a,b,c,d are nonnegative real numbers so that $a+b+c+d=4$, then*

$$(a^2+3)(b^2+3)(c^2+3)(d^2+3)\leq 513.$$

(Vasile C., 2006)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d)\leq \ln 513,$$

where

$$f(u)=\ln(u^2+3),\quad u\in[0,4].$$

From

$$f''(u)=\frac{2(3-u^2)}{(u^2+3)^2},$$

it follows that f is strictly convex on $[0,\sqrt{3}]$ and strictly concave on $[\sqrt{3},4]$. By LCRCF-Theorem, it suffices to consider the cases $a=0$ and $0<a\leq b=c$.

Case 1: $a=0$. We need to show that $b+c+d=4$ involves

$$(b^2+3)(c^2+3)(d^2+3)\leq 171.$$

Substituting b,c,d by $4b/3,4c/3,4d/3$, respectively, we need to show that $b+c+d=3$ involves

$$(b^2+k)(c^2+k)(d^2+k)\leq k^2(k+9),$$

where $k = 27/16$. According to Remark from the proof of P 1.86, this inequality holds for all $k \ge 9/8$.

Case 2: $0 < a \le b = c = d$. We need to show that $a + 3b = 4$ $(0 < a \le 1)$ involves

$$(a^2+3)(b^2+3)^3 \le 513,$$

which is equivalent to $g(a) \le 0$, where

$$g(a) = \ln(a^2+3) + 3\ln(b^2+3) - \ln 513, \quad b = \frac{4-a}{3}, \quad a \in (0,1].$$

Since $b'(a) = -1/3$, we have

$$\begin{aligned} g'(a) &= \frac{2a}{a^2+3} - \frac{2b}{b^2+3} = \frac{2(a-b)(3-ab)}{(a^2+3)(b^2+3)} \\ &= \frac{8(a-1)(a^2-4a+9)}{9(a^2+3)(b^2+3)}. \end{aligned}$$

Because

$$a^2 - 4a + 9 = (a-2)^2 + 5 > 0,$$

we have $g'(a) > 0$ for $a \in [0,1)$, g is strictly decreasing on $[0,1]$, hence it suffices to show that $g(0) \le 0$. This reduces to show that the original inequality holds for $a = 0$ and $b = c = d = 4/3$, which follows immediately from the case 1.

The equality holds for $a = b = c = 0$ and $d = 4$ (or any cyclic permutation).

□

P 1.91. *If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 4$, then*

$$(a^2+2)(b^2+2)(c^2+2)(d^2+2) \le 144.$$

(Vasile C., 2006)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) \le \ln 144,$$

where

$$f(u) = \ln(u^2+2), \quad u \in [0,4].$$

From

$$f''(u) = \frac{2(2-u^2)}{(u^2+2)^2},$$

it follows that f is strictly convex on $[0, \sqrt{2}]$ and strictly concave on $[\sqrt{2}, 4]$. By LCRCF-Theorem, it suffices to consider the cases $a = 0$ and $0 < a \le b = c$.

Case 1: $a = 0$. We need to show that $b + c + d = 4$ involves

$$(b^2 + 2)(c^2 + 2)(d^2 + 2) \le 72.$$

Substituting b, c, d by $4b/3, 4c/3, 4d/3$, respectively, we need to show that $b + c + d = 3$ involves

$$(8b^2 + 9)(8c^2 + 9)(8d^2 + 9) \le 9^4.$$

This is true according to Remark from the proof of P 1.86.

Case 2: $0 < a \le b = c = d$. We need to show that $a + 3b = 4$ $(0 < a \le 1)$ involves

$$(a^2 + 2)(b^2 + 2)^3 \le 144,$$

which is equivalent to $g(a) \le 0$, where

$$g(a) = \ln(a^2 + 2) + 3\ln(b^2 + 2) - \ln 144, \quad b = \frac{4-a}{3}, \quad a \in (0, 1].$$

Since $b'(a) = -1/3$, we have

$$\begin{aligned} g'(a) &= \frac{2a}{a^2+2} - \frac{2b}{b^2+2} = \frac{2(a-b)(2-ab)}{(a^2+2)(b^2+2)} \\ &= \frac{8(a-1)(a^2-4a+6)}{9(a^2+2)(b^2+2)}. \end{aligned}$$

Because

$$a^2 - 4a + 6 = (a-2)^2 + 2 > 0,$$

we have $g'(a) > 0$ for $a \in [0, 1)$, g is strictly decreasing on $[0, 1]$, hence it suffices to show that $g(0) \le 0$. This reduces to show that the original inequality holds for $a = 0$ and $b = c = d = 4/3$, which follows immediately from the case 1.

The equality holds for $a = b = c = 0$ and $d = 4$ (or any cyclic permutation), and also for $a = b = 0$ and $c = d = 2$ (or any permutation).

□

P 1.92. *If a, b, c, d are nonnegative real numbers such that*

$$a + b + c + d = 4,$$

then

$$\frac{a}{3a^3+2} + \frac{b}{3b^3+2} + \frac{c}{3c^3+2} + \frac{d}{3d^3+2} \le \frac{4}{5}.$$

(Vasile Cîrtoaje, 2019)

Solution. Consider the function

$$f(u) = \frac{-u}{3u^3+2} : \mathbb{I} = [0,4].$$

Since

$$f''(u) = \frac{18u^2(4-3u^3)}{(3u^3+2)^3}$$

is positive for $u \in [0,1]$, f is left convex on $\mathbb{I}_{\leq 1}$. According to LHCF-Theorem and Note 1, it is enough to show that $h(x,y) \geq 0$ for $x, y \in [0,4]$ such that $x+3y=4$. We have

$$g(u) = \frac{f(u)-f(1)}{u-1} = \frac{3u^2+3u-2}{3u^3+2},$$

$$h(x,y) = \frac{g(x)-g(y)}{x-y} = \frac{2F(x,y)}{(3x^3+2)(3y^3+2)},$$

where

$$F(x,y) = 2(x^2+xy+y^2)+2(x+y)+2-3x^2y^2-3xy(x+y).$$

From

$$4 = x+3y \geq 2\sqrt{3xy},$$

we get $3xy \leq 4$. Thus, we have

$$F(x,y) \geq 2(x^2+xy+y^2)+2(x+y)+2-4xy-4(x+y) = 26(y-1)^2 \geq 0.$$

The proof is completed. The equality occurs for $a=b=c=d=1$.

□

P 1.93. *If $a_1, a_2, ..., a_n$ are nonnegative real numbers such that $a_1+a_2+\cdots+a_n = 1$, then*

$$a_1^3+a_2^3+\cdots+a_n^3 \leq \frac{1}{8}+a_1^4+a_2^4+\cdots+a_n^4.$$

(Vasile C., 2018)

Solution. We use the induction method. For $n=2$, denoting

$$a_1a_2 = p, \qquad p \leq 1/4,$$

we have

$$a_1^3+a_2^3 = (a_1+a_2)^3 - 3a_1a_2(a_1+a_2) = 1-3p,$$

$$a_1^4+a_2^4 = (a_1^2+a_2^2)^2 - 2a_1^2a_2^2 = 2p^2-4p+1,$$

and the inequality is equivalent to

$$(4p-1)^2 \geq 0.$$

Consider further that $n \ge 3$, $a_1 \le a_2 \le \cdots \le a_n$, and write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \le \frac{1}{8},$$

where

$$f(u) = u^3 - u^4, \qquad u \in [0,1].$$

From

$$f''(u) = 6u(1-2u),$$

it follows that f is strictly convex on $[0,1/2]$ and strictly concave on $[1/2,1]$. By LCRCF-Theorem, it suffices to consider the cases $a_1 = 0$ and $0 < a_1 \le a_2 = \cdots = a_n$.

Case 1: $a_1 = 0$. The inequality follows by the induction hypothesis.

Case 2: $0 < a_1 \le a_2 = \cdots = a_n$. We only need to prove the homogeneous inequality

$$8(a_1^4 + a_2^4 + \cdots + a_n^4) + (a_1 + a_2 + \cdots + a_n)^4 \ge 8(a_1 + a_2 + \cdots + a_n)(a_1^3 + a_2^3 + \cdots + a_n^3)$$

for $a_1 = x$ and $a_2 = \cdots = a_{n-1} = 1$, that is

$$8(x^4 + n - 1) + (x + n - 1)^4 \ge 8(x + n - 1)(x^3 + n - 1),$$

$$x^4 - 4(n-1)x^3 + 6(n-1)^2x^2 + 4(n-1)(n^2-2n-1)x + (n-3)(n-1)(n^2-5) \ge 0,$$

$$x^2(x-2n+2)^2 + 2(n-1)^2x^2 + 4(n-1)(n^2-2n-1)x + (n-3)(n-1)(n^2-5) \ge 0.$$

The equality holds for $a_1 = \cdots = a_{n-2} = 0$ and $a_{n-1} = a_n = 1/2$ (or any permutation).

Remark. The inequality can be also proved by using EV-method (see Corollary 5 from section 5, case $k = 3$ and $m = 4$): If

$$a_1 + a_2 + \cdots + a_n = 1, \quad a_1^3 + a_2^3 + \cdots + a_n^3 = constant,$$

then the sum

$$S_n = a_1^4 + a_2^4 + \cdots + a_n^4$$

is minimum for either $a_1 = 0$ or $0 < a_1 \le a_2 = \cdots = a_n$.

□

Chapter 2

Half Convex Function Method for Ordered Variables

2.1 Theoretical Basis

The following statement is known as the Right Half Convex Function Theorem for Ordered Variables (RHCF-OV Theorem).

RHCF-OV Theorem (*Vasile Cîrtoaje*, 2008). *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\geq s}$, where $s \in int(\mathbb{I})$. The inequality*

$$f(a_1)+f(a_2)+\cdots+f(a_n) \geq nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying

$$a_1+a_2+\cdots+a_n = ns$$

and

$$a_1 \leq a_2 \leq \cdots \leq a_m \leq s, \quad m \in \{1,2,\ldots,n-1\},$$

if and only if

$$f(x)+(n-m)f(y) \geq (1+n-m)f(s)$$

for all $x, y \in \mathbb{I}$ so that

$$x \leq s \leq y, \quad x+(n-m)y = (1+n-m)s.$$

Proof. For

$$a_1 = x, \quad a_2 = \cdots = a_m = s, \quad a_{m+1} = \cdots = a_n = y,$$

the inequality

$$f(a_1)+f(a_2)+\cdots+f(a_n) \geq nf(s)$$

becomes

$$f(x)+(n-m)f(y) \geq (1+n-m)f(s);$$

thus, the necessity is proved. To prove the sufficiency, we assume that

$$a_1 \le a_2 \le \cdots \le a_n.$$

From $a_1 \le a_2 \le \cdots \le a_m \le s$, it follows that there is an integer

$$k \in \{m, m+1, \ldots, n-1\}$$

so that

$$a_1 \le \cdots \le a_k \le s \le a_{k+1} \le \cdots \le a_n.$$

Since f is convex on $\mathbb{I}_{\ge s}$, we may apply Jensen's inequality to get

$$f(a_{k+1}) + \cdots + f(a_n) \ge (n-k)f(z),$$

where

$$z = \frac{a_{k+1} + \cdots + a_n}{n-k}, \quad z \in \mathbb{I}.$$

Therefore, to prove the desired inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge f(s),$$

it suffices to show that

$$f(a_1) + \cdots + f(a_k) + (n-k)f(z) \ge nf(s). \tag{*}$$

Let $b_1, \ldots, b_k$ be defined by

$$a_i + (n-m)b_i = (1+n-m)s, \quad i = 1, \ldots, k.$$

We claim that

$$z \ge b_1 \ge \cdots \ge b_k \ge s, \qquad b_1, \ldots, b_k \in \mathbb{I}.$$

Indeed, we have

$$b_1 \ge \cdots \ge b_k,$$

$$b_k - s = \frac{s - a_k}{n-m} \ge 0,$$

and

$$z \ge b_1$$

because

$$\begin{aligned}(n-m)b_1 &= (1+n-m)s - a_1 \\ &= -(m-1)s + (a_2 + \cdots + a_k) + (a_{k+1} + \cdots + a_n) \\ &\le -(m-1)s + (k-1)s + (a_{k+1} + \cdots + a_n) = \\ &= (k-m)s + (n-k)z \le (n-m)z.\end{aligned}$$

Since $b_1,\ldots,b_k \in \mathbb{I}_{\ge s}$, by hypothesis we have

$$f(a_1)+(n-m)f(b_1) \ge (1+n-m)f(s),$$

$$\cdots$$

$$f(a_k)+(n-m)f(b_k) \ge (1+n-m)f(s),$$

hence

$$f(a_1)+\cdots+f(a_k)+(n-m)[f(b_1)+\cdots+f(b_k)] \ge k(1+n-m)f(s),$$

$$f(a_1)+\cdots+f(a_k) \ge k(1+n-m)f(s)-(n-m)[f(b_1)+\cdots+f(b_k)].$$

According to this result, the inequality (*) is true if

$$k(1+n-m)f(s)-(n-m)[f(b_1)+\cdots+f(b_k)]+(n-k)f(z) \ge nf(s),$$

which is equivalent to

$$pf(z)+(k-p)f(s) \ge f(b_1)+\cdots+f(b_k), \qquad p=\frac{n-k}{n-m} \le 1.$$

By Jensen's inequality, we have

$$pf(z)+(1-p)f(s) \ge f(w), \qquad w=pz+(1-p)s \ge s.$$

Thus, we only need to show that

$$f(w)+(k-1)f(s) \ge f(b_1)+\cdots+f(b_k).$$

Since the decreasingly ordered vector $\vec{A_k}=(w,s,\ldots,s)$ majorizes the decreasingly ordered vector $\vec{B_k}=(b_1,b_2,\ldots,b_k)$, this inequality follows from Karamata's inequality for convex functions.

Similarly, we can prove the Left Half Convex Function Theorem for Ordered Variables (LHCF-OV Theorem).

LHCF-OV Theorem. *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\le s}$, where $s \in int(\mathbb{I})$. The inequality*

$$f(a_1)+f(a_2)+\cdots+f(a_n) \ge nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$$

holds for all $a_1,a_2,\ldots,a_n \in \mathbb{I}$ satisfying

$$a_1+a_2+\cdots+a_n=ns$$

and

$$a_1 \ge a_2 \ge \cdots \ge a_m \ge s, \qquad m \in \{1,2,\ldots,n-1\},$$

if and only if

$$f(x)+(n-m)f(y) \ge (1+n-m)f(s)$$

for all $x, y \in \mathbb{I}$ *so tht*

$$x \ge s \ge y, \quad x+(n-m)y = (1+n-m)s.$$

From the RHCF-OV Theorem and the LHCF-OV Theorem, we find the HCF-OV Theorem (Half Convex Function Theorem for Ordered Variables).

HCF-OV Theorem. *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\ge s}$ $(or\ \mathbb{I}_{\le s})$, where $s \in int(\mathbb{I})$. The inequality*

$$f(a_1)+f(a_2)+\cdots+f(a_n) \ge nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ so that

$$a_1+a_2+\cdots+a_n = ns$$

and at least m of $a_1, a_2, \ldots, a_n$ are smaller (greater) than s, where $m \in \{1, 2, \ldots, n-1\}$, if and only if

$$f(x)+(n-m)f(y) \ge (1+n-m)f(s)$$

for all $x, y \in \mathbb{I}$ satisfying $x+(n-m)y = (1+n-m)s$.

The RHCF-OV Theorem, the LHCF-OV Theorem and the HCF-OV Theorem are respectively generalizations of the RHCF-Theorem, the LHCF Theorem and the HCF-Theorem, because the last theorems can be obtained from the first theorems for $m = 1$.

Note 1. Let us denote

$$g(u) = \frac{f(u)-f(s)}{u-s}, \quad h(x,y) = \frac{g(x)-g(y)}{x-y}.$$

In many applications, it is useful to replace the hypothesis

$$f(x)+(n-m)f(y) \ge (1+n-m)f(s)$$

in the RHCF-OV Theorem and the LHCF-OV Theorem by the equivalent condition

$$h(x,y) \ge 0 \ \textit{ for all } \ x, y \in \mathbb{I} \ \textit{ so that } \ x+(n-m)y = (1+n-m)s.$$

This equivalence is true since

$$\begin{aligned} f(x)+(n-m)f(y)-(1+n-m)f(s) &= [f(x)-f(s)]+(n-m)[f(y)-f(s)] \\ &= (x-s)g(x)+(n-m)(y-s)g(y) \\ &= \frac{n-m}{1+n-m}(x-y)[g(x)-g(y)] \\ &= \frac{n-m}{1+n-m}(x-y)^2h(x,y). \end{aligned}$$

Note 2. Assume that f is differentiable on $\mathbb{I}$, and let

$$H(x,y)=\frac{f'(x)-f'(y)}{x-y}.$$

The desired inequality of Jensen's type in the RHCF-OV Theorem and the LHCF-OV Theorem holds true by replacing the hypothesis

$$f(x)+(n-m)f(y)\ge(1+n-m)f(s)$$

with the more restrictive condition

$$H(x,y)\ge 0 \ \textit{ for all } \ x,y\in\mathbb{I} \ \textit{ so that } \ x+(n-m)y=(1+n-m)s.$$

To prove this, we will show that the new condition implies

$$f(x)+(n-m)f(y)\ge(1+n-m)f(s)$$

for all $x,y\in\mathbb{I}$ so that $x+(n-m)y=(1+n-m)s$. Write this inequality as

$$f_1(x)\ge(1+n-m)f(s),$$

where

$$f_1(x)=f(x)+(n-m)f\left(\frac{(1+n-m)s-x}{n-m}\right).$$

From

$$\begin{aligned}f_1'(x)&=f'(x)-f'\left(\frac{(1+n-m)s-x}{n-m}\right)\\&=f'(x)-f'(y)\\&=\frac{1+n-m}{n-m}(x-s)H(x,y),\end{aligned}$$

it follows that f_1 is decreasing on $\mathbb{I}_{\le s}$ and increasing on $\mathbb{I}_{\ge s}$; therefore,

$$f_1(x)\ge f_1(s)=(1+n-m)f(s).$$

Note 3. The RHCF-OV Theorem and the LHCF-OV Theorem are also valid in the case when f is defined on $\mathbb{I}\setminus\{u_0\}$, where $u_0\in\mathbb{I}_{<s}$ for the RHCF-OV Theorem, and $u_0\in\mathbb{I}_{>s}$ for the LHCF-OV Theorem.

Note 4. The desired inequalities in the RHCF-OV Theorem and the LHCF-OV Theorem become equalities for

$$a_1=a_2=\cdots=a_n=s.$$

In addition, if there exist $x,y\in\mathbb{I}$ so that

$$x+(n-m)y=(1+n-m)s,\quad f(x)+(n-m)f(y)=(1+n-m)f(s),\quad x\ne y,$$

then the equality holds also for

$$a_1 = x, \quad a_2 = \cdots = a_m = s, \quad a_{m+1} = \cdots = a_n = y$$

Notice that these equality conditions are equivalent to

$$x + (n-m)y = (1+n-m)s, \quad h(x,y) = 0$$

($x < y$ for the RHCF-OV Theorem, and $x > y$ for the LHCF-OV Theorem).

Note 5. The WRHCF-OV Theorem and the WLHCF-OV Theorem are extensions of the *weighted* Jensen's inequality to right and left half convex functions with ordered variables (*Vasile Cirtoaje*, 2008).

WRHCF-OV Theorem. *Let $p_1, p_2, \ldots, p_n$ be positive real numbers so that*

$$p_1 + p_2 + \cdots + p_n = 1,$$

and let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\geq s}$, where $s \in int(\mathbb{I})$. The inequality

$$p_1 f(x_1) + p_2 f(x_2) + \cdots + p_n f(x_n) \geq f(p_1x_1 + p_2x_2 + \cdots + p_nx_n)$$

holds for all $x_1, x_2, \ldots, x_n \in \mathbb{I}$ so that $p_1x_1 + p_2x_2 + \cdots + p_nx_n = s$ and

$$x_1 \leq x_2 \leq \cdots \leq x_n, \quad x_m \leq s, \quad m \in \{1,2,\ldots,n-1\},$$

if and only if

$$f(x) + kf(y) \geq (1+k)f(s)$$

for all $x, y \in \mathbb{I}$ satisfying

$$x \leq s \leq y, \quad x + ky = (1+k)s,$$

where

$$k = \frac{p_{m+1} + p_{m+2} + \cdots + p_n}{p_1}.$$

WLHCF-OV Theorem. *Let $p_1, p_2, \ldots, p_n$ be positive real numbers so that*

$$p_1 + p_2 + \cdots + p_n = 1,$$

and let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\leq s}$, where $s \in int(\mathbb{I})$. The inequality

$$p_1 f(x_1) + p_2 f(x_2) + \cdots + p_n f(x_n) \geq f(p_1x_1 + p_2x_2 + \cdots + p_nx_n)$$

holds for all $x_1, x_2, \ldots, x_n \in \mathbb{I}$ so that $p_1x_1 + p_2x_2 + \cdots + p_nx_n = s$ and

$$x_1 \geq x_2 \geq \cdots \geq x_n, \quad x_m \geq s, \quad m \in \{1,2,\ldots,n-1\},$$

if and only if

$$f(x) + kf(y) \geq (1+k)f(s)$$

for all $x, y \in \mathbb{I}$ *satisfying*

$$x \geq s \geq y, \quad x + ky = (1+k)s,$$

where

$$k = \frac{p_{m+1} + p_{m+2} + \cdots + p_n}{p_1}.$$

2.2 Applications

2.1. If a,b,c,d are real numbers so that

$$a \le b \le 1 \le c \le d, \qquad a+b+c+d=4,$$

then

$$(3a^2-2)(a-1)^2+(3b^2-2)(b-1)^2+(3c^2-2)(c-1)^2+(3d^2-2)(d-1)^2 \ge 0.$$

2.2. If a,b,c,d are nonnegative real numbers so that

$$a \ge b \ge 1 \ge c \ge d, \qquad a+b+c+d=4,$$

then

$$\frac{1}{2a^3+5}+\frac{1}{2b^3+5}+\frac{1}{2c^3+5}+\frac{1}{2d^3+5} \le \frac{4}{7}.$$

2.3. If

$$\frac{-2n-1}{n-1} \le a_1 \le \cdots \le a_n \le 1 \le a_{n+1} \le \cdots \le a_{2n}, \qquad a_1+a_2+\cdots+a_{2n}=2n,$$

then

$$a_1^3+a_2^3+\cdots+a_{2n}^3 \ge 2n.$$

2.4. Let $a_1,a_2,\ldots,a_n$ $(n \ge 3)$ be real numbers so that $a_1+a_2+\cdots+a_n=n$. Prove that

(a) if $-3 \le a_1 \le \cdots \le a_{n-2} \le 1 \le a_{n-1} \le a_n$, then

$$a_1^3+a_2^3+\cdots+a_n^3 \ge a_1^2+a_2^2+\cdots+a_n^2;$$

(b) if $-\dfrac{n-1}{n-3} \le a_1 \le a_2 \le 1 \le \cdots \le a_n$, then

$$a_1^3+a_2^3+\cdots+a_n^3+n \ge 2(a_1^2+a_2^2+\cdots+a_n^2).$$

2.5. Let $a_1,a_2,\ldots,a_n$ be nonnegative real numbers so that $a_1+a_2+\cdots+a_n=n$ and let $m \in \{1,2,\ldots,n-1\}$. Prove that

(a) if $a_1 \le a_2 \le \cdots \le a_m \le 1$, then

$$(n-m)(a_1^3+a_2^3+\cdots+a_n^3-n) \ge (2n-2m+1)(a_1^2+a_2^2+\cdots+a_n^2-n);$$

(b) if $a_1 \ge a_2 \ge \cdots \ge a_m \ge 1$, then

$$a_1^3+a_2^3+\cdots+a_n^3-n \le (n-m+2)(a_1^2+a_2^2+\cdots+a_n^2-n).$$

2.6. Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be real numbers so that $a_1 + a_2 + \cdots + a_n = n$. Prove that

(a) if $a_1 \le \cdots \le a_{n-1} \le 1 \le a_n$, then

$$a_1^4 + a_2^4 + \cdots + a_n^4 - n \ge 6(a_1^2 + a_2^2 + \cdots + a_n^2 - n);$$

(b) if $a_1 \le \cdots \le a_{n-2} \le 1 \le a_{n-1} \le a_n$, then

$$a_1^4 + a_2^4 + \cdots + a_n^4 - n \ge \frac{14}{3}(a_1^2 + a_2^2 + \cdots + a_n^2 - n);$$

(c) if $a_1 \le a_2 \le 1 \le a_3 \le \cdots \le a_n$, then

$$a_1^4 + a_2^4 + \cdots + a_n^4 - n \ge \frac{2(n^2 - 3n + 3)}{n^2 - 5n + 7}\,(a_1^2 + a_2^2 + \cdots + a_n^2 - n).$$

2.7. Let a, b, c, d, e be nonnegative real numbers so that $a + b + c + d + e = 5$. Prove that

(a) if $a \ge b \ge 1 \ge c \ge d \ge e$, then

$$21(a^2 + b^2 + c^2 + d^2 + e^2) \ge a^4 + b^4 + c^4 + d^4 + e^4 + 100;$$

(b) if $a \ge b \ge c \ge 1 \ge d \ge e$, then

$$13(a^2 + b^2 + c^2 + d^2 + e^2) \ge a^4 + b^4 + c^4 + d^4 + e^4 + 60.$$

2.8. Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be nonnegative numbers so that $a_1 + a_2 + \cdots + a_n = n$. Prove that

(a) if $a_1 \ge \cdots \ge a_{n-1} \ge 1 \ge a_n$, then

$$7(a_1^3 + a_2^3 + \cdots + a_n^3) \ge 3(a_1^4 + a_2^4 + \cdots + a_n^4) + 4n;$$

(b) if $a_1 \ge \cdots \ge a_{n-2} \ge 1 \ge a_{n-1} \ge a_n$, then

$$13(a_1^3 + a_2^3 + \cdots + a_n^3) \ge 4(a_1^4 + a_2^4 + \cdots + a_n^4) + 9n.$$

2.9. If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1 + a_2 + \cdots + a_n = n$ and

$$a_1 \ge \cdots \ge a_m \ge 1 \ge a_{m+1} \ge \cdots \ge a_n, \qquad m \in \{1, 2, \ldots, n-1\},$$

then

$$(n - m + 1)^2 \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n \right) \ge 4(n - m)(a_1^2 + a_2^2 + \cdots + a_n^2 - n).$$

2.10. If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}=n$ and

$$a_1 \le \cdots \le a_m \le 1 \le a_{m+1} \le \cdots \le a_n, \quad m \in \{1,2,\ldots,n-1\},$$

then

$$a_1^2+a_2^2+\cdots+a_n^2-n \ge 2\left(1+\frac{\sqrt{n-m}}{n-m+1}\right)(a_1+a_2+\cdots+a_n-n).$$

2.11. Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be nonnegative numbers so that $a_1+a_2+\cdots+a_n=n$. Prove that

(a) if $a_1 \le \cdots \le a_{n-1} \le 1 \le a_n$, then

$$\frac{1}{a_1^2+2}+\frac{1}{a_2^2+2}+\cdots+\frac{1}{a_n^2+2} \ge \frac{n}{3};$$

(b) if $a_1 \le \cdots \le a_{n-2} \le 1 \le a_{n-1} \le a_n$, then

$$\frac{1}{2a_1^2+3}+\frac{1}{2a_2^2+3}+\cdots+\frac{1}{2a_n^2+3} \ge \frac{n}{5}.$$

2.12. If $a_1, a_2, \ldots, a_{2n}$ are nonnegative real numbers so that

$$a_1 \ge \cdots \ge a_n \ge 1 \ge a_{n+1} \ge \cdots \ge a_{2n}, \quad a_1+a_2+\cdots+a_{2n}=2n,$$

then

$$\frac{1}{na_1^2+n^2+n+1}+\frac{1}{na_2^2+n^2+n+1}+\cdots+\frac{1}{na_{2n}^2+n^2+n+1} \le \frac{2n}{(n+1)^2}.$$

2.13. If a, b, c, d, e, f are nonnegative real numbers so that

$$a \ge b \ge c \ge 1 \ge d \ge e \ge f, \quad a+b+c+d+e+f=6,$$

then

$$\frac{3a+4}{3a^2+4}+\frac{3b+4}{3b^2+4}+\frac{3c+4}{3c^2+4}+\frac{3d+4}{3d^2+4}+\frac{3e+4}{3e^2+4}+\frac{3f+4}{3f^2+4} \le 6.$$

2.14. If a,b,c,d,e,f are nonnegative real numbers so that

$$a \ge b \ge 1 \ge c \ge d \ge e \ge f, \qquad a+b+c+d+e+f=6,$$

then

$$\frac{a^2-1}{(2a+7)^2}+\frac{b^2-1}{(2b+7)^2}+\frac{c^2-1}{(2c+7)^2}+\frac{d^2-1}{(2d+7)^2}+\frac{e^2-1}{(2e+7)^2}+\frac{f^2-1}{(2f+7)^2} \ge 0.$$

2.15. If a,b,c,d,e,f are nonnegative real numbers so that

$$a \le b \le 1 \le c \le d \le e \le f, \qquad a+b+c+d+e+f=6,$$

then

$$\frac{a^2-1}{(2a+5)^2}+\frac{b^2-1}{(2b+5)^2}+\frac{c^2-1}{(2c+5)^2}+\frac{d^2-1}{(2d+5)^2}+\frac{e^2-1}{(2e+5)^2}+\frac{f^2-1}{(2f+5)^2} \le 0.$$

2.16. If a,b,c are nonnegative real numbers so that

$$a \le b \le 1 \le c, \qquad a+b+c=3,$$

then

$$\sqrt{\frac{2a}{b+c}}+\sqrt{\frac{2b}{c+a}}+\sqrt{\frac{2c}{a+b}} \ge 3.$$

2.17. If $a_1,a_2,\ldots,a_8$ are nonnegative real numbers so that

$$a_1 \ge a_2 \ge a_3 \ge a_4 \ge 1 \ge a_5 \ge a_6 \ge a_7 \ge a_8, \qquad a_1+a_2+\cdots+a_8=8,$$

then

$$(a_1^2+1)(a_2^2+1)\cdots(a_8^2+1) \ge (a_1+1)(a_2+1)\cdots(a_8+1).$$

2.18. If a,b,c,d are real numbers so that

$$\frac{-1}{2} \le a \le b \le 1 \le c \le d, \qquad a+b+c+d=4,$$

then

$$7\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right)+3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \ge 40.$$

2.19. Let a,b,c,d be real numbers. Prove that

(a) if $-1 \le a \le b \le c \le 1 \le d$, then

$$3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right) \ge 8+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d};$$

(b) if $-1 \le a \le b \le 1 \le c \le d$, then

$$2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right) \ge 4+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}.$$

2.20. If a,b,c,d are positive real numbers so that

$$a \ge b \ge 1 \ge c \ge d, \qquad abcd = 1,$$

then

$$a^2+b^2+c^2+d^2-4 \ge 18\left(a+b+c+d-\frac{1}{a}-\frac{1}{b}-\frac{1}{c}-\frac{1}{d}\right).$$

2.21. If a,b,c,d are positive real numbers so that

$$a \le b \le 1 \le c \le d, \qquad abcd = 1,$$

then

$$\sqrt{a^2-a+1}+\sqrt{b^2-b+1}+\sqrt{c^2-c+1}+\sqrt{d^2-d+1} \ge a+b+c+d.$$

2.22. If a,b,c,d are positive real numbers so that

$$a \le b \le c \le 1 \le d, \qquad abcd = 1,$$

then

$$\frac{1}{a^3+3a+2}+\frac{1}{b^3+3b+2}+\frac{1}{c^3+3c+2}+\frac{1}{d^3+3d+2} \ge \frac{2}{3}.$$

2.23. If $a_1,a_2,\dots,a_n$ are positive real numbers so that

$$a_1 \ge \cdots \ge a_{n-1} \ge 1 \ge a_n, \qquad a_1a_2\cdots a_n = 1,$$

then

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n} \ge a_1+a_2+\cdots+a_n.$$

2.24. Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that

$$a_1 \le \cdots \le a_{n-1} \le 1 \le a_n, \qquad a_1a_2\cdots a_n = 1.$$

If $k \ge 1$, then

$$\frac{1}{1+ka_1} + \frac{1}{1+ka_2} + \cdots + \frac{1}{1+ka_n} \ge \frac{n}{1+k}.$$

2.25. If $a_1, a_2, \ldots, a_9$ are positive real numbers so that

$$a_1 \le \cdots \le a_8 \le 1 \le a_9, \qquad a_1a_2\cdots a_9 = 1,$$

then

$$\frac{1}{(a_1+2)^2} + \frac{1}{(a_2+2)^2} + \cdots + \frac{1}{(a_9+2)^2} \ge 1.$$

2.26. Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that

$$a_1 \le \cdots \le a_{n-1} \le 1 \le a_n, \qquad a_1a_2\cdots a_n = 1.$$

If $p, q \ge 0$ so that

$$p + q \ge 1 + \frac{2pq}{p+4q},$$

then

$$\frac{1}{1+pa_1+qa_1^2} + \frac{1}{1+pa_2+qa_2^2} + \cdots + \frac{1}{1+pa_n+qa_n^2} \ge \frac{n}{1+p+q}.$$

2.27. Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that

$$a_1 \le \cdots \le a_{n-1} \le 1 \le a_n, \qquad a_1a_2\cdots a_n = 1.$$

If $m \ge 1$ and $0 < k \le m$, then

$$\frac{1}{(a_1+k)^m} + \frac{1}{(a_2+k)^m} + \cdots + \frac{1}{(a_n+k)^m} \ge \frac{n}{(1+k)^m}.$$

2.28. If $a_1, a_2, \ldots, a_n$ are positive real numbers so that

$$a_1 \le \cdots \le a_{n-1} \le 1 \le a_n, \qquad a_1a_2\cdots a_n = 1,$$

then

$$\frac{1}{\sqrt{1+3a_1}} + \frac{1}{\sqrt{1+3a_2}} + \cdots + \frac{1}{\sqrt{1+3a_n}} \ge \frac{n}{2}.$$

2.29. Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that

$$a_1 \le \cdots \le a_{n-1} \le 1 \le a_n, \qquad a_1 a_2 \cdots a_n = 1.$$

If $0 < m < 1$ and $0 < k \le \dfrac{1}{2^{1/m} - 1}$, then

$$\frac{1}{(a_1+k)^m} + \frac{1}{(a_2+k)^m} + \cdots + \frac{1}{(a_n+k)^m} \ge \frac{n}{(1+k)^m}.$$

2.30. If $a_1, a_2, \ldots, a_n$ $(n \ge 4)$ are positive real numbers so that

$$a_1 \ge a_2 \ge a_3 \ge 1 \ge a_4 \ge \cdots \ge a_n, \qquad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{3a_1+1} + \frac{1}{3a_2+1} + \cdots + \frac{1}{3a_n+1} \ge \frac{n}{4}.$$

2.31. If $a_1, a_2, \ldots, a_n$ $(n \ge 4)$ are positive real numbers so that

$$a_1 \ge a_2 \ge a_3 \ge 1 \ge a_4 \ge \cdots \ge a_n, \qquad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{(a_1+1)^2} + \frac{1}{(a_2+1)^2} + \cdots + \frac{1}{(a_n+1)^2} \ge \frac{n}{4}.$$

2.32. If $a_1, a_2, \ldots, a_n$ are positive real numbers so that

$$a_1 \ge \cdots \ge a_{n-1} \ge 1 \ge a_n, \qquad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{(a_1+3)^2} + \frac{1}{(a_2+3)^2} + \cdots + \frac{1}{(a_n+3)^2} \le \frac{n}{16}.$$

2.33. Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that

$$a_1 \ge \cdots \ge a_{n-1} \ge 1 \ge a_n, \qquad a_1 a_2 \cdots a_n = 1.$$

If $p, q \ge 0$ so that $p + q \le 1$, then

$$\frac{1}{1+pa_1+qa_1^2} + \frac{1}{1+pa_2+qa_2^2} + \cdots + \frac{1}{1+pa_n+qa_n^2} \le \frac{n}{1+p+q}.$$

2.34. Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that

$$a_1 \ge \cdots \ge a_{n-1} \ge 1 \ge a_n, \qquad a_1 a_2 \cdots a_n = 1.$$

If $m > 1$ and $k \ge \dfrac{1}{2^{1/m} - 1}$, then

$$\frac{1}{(a_1+k)^m} + \frac{1}{(a_2+k)^m} + \cdots + \frac{1}{(a_n+k)^m} \le \frac{n}{(1+k)^m}.$$

2.35. If $a_1, a_2, \ldots, a_n$ are positive real numbers so that

$$a_1 \ge \cdots \ge a_{n-1} \ge 1 \ge a_n, \qquad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{\sqrt{1+2a_1}} + \frac{1}{\sqrt{1+2a_2}} + \cdots + \frac{1}{\sqrt{1+2a_n}} \le \frac{n}{\sqrt{3}}.$$

2.36. Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that

$$a_1 \ge \cdots \ge a_{n-1} \ge 1 \ge a_n, \qquad a_1 a_2 \cdots a_n = 1.$$

If $0 < m < 1$ and $k \ge m$, then

$$\frac{1}{(a_1+k)^m} + \frac{1}{(a_2+k)^m} + \cdots + \frac{1}{(a_n+k)^m} \le \frac{n}{(1+k)^m}.$$

2.37. If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are positive real numbers so that

$$a_1 \ge \cdots \ge a_{n-2} \ge 1 \ge a_{n-1} \ge a_n, \qquad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{(a_1+5)^2} + \frac{1}{(a_2+5)^2} + \cdots + \frac{1}{(a_n+5)^2} \le \frac{n}{36}.$$

2.38. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that

$$a_1 \ge \cdots \ge a_{n-1} \ge 1 \ge a_n, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n,$$

then

$$\frac{1}{3-a_1} + \frac{1}{3-a_2} + \cdots + \frac{1}{3-a_n} \le \frac{n}{2}.$$

2.39. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that

$$a_1 \le \cdots \le a_{n-1} \le 1 \le a_n, \qquad a_1 + a_2 + \cdots + a_n = n.$$

Prove that

$$a_1^3 + a_2^3 + \cdots + a_n^3 - n \ge (n-1)^2 \left[\left(\frac{n-a_1}{n-1} \right)^3 + \left(\frac{n-a_2}{n-1} \right)^3 + \cdots + \left(\frac{n-a_n}{n-1} \right)^3 - n \right].$$

2.3 Solutions

P 2.1. *If a,b,c,d are real numbers so that*

$$a \le b \le 1 \le c \le d, \quad a+b+c+d=4,$$

then

$$(3a^2-2)(a-1)^2+(3b^2-2)(b-1)^2+(3c^2-2)(c-1)^2+(3d^2-2)(d-1)^2 \ge 0.$$

(Vasile C., 2007)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d) \ge 4f(s), \quad s=\frac{a+b+c+d}{4}=1,$$

where

$$f(u)=(3u^2-2)(u-1)^2, \quad u \in \mathbb{I}=\mathbb{R}.$$

From

$$f''(u)=2(18u^2-18u+1),$$

it follows that $f''(u)>0$ for $u \ge 1$, hence f is convex on $\mathbb{I}_{\ge s}$. Therefore, we may apply the RHCF-OV Theorem for $n=4$ and $m=2$. Thus, it suffices to show that $f(x)+2f(y) \ge 3f(1)$ for all real x,y so that $x+2y=3$. Using Note 1, we only need to show that $h(x,y) \ge 0$, where

$$h(x,y)=\frac{g(x)-g(y)}{x-y}, \quad g(u)=\frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u)=3(u^3+u^2+u+1)-6(u^2+u+1)+u+1=3u^3-3u^2-2u-2,$$

$$h(x,y)=3(x^2+xy+y^2)-3(x+y)-2=(3y-4)^2 \ge 0.$$

From $x+2y=3$ and $h(x,y)=0$, we get $x=1/3$, $y=4/3$. Therefore, in accordance with Note 4, the equality holds for $a=b=c=d=1$, and also for

$$a=\frac{1}{3}, \quad b=1, \quad c=d=\frac{4}{3}.$$

Remark. Similarly, we can prove the following generalization:

- *Let $a_1,a_2,\dots,a_{2n}$ be real numbers so that*

$$a_1 \le \cdots \le a_n \le 1 \le a_{n+1} \le \cdots \le a_{2n}, \quad a_1+a_2+\cdots+a_{2n}=2n.$$

If $k = \dfrac{n}{n^2-n+1}$, *then*

$$(a_1^2-k)(a_1-1)^2+(a_2^2-k)(a_2-1)^2+\cdots+(a_{2n}^2-k)(a_{2n}-1)^2 \ge 0,$$

with equality for $a_1 = a_2 = \cdots = a_{2n} = 1$, *and also for*

$$a_1 = \frac{1}{n^2-n+1}, \quad a_2 = \cdots = a_n = 1, \quad a_{n+1} = \cdots = a_n = \frac{n^2}{n^2-n+1}.$$

□

P 2.2. *If* a,b,c,d *are nonnegative real numbers so that*

$$a \ge b \ge 1 \ge c \ge d, \qquad a+b+c+d = 4,$$

then

$$\frac{1}{2a^3+5}+\frac{1}{2b^3+5}+\frac{1}{2c^3+5}+\frac{1}{2d^3+5} \le \frac{4}{7}.$$

(Vasile C., 2009)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d) \ge 4f(s), \qquad s = \frac{a+b+c+d}{4} = 1,$$

where

$$f(u) = \frac{-1}{2u^3+5}, \quad u \ge 0.$$

From

$$f''(u) = \frac{12u(5-4u^3)}{(2u^3+5)^3},$$

it follows that $f''(u) \ge 0$ for $u \in [0,1]$, hence f is convex on $[0,s]$. Therefore, we may apply the LHCF-OV Theorem for $n = 4$ and $m = 2$. Using Note 1, we only need to show that $h(x,y) \ge 0$ for $x,y \ge 0$ so that $x+2y = 3$. We have

$$g(u) = \frac{f(u)-f(1)}{u-1} = \frac{2(u^2+u+1)}{7(2u^3+5)},$$

$$h(x,y) = \frac{g(x)-g(y)}{x-y} = \frac{2E}{7(2x^3+5)(2y^3+5)},$$

where

$$E = -2x^2y^2-2xy(x+y)-2(x^2+xy+y^2)+5(x+y)+5.$$

Since

$$E = (1-2y)^2(2+3y-2y^2) = (1-2y)^2(2+xy) \ge 0,$$

the proof is completed. From $x+2y=3$ and $h(x,y)=0$, we get $x=2$, $y=1/2$. Therefore, in accordance with Note 4, the equality holds for $a=b=c=d=1$, and also for

$$a=2, \quad b=1, \quad c=d=\frac{1}{2}.$$

Remark. Similarly, we can prove the following generalization.

- *If $a_1,a_2,\dots,a_{2n}$ are nonnegative real numbers so that*

$$a_1 \ge \cdots \ge a_n \ge 1 \ge a_{n+1} \ge \cdots \ge a_{2n}, \quad a_1+a_2+\cdots+a_{2n}=2n.$$

then

$$\frac{1}{a_1^3+n+\frac{1}{n}}+\frac{1}{a_2^3+n+\frac{1}{n}}+\cdots+\frac{1}{a_{2n}^3+n+\frac{1}{n}} \ge \frac{2n^2}{n^2+n+1},$$

with equality for $a_1=a_2=\cdots=a_{2n}=1$, and also for

$$a_1=n, \quad a_2=\cdots=a_n=1, \quad a_{n+1}=\cdots=a_{2n}=\frac{1}{n}.$$

□

P 2.3. *If*

$$\frac{-2n-1}{n-1} \le a_1 \le \cdots \le a_n \le 1 \le a_{n+1} \le \cdots \le a_{2n}, \qquad a_1+a_2+\cdots+a_{2n}=2n,$$

then

$$a_1^3+a_2^3+\cdots+a_{2n}^3 \ge 2n.$$

(Vasile C., 2007)

Solution. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_{2n}) \ge 2nf(s), \qquad s=\frac{a_1+a_2+\cdots+a_{2n}}{2n}=1,$$

where

$$f(u)=u^3, \quad u \ge \frac{-2n-1}{n-1}.$$

From $f''(u)=6u$, it follows that $f(u)$ is convex for $u \ge s$. Therefore, we may apply the RHCF-OV Theorem for $2n$ numbers and $m=n$. By Note 1, it suffices to show that $h(x,y) \ge 0$ for all $x,y \ge \dfrac{-2n-1}{n-1}$ so that $x+ny=1+n$. We have

$$g(u)=\frac{f(u)-f(1)}{u-1}=u^2+u+1,$$

$$h(x,y) = \frac{g(x)-g(y)}{x-y} = x+y+1 = \frac{(n-1)x+2n+1}{n-1} \ge 0.$$

From $x+ny = 1+n$ and $h(x,y) = 0$, we get

$$x = \frac{-2n-1}{n-1}, \qquad y = \frac{n+2}{n-1}.$$

In accordance with Note 4, the equality holds for $a_1 = a_2 = \cdots = a_{2n} = 1$, and also for

$$a_1 = \frac{-2n-1}{n-1}, \quad a_2 = \cdots = a_n = 1, \quad a_{n+1} = \cdots = a_{2n} = \frac{n+2}{n-1}.$$

□

P 2.4. *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be real numbers so that $a_1 + a_2 + \cdots + a_n = n$. Prove that*

(a) if $-3 \le a_1 \le \cdots \le a_{n-2} \le 1 \le a_{n-1} \le a_n$, then

$$a_1^3 + a_2^3 + \cdots + a_n^3 \ge a_1^2 + a_2^2 + \cdots + a_n^2;$$

(b) if $-\dfrac{n-1}{n-3} \le a_1 \le a_2 \le 1 \le \cdots \le a_n$, then

$$a_1^3 + a_2^3 + \cdots + a_n^3 + n \ge 2(a_1^2 + a_2^2 + \cdots + a_n^2).$$

(Vasile C., 2007)

Solution. (a) Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \qquad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = u^3 - u^2, \quad u \ge -3.$$

For $u \ge 1$, we have

$$f''(u) = 6u - 2 > 0,$$

hence $f(u)$ is convex for $u \ge s$. Thus, we may apply the RHCF-OV Theorem for $m = n-2$. According to this theorem, it suffices to show that

$$f(x) + 2f(y) \ge 3f(1)$$

for $-3 \le x \le y$ satisfying $x + 2y = 3$. Using Note 1, we only need to show that $h(x,y) \ge 0$, where

$$h(x,y) = \frac{g(x)-g(y)}{x-y}, \quad g(u) = \frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u) = u^2,$$

$$h(x, y) = x + y = \frac{x + 3}{2} \geq 0.$$

From $x + 2y = 3$ and $h(x, y) = 0$, we get $x = -3$ and $y = 3$. Therefore, in accordance with Note 4, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = -3, \quad a_2 = \cdots = a_{n-2} = 1, \quad a_{n-1} = a_n = 3.$$

(b) Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = u^3 - 2u^2, \quad u \geq -\frac{n-1}{n-3}.$$

For $u \geq 1$, we have

$$f''(u) = 6u - 4 > 0,$$

hence $f(u)$ is convex for $u \geq s$. Thus, we may apply the RHCF-OV Theorem for $m = 2$. According to this theorem, it suffices to show that

$$f(x) + (n-2)f(y) \geq (n-1)f(1)$$

for $-\dfrac{n-1}{n-3} \leq x \leq y$ satisfying $x + (n-2)y = n-1$. Using Note 1, we only need to show that $h(x, y) \geq 0$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

We have

$$g(u) = u^2 - u - 1,$$

$$h(x, y) = x + y - 1 = \frac{(n-3)x + n - 1}{n-1} \geq 0.$$

From $x + (n-2)y = n-1$ and $h(x, y) = 0$, we get $x = -\dfrac{n-1}{n-3}$ and $y = \dfrac{n-1}{n-3}$. Therefore, in accordance with Note 4, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $n \geq 4$, then the equality holds also for

$$a_1 = -\frac{n-1}{n-3}, \quad a_2 = 1, \quad a_3 = \cdots = a_n = \frac{n-1}{n-3}.$$

□

P 2.5. *Let $a_1,a_2,\dots,a_n$ be nonnegative real numbers so that $a_1+a_2+\cdots+a_n=n$ and let $m\in\{1,2,\dots,n-1\}$. Prove that*

(a) if $a_1\le a_2\le\cdots\le a_m\le 1$, then

$$(n-m)(a_1^3+a_2^3+\cdots+a_n^3-n)\ge(2n-2m+1)(a_1^2+a_2^2+\cdots+a_n^2-n);$$

(b) if $a_1\ge a_2\ge\cdots\ge a_m\ge 1$, then

$$a_1^3+a_2^3+\cdots+a_n^3-n\le(n-m+2)(a_1^2+a_2^2+\cdots+a_n^2-n).$$

(Vasile C., 2007)

Solution. (a) Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf(s),\qquad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=(n-m)u^3-(2n-2m+1)u^2,\quad u\in\mathbb{I}=[0,n].$$

For $u\ge 1$, we have

$$\begin{aligned}f''(u)&=6(n-m)u-2(2n-2m+1)\\&\ge 6(n-m)-2(2n-2m+1)=2(n-m-1)\ge 0,\end{aligned}$$

hence f is convex on $\mathbb{I}_{\ge s}$. Thus, by the RHCF-OV Theorem and Note 1, we only need to show that $h(x,y)\ge 0$ for all nonnegative numbers x,y so that $x+(n-m)y=n-m+1$. We have

$$\begin{aligned}g(u)=\frac{f(u)-f(1)}{u-1}&=(n-m)(u^2+u+1)-(2n-2m+1)(u+1)\\&=(n-m)u^2-(n-m+1)u-n+m-1,\end{aligned}$$

$$h(x,y)=\frac{g(x)-g(y)}{x-y}=(n-m)(x+y)-n+m-1=(n-m-1)x\ge 0.$$

From $x+(n-m)y=1+n-m$ and $h(x,y)=0$, we get $x=0$, $y=(n-m+1)/(n-m)$. Therefore, in accordance with Note 4, the equality holds for $a_1=a_2=\cdots=a_n=1$, and also for

$$a_1=0,\quad a_2=\cdots=a_m=1,\quad a_{m+1}=\cdots=a_n=1+\frac{1}{n-m}.$$

(b) Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf(s),\qquad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=(n-m+2)u^2-u^3,\quad u\in\mathbb{I}=[0,n].$$

For $u \le 1$, we have

$$f''(u) = 2(n-m+2-3u) \ge 2(n-m+2-3) = 2(n-m-1) \ge 0,$$

hence f is convex on $\mathbb{I}_{\le s}$. By the LHCF-OV Theorem and Note 1, it suffices to show that $h(x,y) \ge 0$ for all $x, y \ge 0$ so that $x + (n-m)y = 1 + n - m$. We have

$$g(u) = \frac{f(u) - f(1)}{u-1} = (n-m+2)(u+1) - (u^2+u+1)$$
$$= -u^2 + (n-m+1)u + n - m + 1,$$

$$h(x,y) = \frac{g(x) - g(y)}{x-y} = -(x+y) + n - m + 1 = (n-m-1)y \ge 0.$$

From $x + (n-m)y = 1 + n - m$ and $h(x,y) = 0$, we get $x = n-m+1$, $y = 0$. Therefore, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = n-m+1, \quad a_2 = \cdots = a_m = 1, \quad a_{m+1} = \cdots = a_n = 0.$$

Remark 1. For $m = 1$, we get the following results:

- *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then*

$$(n-1)(a_1^3 + a_2^3 + \cdots + a_n^3 - n) \ge (2n-1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n),$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = 0, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n-1}$$

(or any cyclic permutation).

- *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then*

$$a_1^3 + a_2^3 + \cdots + a_n^3 - n \le (n+1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n),$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = n, \quad a_2 = a_3 = \cdots = a_n = 0$$

(or any cyclic permutation).

Remark 2. For $m = n-1$, we get the following statements:

- *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that*

$$a_1 \le \cdots \le a_{n-1} \le 1 \le a_n, \quad a_1 + a_2 + \cdots + a_n = n,$$

then

$$a_1^3 + a_2^3 + \cdots + a_n^3 + 2n \ge 3(a_1^2 + a_2^2 + \cdots + a_n^2),$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = 0, \quad a_2 = \cdots = a_{n-1} = 1, \quad a_n = 2.$$

- *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that*

$$a_1 \ge \cdots \ge a_{n-1} \ge 1 \ge a_n, \quad a_1 + a_2 + \cdots + a_n = n,$$

then

$$a_1^3 + a_2^3 + \cdots + a_n^3 + 2n \le 3(a_1^2 + a_2^2 + \cdots + a_n^2),$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = 2, \quad a_2 = \cdots = a_{n-1} = 1, \quad a_n = 0.$$

Remark 3. Replacing n with $2n$ and choosing then $m = n$, we get the following results:

- *If $a_1, a_2, \ldots, a_{2n}$ are nonnegative real numbers so that*

$$a_1 \le \cdots \le a_n \le 1 \le a_{n+1} \le \cdots \le a_{2n}, \quad a_1 + a_2 + \cdots + a_{2n} = 2n,$$

then

$$n(a_1^3 + a_2^3 + \cdots + a_{2n}^3 - 2n) \ge (2n+1)(a_1^2 + a_2^2 + \cdots + a_{2n}^2 - 2n),$$

with equality for $a_1 = a_2 = \cdots = a_{2n} = 1$, and also for

$$a_1 = 0, \quad a_2 = \cdots = a_n = 1, \quad a_{n+1} = \cdots = a_{2n} = 1 + \frac{1}{n}.$$

- *If $a_1, a_2, \ldots, a_{2n}$ are nonnegative real numbers so that*

$$a_1 \ge \cdots \ge a_n \ge 1 \ge a_{n+1} \ge \cdots \ge a_{2n}, \quad a_1 + a_2 + \cdots + a_{2n} = 2n,$$

then

$$a_1^3 + a_2^3 + \cdots + a_{2n}^3 - 2n \le (n+2)(a_1^2 + a_2^2 + \cdots + a_{2n}^2 - 2n),$$

with equality for $a_1 = a_2 = \cdots = a_{2n} = 1$, and also for

$$a_1 = n+1, \quad a_2 = \cdots = a_n = 1, \quad a_{n+1} = \cdots = a_{2n} = 0.$$

□

P 2.6. *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be real numbers so that $a_1 + a_2 + \cdots + a_n = n$. Prove that*

(a) *if $a_1 \le \cdots \le a_{n-1} \le 1 \le a_n$, then*

$$a_1^4 + a_2^4 + \cdots + a_n^4 - n \ge 6(a_1^2 + a_2^2 + \cdots + a_n^2 - n);$$

(b) if $a_1 \le \cdots \le a_{n-2} \le 1 \le a_{n-1} \le a_n$, *then*

$$a_1^4 + a_2^4 + \cdots + a_n^4 - n \ge \frac{14}{3}(a_1^2 + a_2^2 + \cdots + a_n^2 - n);$$

(c) if $a_1 \le a_2 \le 1 \le a_3 \le \cdots \le a_n$, *then*

$$a_1^4 + a_2^4 + \cdots + a_n^4 - n \ge \frac{2(n^2-3n+3)}{n^2-5n+7}\,(a_1^2 + a_2^2 + \cdots + a_n^2 - n).$$

(Vasile C., 2009)

Solution. Consider the inequality

$$a_1^4 + a_2^4 + \cdots + a_n^4 - n \ge k(a_1^2 + a_2^2 + \cdots + a_n^2 - n), \qquad k \le 6,$$

and write it as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \qquad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = u^4 - ku^2, \quad u \in \mathbb{R}.$$

From $f''(u) = 2(6u^2 - k)$, it follows that f is convex for $u \ge 1$. Therefore, we may apply the RHCF-OV Theorem for $m = n-1$, $m = n-2$ and $m = 2$, respectively. By Note 1, it suffices to show that $h(x,y) \ge 0$ for all real x, y so that $x + (n-m)y = 1 + n - m$. We have

$$g(u) = \frac{f(u) - f(1)}{u-1} = u^3 + u^2 + u + 1 - k(u+1),$$

$$h(x,y) = \frac{g(x) - g(y)}{x-y} = x^2 + xy + y^2 + x + y + 1 - k.$$

(a) We need to show that $h(x,y) \ge 0$ for $k = 6$, $m = n-1$, $x + y = 2$. Indeed, we have

$$h(x,y) = 1 - xy = \frac{1}{4}(x-y)^2 \ge 0.$$

From $x + y = 2$ and $h(x,y) = 0$, we get $x = y = 1$. Therefore, in accordance with Note 4, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

(b) For $k = 14/3$, $m = n-2$ and $x + 2y = 3$, we have

$$h(x,y) = \frac{1}{3}(3y-5)^2 \ge 0.$$

From $x + 2y = 3$ and $h(x,y) = 0$, we get $x = -1/3$ and $y = 5/3$. Therefore, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = \frac{-1}{3}, \quad a_2 = \cdots = a_{n-2} = 1, \quad a_{n-1} = a_n = \frac{5}{3}.$$

(c) We have $k=\dfrac{2(n^2-3n+3)}{n^2-5n+7}$, $m=2$ and $x+(n-2)y=n-1$, which involve

$$h(x,y)=\frac{[(n^2-5n+7)y-n^2+3n-1]^2}{n^2-5n+7}\geq 0.$$

From $x+(n-2)y=n-1$ and $h(x,y)=0$, we get

$$x=\frac{-n^2+5n-5}{n^2-5n+7},\qquad y=\frac{n^2-3n+1}{n^2-5n+7}.$$

Therefore, the equality holds for $a_1=a_2=\cdots=a_n=1$, and also for

$$a_1=\frac{-n^2+5n-5}{n^2-5n+7},\quad a_2=1,\quad a_3=\cdots=a_n=\frac{n^2-3n+1}{n^2-5n+7}.$$

$\square$

P 2.7. *Let a,b,c,d,e be nonnegative real numbers so that $a+b+c+d+e=5$. Prove that*

(a) if $a\geq b\geq 1\geq c\geq d\geq e$, then

$$21(a^2+b^2+c^2+d^2+e^2)\geq a^4+b^4+c^4+d^4+e^4+100;$$

(b) if $a\geq b\geq c\geq 1\geq d\geq e$, then

$$13(a^2+b^2+c^2+d^2+e^2)\geq a^4+b^4+c^4+d^4+e^4+60.$$

(Vasile C., 2009)

Solution. Consider the inequality

$$k(a^2+b^2+c^2+d^2+e^2-5)\geq a^4+b^4+c^4+d^4+e^4-5,\qquad k\geq 6,$$

and write it as

$$f(a)+f(b)+f(c)+f(d)+f(e)\geq 5f(s),\qquad s=\frac{a+b+c+d+e}{5}=1,$$

where

$$f(u)=ku^2-u^4,\quad u\geq 0.$$

From $f''(u)=2(k-6u^2)$, it follows that f is convex on $[0,1]$. Therefore, we may apply the LHCF-OV Theorem for $m=2$ and $m=3$, respectively. By Note 1, it suffices to show that $h(x,y)\geq 0$ for all $x,y\geq 0$ so that $x+(5-m)y=6-m$. We have

$$g(u)=\frac{f(u)-f(1)}{u-1}=k(u+1)-(u^3+u^2+u+1),$$

$$h(x,y) = \frac{g(x)-g(y)}{x-y} = k-(x^2+xy+y^2+x+y+1).$$

(a) We need to show that $h(x,y) \ge 0$ for $k = 21$, $n = 5$, $m = 2$ and $x+3y = 4$; indeed, we have

$$h(x,y) = 21-(x^2+xy+y^2+x+y+1) = y(22-7y) = y(10+3x+2y) \ge 0.$$

From $x+3y = 4$ and $h(x,y) = 0$, we get $x = 4$ and $y = 0$. Therefore, in accordance with Note 4, the equality holds for $a = b = c = d = e = 1$, and also for

$$a = 4, \quad b = 1, \quad c = d = e = 0.$$

(b) We have $k = 13$, $n = 5$, $m = 3$ and $x+2y = 3$, which involve

$$h(x,y) = 13-(x^2+xy+y^2+x+y+1) = y(10-3y) = y(4+2x+y) \ge 0.$$

From $x+2y = 3$ and $h(x,y) = 0$, we get $x = 3$ and $y = 0$. Therefore, the equality holds for $a = b = c = d = e = 1$, and also for

$$a = 3, \quad b = c = 1, \quad d = e = 0.$$

□

P 2.8. *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be nonnegative numbers so that $a_1+a_2+\cdots+a_n = n$. Prove that*

(a) if $a_1 \ge \cdots \ge a_{n-1} \ge 1 \ge a_n$, then

$$7(a_1^3+a_2^3+\cdots+a_n^3) \ge 3(a_1^4+a_2^4+\cdots+a_n^4)+4n;$$

(b) if $a_1 \ge \cdots \ge a_{n-2} \ge 1 \ge a_{n-1} \ge a_n$, then

$$13(a_1^3+a_2^3+\cdots+a_n^3) \ge 4(a_1^4+a_2^4+\cdots+a_n^4)+9n.$$

(Vasile C., 2009)

Solution. Consider the inequality

$$k(a_1^3+a_2^3+\cdots+a_n^3-n) \ge a_1^4+a_2^4+\cdots+a_n^4-n, \quad k \ge 2,$$

and write it as

$$f(a_1)+f(a_2)+\cdots+f(a_n) \ge nf(s), \qquad s = \frac{a_1+a_2+\cdots+a_n}{n} = 1,$$

where

$$f(u) = ku^3-u^4, \quad u \ge 0.$$

From $f''(u) = 6u(k - 2u^2)$, it follows that f is convex on $[0,1]$. Therefore, we may apply the LHCF-OV Theorem for $m = n-1$ and $m = n-2$, respectively. By Note 1, it suffices to show that $h(x, y) \geq 0$ for $x \geq y \geq 0$ so that $x + my = 1 + m$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = k(u^2 + u + 1) - (u^3 + u^2 + u + 1),$$

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = -(x^2 + xy + y^2) + (k-1)(x + y + 1).$$

(a) We need to show that $h(x, y) \geq 0$ for $k = 7/3$, $m = n-1$, $x + y = 2$. Indeed,

$$h(x, y) = xy \geq 0.$$

From $x > y$, $x + y = 2$ and $h(x, y) = 0$, we get $x = 2$ and $y = 0$. Therefore, in accordance with Note 4, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = 2, \quad a_2 = \cdots = a_{n-1} = 1, \quad a_n = 0.$$

(b) We have $k = 13/4$, $m = n-2$, $x + 2y = 3$, which involve

$$h(x, y) = 3y(9 - 4y) = 3y(3 + 2x) \geq 0.$$

From $x + 2y = 3$ and $h(x, y) = 0$, we get $x = 3$ and $y = 0$. Therefore, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = 3, \quad a_2 = \cdots = a_{n-2} = 1, \quad a_{n-1} = a_n = 0.$$

□

P 2.9. *If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1 + a_2 + \cdots + a_n = n$ and*

$$a_1 \geq \cdots \geq a_m \geq 1 \geq a_{m+1} \geq \cdots \geq a_n, \quad m \in \{1, 2, \ldots, n-1\},$$

then

$$(n-m+1)^2\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n\right) \geq 4(n-m)(a_1^2 + a_2^2 + \cdots + a_n^2 - n).$$

(Vasile C., 2007)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{(n-m+1)^2}{u} - 4(n-m)u^2, \quad u > 0.$$

For $u \in (0,1]$, we have

$$\begin{aligned} f''(u) &= \frac{2(n-m+1)^2}{u^3} - 8(n-m) \\ &\geq 2(n-m+1)^2 - 8(n-m) = 2(n-m-1)^2 \geq 0. \end{aligned}$$

Since f is convex on $(0,s]$, we may apply the LHCF-OV Theorem. By Note 1, it suffices to show that $h(x,y) \geq 0$ for all $x,y>0$ so that $x+(n-m)y = 1+n-m$. We have

$$g(u) = \frac{f(u)-f(1)}{u-1} = \frac{-(n-m+1)^2}{u} - 4(n-m)(u+1),$$

$$h(x,y) = \frac{(n-m+1)^2}{xy} - 4(n-m) = \frac{[n-m+1-2(n-m)y]^2}{xy} \geq 0.$$

From $x+(n-m)y = 1+n-m$ and $h(x,y)=0$, we get

$$x = \frac{n-m+1}{2}, \qquad y = \frac{n-m+1}{2(n-m)}.$$

Therefore, in accordance with Note 4, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = \frac{n-m+1}{2}, \quad a_2 = a_3 = \cdots = a_m = 1, \quad a_{m+1} = \cdots = a_n = \frac{n-m+1}{2(n-m)}.$$

Remark 1. For $m = n-1$, we get the following elegant statement:

- *If $a_1, a_2, \ldots, a_n$ are positive real numbers so that*

$$a_1 \geq \cdots \geq a_{n-1} \geq 1 \geq a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \geq a_1^2 + a_2^2 + \cdots + a_n^2,$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$

Remark 2. Replacing n with $2n$ and choosing then $m = n$, we get the following statement:

- *If $a_1, a_2, \ldots, a_{2n}$ are positive real numbers so that*

$$a_1 \geq \cdots \geq a_n \geq 1 \geq a_{n+1} \geq \cdots \geq a_{2n}, \qquad a_1 + a_2 + \cdots + a_{2n} = 2n,$$

then

$$(n+1)^2\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{2n}} - 2n\right) \geq 4n(a_1^2 + a_2^2 + \cdots + a_{2n}^2 - 2n),$$

with equality for $a_1 = a_2 = \cdots = a_{2n} = 1$, and also for

$$a_1 = \frac{n+1}{2}, \quad a_2 = a_3 = \cdots = a_n = 1, \; a_{n+1} = \cdots = a_{2n} = \frac{n+1}{2n}.$$

$\square$

P 2.10. *If $a_1,a_2,\dots,a_n$ are positive real numbers so that $\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}=n$ and*

$$a_1\le\cdots\le a_m\le 1\le a_{m+1}\le\cdots\le a_n,\quad m\in\{1,2,\dots,n-1\},$$

then

$$a_1^2+a_2^2+\cdots+a_n^2-n\ge 2\left(1+\frac{\sqrt{n-m}}{n-m+1}\right)(a_1+a_2+\cdots+a_n-n).$$

(Vasile C., 2007)

Solution. Replacing each a_i by $1/a_i$, we need to prove that

$$a_1\ge\cdots\ge a_m\ge 1\ge a_{m+1}\ge\cdots\ge a_n,\quad a_1+a_2+\cdots+a_n=n$$

involves

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf(s),\quad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=\frac{1}{u^2}-\frac{2k}{u},\quad k=1+\frac{\sqrt{m-n}}{n-m+1},\quad u>0.$$

For $u\in(0,1]$, we have

$$f''(u)=\frac{6-4ku}{u^4}\ge\frac{6-4k}{u^4}=\frac{2(\sqrt{n-m}-1)^2}{(n-m+1)u^4}\ge 0.$$

Thus, f is convex on $(0,1]$. By the LHCF-OV Theorem and Note 1, it suffices to show that $h(x,y)\ge 0$ for $x,y>0$ so that $x+(n-m)y=1+n-m$, where

$$h(x,y)=\frac{g(x)-g(y)}{x-y},\quad g(u)=\frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u)=\frac{-1}{u^2}+\frac{2k-1}{u}$$

and

$$h(x,y)=\frac{1}{xy}\left(\frac{1}{x}+\frac{1}{y}+1-2k\right).$$

We only need to show that

$$\frac{1}{x}+\frac{1}{y}\ge 1+\frac{2\sqrt{n-m}}{n-m+1}.$$

Indeed, using the Cauchy-Schwarz inequality, we get

$$\frac{1}{x}+\frac{1}{y}\ge\frac{(1+\sqrt{n-m})^2}{x+(n-m)y}=\frac{(1+\sqrt{n-m})^2}{n-m+1}=1+\frac{2\sqrt{n-m}}{n-m+1}.$$

From $x+(n-m)y=1+n-m$ and $h(x,y)=0$, we get

$$x=\frac{n-m+1}{1+\sqrt{n-m}}, \qquad y=\frac{n-m+1}{n-m+\sqrt{n-m}}.$$

By Note 4, we have

$$f(a_1)+f(a_2)+\cdots+f(a_n)=nf(1)$$

for $a_1=a_2=\cdots=a_n=1$, and also for

$$a_1=\frac{n-m+1}{1+\sqrt{n-m}}, \quad a_2=a_3=\cdots=a_m=1, \quad a_{m+1}=\cdots=a_n=\frac{n-m+1}{n-m+\sqrt{n-m}}.$$

Therefore, the original inequality becomes an equality for $a_1=a_2=\cdots=a_n=1$, and also for

$$a_1=\frac{1+\sqrt{n-m}}{n-m+1}, \quad a_2=a_3=\cdots=a_m=1, \quad a_{m+1}=\cdots=a_n=\frac{n-m+\sqrt{n-m}}{n-m+1}.$$

Remark. Replacing n with $2n$ and choosing then $m=n$, we get the statement below.

- *If $a_1,a_2,\ldots,a_{2n}$ are positive real numbers so that*

$$a_1\le\cdots\le a_n\le 1\le a_{n+1}\le\cdots\le a_{2n}, \qquad \frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_{2n}}=2n,$$

then

$$a_1^2+a_2^2+\cdots+a_{2n}^2-2n\ge 2\left(1+\frac{\sqrt{n}}{n+1}\right)(a_1+a_2+\cdots+a_{2n}-2n).$$

with equality for $a_1=a_2=\cdots=a_{2n}=1$, and also for

$$a_1=\frac{1+\sqrt{n}}{n+1}, \quad a_2=a_3=\cdots=a_n=1, \quad a_{n+1}=\cdots=a_{2n}=\frac{n+\sqrt{n}}{n+1}.$$

□

P 2.11. *Let $a_1,a_2,\ldots,a_n$ $(n\ge 3)$ be nonnegative numbers so that $a_1+a_2+\cdots+a_n=n$. Prove that*

(a) if $a_1\le\cdots\le a_{n-1}\le 1\le a_n$, then

$$\frac{1}{a_1^2+2}+\frac{1}{a_2^2+2}+\cdots+\frac{1}{a_n^2+2}\ge\frac{n}{3};$$

(b) if $a_1\le\cdots\le a_{n-2}\le 1\le a_{n-1}\le a_n$, then

$$\frac{1}{2a_1^2+3}+\frac{1}{2a_2^2+3}+\cdots+\frac{1}{2a_n^2+3}\ge\frac{n}{5}.$$

(Vasile C., 2007)

Solution. Consider the inequality

$$\frac{1}{a_1^2+k}+\frac{1}{a_2^2+k}+\cdots+\frac{1}{a_n^2+k}\ge\frac{n}{1+k},\quad k\in[0,3];$$

and write it as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf(s),\qquad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

and

$$f(u)=\frac{1}{u^2+k},\qquad u\ge 0.$$

For $u\ge 1$, we have

$$f''(u)=\frac{2(3u^2-k)}{(u^2+k)^3}\ge\frac{2(3-k)}{(u^2+k)^3}\ge 0,$$

hence $f(u)$ is convex for $u\ge s$. Therefore, we may apply the RHCF-OV Theorem for $m=n-1$ and $m=n-2$, respectively. By Note 1, it suffices to show that $h(x,y)\ge 0$ for all $x,y\ge 0$ so that $x+(n-m)y=1+n-m$. Since

$$g(u)=\frac{f(u)-f(1)}{u-1}=\frac{-u-1}{(1+k)(u^2+k)},$$

$$h(x,y)=\frac{g(x)-g(y)}{x-y}=\frac{xy+x+y-k}{(1+k)(x^2+k)(y^2+k)},$$

we only need to show that

$$xy+x+y-k\ge 0.$$

(a) We need to show that $xy+x+y-k\ge 0$ for $k=2$, $m=n-1$, $x+y=2$; indeed, we have

$$xy+x+y-k=xy\ge 0.$$

From $x<y$, $x+y=2$ and $xy+x+y-k=0$, we get $x=0$ and $y=2$. Therefore, by Note 4, the equality holds for $a_1=a_2=\cdots=a_n=1$, and also for

$$a_1=0,\qquad a_2=\cdots=a_{n-1}=1,\qquad a_n=2.$$

(b) We have $k=3/2$, $m=n-2$, $x+2y=3$, hence

$$xy+x+y-k=\frac{x(4-x)}{2}=\frac{x(1+2y)}{2}\ge 0.$$

From $x+2y=3$ and $xy+x+y-k=0$, we get $x=0$ and $y=3/2$. Therefore, the equality holds for $a_1=a_2=\cdots=a_n=1$, and also for

$$a_1=0,\qquad a_2=\cdots=a_{n-2}=1,\qquad a_{n-1}=a_n=\frac{3}{2}.$$

$\square$

P 2.12. *If $a_1, a_2, \ldots, a_{2n}$ are nonnegative real numbers so that*

$$a_1 \geq \cdots \geq a_n \geq 1 \geq a_{n+1} \geq \cdots \geq a_{2n}, \quad a_1 + a_2 + \cdots + a_{2n} = 2n,$$

then

$$\frac{1}{na_1^2 + n^2 + n + 1} + \frac{1}{na_2^2 + n^2 + n + 1} + \cdots + \frac{1}{na_{2n}^2 + n^2 + n + 1} \leq \frac{2n}{(n+1)^2}.$$

(Vasile C., 2007)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_{2n}) \geq 2nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_{2n}}{2n} = 1,$$

where

$$f(u) = \frac{-1}{nu^2 + n^2 + n + 1}, \quad u \geq 0.$$

For $u \in [0, 1]$, we have

$$f''(u) = \frac{2nu(n^2 + n + 1 - 3nu^2)}{(nu^2 + n^2 + n + 1)^3} \geq \frac{2nu(n^2 + n + 1 - 3n)}{(nu^2 + n^2 + n + 1)^3} \geq 0,$$

hence f is convex on $[0, s]$. Therefore, we may apply the LHCF-OV Theorem for $2n$ numbers and $m = n$. By Note 1, it suffices to show that $h(x, y) \geq 0$ for all $x, y \geq 0$ so that $x + ny = 1 + n$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{n(u + 1)}{(n+1)^2(nu^2 + n^2 + n + 1)},$$

$$\begin{aligned} h(x, y) &= \frac{g(x) - g(y)}{x - y} \\ &= \frac{n(n^2 + n + 1 - nx - ny - nxy)}{(n+1)^2(nx^2 + n^2 + n + 1)(ny^2 + n^2 + n + 1)} \\ &= \frac{n(ny - 1)^2}{(n+1)^2(nx^2 + n^2 + n + 1)(ny^2 + n^2 + n + 1)} \geq 0. \end{aligned}$$

From $x + ny = 1 + n$ and $h(x, y) = 0$, we get $x = n$ and $y = 1/n$. Therefore, the equality holds for $a_1 = a_2 = \cdots = a_{2n} = 1$, and also for

$$a_1 = n, \quad a_2 = \cdots = a_n = 1, \quad a_{n+1} = \cdots = a_n = frac1n.$$

□

P 2.13. *If a,b,c,d,e,f are nonnegative real numbers so that*

$$a \ge b \ge c \ge 1 \ge d \ge e \ge f, \qquad a+b+c+d+e+f=6,$$

then

$$\frac{3a+4}{3a^2+4}+\frac{3b+4}{3b^2+4}+\frac{3c+4}{3c^2+4}+\frac{3d+4}{3d^2+4}+\frac{3e+4}{3e^2+4}+\frac{3f+4}{3f^2+4} \le 6.$$

(Vasile C., 2009)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d)+f(e)+f(f) \ge 6f(s), \qquad s=\frac{a+b+c+d+e+f}{6}=1,$$

where

$$f(u)=\frac{-3u-4}{3u^2+4}, \qquad u \ge 0.$$

For $u \in [0,1]$, we have

$$f''(u)=\frac{6(16-9u^3)+216u(1-u)}{(3u^2+4)^3}>0,$$

hence f is convex on $[0,s]$. Therefore, we may apply the LHCF-OV Theorem for $n=6$ and $m=3$. By Note 1, it suffices to show that $h(x,y) \ge 0$ for all $x,y \ge 0$ so that $x+3y=4$. We have

$$g(u)=\frac{f(u)-f(1)}{u-1}=\frac{3u}{3u^2+4},$$

$$h(x,y)=\frac{g(x)-g(y)}{x-y}=\frac{3(4-3xy)}{(3x^2+4)(3y^2+4)}$$
$$=\frac{3(x-2)^2}{(3x^2+4)(3y^2+4)} \ge 0.$$

From $x+3y=4$ and $h(x,y)=0$, we get $x=2$ and $y=2/3$. Therefore, in accordance with Note 4, the equality holds for $a=b=c=d=e=f=1$, and also for

$$a=2, \quad b=c=1, \quad d=e=f=\frac{2}{3}.$$

□

P 2.14. *If a,b,c,d,e,f are nonnegative real numbers so that*

$$a \ge b \ge 1 \ge c \ge d \ge e \ge f, \qquad a+b+c+d+e+f=6,$$

then

$$\frac{a^2-1}{(2a+7)^2}+\frac{b^2-1}{(2b+7)^2}+\frac{c^2-1}{(2c+7)^2}+\frac{d^2-1}{(2d+7)^2}+\frac{e^2-1}{(2e+7)^2}+\frac{f^2-1}{(2f+7)^2} \ge 0.$$

(Vasile C., 2009)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d)+f(e)+f(f) \geq 6f(s), \quad s = \frac{a+b+c+d+e+f}{6} = 1,$$

where

$$f(u) = \frac{u^2-1}{(2u+7)^2}, \quad u \geq 0.$$

For $u \in [0,1]$, we have

$$f''(u) = \frac{2(37-28u)}{(2u+7)^4} > 0,$$

hence f is convex on $[0,s]$. Therefore, we may apply the LHCF-OV Theorem for $n=6$ and $m=2$. By Note 1, it suffices to show that $h(x,y) \geq 0$ for all $x,y \geq 0$ so that $x+4y=5$. We have

$$g(u) = \frac{f(u)-f(1)}{u-1} = \frac{u+1}{(2u+7)^2},$$

$$\begin{aligned} h(x,y) = \frac{g(x)-g(y)}{x-y} &= \frac{21-4x-4y-4xy}{(2x+7)^2(2y+7)^2} \\ &= \frac{(x-4)^2}{(2x+7)^2(2y+7)^2} \geq 0. \end{aligned}$$

From $x+4y=5$ and $h(x,y)=0$, we get $x=4$ and $y=1/4$. Therefore, the equality holds only for $a=b=c=d=e=f=1$, and also for

$$a=4, \quad b=1, \quad c=d=e=f=\frac{1}{4}.$$

□

P 2.15. *If a,b,c,d,e,f are nonnegative real numbers so that*

$$a \leq b \leq 1 \leq c \leq d \leq e \leq f, \quad a+b+c+d+e+f=6,$$

then

$$\frac{a^2-1}{(2a+5)^2}+\frac{b^2-1}{(2b+5)^2}+\frac{c^2-1}{(2c+5)^2}+\frac{d^2-1}{(2d+5)^2}+\frac{e^2-1}{(2e+5)^2}+\frac{f^2-1}{(2f+5)^2} \leq 0.$$

(Vasile C., 2009)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d)+f(e)+f(f)\ge 6f(s), \qquad s=\frac{a+b+c+d+e+f}{6}=1,$$

where

$$f(u)=\frac{1-u^2}{(2u+5)^2}, \quad u\ge 0.$$

For $u\ge 1$, we have

$$f''(u)=\frac{2(20u-13)}{(2u+5)^4}>0,$$

hence $f(u)$ is convex for $u\ge s$. Therefore, we may apply the RHCF-OV Theorem for $n=6$ and $m=2$. By Note 1, it suffices to show that $h(x,y)\ge 0$ for all $x,y\ge 0$ so that $x+4y=5$. We have

$$g(u)=\frac{f(u)-f(1)}{u-1}=\frac{-u-1}{(2u+5)^2},$$

$$\begin{aligned} h(x,y)&=\frac{g(x)-g(y)}{x-y}\\ &=\frac{4xy+4x+4y-5}{(2x+5)^2(2y+5)^2}\\ &=\frac{4xy+3x}{(2x+5)^2(2y+5)^2}\ge 0. \end{aligned}$$

From $x+4y=5$ and $h(x,y)=0$, we get $x=0$ and $y=5/4$. Therefore, in accordance with Note 4, the equality holds only for $a=b=c=d=e=f=1$, and also for

$$a=0, \quad b=1, \quad c=d=e=f=\frac{5}{4}.$$

□

P 2.16. *If a,b,c are nonnegative real numbers so that*

$$a\le b\le 1\le c, \qquad a+b+c=3,$$

then

$$\sqrt{\frac{2a}{b+c}}+\sqrt{\frac{2b}{c+a}}+\sqrt{\frac{2c}{a+b}}\ge 3.$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)\ge 3f(s),\qquad s=\frac{a+b+c}{3}=1,$$

where

$$f(u)=\sqrt{\frac{u}{3-u}},\qquad u\in[0,3).$$

From

$$f''(u)=\frac{3(4u-3)}{4u^{3/2}(3-u)^{5/2}},$$

it follows that $f(u)$ is convex for $u\ge s$. Therefore, we may apply the RHCF-OV Theorem for $n=3$ and $m=2$. So, it suffices to show that

$$f(x)+f(y)\ge 2f(1)$$

for $x+y=2$, $0\le x\le 1\le y$. This inequality is true if $g(x)\ge 0$, where

$$g(x)=f(x)+f(y)-2f(1),\qquad y=2-x,\qquad x\in[0,1].$$

Since $y'=-1$, we have

$$g'(x)=f'(x)-f'(y)=\frac{3}{2}\left[\frac{1}{\sqrt{x(3-x)^3}}-\frac{1}{\sqrt{y(3-y)^3}}\right].$$

The derivative $f'(x)$ has the same sign as h(x), where

$$\begin{aligned}h(x)&=y(3-y)^3-x(3-x)^3=(2-x)(1+x)^3-x(3-x)^3\\&=2(1-11x+15x^2-5x^3)=2(1-x)(1-10x+5x^2).\end{aligned}$$

Let

$$x_1=1-\frac{2}{\sqrt{5}}.$$

Since $h(x_1)=0$, $h(x)>0$ for $x\in[0,x_1)$ and $h(x)<0$ for $x\in(x_1,1)$, it follows that g is increasing on $[0,x_1]$ and decreasing on $[x_1,1]$. From

$$g(0)=f(0)+f(2)-2f(1)=0,$$

$$g(1)=f(1)+f(1)-2f(1)=0,$$

it follows that $g(x)\ge 0$ for $x\in[0,1]$.

The equality holds for $a=b=c=1$, and also for $a=0$, $b=1$ and $c=2$.

□

P 2.17. *If $a_1, a_2, \dots, a_8$ are nonnegative real numbers so that*

$$a_1 \ge a_2 \ge a_3 \ge a_4 \ge 1 \ge a_5 \ge a_6 \ge a_7 \ge a_8, \quad a_1 + a_2 + \cdots + a_8 = 8,$$

then

$$(a_1^2+1)(a_2^2+1)\cdots(a_8^2+1) \ge (a_1+1)(a_2+1)\cdots(a_8+1).$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_8) \ge 8f(s), \qquad s = \frac{a_1+a_2+\cdots+a_8}{8} = 1,$$

where

$$f(u) = \ln(u^2+1) - \ln(u+1), \quad u \ge 0.$$

For $u \in [0,1]$, we have

$$f''(u) = \frac{2(1-u^2)}{(u^2+1)^2} + \frac{1}{(u+1)^2} = \frac{(u^2-u^4)+4u(1-u^2)+u^2+3}{(u^2+1)^2(u+1)^2} > 0.$$

Therefore, f is convex on $[0,s]$. According to the LHCF-OV Theorem applied for $n=8$ and $m=4$, it suffices to show that $f(x)+4f(y) \ge 5f(1)$ for $x,y \ge 0$ so that $x+4y=5$. Using Note 2, we only need to show that $H(x,y) \ge 0$ for $x,y \ge 0$ so that $x+4y=5$, where

$$H(x,y) = \frac{f'(x)-f'(y)}{x-y} = \frac{2(1-xy)}{(x^2+1)(y^2+1)} + \frac{1}{(x+1)(y+1)}.$$

The inequality $H(x,y) \ge 0$ is equivalent to

$$2(1-xy)(x+1)(y+1) + (x^2+1)(y^2+1) \ge 0.$$

Since $2(x^2+1) \ge (x+1)^2$ and $2(y^2+1) \ge (y+1)^2$, it suffices to prove that

$$8(1-xy) + (x+1)(y+1) \ge 0.$$

Indeed,

$$8(1-xy)+(x+1)(y+1) = 28x^2 - 38x + 14 = 28(x-19/28)^2 + 31/28 > 0.$$

The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_8$.

$\square$

P 2.18. *If a, b, c, d are real numbers so that*

$$\frac{-1}{2} \le a \le b \le 1 \le c \le d, \qquad a+b+c+d=4,$$

then

$$7\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right)+3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \ge 40.$$

(Vasile C., 2011)

Solution. We have

$$d = 4-a-b-c \le 4+\frac{1}{2}+\frac{1}{2}-1 = 4.$$

Write the inequality as

$$f(a)+f(b)+f(c)+f(d) \ge 4f(s), \qquad s=\frac{a+b+c+d}{4}=1,$$

where

$$f(u)=\frac{7}{u^2}+\frac{3}{u}, \quad u \in \mathbb{I}=\left[\frac{-1}{2},4\right]\setminus\{0\}.$$

Clearly, $f(u)$ is convex for $u \ge 1$ (because $\frac{7}{u^2}$ and $\frac{3}{u}$ are convex). According to Note 3, we may apply the RHCF-OV Theorem for $n=4$ and $m=2$. By Note 1, we only need to show that $h(x,y) \ge 0$ for $x, y \in \mathbb{I}$ so that $x+2y=3$, where

$$h(x,y)=\frac{g(x)-g(y)}{x-y}, \quad g(u)=\frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u) = -\frac{7}{u^2}-\frac{10}{u},$$

$$h(x,y)=\frac{7(x+y)+10xy}{x^2y^2}=\frac{(2x+1)(-5x+21)}{2x^2y^2} \ge 0.$$

From $x+2y=3$ and $h(x,y)=0$, we get $x=-1/2$, $y=7/3$. Therefore, in accordance with Note 4, the equality holds for $a=b=c=d=1$, and also for

$$a=\frac{-1}{2}, \quad b=1, \quad c=d=\frac{7}{4}.$$

□

P 2.19. *Let a, b, c, d be real numbers. Prove that*

(a) *if $-1 \le a \le b \le c \le 1 \le d$, then*

$$3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right) \ge 8+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d};$$

(b) *if $-1 \le a \le b \le 1 \le c \le d$, then*

$$2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right) \ge 4+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}.$$

(Vasile C., 2011)

Solution. (a) We have

$$d = 4-a-b-c \le 4+1+1+1 = 7.$$

Write the desired inequality as

$$f(a)+f(b)+f(c)+f(d) \ge 4f(s), \qquad s = \frac{a+b+c+d}{4} = 1,$$

where

$$f(u) = \frac{3}{u^2}-\frac{1}{u}, \qquad u \in \mathbb{I} = [-1,7]\setminus\{0\}.$$

From

$$f''(u) = \frac{2(9-u)}{u^4} > 0,$$

it follows that f is convex on $\mathbb{I}_{\ge s}$. According to Note 3, we may apply the RHCF-OV Theorem for $n = 4$ and $m = 3$. By Note 1, it suffices to show that $h(x,y) \ge 0$ for all $x, y \in \mathbb{I}$ so that $x+y = 2$. We have

$$g(u) = \frac{f(u)-f(1)}{u-1} = -\frac{2}{u}-\frac{3}{u^2},$$

$$\begin{aligned} h(x,y) &= \frac{g(x)-g(y)}{x-y} = \frac{3(x+y)+2xy}{x^2y^2} \\ &= \frac{2(x+1)(3-x)}{x^2y^2} = \frac{2(x+1)(y+1)}{x^2y^2} \ge 0. \end{aligned}$$

From $x < y$, $x+y = 2$ and $h(x,y) = 0$, we get $x = -1$ and $y = 3$. Therefore, in accordance with Note 4, the equality holds for $a = b = c = d = 1$, and also for

$$a = -1, \quad b = c = 1, \quad d = 3.$$

(b) We have

$$d = 4-a-b-c \le 4+1+1-1 = 5.$$

Write the desired inequality as

$$f(a)+f(b)+f(c)+f(d) \geq 4f(s), \qquad s=\frac{a+b+c+d}{4}=1,$$

where

$$f(u)=\frac{2}{u^2}-\frac{1}{u}, \quad u \in \mathbb{I}=[-1,5]\setminus\{0\}.$$

From

$$f''(u)=\frac{2(6-u)}{u^4}>0,$$

it follows that f is convex on $\mathbb{I}_{\geq s}$. According to Note 3, we may apply the RHCF-OV Theorem for $n=4$ and $m=2$. By Note 1, it suffices to show that $h(x,y) \geq 0$ for all $x,y \in \mathbb{I}$ so that $x+2y=3$. We have

$$g(u)=\frac{f(u)-f(1)}{u-1}=-\frac{1}{u}-\frac{2}{u^2},$$

$$\begin{aligned} h(x,y)=\frac{g(x)-g(y)}{x-y} &= \frac{2(x+y)+xy}{x^2y^2} \\ &= \frac{(x+1)(6-x)}{2x^2y^2} \geq 0. \end{aligned}$$

From $x+2y=3$ and $h(x,y)=0$, we get $x=-1$ and $y=2$. Therefore, the equality holds for $a=b=c=d=1$, and also for

$$a=-1, \quad b=1, \quad c=d=2.$$

□

P 2.20. *If a,b,c,d are positive real numbers so that*

$$a \geq b \geq 1 \geq c \geq d, \qquad abcd=1,$$

then

$$a^2+b^2+c^2+d^2-4 \geq 18\left(a+b+c+d-\frac{1}{a}-\frac{1}{b}-\frac{1}{c}-\frac{1}{d}\right).$$

(Vasile C., 2008)

Solution. Using the substitution

$$a=e^x, \quad b=e^y, \quad c=e^z, \quad d=e^w,$$

we need to show that

$$f(x)+f(y)+f(z)+f(w) \geq 4f(s),$$

where

$$x \ge y \ge 0 \ge z \ge w, \qquad s = \frac{x+y+z+w}{4} = 0,$$

$$f(u) = e^{2u} - 1 - 18(e^u - e^{-u}), \quad u \in \mathbb{R}.$$

For $u \le 0$, we have

$$f''(u) = 4e^{2u} + 18(e^{-u} - e^u) > 0,$$

hence f is convex on $(-\infty, s]$. By the LHCF-OV Theorem applied for $n = 4$ and $m = 2$, it suffices to show that $f(x) + 2f(y) \ge 3f(0)$ for all real x, y so that $x + 2y = 0$; that is, to show that

$$a^2 + 2b^2 - 3 - 18\left(a + 2b - \frac{1}{a} - \frac{2}{b}\right) \ge 0$$

for all $a, b > 0$ so that $ab^2 = 1$. This inequality is equivalent to

$$\frac{(b^2-1)^2(2b^2+1)}{b^4} + \frac{18(b-1)^3(b+1)}{b^2} \ge 0,$$

$$\frac{(b-1)^2(2b-1)^2(b+1)(5b+1)}{b^4} \ge 0.$$

The proof is completed. The equality holds for $a = b = c = d = 1$, and also for

$$a = 4, \quad b = 1, \quad c = d = 1/2.$$

□

P 2.21. *If a, b, c, d are positive real numbers so that*

$$a \le b \le 1 \le c \le d, \qquad abcd = 1,$$

then

$$\sqrt{a^2 - a + 1} + \sqrt{b^2 - b + 1} + \sqrt{c^2 - c + 1} + \sqrt{d^2 - d + 1} \ge a + b + c + d.$$

(Vasile C., 2008)

Solution. Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z, \quad d = e^w,$$

we need to show that

$$f(x) + f(y) + f(z) + f(w) \ge 4f(s),$$

where

$$x \le y \le 0 \le z \le w, \qquad s = \frac{x+y+z+w}{4} = 0,$$

$$f(u) = \sqrt{e^{2u} - e^u + 1} - e^u, \quad u \in \mathbb{R}.$$

We claim that f is convex for $u \ge 0$. Since

$$e^{-u} f''(u) = \frac{4e^{3u} - 6e^{2u} + 9e^u - 2}{4(e^{2u} - e^u + 1)^{3/2}} - 1,$$

we need to show that

$$4t^3 - 6t^2 + 9t - 2 \ge 0$$

and

$$(4t^3 - 6t^2 + 9t - 2)^2 \ge 16(t^2 - t + 1)^3,$$

where $t = e^u \ge 1$. Indeed, we have

$$4t^3 - 6t^2 + 9t - 2 \ge 4t^3 - 6t^2 + 7t > 4t^3 - 6t^2 + 2t = 2t(t-1)(2t-1) \ge 0$$

and

$$(4t^3 - 6t^2 + 9t - 2)^2 - 16(t^2 - t + 1)^3 = 12t^3(t-1) + 9t^2 + 12(t-1) > 0.$$

By the RHCF-OV Theorem applied for $n = 4$ and $m = 2$, it suffices to show that $f(x) + 2f(y) \ge 3f(0)$ for all real x, y so that $x + 2y = 0$; that is, to show that

$$\sqrt{a^2 - a + 1} + 2\sqrt{b^2 - b + 1} \ge a + 2b$$

for all $a, b > 0$ so that $ab^2 = 1$. This inequality is equivalent to

$$\frac{\sqrt{b^4 - b^2 + 1}}{b^2} + 2\sqrt{b^2 - b + 1} \ge \frac{1}{b^2} + 2b,$$

$$\frac{\sqrt{b^4 - b^2 + 1} - 1}{b^2} + 2(\sqrt{b^2 - b + 1} - 1) \ge 0,$$

$$\frac{b^2 - 1}{\sqrt{b^4 - b^2 + 1} + 1} + \frac{2(1-b)}{\sqrt{b^2 - b + 1} + b} \ge 0.$$

Since

$$\frac{b^2 - 1}{\sqrt{b^4 - b^2 + 1} + 1} \ge \frac{b^2 - 1}{b^2 + 1},$$

it suffices to show that

$$\frac{b^2 - 1}{b^2 + 1} + \frac{2(1-b)}{\sqrt{b^2 - b + 1} + b} \ge 0,$$

which is equivalent to

$$(b-1)\left[\frac{b+1}{b^2+1} - \frac{2}{\sqrt{b^2 - b + 1} + b}\right] \ge 0,$$

$$(b-1)\left[(b+1)\sqrt{b^2-b+1}-b^2+b-2\right]\geq 0,$$

$$\frac{(b-1)^2(3b^2-2b+3)}{(b+1)\sqrt{b^2-b+1}+b^2-b+2}\geq 0.$$

The last inequality is clearly true. The equality holds for $a=b=c=d=1$.

□

P 2.22. *If a,b,c,d are positive real numbers so that*

$$a\leq b\leq c\leq 1\leq d, \qquad abcd=1,$$

then

$$\frac{1}{a^3+3a+2}+\frac{1}{b^3+3b+2}+\frac{1}{c^3+3c+2}+\frac{1}{d^3+3d+2}\geq\frac{2}{3}.$$

(Vasile C., 2007)

Solution. Using the substitution

$$a=e^x, \quad b=e^y, \quad c=e^z, \quad d=e^w,$$

we need to show that

$$f(x)+f(y)+f(z)+f(w)\geq 4f(s),$$

where

$$x\leq y\leq z\leq 0\leq w, \qquad s=\frac{x+y+z+w}{4}=0,$$

$$f(u)=\frac{1}{e^{3u}+3e^u+2}, \quad u\in\mathbb{R}.$$

We claim that f is convex for $u\geq 0$. Indeed, denoting $t=e^u$, $t\geq 1$, we have

$$f''(u)=\frac{3t(3t^5+2t^3-6t^2+3t-2)}{(t^3+3t+2)^3}$$
$$=\frac{3t(t-1)(3t^4+3t^3+5t^2-t+2)}{(t^3+3t+2)^3}\geq 0.$$

By the RHCF-OV Theorem applied for $n=4$ and $m=3$, it suffices to show that $f(x)+f(y)\geq 2f(0)$ for all real x,y so that $x+y=0$; that is, to show that

$$\frac{1}{a^3+3a+2}+\frac{1}{b^3+3b+2}\geq\frac{1}{3}$$

for all $a,b>0$ so that $ab=1$. This inequality is equivalent to

$$(a-1)^4(a^2+a+1)\geq 0.$$

The equality holds for $a=b=c=d=1$.

□

P 2.23. *If $a_1, a_2, \ldots, a_n$ are positive real numbers so that*

$$a_1 \geq \cdots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \geq a_1 + a_2 + \cdots + a_n.$$

(Vasile C., 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \quad i = 1, 2, \ldots, n,$$

we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq n f(s),$$

where

$$x_1 \geq \cdots \geq x_{n-1} \geq 0 \geq x_n, \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

$$f(u) = e^{-u} - e^u, \quad u \in \mathbb{R}.$$

For $u \leq 0$, we have

$$f''(u) = e^{-u} - e^u \geq 0,$$

therefore $f(u)$ is convex for $u \leq s$. By the LHCF-OV Theorem applied for $m = n-1$, it suffices to show that $f(x) + f(y) \geq 2f(0)$ for all real x, y so that $x + y = 0$; that is, to show that

$$\frac{1}{a} - a + \frac{1}{b} - b \geq 0$$

for all $a, b > 0$ so that $ab = 1$. This is true since

$$\frac{1}{a} - a + \frac{1}{b} - b = \frac{1}{a} - a + a - \frac{1}{a} = 0.$$

The equality holds for

$$a_1 \geq 1, \quad a_2 = \cdots = a_{n-1} = 1, \quad a_n = 1/a_1.$$

□

P 2.24. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that*

$$a_1 \leq \cdots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1 a_2 \cdots a_n = 1.$$

If $k \geq 1$, then

$$\frac{1}{1 + ka_1} + \frac{1}{1 + ka_2} + \cdots + \frac{1}{1 + ka_n} \geq \frac{n}{1 + k}.$$

(Vasile C., 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \qquad i = 1, 2, \ldots, n,$$

we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge nf(s),$$

where

$$x_1 \le \cdots \le x_{n-1} \le 0 \le x_n, \qquad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

$$f(u) = \frac{1}{1 + ke^u}, \quad u \in \mathbb{R}.$$

For $u \ge 0$, we have

$$f''(u) = \frac{ke^u(ke^u - 1)}{(1 + ke^u)^3} \ge 0,$$

therefore $f(u)$ is convex for $u \ge s$. By the RHCF-OV Theorem applied for $m = n-1$, it suffices to show that $f(x) + f(y) \ge 2f(0)$ for all real x, y so that $x + y = 0$; that is, to show that

$$\frac{1}{1 + ka} + \frac{1}{1 + kb} \ge \frac{2}{1 + k}$$

for all $a, b > 0$ so that $ab = 1$. This is true since

$$\frac{1}{1 + ka} + \frac{1}{1 + kb} - \frac{2}{1 + k} = \frac{k(k-1)(a-1)^2}{(1 + ka)(a + k)} \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $k = 1$, then the equality holds for

$$a_1 \le 1, \quad a_2 = \cdots = a_{n-1} = 1, \quad a_n = 1/a_1.$$

$\square$

P 2.25. *If $a_1, a_2, \ldots, a_9$ are positive real numbers so that*

$$a_1 \le \cdots \le a_8 \le 1 \le a_9, \qquad a_1 a_2 \cdots a_9 = 1,$$

then

$$\frac{1}{(a_1 + 2)^2} + \frac{1}{(a_2 + 2)^2} + \cdots + \frac{1}{(a_9 + 2)^2} \ge 1.$$

(Vasile C., 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \qquad i = 1, 2, \ldots, 9,$$

we can write the inequality as

$$f(x_1) + f(x_2) + \cdots + f(x_9) \ge 9f(s),$$

where

$$x_1 \le \cdots \le x_8 \le 0 \le x_9, \qquad s = \frac{x_1 + x_2 + \cdots + x_9}{9} = 0,$$

$$f(u) = \frac{1}{(e^u + 2)^2}, \qquad u \in \mathbb{R}.$$

For $u \in [0, \infty)$, we have

$$f''(u) = \frac{4e^u(e^u - 1)}{(e^u + 2)^4} \ge 0,$$

hence f is convex on $[s, \infty)$. According to the RHCF-OV Theorem (case $n = 9$ and $m = 8$), it suffices to show that $f(x) + f(y) \ge 2f(0)$ for all real x, y so that $x + y = 0$; that is, to show that

$$\frac{1}{(a+2)^2} + \frac{1}{(b+2)^2} \ge \frac{2}{9}$$

for all $a, b > 0$ so that $ab = 1$. Write this inequality as

$$\frac{b^2}{(2b+1)^2} + \frac{1}{(b+2)^2} \ge \frac{2}{9},$$

which is equivalent to the obvious inequality

$$(b-1)^4 \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_9 = 1$.

□

P 2.26. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that*

$$a_1 \le \cdots \le a_{n-1} \le 1 \le a_n, \qquad a_1 a_2 \cdots a_n = 1.$$

If $p, q \ge 0$ so that

$$p + q \ge 1 + \frac{2pq}{p + 4q},$$

then

$$\frac{1}{1 + pa_1 + qa_1^2} + \frac{1}{1 + pa_2 + qa_2^2} + \cdots + \frac{1}{1 + pa_n + qa_n^2} \ge \frac{n}{1 + p + q}.$$

(Vasile C., 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \qquad i = 1, 2, \ldots, n,$$

we can write the inequality as

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s),$$

where

$$x_1 \leq \cdots \leq x_{n-1} \leq 0 \leq x_n, \qquad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

$$f(u) = \frac{1}{1 + pe^u + qe^{2u}}, \quad u \in \mathbb{R}.$$

We have

$$f''(u) = \frac{e^u f_1(u)}{(1 + pe^u + qe^{2u})^3},$$

where

$$f_1(u) = 4q^2e^{3u} + 3pqe^{2u} + (p^2 - 4q)e^u - p.$$

The hypothesis $p + q \geq 1 + \dfrac{2pq}{p + 4q}$ is equivalent to

$$p^2 + 3pq + 4q^2 \geq p + 4q.$$

For $u \in [0, \infty)$, we have

$$f_1(u) \geq 4q^2e^u + 3pqe^u + (p^2 - 4q)e^u - p \geq p(e^u - 1) \geq 0,$$

hence f is convex on $[s, \infty)$. According to the RHCF-OV Theorem (case $m = n-1$), it suffices to show that $f(x) + f(y) \geq 2f(0)$ for all real x, y so that $x + y = 0$; that is, to show that

$$\frac{1}{1 + pa + qa^2} + \frac{1}{1 + pb + qb^2} \geq \frac{2}{1 + p + q}$$

for all $a, b > 0$ so that $ab = 1$. Write this inequality as

$$\frac{1}{1 + pa + qa^2} + \frac{a^2}{a^2 + pa + q} \geq \frac{2}{1 + p + q}$$

which is equivalent to

$$(a-1)^2 h(a) \geq 0,$$

where

$$\begin{aligned} h(a) &= q(p + q - 1)(a^2 + 1) + (p^2 + pq + 2q^2 - p - 2q)a \\ &\geq 2q(p + q - 1)a + (p^2 + pq + 2q^2 - p - 2q)a \\ &= (p^2 + 3pq + 4q^2 - p - 4q)a \geq 0. \end{aligned}$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark. For $p = 1$, $q = 1/4$ and $n = 9$, we get the preceding P 2.25.

□

P 2.27. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that*

$$a_1 \le \cdots \le a_{n-1} \le 1 \le a_n, \qquad a_1 a_2 \cdots a_n = 1.$$

If $m \ge 1$ and $0 < k \le m$, then

$$\frac{1}{(a_1+k)^m} + \frac{1}{(a_2+k)^m} + \cdots + \frac{1}{(a_n+k)^m} \ge \frac{n}{(1+k)^m}.$$

(Vasile C., 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \qquad i = 1, 2, \ldots, n,$$

we can write the inequality as

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge nf(s),$$

where

$$x_1 \le \cdots \le x_{n-1} \le 0 \le x_n, \qquad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

$$f(u) = \frac{1}{(e^u+k)^m}, \qquad u \in \mathbb{R}.$$

For $u \in [0, \infty)$, we have

$$f''(u) = \frac{me^u(me^u - k)}{(e^u+k)^{m+2}} \ge 0,$$

hence f is convex on $[s, \infty)$. According to the RHCF-OV Theorem (case $m = n-1$), it suffices to show that $f(x) + f(y) \ge 2f(0)$ for all real x, y so that $x \le y$ and $x + y = 0$; that is, to show that

$$\frac{1}{(a+k)^m} + \frac{1}{(b+k)^m} \ge \frac{2}{(1+k)^m}$$

for all $a, b > 0$ so that $a \in (0, 1]$ and $ab = 1$. Write this inequality as $g(a) \ge 0$, where

$$g(a) = \frac{1}{(a+k)^m} + \frac{a^m}{(ka+1)^m} - \frac{2}{(1+k)^m},$$

with

$$\frac{g'(a)}{m} = \frac{a^{m-1}(a+k)^{m+1} - (ka+1)^{m+1}}{(a+k)^{m+1}(ka+1)^{m+1}}.$$

If $g'(a) \le 0$ for $a \in (0, 1]$, then g is decreasing, hence $g(a) \ge g(1) = 0$. Thus, it suffices to show that

$$a^{m-1} \le \left(\frac{ka+1}{a+k}\right)^{m+1}.$$

Since
$$\frac{ka+1}{a+k}-\frac{ma+1}{a+m}=\frac{(m-k)(1-a^2)}{(a+k)(a+m)}\geq 0,$$
we only need to show that
$$a^{m-1}\leq\left(\frac{ma+1}{a+m}\right)^{m+1},$$
which is equivalent to $h(a)\leq 0$ for $a\in(0,1]$, where
$$h(a)=(m-1)\ln a+(m+1)\ln(a+m)-(m+1)\ln(ma+1),$$
with
$$h'(a)=\frac{m-1}{a}+\frac{m+1}{a+m}-\frac{m(m+1)}{ma+1}=\frac{m(m-1)(a-1)^2}{a(a+m)(ma+1)}.$$
Since $h'(a)\geq 0$, $h(a)$ is increasing for $a\in(0,1]$, therefore $h(a)\leq h(1)=0$. The equality holds for $a_1=a_2=\cdots=a_n=1$.

Remark. For $k=m=2$ and $n=9$, we get the inequality in P 2.25.

□

P 2.28. *If $a_1,a_2,\ldots,a_n$ are positive real numbers so that*
$$a_1\leq\cdots\leq a_{n-1}\leq 1\leq a_n,\qquad a_1a_2\cdots a_n=1,$$
then
$$\frac{1}{\sqrt{1+3a_1}}+\frac{1}{\sqrt{1+3a_2}}+\cdots+\frac{1}{\sqrt{1+3a_n}}\geq\frac{n}{2}.$$
(Vasile C., 2007)

Solution. Using the substitution
$$a_i=e^{x_i},\qquad i=1,2,\ldots,n,$$
we can write the inequality as
$$f(x_1)+f(x_2)+\cdots+f(x_n)\geq nf(s),$$
where
$$x_1\leq\cdots\leq x_{n-1}\leq 0\leq x_n,\qquad s=\frac{x_1+x_2+\cdots+x_n}{n}=0,$$
$$f(u)=\frac{1}{\sqrt{1+3e^u}},\qquad u\in\mathbb{R}.$$
For $u\geq 0$, we have
$$f''(u)=\frac{3e^u(3e^u-2)}{4(1+3e^u)^{5/2}}>0,$$

hence f is convex on $[s,\infty)$. According to the RHCF-OV Theorem (case $m=n-1$), it suffices to show that $f(x)+f(y)\ge 2f(0)$ for all real x,y so that $x+y=0$; that is, to show that
$$\frac{1}{\sqrt{1+3a}}+\frac{1}{\sqrt{1+3b}}\ge 1$$
for all $a,b>0$ so that $ab=1$. Write this inequality as
$$\frac{1}{\sqrt{1+3a}}+\sqrt{\frac{a}{a+3}}\ge 1.$$
Substituting $\dfrac{1}{\sqrt{1+3a}}=t$, $0<t<1$, the inequality becomes
$$\sqrt{\frac{1-t^2}{8t^2+1}}\ge 1-t.$$
By squaring, we get
$$t(1-t)(2t-1)^2\ge 0,$$
which is true. The equality holds for $a_1=a_2=\cdots=a_n=1$.

□

P 2.29. *Let $a_1,a_2,\ldots,a_n$ be positive real numbers so that*
$$a_1\le\cdots\le a_{n-1}\le 1\le a_n,\qquad a_1a_2\cdots a_n=1.$$
If $0<m<1$ and $0<k\le\dfrac{1}{2^{1/m}-1}$, then
$$\frac{1}{(a_1+k)^m}+\frac{1}{(a_2+k)^m}+\cdots+\frac{1}{(a_n+k)^m}\ge\frac{n}{(1+k)^m}.$$

(Vasile C., 2007)

Solution. By Bernoulli's inequality, we have
$$2^{1/m}>1+\frac{1}{m},$$
hence
$$k\le\frac{1}{2^{1/m}-1}<m<1.$$
Using the substitution
$$a_i=e^{x_i},\qquad i=1,2,\ldots,n,$$
we can write the inequality as
$$f(x_1)+f(x_2)+\cdots+f(x_n)\ge nf(s),$$

where

$$x_1 \le \cdots \le x_{n-1} \le 0 \le x_n, \qquad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

$$f(u) = \frac{1}{(e^u + k)^m}, \qquad u \in \mathbb{R}.$$

For $u \in [0, \infty)$, we have

$$f''(u) = \frac{me^u(me^u - k)}{(e^u + k)^{m+2}} \ge 0,$$

hence f is convex on $[s, \infty)$. According to the RHCF-OV Theorem (case $m = n-1$), it suffices to show that $f(x) + f(y) \ge 2f(0)$ for all real x, y so that $x + y = 0$; that is, to show that

$$\frac{1}{(a+k)^m} + \frac{1}{(b+k)^m} \ge \frac{2}{(1+k)^m}$$

for all $a, b > 0$ so that $ab = 1$. Write this inequality as $g(a) \ge 0$ for $a \ge 1$, where

$$g(a) = \frac{1}{(a+k)^m} + \frac{a^m}{(ka+1)^m} - \frac{2}{(1+k)^m}.$$

The derivative

$$\frac{g'(a)}{m} = \frac{a^{m-1}(a+k)^{m+1} - (ka+1)^{m+1}}{(a+k)^{m+1}(ka+1)^{m+1}}$$

has the same sign as the function

$$h(a) = (m-1)\ln a + (m+1)\ln(a+k) - (m+1)\ln(ka+1).$$

We have

$$h'(a) = \frac{m-1}{a} + (m+1)\left(\frac{1}{a+k} - \frac{k}{ka+1}\right) = \frac{kh_1(a)}{a(a+k)(ka+1)},$$

where

$$h_1(a) = (m-1)(a^2+1) - 2\left(k - \frac{m}{k}\right)a.$$

The discriminant D of the quadratic function $h_1(a)$ is

$$\frac{D}{4} = \left(k - \frac{m}{k}\right)^2 - (m-1)^2 = (1-k^2)\left(\frac{m^2}{k^2} - 1\right).$$

Since $D > 0$, the roots a_1 and a_2 of $h_1(a)$ are real and unequal. If $a_1 < a_2$, then $h_1(a) \ge 0$ for $a \in [a_1, a_2]$ and $h_1(a) \le 0$ for $a \in (-\infty, a_1] \cup [a_2, \infty)$. Since

$$h_1(1) = \frac{2(k+1)(m-k)}{k} > 0,$$

it follows that $a_1 < 1 < a_2$, therefore $h_1(a)$ and $h'(a)$ are positive for $a \in [1, a_2)$ and negative for $a \in (a_2, \infty)$, h is increasing on $[1, a_2]$ and decreasing on $[a_2, \infty)$. From $h(1) = 0$ and

$$\lim_{a \to \infty} h(a) = -\infty,$$

it follows that there is $a_3 > a_2$ so that $h(a)$ and $g'(a)$ are positive for $a \in (1, a_3)$ and negative for $a \in (a_3, \infty)$. As a result, g is increasing on $[1, a_3]$ and decreasing on $[a_3, \infty)$. Since $g(1) = 0$ and

$$\lim_{a \to \infty} g(a) = \frac{1}{k^m} - \frac{2}{(1+k)^m} \ge 0,$$

it follows that $g(a) \ge 0$ for $a \ge 1$. This completes the proof. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark. For $k = \frac{1}{3}$ and $m = \frac{1}{2}$, we get the preceding P 2.28.

□

P 2.30. *If $a_1, a_2, \ldots, a_n$ $(n \ge 4)$ are positive real numbers so that*

$$a_1 \ge a_2 \ge a_3 \ge 1 \ge a_4 \ge \cdots \ge a_n, \qquad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{3a_1+1} + \frac{1}{3a_2+1} + \cdots + \frac{1}{3a_n+1} \ge \frac{n}{4}.$$

(Vasile C., 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \qquad i = 1, 2, \ldots, n,$$

we can write the inequality as

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge n f(s),$$

where

$$x_1 \ge x_2 \ge x_3 \ge 0 \ge x_4 \ge \cdots \ge x_n, \qquad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

$$f(u) = \frac{1}{3e^u + 1}, \qquad u \in \mathbb{R}.$$

For $u \in [0, \infty)$, we have

$$f''(u) = \frac{3e^u(3e^u - 1)}{(3e^u + 1)^3} > 0,$$

hence f is convex on $[s,\infty)$. According to the RHCF-OV Theorem (case $m=n-3$), it suffices to show that $f(x)+3f(y)\ge 4f(0)$ for all real x,y so that $x+3y=0$; that is, to show that

$$\frac{1}{3a+1}+\frac{3}{3b+1}\ge 1$$

for all $a,b>0$ so that $ab^3=1$. The inequality is equivalent to

$$\frac{b^3}{b^3+3}+\frac{3}{3b+1}\ge 1,$$

$$(b-1)^2(b+2)\ge 0.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$.

□

P 2.31. *If $a_1,a_2,\ldots,a_n$ $(n\ge 4)$ are positive real numbers so that*

$$a_1\ge a_2\ge a_3\ge 1\ge a_4\ge\cdots\ge a_n,\qquad a_1a_2\cdots a_n=1,$$

then

$$\frac{1}{(a_1+1)^2}+\frac{1}{(a_2+1)^2}+\cdots+\frac{1}{(a_n+1)^2}\ge\frac{n}{4}.$$

(Vasile C., 2007)

Solution. Using the substitution

$$a_i=e^{x_i},\qquad i=1,2,\ldots,n,$$

we can write the inequality as

$$f(x_1)+f(x_2)+\cdots+f(x_n)\ge nf(s),$$

where

$$x_1\ge x_2\ge x_3\ge 0\ge x_4\ge\cdots\ge x_n,\qquad s=\frac{x_1+x_2+\cdots+x_n}{n}=0,$$

$$f(u)=\frac{1}{(e^u+1)^2},\qquad u\in\mathbb{R}.$$

For $u\in[0,\infty)$, we have

$$f''(u)=\frac{2e^u(2e^u-1)}{(e^u+1)^4}>0,$$

hence f is convex on $[s,\infty)$. According to the RHCF-OV Theorem (case $m=3$), it suffices to show that $f(x)+3f(y)\ge 4f(0)$ for all real x,y so that $x+3y=0$; that is, to show that

$$\frac{1}{(a+1)^2}+\frac{3}{(b+1)^2}\ge 1$$

for all $a, b > 0$ so that $ab^3 = 1$. The inequality is equivalent to

$$\frac{b^6}{(b^3+1)^2} + \frac{3}{(b+1)^2} \geq 1.$$

Using the Cauchy-Schwarz inequality, it suffices to show that

$$\frac{(b^3+3)^2}{(b^3+1)^2+3(b+1)^2} \geq 1,$$

which is equivalent to the obvious inequality

$$(b-1)^2(4b+5) \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 2.32. *If $a_1, a_2, \ldots, a_n$ are positive real numbers so that*

$$a_1 \geq \cdots \geq a_{n-1} \geq 1 \geq a_n, \qquad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{(a_1+3)^2} + \frac{1}{(a_2+3)^2} + \cdots + \frac{1}{(a_n+3)^2} \leq \frac{n}{16}.$$

(Vasile C., 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \qquad i = 1, 2, \ldots, n,$$

we can write the inequality as

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s),$$

where

$$x_1 \geq \cdots \geq x_{n-1} \geq 0 \geq x_n, \qquad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

$$f(u) = \frac{-1}{(e^u+3)^2}, \qquad u \in \mathbb{R}.$$

For $u \in (-\infty, 0]$, we have

$$f''(u) = \frac{2e^u(3-2e^u)}{(e^u+3)^4} > 0,$$

hence f is convex on $(-\infty, s]$. According to the LHCF-OV Theorem (case $m = n-1$), it suffices to show that $f(x)+f(y) \geq 2f(0)$ for all real x, y so that $x+y = 0$; that is, to show that

$$\frac{1}{(a+3)^2} + \frac{1}{(b+3)^2} \leq \frac{1}{8}$$

for all $a, b > 0$ so that $ab = 1$. Write this inequality as

$$\frac{b^2}{(3b+1)^2} + \frac{1}{(b+3)^2} \le \frac{1}{8},$$

which is equivalent to the obvious inequality

$$(b^2-1)^2 + 12b(b-1)^2 \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark. Similarly, we can prove the following generalization:

- *Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that*

$$a_1 \ge \cdots \ge a_{n-1} \ge 1 \ge a_n, \qquad a_1a_2\cdots a_n = 1,$$

If $k \ge 1+\sqrt{2}$, then

$$\frac{1}{(a_1+k)^2} + \frac{1}{(a_2+k)^2} + \cdots + \frac{1}{(a_n+k)^2} \le \frac{n}{(1+k)^2},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 2.33. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that*

$$a_1 \ge \cdots \ge a_{n-1} \ge 1 \ge a_n, \qquad a_1a_2\cdots a_n = 1.$$

If $p, q \ge 0$ so that $p+q \le 1$, then

$$\frac{1}{1+pa_1+qa_1^2} + \frac{1}{1+pa_2+qa_2^2} + \cdots + \frac{1}{1+pa_n+qa_n^2} \le \frac{n}{1+p+q}.$$

(Vasile C., 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \qquad i = 1, 2, \ldots, n,$$

we can write the inequality as

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge nf(s),$$

where

$$x_1 \ge \cdots \ge x_{n-1} \ge 0 \ge x_n, \qquad s = \frac{x_1+x_2+\cdots+x_n}{n} = 0,$$

$$f(u)=\frac{-1}{1+pe^u+qe^{2u}}, \quad u\in\mathbb{R}.$$

For $u\le 0$, we have

$$\begin{aligned}f''(u)&=\frac{e^u[-4q^2e^{3u}-3pqe^{2u}+(4q-p^2)e^u+p]}{(1+pe^u+qe^{2u})^3}\\&\ge\frac{e^{2u}[-4q^2-3pq+(4q-p^2)+p]}{(1+pe^u+qe^{2u})^3}\\&=\frac{e^{2u}[(p+4q)(1-p-q)+2pq]}{(1+pe^u+qe^{2u})^3}\ge 0,\end{aligned}$$

therefore $f(u)$ is convex for $u\le s$. According to the LHCF-OV Theorem (case $m=n-1$), it suffices to show that $f(x)+f(y)\ge 2f(0)$ for all real x,y so that $x+y=0$; that is, to show that

$$\frac{1}{1+pa+qa^2}+\frac{1}{1+pb+qb^2}\le\frac{2}{1+p+q}$$

for all $a,b>0$ so that $ab=1$. Write this inequality as

$$(a-1)^2[q(1-p-q)a^2+(p+2q-p^2-pq-2q^2)a+q(1-p-q)]\ge 0,$$

which is true because

$$p+2q-p^2-pq-2q^2\ge(p+2q)(p+q)-p^2-pq-2q^2=2pq\ge 0.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$.

□

P 2.34. *Let $a_1,a_2,\ldots,a_n$ be positive real numbers so that*

$$a_1\ge\cdots\ge a_{n-1}\ge 1\ge a_n, \quad a_1a_2\cdots a_n=1.$$

If $m>1$ and $k\ge\dfrac{1}{2^{1/m}-1}$, then

$$\frac{1}{(a_1+k)^m}+\frac{1}{(a_2+k)^m}+\cdots+\frac{1}{(a_n+k)^m}\le\frac{n}{(1+k)^m}.$$

(Vasile C., 2007)

Solution. By Bernoulli's inequality, we have

$$2^{1/m}<1+\frac{1}{m},$$

hence

$$k \ge \frac{1}{2^{1/m}-1} > m > 1.$$

Using the substitution

$$a_i = e^{x_i}, \qquad i = 1, 2, \ldots, n,$$

we can write the inequality as

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge nf(s),$$

where

$$x_1 \ge \cdots \ge x_{n-1} \ge 0 \ge x_n, \qquad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

$$f(u) = \frac{-1}{(e^u + k)^m}, \quad u \in \mathbb{R}.$$

For $u \le 0$, we have

$$f''(u) = \frac{me^u(k - me^u)}{(e^u + k)^{m+2}} \ge 0,$$

hence f is convex $u \le s$. By the LHCF-OV Theorem (case $m = n-1$), it suffices to show that $f(x) + f(y) \ge 2f(0)$ for all real x, y so that $x + y = 0$; that is, to show that

$$\frac{1}{(a+k)^m} + \frac{1}{(b+k)^m} \le \frac{2}{(1+k)^m}$$

for all $a, b > 0$ so that $ab = 1$. Write this inequality as $g(a) \le 0$ for $a \ge 1$, where

$$g(a) = \frac{1}{(a+k)^m} + \frac{a^m}{(ka+1)^m} - \frac{2}{(1+k)^m}.$$

The derivative

$$\frac{g'(a)}{m} = \frac{a^{m-1}(a+k)^{m+1} - (ka+1)^{m+1}}{(a+k)^{m+1}(ka+1)^{m+1}}$$

has the same sign as the function

$$h(a) = (m-1)\ln a + (m+1)\ln(a+k) - (m+1)\ln(ka+1).$$

We have

$$h'(a) = \frac{m-1}{a} + (m+1)\left(\frac{1}{a+k} - \frac{k}{ka+1}\right) = \frac{kh_1(a)}{a(a+k)(ka+1)},$$

where

$$h_1(a) = (m-1)(a^2+1) - 2\left(k - \frac{m}{k}\right)a.$$

The discriminant D of the quadratic function $h_1(a)$ is

$$\frac{D}{4} = \left(k - \frac{m}{k}\right)^2 - (m-1)^2 = (k^2-1)\left(1 - \frac{m^2}{k^2}\right).$$

Since $D > 0$, the roots a_1 and a_2 of $h_1(a)$ are real and unequal. If $a_1 < a_2$, then $h_1(a) \le 0$ for $a \in [a_1, a_2]$ and $h_1(a) \ge 0$ for $a \in (-\infty, a_1] \cup [a_2, \infty)$. Since

$$h_1(1) = \frac{2(k+1)(m-k)}{k} < 0,$$

it follows that $a_1 < 1 < a_2$, therefore $h_1(a)$ and $h'(a)$ are negative for $a \in [1, a_2)$ and positive for $a \in (a_2, \infty)$, $h(a)$ is decreasing for $a \in [1, a_2]$ and increasing for $a \in [a_2, \infty)$. From $h(1) = 0$ and

$$\lim_{a\to\infty} h(a) = \infty,$$

it follows that there is $a_3 > a_2$ so that $h(a)$ and $g'(a)$ are negative for $a \in (1, a_3)$ and positive for $a \in (a_3, \infty)$. As a result, g is decreasing on $[1, a_3]$ and increasing on $[a_3, \infty)$. Since $g(1) = 0$ and

$$\lim_{a\to\infty} g(a) = \frac{1}{k^m} - \frac{2}{(1+k)^m} \le 0,$$

it follows that $g(a) \le 0$ for $a \ge 1$. This completes the proof. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 2.35. *If $a_1, a_2, \ldots, a_n$ are positive real numbers so that*

$$a_1 \ge \cdots \ge a_{n-1} \ge 1 \ge a_n, \qquad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{\sqrt{1+2a_1}} + \frac{1}{\sqrt{1+2a_2}} + \cdots + \frac{1}{\sqrt{1+2a_n}} \le \frac{n}{\sqrt{3}}.$$

(Vasile C., 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \qquad i = 1, 2, \ldots, n,$$

we can write the inequality as

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge nf(s),$$

where

$$x_1 \ge \cdots \ge x_{n-1} \ge 0 \ge x_n, \qquad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

$$f(u) = \frac{-1}{\sqrt{1+2e^u}}, \qquad u \in \mathbb{R}.$$

For $u \le 0$, we have

$$f''(u) = \frac{e^u(1-e^u)}{(1+2e^u)^{5/2}} > 0,$$

hence f is convex on $(-\infty, s]$. According to the LHCF-OV Theorem (case $m = n-1$), it suffices to show that $f(x)+f(y) \ge 2f(0)$ for all real x, y so that $x+y=0$; that is, to show that

$$\sqrt{\frac{3}{1+2a}} + \sqrt{\frac{3}{1+2b}} \le 2$$

for all $a, b > 0$ so that $ab = 1$. By the Cauchy-Schwarz inequality, we get

$$\sqrt{\frac{3}{1+2a}} + \sqrt{\frac{3}{1+2b}} \le \sqrt{\left(\frac{3}{1+2a}+1\right)\left(1+\frac{3}{1+2b}\right)} = 2.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 2.36. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that*

$$a_1 \ge \cdots \ge a_{n-1} \ge 1 \ge a_n, \qquad a_1a_2\cdots a_n = 1.$$

If $0 < m < 1$ and $k \ge m$, then

$$\frac{1}{(a_1+k)^m} + \frac{1}{(a_2+k)^m} + \cdots + \frac{1}{(a_n+k)^m} \le \frac{n}{(1+k)^m}.$$

(Vasile C., 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \qquad i = 1, 2, \ldots, n,$$

we can write the inequality as

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge nf(s),$$

where

$$x_1 \ge \cdots \ge x_{n-1} \ge 0 \ge x_n, \qquad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

$$f(u) = \frac{-1}{(e^u+k)^m}, \qquad u \in \mathbb{R}.$$

For $u \le 0$, we have

$$f''(u) = \frac{me^u(k-me^u)}{(e^u+k)^{m+2}} \ge 0,$$

hence f is convex on $(-\infty, s]$. According to the LHCF-OV Theorem (case $m = n-1$), it suffices to show that $f(x)+f(y) \ge 2f(0)$ for all real x, y so that $x+y=0$; that is, to show that

$$\frac{1}{(a+k)^m}+\frac{1}{(b+k)^m} \le \frac{2}{(1+k)^m}$$

for all $a, b > 0$ so that $ab = 1$. Write this inequality as $g(a) \le 0$ for $a \ge 1$, where

$$g(a) = \frac{1}{(a+k)^m}+\frac{a^m}{(ka+1)^m}-\frac{2}{(1+k)^m},$$

with

$$\frac{g'(a)}{m} = \frac{a^{m-1}(a+k)^{m+1}-(ka+1)^{m+1}}{(a+k)^{m+1}(ka+1)^{m+1}}.$$

If $g'(a) \le 0$ for $a \ge 1$, then g is decreasing, hence $g(a) \le g(1) = 0$. Thus, it suffices to show that

$$a^{m-1} \le \left(\frac{ka+1}{a+k}\right)^{m+1}.$$

Since

$$\frac{ka+1}{a+k}-\frac{ma+1}{a+m} = \frac{(k-m)(a^2-1)}{(a+k)(a+m)} \ge 0,$$

we only need to show that

$$a^{m-1} \le \left(\frac{ma+1}{a+m}\right)^{m+1},$$

which is equivalent to $h(a) \le 0$ for $a \ge 1$, where

$$h(a) = (m-1)\ln a+(m+1)\ln(a+m)-(m+1)\ln(ma+1),$$

$$h'(a) = \frac{m-1}{a}+\frac{m+1}{a+m}-\frac{m(m+1)}{ma+1} = \frac{m(m-1)(a-1)^2}{a(a+m)(ma+1)}.$$

Since $h'(a) \le 0$, $h(a)$ is decreasing for $a \ge 1$, hence

$$h(a) \le h(1) = 0.$$

This completes the proof. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark. For $k = \dfrac{1}{2}$ and $m = \dfrac{1}{2}$, we get the preceding P 2.35.

□

P 2.37. *If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$are positive real numbers so that*

$$a_1 \ge \cdots \ge a_{n-2} \ge 1 \ge a_{n-1} \ge a_n, \qquad a_1a_2\cdots a_n = 1,$$

then

$$\frac{1}{(a_1+5)^2}+\frac{1}{(a_2+5)^2}+\cdots+\frac{1}{(a_n+5)^2} \le \frac{n}{36}.$$

(Vasile C., 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \qquad i = 1, 2, \ldots, n,$$

we can write the inequality as

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s),$$

where

$$x_1 \geq \cdots \geq x_{n-2} \geq 0 \geq x_{n-1} \geq x_n, \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

$$f(u) = \frac{-1}{(e^u + 5)^2}, \quad u \in \mathbb{R}.$$

For $u \in (-\infty, 0]$, we have

$$f''(u) = \frac{2e^u(5 - 2e^u)}{(e^u + 5)^4} > 0,$$

hence f is convex on $(-\infty, s]$. According to the LHCF-OV Theorem (case $m = n - 2$), it suffices to show that $f(x) + 2f(y) \geq 3f(0)$ for all real x, y so that $x + 2y = 0$; that is, to show that

$$\frac{1}{(a+5)^2} + \frac{2}{(b+5)^2} \leq \frac{1}{12}$$

for all $a, b > 0$ so that $ab^2 = 1$. Since

$$\frac{1}{(a+5)^2} = \frac{b^4}{(5b^2+1)^2} \leq \frac{b^4}{(4b^2+2b)^2} = \frac{b^2}{4(2b+1)^2},$$

it suffices to show that

$$\frac{b^2}{4(2b+1)^2} + \frac{2}{(b+5)^2} \leq \frac{1}{12},$$

which is equivalent to the obvious inequality

$$(b-1)^2(b^2 + 16b + 1) \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark. Similarly, we can prove the following refinement:

- *Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that*

$$a_1 \geq \cdots \geq a_{n-2} \geq 1 \geq a_{n-1} \geq a_n, \qquad a_1a_2\cdots a_n = 1.$$

If $k \geq 2 + \sqrt{6}$, then

$$\frac{1}{(a_1+k)^2} + \frac{1}{(a_2+k)^2} + \cdots + \frac{1}{(a_n+k)^2} \leq \frac{n}{(1+k)^2},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 2.38. *If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that*

$$a_1 \ge \cdots \ge a_{n-1} \ge 1 \ge a_n, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n,$$

then

$$\frac{1}{3-a_1} + \frac{1}{3-a_2} + \cdots + \frac{1}{3-a_n} \le \frac{n}{2}.$$

(Vasile C., 2007)

Solution. From

$$n = a_1^2 + (a_2^2 + \cdots + a_{n-1}^2) + a_n^2 \ge a_1^2 + (n-2) + 0,$$

we get

$$a_1 \le \sqrt{2}.$$

Replacing $a_1, a_2, \dots, a_n$ by $\sqrt{a_1}, \sqrt{a_2}, \dots, \sqrt{a_n}$, we have to prove that

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s),$$

where

$$2 \ge a_1 \ge \cdots \ge a_{n-1} \ge 1 \ge a_n, \qquad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

$$f(u) = \frac{1}{\sqrt{u}-3}, \quad u \in [0,2].$$

For $u \in [0,1]$, we have

$$f''(u) = \frac{3(1-\sqrt{u})}{4u\sqrt{u}(3-\sqrt{u})^3} \ge 0.$$

Therefore, f is convex on $[0,s]$. According to the LHCF-OV Theorem and Note 1 (case $m = n-1$), it suffices to show that $h(x,y) \ge 0$ for $x, y \ge 0$ so that $x + y = 2$. Since

$$g(u) = \frac{f(u)-f(1)}{u-1} = \frac{-1}{2(3-\sqrt{u})(1+\sqrt{u})}$$

and

$$h(x,y) = \frac{g(x)-g(y)}{x-y} = \frac{2-\sqrt{x}-\sqrt{y}}{2(\sqrt{x}+\sqrt{y})(1+\sqrt{x})(1+\sqrt{y})(3-\sqrt{x})(3-\sqrt{y})},$$

we need to show that

$$\sqrt{x} + \sqrt{y} \le 2.$$

Indeed, we have

$$\sqrt{x} + \sqrt{y} \le \sqrt{2(x+y)} = 2.$$

This completes the proof. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 2.39. *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that*

$$a_1 \le \cdots \le a_{n-1} \le 1 \le a_n, \qquad a_1 + a_2 + \cdots + a_n = n.$$

Prove that

$$a_1^3 + a_2^3 + \cdots + a_n^3 - n \ge (n-1)^2 \left[\left(\frac{n-a_1}{n-1} \right)^3 + \left(\frac{n-a_2}{n-1} \right)^3 + \cdots + \left(\frac{n-a_n}{n-1} \right)^3 - n \right].$$

(Vasile C., 2010)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \qquad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = u^3 - (n-1)^2 \left(\frac{n-u}{n-1} \right)^3, \quad u \ge 0.$$

For $u \ge 1$, we have

$$f''(u) = \frac{6n(u-1)}{n-1} \ge 0.$$

Therefore, $f(u)$ is convex for $u \ge s$. Thus, by the RHCF-OV Theorem (case $m = n-1$), it suffices to show that $f(x) + f(y) \ge 2f(1)$ for $x, y \ge 0$ so that $x + y = 2$. We have

$$\begin{aligned} f(x) + f(y) - 2f(1) &= x^3 + y^3 - 2 - (n-1)^2 \left[\left(\frac{n-x}{n-1} \right)^3 + \left(\frac{n-y}{n-1} \right)^3 - 2 \right] \\ &= 6(1-xy) - 6(n-1)^2 \left[1 - \frac{(n-x)(n-y)}{(n-1)^2} \right] = 0. \end{aligned}$$

This completes the proof. The equality holds for

$$a_1 \le 1, \quad a_2 = \cdots = a_{n-1} = 1, \quad a_n = 2 - a_1.$$

□

Chapter 3

Partially Convex Function Method

3.1 Theoretical Basis

The following statement is known as the Right Partially Convex Function Theorem (RPCF-Theorem).

Right Partially Convex Function Theorem (*Vasile Cîrtoaje*, 2012). *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $[s, s_0]$, where $s, s_0 \in \mathbb{I}$, $s < s_0$. In addition, f is decreasing on $\mathbb{I}_{\le s_0}$ and $f(u) \ge f(s_0)$ for $u \in \mathbb{I}$. The inequality*

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying

$$a_1 + a_2 + \cdots + a_n = ns$$

if and only if

$$f(x) + (n-1)f(y) \ge nf(s)$$

for all $x, y \in \mathbb{I}$ so that $x \le s \le y$ and $x + (n-1)y = ns$.

Proof. For

$$a_1 = x, \quad a_2 = a_3 = \cdots = a_n = y,$$

the inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge f(s)$$

becomes

$$f(x) + (n-1)f(y) \ge nf(s);$$

therefore, the necessity is obvious.
The proof of sufficiency is based on Lemma below. According to this lemma, it suffices to consider that $a_1, a_2, \ldots, a_n \in \mathbb{J}$, where

$$\mathbb{J} = \mathbb{I}_{\le s_0}.$$

Because $f(u)$ is convex on $\mathbb{J}_{\geq s}$, the desired inequality follows from the RHCF Theorem (see Chapter 1) applied to the interval $\mathbb{J}$.

Lemma. *Let f be a real function defined on an interval $\mathbb{I}$. In addition, f is decreasing on $\mathbb{I}_{\leq s_0}$, and $f(u) \geq f(s_0)$ for $u \in \mathbb{I}$, where $s, s_0 \in \mathbb{I}$, $s < s_0$. If the inequality*

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}_{\leq s_0}$ so that $a_1 + a_2 + \cdots + a_n = ns$, then it holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ so that $a_1 + a_2 + \cdots + a_n = ns$.

Proof. For $i = 1, 2, \ldots, n$, define the numbers

$$b_i = \begin{cases} a_i, & a_i \leq s_0 \\ s_0, & a_i > s_0. \end{cases}$$

Clearly, $b_i \in \mathbb{I}_{\leq s_0}$ and $b_i \leq a_i$. Since $f(u) \geq f(s_0)$ for $u \in \mathbb{I}_{\geq s_0}$, it follows that $f(b_i) \leq f(a_i)$ for $i = 1, 2, \ldots, n$. Therefore,

$$b_1 + b_2 + \cdots + b_n \leq a_1 + a_2 + \cdots + a_n = ns$$

and

$$f(b_1) + f(b_2) + \cdots + f(b_n) \leq f(a_1) + f(a_2) + \cdots + f(a_n).$$

Thus, it suffices to show that

$$f(b_1) + f(b_2) + \cdots + f(b_n) \geq nf(s)$$

for all $b_1, b_2, \ldots, b_n \in \mathbb{I}_{\leq s_0}$ so that $b_1 + b_2 + \cdots + b_n \leq ns$. By hypothesis, this inequality is true for $b_1, b_2, \ldots, b_n \in \mathbb{I}_{\leq s_0}$ and $b_1 + b_2 + \cdots + b_n = ns$. Since $f(u)$ is decreasing on $\mathbb{I}_{\leq s_0}$, the more we have $f(b_1) + f(b_2) + \cdots + f(b_n) \geq nf(s)$ for $b_1, b_2, \ldots, b_n \in \mathbb{I}_{\leq s_0}$ and $b_1 + b_2 + \cdots + b_n \leq ns$.

Similarly, we can prove the Left Partially Convex Function Theorem (LPCF-Theorem).

Left Partially Convex Function Theorem (*Vasile Cîrtoaje*, 2012). *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $[s_0, s]$, where $s_0, s \in \mathbb{I}$, $s_0 < s$. In addition, f is increasing on $\mathbb{I}_{\geq s_0}$ and $f(u) \geq f(s_0)$ for $u \in \mathbb{I}$. The inequality*

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying

$$a_1 + a_2 + \cdots + a_n = ns$$

if and only if

$$f(x) + (n-1)f(y) \geq nf(s)$$

for all $x, y \in \mathbb{I}$ so that $x \geq s \geq y$ and $x + (n-1)y = ns$.

From the RPCF-Theorem and the LPCF-Theorem, we find the PCF-Theorem (Partially Convex Function Theorem).

Partially Convex Function Theorem (*Vasile Cîrtoaje*, 2012). *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $[s_0, s]$ or $[s, s_0]$, where $s_0, s \in \mathbb{I}$. In addition, f is decreasing on $\mathbb{I}_{\le s_0}$ and increasing on $\mathbb{I}_{\ge s_0}$. The inequality*

$$f(a_1)+f(a_2)+\cdots+f(a_n) \ge nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying

$$a_1+a_2+\cdots+a_n = ns$$

if and only if

$$f(x)+(n-1)f(y) \ge nf(s)$$

for all $x, y \in \mathbb{I}$ so that $x+(n-1)y = ns$.

Note 1. Let us denote

$$g(u) = \frac{f(u)-f(s)}{u-s}, \quad h(x,y) = \frac{g(x)-g(y)}{x-y}.$$

As shown in Note 1 from Chapter 1, we may replace the hypothesis condition in the RPCF-Theorem and the LPCF-Theorem), namely

$$f(x)+(n-1)f(y) \ge nf(s),$$

by the condition

$$h(x,y) \ge 0 \ \textit{ for all } \ x, y \in \mathbb{I} \ \textit{ so that } \ x+(n-1)y = ns.$$

Note 2. Assume that f is differentiable on $\mathbb{I}$, and let

$$H(x,y) = \frac{f'(x)-f'(y)}{x-y}.$$

As shown in Note 2 from Chapter 1, the inequalities in the RPCF-Theorem and the LPCF-Theorem hold true by replacing the hypothesis

$$f(x)+(n-1)f(y) \ge nf(s)$$

with the more restrictive condition

$$H(x,y) \ge 0 \ \text{ for all } \ x, y \in \mathbb{I} \ \text{ so that } \ x+(n-1)y = ns.$$

Note 3. The desired inequalities in the RPCF-Theorem and the LPCF-Theorem become equalities for

$$a_1 = a_2 = \cdots = a_n = s.$$

In addition, if there exist $x,y \in \mathbb{I}$ so that

$$x+(n-1)y = ns, \quad f(x)+(n-1)f(y) = nf(s), \quad x \neq y,$$

then the equality holds also for

$$a_1 = x, \quad a_2 = \cdots = a_n = y$$

(or any cyclic permutation). Notice that these equality conditions are equivalent to

$$x+(n-1)y = ns, \quad h(x,y) = 0$$

($x < y$ for the RPCF-Theorem, and $x > y$ for the LPCF-Theorem).

Note 4. From the proof of the RPCF-Theorem, it follows that this theorem is also valid in the case when f is defined on $\mathbb{I} \setminus \{u_0\}$, where $u_0 \in \mathbb{I}_{>s_0}$. Similarly, the LPCF-Theorem is also valid in the case when f is defined on $\mathbb{I} \setminus \{u_0\}$, where $u_0 \in \mathbb{I}_{<s_0}$.

Note 5. The RPCF-Theorem holds true by replacing the condition

f is decreasing on $\mathbb{I}_{\le s_0}$

with

$$ns-(n-1)s_0 \le \inf \mathbb{I}.$$

More precisely, the following theorem holds:

Theorem 1. *Let f be a function defined on a real interval $\mathbb{I}$, convex on $[s,s_0]$ and satisfying*

$$\min_{u \in \mathbb{I}_{\ge s}} f(u) = f(s_0),$$

where

$$s,s_0 \in \mathbb{I}, \;\; s < s_0, \;\; ns-(n-1)s_0 \le \inf \mathbb{I}.$$

If

$$f(x)+(n-1)f(y) \ge nf(s)$$

for all $x,y \in \mathbb{I}$ so that $x \le s \le y$ and $x+(n-1)y = ns$, then

$$f(x_1)+f(x_2)+\cdots+f(x_n) \ge nf\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)$$

for all $x_1,x_2,\ldots,x_n \in \mathbb{I}$ satisfying $x_1+x_2+\cdots+x_n = ns$.

In order to prove Theorem 1, we define the function

$$f_0(u) = \begin{cases} f(u), & u \le s_0, \; u \in \mathbb{I} \\ f(s_0), & u \ge s_0, \; u \in \mathbb{I}, \end{cases}$$

which is convex on $\mathbb{I}_{\ge s}$. Taking into account that $f_0(s) = f(s)$ and $f_0(u) \le f(u)$ for all $u \in \mathbb{I}$, it suffices to prove that

$$f_0(x_1)+f_0(x_2)+\cdots+f_0(x_n) \ge nf_0(s)$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{I}$ satisfying $x_1 + x_2 + \cdots + x_n = ns$. According to the HCF-Theorem and Note 5 from Chapter 1, we only need to show that

$$f_0(x) + (n-1)f_0(y) \geq nf_0(s)$$

for all $x, y \in \mathbb{I}$ so that $x \leq s \leq y$ and $x + (n-1)y = ns$. Since

$$y - s_0 = \frac{ns - x}{n-1} - s_0 = \frac{ns - (n-1)s_0 - x}{n-1} \leq \frac{ns - (n-1)s_0 - \inf\mathbb{I}}{n-1} \leq 0,$$

the inequality $f_0(x) + (n-1)f_0(y) \geq nf_0(s)$ turns into $f(x) + (n-1)f(y) \geq nf(s)$, which holds (by hypothesis) for all $x, y \in \mathbb{I}$ so that $x \leq s \leq y$ and $x + (n-1)y = ns$.

Similarly, the LPCF-Theorem holds true by replacing the condition

f is increasing on $\mathbb{I}_{\geq s_0}$

with

$$ns - (n-1)s_0 \geq \sup\mathbb{I}.$$

More precisely, the following theorem holds:

Theorem 2. *Let f be a function defined on a real interval $\mathbb{I}$, convex on $[s_0, s]$ and satisfying*

$$\min_{u \in \mathbb{I}_{\leq s}} f(u) = f(s_0),$$

where

$$s, s_0 \in \mathbb{I}, \quad s > s_0, \quad ns - (n-1)s_0 \geq \sup\mathbb{I}.$$

If

$$f(x) + (n-1)f(y) \geq nf(s)$$

for all $x, y \in \mathbb{I}$ so that $x \geq s \geq y$ and $x + (n-1)y = ns$, then

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{I}$ satisfying $x_1 + x_2 + \cdots + x_n = ns$.

The proof of Theorem 2 is similar to the proof of Theorem 1.

Note 6. From the proof of Theorem 1, it follows that Theorem 1 is also valid in the case in which f is defined on $\mathbb{I} \setminus \{u_0\}$, where u_0 is an interior point of $\mathbb{I}$ so that $u_0 \notin [s, s_0]$. Similarly, Theorem 2 is also valid in the case in which f is defined on $\mathbb{I} \setminus \{u_0\}$, where u_0 is an interior point of $\mathbb{I}$ so that $u_0 \notin [s_0, s]$.

Note 7. In the same manner, we can extend *weighted* Jensen's inequality to right and left partially convex functions establishing the WRPCF-Theorem, the WLPCF-Theorem and the WPCF-Theorem (*Vasile Cîrtoaje*, 2014).

WRPCF-Theorem. *Let $p_1, p_2, \ldots, p_n$ be positive real numbers so that*

$$p_1 + p_2 + \cdots + p_n = 1, \qquad p = \min\{p_1, p_2, \ldots, p_n\},$$

and let f be a real function defined on an interval $\mathbb{I}$ and convex on $[s, s_0]$, where $s, s_0 \in \mathbb{I}$, $s < s_0$. In addition, f is decreasing on $\mathbb{I}_{\leq s_0}$ and $f(u) \geq f(s_0)$ for $u \in \mathbb{I}$. The inequality

$$p_1 f(a_1) + p_2 f(a_2) + \cdots + p_n f(a_n) \geq f(p_1 a_1 + p_2 a_2 + \cdots + p_n a_n)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying

$$p_1 a_1 + p_2 a_2 + \cdots + p_n a_n = s,$$

if and only if

$$pf(x) + (1-p)f(y) \geq f(s)$$

for all $x, y \in \mathbb{I}$ so that $x \leq s \leq y$ and $px + (1-p)y = s$.

WLPCF-Theorem. *Let $p_1, p_2, \ldots, p_n$ be positive real numbers so that*

$$p_1 + p_2 + \cdots + p_n = 1, \qquad p = \min\{p_1, p_2, \ldots, p_n\},$$

and let f be a real function defined on an interval $\mathbb{I}$ and convex on $[s_0, s]$, where $s_0, s \in \mathbb{I}$, $s_0 < s$. In addition, f is increasing on $\mathbb{I}_{\geq s_0}$ and $f(u) \geq f(s_0)$ for $u \in \mathbb{I}$. The inequality

$$p_1 f(a_1) + p_2 f(a_2) + \cdots + p_n f(a_n) \geq f(p_1 a_1 + p_2 a_2 + \cdots + p_n a_n)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying

$$p_1 a_1 + p_2 a_2 + \cdots + p_n a_n = s,$$

if and only if

$$pf(x) + (1-p)f(y) \geq f(s)$$

for all $x, y \in \mathbb{I}$ so that $x \geq s \geq y$ and $px + (1-p)y = s$.

3.2 Applications

3.1. If a, b, c are real numbers so that $a+b+c=3$, then

$$\frac{16a-5}{32a^2+1}+\frac{16b-5}{32b^2+1}+\frac{16c-5}{32c^2+1}\le 1.$$

3.2. If a, b, c, d are real numbers so that $a+b+c+d=4$, then

$$\frac{18a-5}{12a^2+1}+\frac{18b-5}{12b^2+1}+\frac{18c-5}{12c^2+1}+\frac{18d-5}{12d^2+1}\le 4.$$

3.3. If a, b, c, d, e, f are real numbers so that $a+b+c+d+e+f=6$, then

$$\frac{5a-1}{5a^2+1}+\frac{5b-1}{5b^2+1}+\frac{5c-1}{5c^2+1}+\frac{5d-1}{5d^2+1}+\frac{5e-1}{5e^2+1}+\frac{5f-1}{5f^2+1}\le 4.$$

3.4. If $a_1, a_2, \ldots, a_n$ $(n\ge 3)$ are real numbers so that $a_1+a_2+\cdots+a_n=n$, then

$$\frac{n(n+1)-2a_1}{n^2+(n-2)a_1^2}+\frac{n(n+1)-2a_2}{n^2+(n-2)a_2^2}+\cdots+\frac{n(n+1)-2a_n}{n^2+(n-2)a_n^2}\le n.$$

3.5. If a, b, c, d are real numbers so that $a+b+c+d=4$, then

$$\frac{a(a-1)}{3a^2+4}+\frac{b(b-1)}{3b^2+4}+\frac{c(c-1)}{3c^2+4}+\frac{d(d-1)}{3d^2+4}\ge 0.$$

3.6. If a, b, c are real numbers so that $a+b+c=3$, then

$$\frac{1}{9a^2-10a+9}+\frac{1}{9b^2-10b+9}+\frac{1}{9c^2-10c+9}\le\frac{3}{8}.$$

3.7. If a, b, c, d are real numbers so that $a+b+c+d=4$, then

$$\frac{1}{4a^2-5a+4}+\frac{1}{4b^2-5b+4}+\frac{1}{4c^2-5c+4}+\frac{1}{4d^2-5d+4}\le\frac{4}{3}.$$

3.8. Let $a_1, a_2, \dots, a_n \neq -k$ be real numbers so that $a_1 + a_2 + \cdots + a_n = n$, where

$$k \geq \frac{n}{2\sqrt{n-1}}.$$

Then,

$$\frac{a_1(a_1-1)}{(a_1+k)^2} + \frac{a_2(a_2-1)}{(a_2+k)^2} + \cdots + \frac{a_n(a_n-1)}{(a_n+k)^2} \geq 0.$$

3.9. Let $a_1, a_2, \dots, a_n \neq -k$ be real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If

$$k \geq 1 + \frac{n}{\sqrt{n-1}},$$

then

$$\frac{a_1^2-1}{(a_1+k)^2} + \frac{a_2^2-1}{(a_2+k)^2} + \cdots + \frac{a_n^2-1}{(a_n+k)^2} \geq 0.$$

3.10. Let a_1, a_2, a_3, a_4, a_5 be real numbers so that $a_1 + a_2 + a_3 + a_4 + a_5 \geq 5$. If

$$k \in \left[\frac{1}{6}, \frac{25}{14}\right],$$

then

$$\sum \frac{1}{ka_1^2 + a_2 + a_3 + a_4 + a_5} \leq \frac{5}{k+4}.$$

3.11. Let $a_1, a_2, \dots, a_5$ be nonnegative numbers so that $a_1 + a_2 + a_3 + a_4 + a_5 \geq 5$. If $k \in [k_1, k_2]$, where

$$k_1 = \frac{29 - \sqrt{761}}{10} \approx 0.1414, \qquad k_2 = \frac{25}{14} \approx 1.7857,$$

then

$$\sum \frac{1}{ka_1^2 + a_2 + a_3 + a_4 + a_5} \leq \frac{5}{k+4}.$$

3.12. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n \geq n$. If $k > 1$, then

$$\frac{1}{a_1^k + a_2 + \cdots + a_n} + \frac{1}{a_1 + a_2^k + \cdots + a_n} + \cdots + \frac{1}{a_1 + a_2 + \cdots + a_n^k} \leq 1.$$

3.13. Let $a_1, a_2, \ldots, a_5$ be nonnegative numbers so that $a_1+a_2+a_3+a_4+a_5 \ge 5$. If

$$k \in \left[\frac{4}{9}, \frac{61}{5}\right],$$

then

$$\sum \frac{a_1}{ka_1^2+a_2+a_3+a_4+a_5} \le \frac{5}{k+4}.$$

3.14. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1+a_2+\cdots+a_n \ge n$. If $k > 1$, then

$$\frac{a_1}{a_1^k+a_2+\cdots+a_n}+\frac{a_2}{a_1+a_2^k+\cdots+a_n}+\cdots+\frac{a_n}{a_1+a_2+\cdots+a_n^k} \le 1.$$

3.15. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1+a_2+\cdots+a_n \le n$. If $k \ge 1-\dfrac{1}{n}$, then

$$\frac{1-a_1}{ka_1^2+a_2+\cdots+a_n}+\frac{1-a_2}{a_1+ka_2^2+\cdots+a_n}+\cdots+\frac{1-a_n}{a_1+a_2+\cdots+ka_n^2} \ge 0.$$

3.16. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1+a_2+\cdots+a_n \le n$. If $k \ge 1-\dfrac{1}{n}$, then

$$\frac{1-a_1}{1-a_1+ka_1^2}+\frac{1-a_2}{1-a_2+ka_2^2}+\cdots+\frac{1-a_n}{1-a_n+ka_n^2} \ge 0.$$

3.17. Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that $a_1+a_2+\cdots+a_n = n$. If $0 < k \le \dfrac{n}{n-1}$, then

$$a_1^{k/a_1}+a_2^{k/a_2}+\cdots+a_n^{k/a_n} \le n.$$

3.18. If a, b, c, d, e are nonzero real numbers so that $a+b+c+d+e=5$, then

$$\left(7-\frac{5}{a}\right)^2+\left(7-\frac{5}{b}\right)^2+\left(7-\frac{5}{c}\right)^2+\left(7-\frac{5}{d}\right)^2+\left(7-\frac{5}{e}\right)^2 \ge 20.$$

3.19. If If $a_1, a_2, \ldots, a_7$ are real numbers so that $a_1 + a_2 + \cdots + a_7 = 7$, then

$$(a_1^2 + 2)(a_2^2 + 2)\cdots(a_7^2 + 2) \geq 3^7.$$

3.20. Let $a_1, a_2, \ldots, a_n$ be real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If $k \geq \dfrac{n^2}{4(n-1)}$, then

$$(a_1^2 + k)(a_2^2 + k)\cdots(a_n^2 + k) \geq (1 + k)^n.$$

3.21. Let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 + a_2 + \cdots + a_n = n$. If $n \leq 10$, then

$$(a_1^2 - a_1 + 1)(a_2^2 - a_2 + 1)\cdots(a_n^2 - a_n + 1) \geq 1.$$

3.22. Let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 + a_2 + \cdots + a_n = n$. If $n \leq 26$, then

$$(a_1^2 - a_1 + 2)(a_2^2 - a_2 + 2)\cdots(a_n^2 - a_n + 2) \geq 2^n.$$

3.23. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$(1 - a + a^4)(1 - b + b^4)(1 - c + c^4) \geq 1.$$

3.24. If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 4$, then

$$(1 - a + a^3)(1 - b + b^3)(1 - c + c^3)(1 - d + d^3) \geq 1.$$

3.25. If a, b, c, d, e are nonzero real numbers so that $a + b + c + d + e = 5$, then

$$5\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{e^2}\right) + 45 \geq 14\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}\right).$$

3.26. If a, b, c are positive real numbers so that $abc = 1$, then

$$\frac{7 - 6a}{2 + a^2} + \frac{7 - 6b}{2 + b^2} + \frac{7 - 6c}{2 + c^2} \geq 1.$$

3.27. If a, b, c are positive real numbers so that $abc = 1$, then

$$\frac{1}{a+5bc} + \frac{1}{b+5ca} + \frac{1}{c+5ab} \le \frac{1}{2}.$$

3.28. If a, b, c are positive real numbers so that $abc = 1$, then

$$\frac{1}{4-3a+4a^2} + \frac{1}{4-3b+4b^2} + \frac{1}{4-3c+4c^2} \le \frac{3}{5}.$$

3.29. If a, b, c are positive real numbers so that $abc = 1$, then

$$\frac{1}{(3a+1)(3a^2-5a+3)} + \frac{1}{(3b+1)(3b^2-5b+3)} + \frac{1}{(3c+1)(3c^2-5c+3)} \le \frac{3}{4}.$$

3.30. Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be positive real numbers so that $a_1 a_2 \cdots a_n = 1$. If $p, q \ge 0$ so that $p + 4q \ge n - 1$, then

$$\frac{1-a_1}{1+pa_1+qa_1^2} + \frac{1-a_2}{1+pa_2+qa_2^2} + \cdots + \frac{1-a_n}{1+pa_n+qa_n^2} \ge 0.$$

3.31. If a, b, c are positive real numbers so that $abc = 1$, then

$$\frac{1-a}{17+4a+6a^2} + \frac{1-b}{17+4b+6b^2} + \frac{1-c}{17+4c+6c^2} \ge 0.$$

3.32. If $a_1, a_2, \ldots, a_8$ are positive real numbers so that $a_1 a_2 \cdots a_8 = 1$, then

$$\frac{1-a_1}{(1+a_1)^2} + \frac{1-a_2}{(1+a_2)^2} + \cdots + \frac{1-a_8}{(1+a_8)^2} \ge 0.$$

3.33. Let a, b, c be positive real numbers so that $abc = 1$. If $k \in \left[\frac{-13}{3\sqrt{3}}, \frac{13}{3\sqrt{3}}\right]$, then

$$\frac{a+k}{a^2+1} + \frac{b+k}{b^2+1} + \frac{c+k}{c^2+1} \le \frac{3(1+k)}{2}.$$

3.34. If a,b,c are positive real numbers and $0<k\le 2+2\sqrt{2}$, then

$$\frac{a^3}{ka^2+bc}+\frac{b^3}{kb^2+ca}+\frac{c^3}{kc^2+ab}\ge\frac{a+b+c}{k+1}.$$

3.35. If a,b,c,d,e are positive real numbers so that $abcde=1$, then

$$2\left(\frac{1}{a+1}+\frac{1}{b+1}+\cdots+\frac{1}{e+1}\right)\ge 3\left(\frac{1}{a+2}+\frac{1}{b+2}+\cdots+\frac{1}{e+2}\right).$$

3.36. If $a_1,a_2,\dots,a_{14}$ are positive real numbers so that $a_1a_2\cdots a_{14}=1$, then

$$3\left(\frac{1}{2a_1+1}+\frac{1}{2a_2+1}+\cdots+\frac{1}{2a_{14}+1}\right)\ge 2\left(\frac{1}{a_1+1}+\frac{1}{a_2+1}+\cdots+\frac{1}{a_{14}+1}\right).$$

3.37. Let $a_1,a_2,\dots,a_8$ be positive real numbers so that $a_1a_2\cdots a_8=1$. If $k>1$, then

$$(k+1)\left(\frac{1}{ka_1+1}+\frac{1}{ka_2+1}+\cdots+\frac{1}{ka_8+1}\right)\ge 2\left(\frac{1}{a_1+1}+\frac{1}{a_2+1}+\cdots+\frac{1}{a_8+1}\right).$$

3.38. If $a_1,a_2,\dots,a_9$ are positive real numbers so that $a_1a_2\cdots a_9=1$, then

$$\frac{1}{2a_1+1}+\frac{1}{2a_2+1}+\cdots+\frac{1}{2a_9+1}\ge\frac{1}{a_1+2}+\frac{1}{a_2+2}+\cdots+\frac{1}{a_9+2}.$$

3.39. If $a_1,a_2,\dots,a_n$ are real numbers so that

$$a_1,a_2,\dots,a_n\le\pi,\qquad a_1+a_2+\cdots+a_n=\pi,$$

then

$$\cos a_1+\cos a_2+\cdots+\cos a_n\le n\cos\frac{\pi}{n}.$$

3.40. If $a_1,a_2,\dots,a_n$ ($n\ge 3$) are real numbers so that

$$a_1,a_2,\dots,a_n\ge\frac{-1}{n-2},\qquad a_1+a_2+\cdots+a_n=n,$$

then

$$\frac{a_1^2}{a_1^2-a_1+1}+\frac{a_2^2}{a_2^2-a_2+1}+\cdots+\frac{a_n^2}{a_n^2-a_n+1}\le n.$$

3.41. If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are nonzero real numbers so that

$$a_1, a_2, \ldots, a_n \ge \frac{-n}{n-2}, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2} \ge \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}.$$

3.42. If $a_1, a_2, \ldots, a_n \ge -1$ so that $a_1 + a_2 + \cdots + a_n = n$, then

$$(n+1)\left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2}\right) \ge 2n + (n-1)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

3.43. If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are real numbers so that

$$a_1, a_2, \ldots, a_n \ge \frac{-(3n-2)}{n-2}, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$\frac{1-a_1}{(1+a_1)^2} + \frac{1-a_2}{(1+a_2)^2} + \cdots + \frac{1-a_n}{(1+a_n)^2} \ge 0.$$

3.44. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If $n \ge 3$ and $k \ge 2 - \dfrac{2}{n}$, then

$$\frac{1-a_1}{(1-ka_1)^2} + \frac{1-a_2}{(1-ka_2)^2} + \cdots + \frac{1-a_n}{(1-ka_n)^2} \ge 0.$$

3.3 Solutions

P 3.1. *If a, b, c are real numbers so that $a+b+c=3$, then*

$$\frac{16a-5}{32a^2+1}+\frac{16b-5}{32b^2+1}+\frac{16c-5}{32c^2+1} \le 1.$$

(Vasile C., 2012)

Solution. Write the inequality as

$$f(a)+f(b)+f(c) \ge 3f(s), \qquad s=\frac{a+b+c}{3}=1,$$

where

$$f(u)=\frac{5-16u}{32u^2+1}, \qquad u \in \mathbb{R}.$$

From

$$f'(u)=\frac{16(32u^2-20u-1)}{(32u^2+1)^2},$$

it follows that f is increasing on

$$\left(-\infty, \frac{5-\sqrt{33}}{16}\right] \cup [s_0, \infty)$$

and decreasing on

$$\left[\frac{5-\sqrt{33}}{16}, s_0\right],$$

where

$$s_0=\frac{5+\sqrt{33}}{16} \approx 0.6715.$$

Also, from

$$\lim_{u \to -\infty} f(u)=0$$

and

$$f(s_0)<0,$$

it follows that $f(u) \ge f(s_0)$ for $u \in \mathbb{R}$. In addition, for $u \in [s_0, 1]$, we have

$$\begin{aligned}\frac{1}{64}f''(u) &= \frac{-512u^3+480u^2+48u-5}{(32u^2+1)^3}\\ &= \frac{512u^2(1-u)+32u(1-u)+(16u-5)}{(32u^2+1)^3}>0,\end{aligned}$$

hence f is convex on $[s_0, s]$. According to the LPCF-Theorem, we only need to show that $f(x)+2f(y) \geq 3f(1)$ for all real x, y so that $x+2y=3$. Using Note 1, it suffices to prove that $h(x,y) \geq 0$, where

$$h(x,y)=\frac{g(x)-g(y)}{x-y}, \quad g(u)=\frac{f(u)-f(1)}{u-1}.$$

Indeed, we have

$$g(u)=\frac{32(2u-1)}{3(32u^2+1)},$$

$$h(x,y)=\frac{64(1+16x+16y-32xy)}{3(32x^2+1)(32y^2+1)}=\frac{64(4x-5)^2}{3(32x^2+1)(32y^2+1)} \geq 0.$$

Thus, the proof is completed. From $x+2y=3$ and $h(x,y)=0$, we get

$$x=\frac{5}{4}, \qquad y=\frac{7}{8}.$$

Therefore, in accordance with Note 3, the equality holds for $a=b=c=1$, and also for

$$a=\frac{5}{4}, \qquad b=c=\frac{7}{8}$$

(or any cyclic permutation).

□

P 3.2. *If a,b,c,d are real numbers so that $a+b+c+d=4$, then*

$$\frac{18a-5}{12a^2+1}+\frac{18b-5}{12b^2+1}+\frac{18c-5}{12c^2+1}+\frac{18d-5}{12d^2+1} \leq 4.$$

(Vasile C., 2012)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d) \geq 4f(s), \qquad s=\frac{a+b+c+d}{4}=1,$$

where

$$f(u)=\frac{5-18u}{12u^2+1}, \quad u \in \mathbb{R}.$$

From

$$f'(u)=\frac{6(36u^2-20u-3)}{(12u^2+1)^2},$$

it follows that f is increasing on

$$\left(-\infty, \frac{5-\sqrt{52}}{18}\right] \cup [s_0, \infty)$$

and decreasing on

$$\left[\frac{5-\sqrt{52}}{18}, s_0\right], \quad s_0 = \frac{5+\sqrt{52}}{18} \approx 0.678.$$

Also, from

$$\lim_{u \to -\infty} f(u) = 0$$

and

$$f(s_0) < 0,$$

it follows that $f(u) \geq f(s_0)$ for $u \in \mathbb{R}$. In addition, for $u \in [s_0, 1]$, we have

$$\begin{aligned}\frac{1}{24} f''(u) &= \frac{-216u^3 + 180u^2 + 54u - 5}{(12u^2+1)^3} \\ &= \frac{216u^2(1-u) + 36u(1-u) + (18u-5)}{(32u^2+1)^3} > 0,\end{aligned}$$

hence f is convex on $[s_0, s]$. According to the LPCF-Theorem and Note 1, we only need to show that $h(x,y) \geq 0$ for $x, y \in \mathbb{R}$ so that $x + 3y = 4$. We have

$$g(u) = \frac{f(u) - f(1)}{u-1} = \frac{6(2u-1)}{12u^2+1},$$

$$h(x,y) = \frac{g(x)-g(y)}{x-y} = \frac{12(1+6x+6y-12xy)}{(12x^2+1)(12y^2+1)} = \frac{12(2x-3)^2}{(12x^2+1)(12y^2+1)} \geq 0.$$

Thus, the proof is completed. From $x + 3y = 4$ and $h(x,y) = 0$, we get $x = 3/2$ and $y = 5/6$. Therefore, in accordance with Note 3, the equality holds for $a = b = c = d = 1$, and also for

$$a = \frac{3}{2}, \quad b = c = d = \frac{5}{6}$$

(or any cyclic permutation).

□

P 3.3. *If a, b, c, d, e, f are real numbers so that $a + b + c + d + e + f = 6$, then*

$$\frac{5a-1}{5a^2+1} + \frac{5b-1}{5b^2+1} + \frac{5c-1}{5c^2+1} + \frac{5d-1}{5d^2+1} + \frac{5e-1}{5e^2+1} + \frac{5f-1}{5f^2+1} \leq 4.$$

(Vasile C., 2012)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) + f(e) + f(f) \geq 4f(s), \quad s = \frac{a+b+c+d+e+f}{6} = 1,$$

where

$$f(u)=\frac{1-5u}{5u^2+1}, \quad u\in\mathbb{R}.$$

From

$$f'(u)=\frac{5(5u^2-2u-1)}{(5u^2+1)^2},$$

it follows that f is increasing on

$$\left(-\infty,\frac{1-\sqrt{6}}{5}\right]\cup[s_0,\infty)$$

and decreasing on

$$\left[\frac{1-\sqrt{6}}{5},s_0\right], \quad s_0=\frac{1+\sqrt{6}}{5}\approx 0.69.$$

Also, from

$$\lim_{u\to-\infty}f(u)=0$$

and

$$f(s_0)<0,$$

it follows that $f(u)\geq f(s_0)$ for $u\in\mathbb{R}$. In addition, for $u\in[s_0,1]$, we have

$$\begin{aligned}\frac{1}{24}f''(u)&=\frac{-216u^3+180u^2+54u-5}{(12u^2+1)^3}\\&=\frac{216u^2(1-u)+36u(1-u)+(18u-5)}{(32u^2+1)^3}>0,\end{aligned}$$

hence f is convex on $[s_0,s]$. According to the LPCF-Theorem and Note 1, we only need to show that $h(x,y)\geq 0$ for $x,y\in\mathbb{R}$ so that $x+5y=6$. We have

$$g(u)=\frac{f(u)-f(1)}{u-1}=\frac{5(2u-1)}{3(5u^2+1)},$$

$$h(x,y)=\frac{g(x)-g(y)}{x-y}=\frac{5(2+5x+5y-10xy)}{3(5x^2+1)(5y^2+1)}=\frac{10(x-2)^2}{3(5x^2+1)(5y^2+1)}\geq 0.$$

In accordance with Note 3, the equality holds for $a=b=c=d=e=f=1$, and also for

$$a=2, \quad b=c=d=e=f=\frac{4}{5}$$

(or any cyclic permutation).

□

P 3.4. *If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then*

$$\frac{n(n+1)-2a_1}{n^2+(n-2)a_1^2} + \frac{n(n+1)-2a_2}{n^2+(n-2)a_2^2} + \cdots + \frac{n(n+1)-2a_n}{n^2+(n-2)a_n^2} \le n.$$

(Vasile C., 2008)

Solution. The desired inequality is true for $a_1 > \dfrac{n(n+1)}{2}$ since

$$\frac{n(n+1)-2a_1}{n^2+(n-2)a_1^2} < 0$$

and

$$\frac{n(n+1)-2a_i}{n^2+(n-2)a_i^2} < \frac{n}{n-1}, \quad i = 2, 3, \ldots, n.$$

The last inequalities are equivalent to

$$n(n-2)a_i^2 + 2(n-1)a_i + n > 0,$$

which are true because

$$n(n-2)a_i^2 + 2(n-1)a_i + n \ge (n-1)a_i^2 + 2(n-1)a_i + n > (n-1)(a_i+1)^2 \ge 0.$$

Consider further that

$$a_1, a_2, \ldots, a_n \le \frac{n(n+1)}{2},$$

and rewrite the desired inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{2u - n(n+1)}{(n-2)u^2 + n^2}, \quad u \in \mathbb{I} = \left(-\infty, \frac{n(n+1)}{2}\right].$$

We have

$$\frac{f'(u)}{2(n-2)} = \frac{n^2 + n(n+1)u - u^2}{[(n-2)u^2 + n^2]^2}$$

and

$$\frac{f''(u)}{2(n-2)} = \frac{f_1(u)}{[(n-2)u^2 + n^2]^3},$$

where

$$f_1(u) = 2(n-2)u^3 - 3n(n+1)(n-2)u^2 - 2n^2(2n-3)u + n^3(n+1).$$

From the expression of f', it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $\left[s_0, \dfrac{n(n+1)}{2}\right]$, where

$$s_0 = \frac{n}{2}\left(n + 1 - \sqrt{n^2 + 2n + 5}\,\right) \in (-1, 0);$$

therefore,

$$\min_{u\in\mathbb{I}} f(u) = f(s_0).$$

On the other hand, for $-1 \le u \le 1$, we have

$$\begin{aligned} f_1(u) &> -2(n-2) - 3n(n+1)(n-2) - 2n^2(2n-3) + n^3(n+1) \\ &= n^2(n-3)^2 + 4(n+1) > 0, \end{aligned}$$

hence $f''(u) > 0$. Since $[s_0, s] \subset [-1, 1]$, f is convex on $[s_0, s]$. By the LPCF-Theorem and Note 1, we only need to show that $h(x, y) \ge 0$ for $x, y \in \mathbb{R}$ and $x + (n-1)y = n$, where

$$h(x,y) = \frac{g(x)-g(y)}{x-y}, \quad g(u) = \frac{f(u)-f(1)}{u-1}.$$

Indeed, we have

$$g(u) = \frac{(n-2)u+n}{(n-2)u^2+n^2}$$

and

$$\begin{aligned} \frac{h(x,y)}{n-2} &= \frac{n^2 - n(x+y) - (n-2)xy}{[(n-2)x^2+n^2][(n-2)y^2+n^2]} \\ &= \frac{(n-1)(n-2)y^2}{[(n-2)x^2+n^2][(n-2)y^2+n^2]} \ge 0. \end{aligned}$$

The proof is completed. By Note 3, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = n, \qquad a_2 = \cdots = a_n = 0$$

(or any cyclic permutation).

□

P 3.5. *If a, b, c, d are real numbers so that $a + b + c + d = 4$, then*

$$\frac{a(a-1)}{3a^2+4} + \frac{b(b-1)}{3b^2+4} + \frac{c(c-1)}{3c^2+4} + \frac{d(d-1)}{3d^2+4} \ge 0.$$

(Vasile C., 2012)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) \ge 4f(s), \quad s = \frac{a+b+c+d}{4} = 1,$$

where

$$f(u) = \frac{u^2-u}{3u^2+4}, \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{3u^2 + 8u - 4}{(3u^2 + 4)^2},$$

it follows that f is increasing on $\left(-\infty, \frac{-4-2\sqrt{7}}{3}\right] \cup [s_0, \infty)$ and decreasing on $\left[\frac{-4-2\sqrt{7}}{3}, s_0\right]$, where

$$s_0 = \frac{-4+2\sqrt{7}}{3} \approx 0.43.$$

Since

$$\lim_{u \to -\infty} f(u) = \frac{1}{3}$$

and $f(s_0) < 0$, it follows that

$$\min_{u \in \mathbb{R}} f(u) = f(s_0).$$

For $u \in [0, 1]$, we have

$$\begin{aligned} \frac{1}{2} f''(u) &= \frac{-9u^3 - 36u^2 + 36u + 14}{(3u^2 + 4)^3} \\ &= \frac{9u^2(1-u) + 45u(1-u) + (16 - 9u)}{(3u^2 + 4)^3} > 0. \end{aligned}$$

Therefore, f is convex on $[0, 1]$, hence on $[s_0, s]$. According to the LPCF-Theorem and Note 1, we only need to show that $h(x, y) \geq 0$ for $x, y \in \mathbb{R}$ so that $x + 3y = 4$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{u}{3u^2 + 4},$$

$$\begin{aligned} h(x, y) = \frac{g(x) - g(y)}{x - y} &= \frac{4 - 3xy}{(3x^2 + 4)(3y^2 + 4)} \\ &= \frac{(x-2)^2}{(3x^2 + 4)(3y^2 + 4)} \geq 0. \end{aligned}$$

The proof is completed. From $x + 3y = 4$ and $h(x, y) = 0$, we get $x = 2$ and $y = 2/3$. By Note 3, the equality holds for $a = b = c = d = 1$, and also for

$$a = 2, \qquad b = c = d = \frac{2}{3}$$

(or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

- *If $a_1, a_2, \ldots, a_n$ are real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then*

$$\frac{a_1(a_1 - 1)}{4(n-1)a_1^2 + n^2} + \frac{a_2(a_2 - 1)}{4(n-1)a_2^2 + n^2} + \cdots + \frac{a_n(a_n - 1)}{4(n-1)a_n^2 + n^2} \geq 0,$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = \frac{n}{2}, \qquad a_2 = a_3 = \cdots = a_n = \frac{n}{2(n-1)}$$

(or any cyclic permutation).

□

P 3.6. *If a, b, c are real numbers so that $a + b + c = 3$, then*

$$\frac{1}{9a^2 - 10a + 9} + \frac{1}{9b^2 - 10b + 9} + \frac{1}{9c^2 - 10c + 9} \le \frac{3}{8}.$$

(Vasile C., 2015)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) \ge 3f(s), \qquad s = \frac{a+b+c}{3} = 1,$$

where

$$f(u) = \frac{-1}{9u^2 - 10u + 9}, \qquad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{2(9u-5)}{(9u^2 - 10u + 9)^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$ and , where

$$s_0 = \frac{5}{9} < 1 = s.$$

For $u \in [s_0, s] = [5/9, 1]$, we have

$$\begin{aligned} f''(u) &= \frac{2(-243u^2 + 270u - 19)}{(9u^2 - 10u + 9)^3} > \frac{2(-243u^2 + 270u - 27)}{(9u^2 - 10u + 9)^3} \\ &= \frac{54(-9u^2 + 10u - 1)}{(9u^2 - 10u + 9)^3} = \frac{54(1-u)(9u-1)}{(9u^2 - 10u + 9)^3} \ge 0. \end{aligned}$$

Therefore, f is convex on $[s_0, s]$. According to the LPCF-Theorem and Note 1, we only need to show that $h(x, y) \ge 0$ for $x, y \in \mathbb{R}$ so that $x + 2y = 3$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{9u - 1)}{8(9u^2 - 10u + 9)},$$

$$\begin{aligned} h(x, y) = \frac{g(x) - g(y)}{x - y} &= \frac{9(x + y) - 81xy + 71}{8(9x^2 - 10x + 9)(9y^2 - 10y + 9)} \\ &= \frac{2(9y - 7)^2}{8(9x^2 - 10x + 9)(9y^2 - 10y + 9)} \ge 0. \end{aligned}$$

The proof is completed. From $x+2y=3$ and $h(x,y)=0$, we get

$$x=\frac{13}{9},\qquad y=\frac{7}{9}.$$

Thus, the equality holds for $a=b=c=d=1$, and also for

$$a=\frac{13}{9},\qquad b=c=\frac{7}{9}$$

(or any cyclic permutation).

□

P 3.7. *If a,b,c,d are real numbers so that $a+b+c+d=4$, then*

$$\frac{1}{4a^2-5a+4}+\frac{1}{4b^2-5b+4}+\frac{1}{4c^2-5c+4}+\frac{1}{4d^2-5d+4}\le\frac{4}{3}.$$

(Vasile C., 2015)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d)\ge 4f(s),\qquad s=\frac{a+b+c+d}{4}=1,$$

where

$$f(u)=\frac{-1}{4u^2-5u+4},\qquad u\in\mathbb{R}.$$

From

$$f'(u)=\frac{2(8u-5)}{(4u^2-5u+4)^2},$$

it follows that f is decreasing on $(-\infty,s_0]$ and increasing on $[s_0,\infty)$, where

$$s_0=\frac{5}{8}<1=s.$$

For $u\in[s_0,s]=[5/8,1]$, we have

$$\begin{aligned}f''(u)&=\frac{4(-48u^2+60u-9)}{(4u^2-5u+4)^3}>\frac{4(-48u^2+60u-12)}{(4u^2-5u+4)^3}\\&=\frac{48(-4u^2+5u-1)}{(4u^2-5u+4)^3}=\frac{48(1-u)(4u-1)}{(4u^2-5u+4)^3}\ge 0.\end{aligned}$$

Therefore, f is convex on $[s_0,s]$. According to the LPCF-Theorem and Note 1, we only need to show that $h(x,y)\ge 0$ for $x,y\in\mathbb{R}$ so that $x+3y=4$. We have

$$g(u)=\frac{f(u)-f(1)}{u-1}=\frac{4u-1)}{3(4u^2-5u+4)},$$

$$h(x,y)=\frac{g(x)-g(y)}{x-y}=\frac{4(x+y)-16xy+11}{3(4x^2-5x+4)(4y^2-5y+4)}$$
$$=\frac{(4y-3)^2}{(4x^2-5x+4)(4y^2-5y+4)}\geq 0.$$

From $x+3y=4$ and $h(x,y)=0$, we get

$$x=\frac{7}{4},\qquad y=\frac{3}{4}.$$

In accord with Note 3, the equality holds for $a=b=c=1$, and also for

$$a=\frac{7}{4},\qquad b=c=d=\frac{3}{4}$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- *Let $a_1,a_2,\ldots,a_n$ be real numbers so that $a_1+a_2+\cdots+a_n=n$. If*

$$k=1-\frac{2(n-1)}{n^2},$$

then

$$\frac{1}{a_1^2-2ka_1+1}+\frac{1}{a_2^2-2ka_2+1}+\cdots+\frac{1}{a_n^2-2ka_n+1}\geq\frac{n}{2(1-k)},$$

with equality for $a_1=a_2=\cdots=a_n=1$, and also for

$$a_1=\frac{3n^2-6n+4}{n^2},\qquad a_2=a_3=\cdots=a_n=\frac{n^2-2n+4}{n^2}$$

(or any cyclic permutation).

□

P 3.8. *Let $a_1,a_2,\ldots,a_n\neq -k$ be real numbers so that $a_1+a_2+\cdots+a_n=n$, where*

$$k\geq\frac{n}{2\sqrt{n-1}}.$$

Then,

$$\frac{a_1(a_1-1)}{(a_1+k)^2}+\frac{a_2(a_2-1)}{(a_2+k)^2}+\cdots+\frac{a_n(a_n-1)}{(a_n+k)^2}\geq 0.$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\geq nf(s), \quad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=\frac{u(u-1)}{(u+k)^2}, \quad u\in\mathbb{I}=\mathbb{R}\setminus\{-k\}.$$

From

$$f'(u)=\frac{(2k+1)u-k}{(u+k)^3},$$

it follows that f is increasing on $(-\infty,-k)\cup[s_0,\infty)$ and decreasing on $(-k,s_0]$, where

$$s_0=\frac{k}{2k+1}<1=s.$$

Since

$$\lim_{u\to-\infty}f(u)=1$$

and $f(s_0)<0$, we have

$$\min_{u\in\mathbb{I}}f(u)=f(s_0).$$

From

$$\frac{1}{2}f''(u)=\frac{k(k+2)-(2k+1)u}{(u+k)^4},$$

it follows that f is convex on $\left[0,\dfrac{k(k+2)}{2k+1}\right]$, hence on $[s_0,1]$. According to the LPCF-Theorem, Note 4 and Note 1, it suffices to show that $h(x,y)\geq 0$ for all $x,y\in\mathbb{I}$ which satisfy $x+(n-1)y=n$, where

$$h(x,y)=\frac{g(x)-g(y)}{x-y}, \qquad g(u)=\frac{f(u)-f(1)}{u-1}.$$

Indeed, we have

$$g(u)=\frac{u}{(u+k)^2}$$

and

$$\begin{aligned}h(x,y)&=\frac{k^2-xy}{(x+k)^2(y+k)^2}\geq\frac{\frac{n^2}{4(n-1)}-xy}{(x+k)^2(y+k)^2}\\&=\frac{[2(n-1)y-n]^2}{4(n-1)(x+k)^2(y+k)^2}\geq 0.\end{aligned}$$

The equality holds for $a_1=a_2=\cdots=a_n=1$. If $k=\dfrac{n}{2\sqrt{n-1}}$, then the equality holds also for

$$a_1=\frac{n}{2}, \qquad a_2=\cdots=a_n=\frac{n}{2(n-1)}$$

(or any cyclic permutation).

$\square$

P 3.9. *Let $a_1, a_2, \ldots, a_n \neq -k$ be real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If*

$$k \geq 1 + \frac{n}{\sqrt{n-1}},$$

then

$$\frac{a_1^2 - 1}{(a_1 + k)^2} + \frac{a_2^2 - 1}{(a_2 + k)^2} + \cdots + \frac{a_n^2 - 1}{(a_n + k)^2} \geq 0.$$

(Vasile C., 2008)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{u^2 - 1}{(u + k)^2}, \quad u \in \mathbb{I} = \mathbb{R} \setminus \{-k\}.$$

From

$$f'(u) = \frac{2(ku + 1)}{(u + k)^3},$$

it follows that f is increasing on $(-\infty, -k) \cup [s_0, \infty)$ and decreasing on $(-k, s_0]$, where

$$s_0 = \frac{-1}{k} < 0 = s, \quad s_0 > -1.$$

Since

$$\lim_{u \to -\infty} f(u) = 1$$

and $f(s_0) < 0$, we have

$$\min_{u \in \mathbb{I}} f(u) = f(s_0).$$

For $u \in [-1, 1]$, we have

$$f''(u) = \frac{2(k^2 - 3 - 2ku)}{(u + k)^4} \geq \frac{2(k^2 - 3 - 2k)}{(u + k)^4} = \frac{2(k + 1)(k - 3)}{(u + k^4} \geq 0,$$

hence f is convex on $[s_0, 1]$. According to the LPCF-Theorem, Note 4 and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y \in \mathbb{I}$ which satisfy $x + (n - 1)y = n$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{u + 1}{(u + k)^2},$$

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{(k - 1)^2 - 1 - x - y - xy}{(x + k)^2 (y + k)^2} \geq 0,$$

since

$$(k - 1)^2 - 1 - x - y - xy \geq \frac{n^2}{n - 1} - 1 - x - y - xy = \frac{[(n - 1)y - 1]^2}{n - 1} \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $k = 1 + \dfrac{n}{\sqrt{n-1}}$, then the equality holds also for

$$a_1 = n-1, \qquad a_2 = \cdots = a_n = \frac{1}{n-1}$$

(or any cyclic permutation).

□

P 3.10. *Let* a_1, a_2, a_3, a_4, a_5 *be real numbers so that* $a_1 + a_2 + a_3 + a_4 + a_5 \ge 5$. *If*

$$k \in \left[\frac{1}{6}, \frac{25}{14}\right],$$

then

$$\sum \frac{1}{ka_1^2 + a_2 + a_3 + a_4 + a_5} \le \frac{5}{k+4}.$$

(Vasile C., 2006)

Solution. We see that

$$ka_i^2 - a_i + (a_1 + a_2 + a_3 + a_4 + a_5) > \frac{1}{6}a_i^2 - a_i + \frac{3}{2} = \frac{(a_1-3)^2}{6} \ge 0$$

for all $i \in \{1, 2, \ldots, n\}$. Since each term of the left hand side of the inequality decreases by increasing any number a_i, it suffices to consider the case

$$a_1 + a_2 + a_3 + a_4 + a_5 = 5,$$

when the desired inequality can be written as

$$f(a_1) + f(a_2) + f(a_3) + f(a_4) + f(a_5) \ge 5f(s), \qquad s = \frac{a_1 + a_2 + a_3 + a_4 + a_5}{5} = 1,$$

where

$$f(u) = \frac{-1}{ku^2 - u + 5}, \qquad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{2ku - 1}{(ku^2 - u + 5)^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \frac{1}{2k}.$$

We have

$$f''(u) = \frac{2g(u)}{(ku^2 - u + 5)^3}, \qquad g(u) = -3k^2u^2 + 3ku + 5k - 1.$$

For
$$\frac{1}{2} \le k \le \frac{25}{14},$$
we have
$$s_0 = \frac{1}{2k} \le 1 = s,$$
and for $u \in [s_0, s]$, that is
$$\frac{1}{2k} \le u \le 1,$$
we have
$$(1-u)(2ku-1) \ge 0,$$
$$-2ku^2 \ge (2k+1)u + 1,$$
$$-2k^2u^2 \ge k(2k+1)u + k,$$
therefore
$$g(u) \ge \frac{3}{2}[k(2k+1)u + k] + 3ku + 5k - 1 = \frac{-3k(2k-1)u + 13k - 2}{2}$$
$$\ge \frac{-3k(2k-1) + 13k - 2}{2} = -3k^2 + 8k - 1 = 3k(2-k) + (2k-1) > 0.$$
Consequently, f is convex on $[s_0, s]$.

For
$$\frac{1}{6} \le k \le \frac{1}{2},$$
we have
$$s_0 = \frac{1}{2k} \ge 1 = s,$$
and for $u \in [s, s_0]$, that is
$$1 \le u \le \frac{1}{2k},$$
we have
$$g(u) = -3k^2u^2 + 3ku + 5k - 1 \ge 3ku(1-k) + 5k - 1$$
$$\ge 3k(1-k) + 5k - 1 = -3k^2 + 8k - 1$$
$$> -6k^2 + 7k - 1 = (1-k)(6k-1) \ge 0.$$
Consequently, f is convex on $[s, s_0]$.

In both cases, by the PCF-Theorem, it suffices to show that
$$\frac{1}{kx^2 - x + 5} + \frac{4}{ky^2 - y + 5} \le \frac{5}{k+4}$$
for
$$x + 4y = 5, \qquad x, y \in \mathbb{R}.$$

Write this inequality as follows:

$$\frac{1}{k+4}-\frac{1}{kx^2-x+5}+4\left[\frac{1}{k+4}-\frac{1}{ky^2-y+5}\right]\geq 0,$$

$$\frac{(x-1)(kx+k-1)}{kx^2-x+5}+\frac{4(y-1)(ky+k-1)}{ky^2-y+5}\geq 0.$$

Since

$$4(y-1)=1-x,$$

the inequality is equivalent to

$$(x-1)\left(\frac{kx+k-1}{kx^2-x+5}-\frac{ky+k-1}{ky^2-y+5}\right)\geq 0,$$

$$\frac{5(x-1)^2h(x,y)}{4(kx^2-x+5)(ky^2-y+5)}\geq 0,$$

where

$$\begin{aligned}h(x,y)&=-k^2xy-k(k-1)(x+y)+6k-1\\&=4k^2y^2-k(2k+3)y-5k^2+11k-1\\&=\left(2ky-\frac{2k+3}{4}\right)^2+\frac{(25-14k)(6k-1)}{16}\geq 0.\end{aligned}$$

The equality holds for $a_1=a_2=a_3=a_4=a_5=1$. If $k=\dfrac{1}{6}$, then the equality holds also for

$$a_1=-5,\qquad a_2=a_3=a_4=a_5=\frac{5}{2}$$

(or any cyclic permutation). If $k=\dfrac{25}{14}$, then the equality holds also for

$$a_1=\frac{79}{25},\qquad a_2=a_3=a_4=a_5=\frac{23}{50}$$

(or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

• *Let* $a_1,a_2,\ldots,a_n$ *be real numbers so that* $a_1+a_2+\cdots+a_n\leq n$. *If* $k\in[k_1,k_2]$, *where*

$$k_1=\frac{(n-1)(\sqrt{53n^2-54n+101}-5n+11)}{2(7n^2+14n-5)},$$

$$k_2=\frac{2n^2-2n+1+\sqrt{(n-1)(3n^3-4n^2+3n-1)}}{2(n^2-n+1)},$$

then

$$\sum \frac{1}{ka_1^2+a_2+\cdots+a_n} \le \frac{n}{k+n-1},$$

with equality for $a_1=a_2=\cdots=a_n=1$. If $k=k_1$, then the equality holds also for

$$a_1=-n, \qquad a_2=\cdots=a_n=\frac{2n}{n-1}$$

(or any cyclic permutation). If $k=k_2$, then the equality holds also for

$$a_1=\frac{(2k-1)(n-1)+1}{2k}, \qquad a_2=\cdots=a_n=\frac{2k+n-2}{2k(n-1)}$$

(or any cyclic permutation).

□

P 3.11. *Let $a_1,a_2,\ldots,a_5$ be nonnegative numbers so that $a_1+a_2+a_3+a_4+a_5\ge 5$. If $k\in[k_1,k_2]$, where*

$$k_1=\frac{29-\sqrt{761}}{10}\approx 0.1414, \qquad k_2=\frac{25}{14}\approx 1.7857,$$

then

$$\sum \frac{1}{ka_1^2+a_2+a_3+a_4+a_5} \le \frac{5}{k+4}.$$

(Vasile C., 2006)

Solution. Since all terms of the left hand side of the inequality decrease by increasing any number a_i, it suffices to consider the case

$$a_1+a_2+a_3+a_4+a_5=5.$$

The proof is similar to the one of the preceding P 3.10. Having in view P 3.10, it suffices to consider the case

$$k\in\left[k_1,\frac{1}{6}\right],$$

when

$$s_0=\frac{1}{2k}>1=s.$$

For $u\in[s,s_0]$, that is

$$1\le u\le\frac{1}{2k},$$

f is convex because

$$\begin{aligned} g(u) &= -3k^2u^2+3ku+5k-1 \ge 3ku(1-k)+5k-1 \\ &\ge 3k(1-k)+5k-1=-3k^2+8k-1 \\ &> -\frac{15}{4}k^2+87k-1=\frac{(2-k)(15k-2)}{4}>0. \end{aligned}$$

Thus, by the RPCF-Theorem, it suffices to show that

$$\frac{1}{kx^2-x+5}+\frac{4}{ky^2-y+5}\le\frac{5}{k+4}$$

for

$$x+4y=5,\qquad 0\le x\le 1\le y\le\frac{5}{4}.$$

As shown at P 3.10, this inequality is true if $h(x,y)\ge 0$, where

$$h(x,y)=-k^2xy-k(k-1)(x+y)+6k-1.$$

We have

$$\begin{aligned}h(x,y)&=4k^2y^2-k(2k+3)y-5k^2+11k-1\\&=(5-4y)(A-k^2y)+B=x(A-k^2y)+B,\end{aligned}$$

where

$$A=\frac{3k(1-k)}{4},\qquad B=\frac{-5k^2+29k-4}{4}.$$

Since $B\ge 0$, it suffices to show that $A-k^2y\ge 0$. Indeed, we have

$$A-k^2y\ge\frac{3k(1-k)}{4}-\frac{5k^2}{4}=\frac{k(3-8k)}{4}>0.$$

The equality holds for $a_1=a_2=a_3=a_4=a_5=1$. If $k=k_1$, then the equality holds also for

$$a_1=0,\qquad a_2=a_3=a_4=a_5=\frac{5}{4}$$

(or any cyclic permutation). If $k=k_2$, then the equality holds also for

$$a_1=\frac{79}{25},\qquad a_2=a_3=a_4=a_5=\frac{23}{50}$$

(or any cyclic permutatio

Remark. Similarly, we can prove the following generalization:

- *Let $a_1,a_2,\dots,a_n$ be nonnegative real numbers so that $a_1+a_2+\cdots+a_n\le n$. If $k\in[k_1,k_2]$, where*

$$k_1=\frac{n^2+n-1-\sqrt{n^4+2n^3-5n^2+2n+1}}{2n},$$

$$k_2=\frac{2n^2-2n+1+\sqrt{(n-1)(3n^3-4n^2+3n-1)}}{2(n^2-n+1)},$$

then

$$\sum\frac{1}{ka_1^2+a_2+\cdots+a_n}\le\frac{n}{k+n-1},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$. If $k = k_1$, then the equality holds also for

$$a_1 = 0, \qquad a_2 = \cdots = a_n = \frac{n}{n-1}$$

(or any cyclic permutation). If $k = k_2$, then the equality holds also for

$$a_1 = \frac{(2k-1)(n-1)+1}{2k}, \qquad a_2 = \cdots = a_n = \frac{2k+n-2}{2k(n-1)}$$

(or any cyclic permutation).

□

P 3.12. *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n \geq n$. If $k > 1$, then*

$$\frac{1}{a_1^k + a_2 + \cdots + a_n} + \frac{1}{a_1 + a_2^k + \cdots + a_n} + \cdots + \frac{1}{a_1 + a_2 + \cdots + a_n^k} \leq 1.$$

(Vasile C., 2006)

Solution. It suffices to consider the case $a_1 + a_2 + \cdots + a_n = n$, when the desired inequality can be written as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{-1}{u^k - u + n}, \quad u \in [0, n].$$

From

$$f'(u) = \frac{ku^{k-1} - 1}{(u^k - u + n)^2},$$

it follows that f is decreasing on $[0, s_0]$ and increasing on $[s_0, n]$, where

$$s_0 = k^{\frac{1}{1-k}} < 1 = s.$$

We will show that f is convex on $[s_0, 1]$. For $u \in [s_0, 1]$, we have

$$f''(u) = \frac{-k(k+1)u^{2k-2} + k(k+3)u^{k-1} + nk(k-1)u^{k-2} - 2}{(u^k - u + n)^3} > \frac{g(u)}{(u^k - u + n)^3},$$

where

$$g(u) = -k(k+1)u^{2k-2} + k(k+3)u^{k-1} - 2.$$

Denoting

$$t = ku^{k-1}, \qquad 1 \leq t \leq k,$$

we get

$$\begin{aligned} kg(u) &= -(k+1)t^2 + k(k+3)t - 2k \\ &= (k+1)(t-1)(k-t) + (k-1)(t+k) > 0. \end{aligned}$$

By the LPCF-Theorem, it suffices to show that

$$\frac{1}{x^k - x + n} + \frac{n-1}{y^k - y + n} \le 1$$

for $x \ge 1 \ge y \ge 0$ and $x+(n-1)y = n$. Since this inequality is trivial for $x = y = 1$, assume next that $x > 1 > y \ge 0$, and write the desired inequality as follows:

$$x^k - x + n \ge \frac{y^k - y + n}{y^k - y + 1},$$

$$x^k - x \ge \frac{(n-1)(y - y^k)}{y^k - y + 1},$$

$$\frac{x^k - x}{x - 1} \ge \frac{y - y^k}{(1-y)(y^k - y + 1)}.$$

Let $h(x) = \dfrac{x^k - x}{x-1}$, $x > 1$. By the weighted AM-GM inequality, we have

$$h'(x) = \frac{(k-1)x^k + 1 - kx^{k-1}}{(x-1)^2} > 0.$$

Therefore, h is increasing. Since

$$x - 1 = (n-1)(1-y) \ge 1 - y, \qquad x \ge 2 - y > 1,$$

we get

$$h(x) \ge h(2-y) = \frac{(2-y)^k + y - 2}{1-y}.$$

Thus, it suffices to show that

$$(2-y)^k + y - 2 \ge \frac{y - y^k}{y^k - y + 1},$$

which is equivalent to

$$(2-y)^k + y - 1 \ge \frac{1}{y^k - y + 1}.$$

Using the substitution

$$t = 1 - y, \qquad 0 < t \le 1,$$

the inequality becomes

$$(1+t)^k - t \geq \frac{1}{(1-t)^k + t},$$

$$(1-t^2)^k + t(1+t)^k \geq 1 + t^2 + t(1-t)^k.$$

By Bernoulli's inequality,

$$(1-t^2)^k + t(1+t)^k \geq 1 - kt^2 + t(1+kt) = 1 + t.$$

So, we only need to show that

$$1 + t \geq 1 + t^2 + t(1-t)^k,$$

which is equivalent to the obvious inequality

$$t(1-t)\left[1-(1-t)^{k-1}\right] \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark. Using this result, we can formulate the following statement:

• *Let $x_1, x_2, \ldots, x_n$ be nonnegative real numbers so that $x_1 + x_2 + \cdots + x_n \geq n$. If $k > 1$, then*

$$\frac{x_1^k - x_1}{x_1^k + x_2 + \cdots + x_n} + \frac{x_2^k - x_2}{x_1 + x_2^k + \cdots + x_n} + \cdots + \frac{x_n^k - x_n}{x_1 + x_2 + \cdots + x_n^k} \geq 0.$$

This inequality is equivalent to

$$\frac{1}{x_1^k + x_2 + \cdots + x_n} + \frac{1}{x_1 + x_2^k + \cdots + x_n} + \cdots + \frac{1}{x_1 + x_2 + \cdots + x_n^k} \leq \frac{n}{x_1 + x_2 + \cdots + x_n}.$$

Using the substitutions

$$s = \frac{x_1 + x_2 + \cdots + x_n}{n}, \quad s \geq 1,$$

and

$$a_i = \frac{x_i}{s}, \quad i = 1, 2, \ldots, n,$$

which yields $a_1 + a_2 + \cdots + a_n = n$, the desired inequality becomes

$$\sum \frac{1}{s^{k-1}a_1^k + a_2 + \cdots + a_n} \leq 1.$$

Since $s^{k-1} \geq 1$, it suffices to show that

$$\sum \frac{1}{a_1^k + a_2 + \cdots + a_n} \leq 1,$$

which follows immediately from the inequality in P 3.12.

Since $x_1x_2\cdots x_n \ge 1$ involves $x_1+x_2+\cdots+x_n \ge n$, the inequality is also true under the more restrictive condition $x_1x_2\cdots x_n \ge 1$. For $n=3$ and $k=5/2$, we get the inequality from IMO-2005:

- *If x,y,z are nonnegative real numbers so that $xyz \ge 1$, then*

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{x^2+y^5+z^2}+\frac{z^5-z^2}{x^2+y^2+z^5} \ge 0.$$

□

P 3.13. *Let $a_1,a_2,\ldots,a_5$ be nonnegative numbers so that $a_1+a_2+a_3+a_4+a_5 \ge 5$. If*

$$k \in \left[\frac{4}{9}, \frac{61}{5}\right],$$

then

$$\sum \frac{a_1}{ka_1^2+a_2+a_3+a_4+a_5} \le \frac{5}{k+4}.$$

(Vasile C., 2006)

Solution. Using the substitution

$$x_1=\frac{a_1}{s},\ x_2=\frac{a_2}{s},\ x_3=\frac{a_3}{s},\ x_4=\frac{a_4}{s},\ x_5=\frac{a_5}{s},$$

where

$$s=\frac{a_1+a_2+a_3+a_4+a_5}{5} \ge 1,$$

we need to show that $x_1+x_2+x_3+x_4+x_5=5$ involves

$$\frac{x_1}{ksx_1^2+x_2+x_3+x_4+x_5}+\cdots+\frac{x_5}{x_1+x_2+x_3+x_4+ksx_5^2} \le \frac{5}{k+4}.$$

Since $s \ge 1$, it suffices to prove the inequality for $s=1$; that is, to show that

$$\frac{a_1}{ka_1^2-a_1+5}+\frac{a_2}{ka_2^2-a_1+5}+\cdots+\frac{a_5}{ka_5^2-a_n+5} \le \frac{5}{k+4}$$

for

$$a_1+a_2+a_3+a_4+a_5=5.$$

Write the desired inequality as

$$f(a_1)+f(a_2)+f(a_3)+f(a_4)+f(a_5) \ge 5f(s),$$

where
$$s=\frac{a_1+a_2+a_3+a_4+a_5}{5}=1$$
and
$$f(u)=\frac{-u}{ku^2-u+5}, \quad u\in[0,5].$$
From
$$f'(u)=\frac{ku^2-5}{(ku^2-u+5)^2},$$
it follows that f is decreasing on $[0,s_0]$ and increasing on $[s_0,5]$, where
$$s_0=\sqrt{\frac{5}{k}}.$$
We have
$$f''(u)=\frac{2g(u)}{(u^2-u+5)^3}, \qquad g(u)=-k^2u^3+15ku-5, \qquad g'(u)=3k(5-ku^2).$$
Case 1: $\dfrac{4}{9}\le k\le 5$. We have
$$s_0=\sqrt{\frac{5}{k}}\ge 1=s.$$
For $u\in[1,s_0]$, the derivative g' is nonnegative, g is increasing, hence
$$g(u)\ge g(1)=-k^2+15k-5=\left(k-\frac{4}{9}\right)(5-k)+\frac{86k-25}{9}>0.$$
Consequently, $f''(u)>0$ for $u\in[1,s_0]$, hence f is convex on $[s,s_0]$.

Case 2: $5\le k\le\dfrac{61}{5}$. We have
$$s_0=\sqrt{\frac{5}{k}}<1=s.$$
For $u\in[s_0,1]$, we have $g'(u)\le 0$, $g(u)$ is decreasing, hence
$$g(u)\ge g(1)=-k^2+15k-5=(k-1)(13-k)+k+8>0.$$
Consequently, $f''(u)>0$ for $u\in[s_0,1]$, hence f is convex on $[s_0,s]$.

In both cases, by the PCF-Theorem, it suffices to show that
$$\frac{x}{kx^2-x+5}+\frac{4y}{ky^2-y+5}\le\frac{5}{k+4}$$

for

$$x + 4y = 5, \qquad x, y \geq 0.$$

Write this inequality as follows:

$$\frac{1}{k+4} - \frac{x}{kx^2 - x + 5} + 4\left[\frac{1}{k+4} - \frac{y}{ky^2 - y + 5}\right] \geq 0,$$

$$\frac{(x-1)(kx-5)}{kx^2 - x + 5} + \frac{4(y-1)(ky-5)}{ky^2 - y + 5} \geq 0.$$

Since

$$4(y-1) = 1 - x,$$

the inequality is equivalent to

$$(x-1)\left(\frac{kx-5}{kx^2 - x + 5} - \frac{ky-5}{ky^2 - y + 5}\right) \geq 0,$$

$$\frac{(x-1)^2 h(x,y)}{(kx^2 - x + 5)(ky^2 - y + 5)} \geq 0,$$

where

$$\begin{aligned} h(x,y) &= -k^2xy + 5k(x+y) + 5k - 5 \\ &= 4k^2y^2 - 5k(k+3)y + 5(6k-1). \end{aligned}$$

We need to show that $h(x,y) \geq 0$ for $k \in \left[\frac{4}{9}, \frac{61}{5}\right]$. For $k \in \left[\frac{4}{9}, 1\right]$, we have

$$\begin{aligned} h(x,y) &= (5-4y)\left(-k^2y + \frac{15k}{4}\right) + \frac{5(9k-4)}{4} \\ &= \frac{kx(15-4ky)}{4} + \frac{5(9k-4)}{4} \\ &= \frac{kx(kx+15-5k)}{4} + \frac{5(9k-4)}{4} \geq 0, \end{aligned}$$

while for $k \in \left[1, \frac{61}{5}\right]$, we have

$$h(x,y) = \left(2ky - \frac{5k+15}{4}\right)^2 + \frac{(61-5k)(k-1)}{16} \geq 0.$$

The equality holds for $a_1 = a_2 = a_3 = a_4 = a_5 = 1$. If $k = \frac{4}{9}$, then the equality holds also for

$$a_1 = 0, \qquad a_2 = a_3 = a_4 = a_5 = \frac{5}{4}$$

(or any cyclic permutation). If $k = \dfrac{61}{5}$, then the equality holds also for

$$a_1 = \frac{115}{61}, \qquad a_2 = a_3 = a_4 = a_5 = \frac{95}{122}$$

(or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

• *Let* $a_1, a_2, \dots, a_n$ *be real numbers so that* $a_1 + a_2 + \cdots + a_n \le n$. *If* $k \in [k_1, k_2]$, *where*

$$k_1 = \frac{n-1}{2n-1},$$

$$k_2 = \frac{n^2 + 2n - 2 + 2\sqrt{(n-1)(2n^2-1)}}{n},$$

then

$$\sum \frac{a_1}{ka_1^2 + a_2 + \cdots + a_n} \le \frac{n}{k+n-1},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$. *If* $k = k_1$, *then the equality holds also for*

$$a_1 = 0, \qquad a_2 = a_3 = a_4 = a_5 = \frac{n}{n-1}$$

(or any cyclic permutation). If $k = k_2$, *then the equality holds also for*

$$a_1 = \frac{n(k-n+2)}{2k}, \qquad a_2 = \cdots = a_n = \frac{n(k+n-2)}{2k(n-1)}$$

(or any cyclic permutation).

□

P 3.14. *Let* $a_1, a_2, \dots, a_n$ *be nonnegative real numbers so that* $a_1 + a_2 + \cdots + a_n \ge n$. *If* $k > 1$, *then*

$$\frac{a_1}{a_1^k + a_2 + \cdots + a_n} + \frac{a_2}{a_1 + a_2^k + \cdots + a_n} + \cdots + \frac{a_n}{a_1 + a_2 + \cdots + a_n^k} \le 1.$$

(Vasile C., 2006)

Solution. Using the substitution

$$x_1 = \frac{a_1}{s},\ x_2 = \frac{a_2}{s},\ \dots,\ x_n = \frac{a_n}{s},$$

where

$$s = \frac{a_1 + a_2 + \cdots + a_n}{n} \ge 1,$$

we need to show that $x_1 + x_2 + \cdots + x_n = n$ involves

$$\frac{x_1}{s^{k-1}x_1^k + x_2 + \cdots + x_n} + \cdots + \frac{x_n}{x_1 + x_2 + \cdots + s^{k-1}x_n^k} \leq 1.$$

Since $s^{k-1} \geq 1$, it suffices to prove the inequality for $s = 1$; that is, to show that

$$\frac{a_1}{a_1^k - a_1 + n} + \frac{a_2}{a_2^k - a_2 + n} + \cdots + \frac{a_n}{a_n^k - a_n + n} \leq 1$$

for

$$a_1 + a_2 + \cdots + a_n = n.$$

Case 1: $1 < k \leq n+1$. By Bernoulli's inequality, we have

$$a_1^k \geq 1 + k(a_1 - 1), \qquad a_1^k - a_1 + n \geq (k-1)a_1 + n - k + 1.$$

Thus, it suffices to show that

$$\sum \frac{a_1}{(k-1)a_1 + n - k + 1} \leq 1.$$

This is an equality for $k = n - 1$. If $1 < k < n + 1$, then the inequality is equivalent to

$$\sum \frac{1}{(k-1)a_1 + n - k + 1} \geq 1,$$

which follows from the the AM-HM inequality

$$\sum \frac{1}{(k-1)a_1 + n - k + 1} \geq \frac{n^2}{\sum[(k-1)a_1 + n - k + 1]}.$$

Case 2: $k > n + 1$. Write the desired inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{-u}{u^k - u + n}, \qquad u \in [0, n].$$

We have

$$f'(u) = \frac{(k-1)u^k - n}{(u^k - u + n)^2}$$

and

$$f''(u) = \frac{f_1(u)}{(u^k - u + n)^3},$$

where

$$f_1(u) = k(k-1)u^{k-1}(u^k - u + n) - 2(ku^{k-1} - 1)[(k-1)u^k - n].$$

From the expression of f', it follows that f is decreasing on $[0,s_0]$ and increasing on $[s_0,n]$, where

$$s_0=\left(\frac{n}{k-1}\right)^{1/k}<1=s.$$

For $u\in[s_0,1]$, we have

$$(k-1)u^k-n\ge(k-1)s_0^k-n=0,$$

hence

$$\begin{aligned}f_1(u)&\ge k(k-1)u^{k-1}(u^k-u+n)-2ku^{k-1}[(k-1)u^k-n]\\&=ku^{k-1}[-(k-1)(u^k+u)+n(k+1)]\\&\ge ku^{k-1}[-2(k-1)+2(k+1)]=4ku^{k-1}>0.\end{aligned}$$

Since $f''(u)>0$, it follows that f is convex on $[s_0,s]$. By the LPCF-Theorem, we need to show that

$$f(x)+(n-1)f(y)\ge nf(1)$$

for

$$x\ge1\ge y\ge0,\qquad x+(n-1)y=n.$$

Consider the nontrivial case where $x>1>y\ge0$ and write the required inequality as follows:

$$\frac{x}{x^k-x+n}+\frac{(n-1)y}{y^k-y+n}\le1,$$

$$x^k-x+n\ge\frac{x(y^k-y+n)}{y^k-ny+n},$$

$$x^k-x\ge\frac{(n-1)y(y-y^k)}{y^k-ny+n}.$$

Since $y<1$ and $y^k-ny+n>y^k-y+1$, it suffices to show that

$$x^k-x\ge\frac{(n-1)(y-y^k)}{y^k-y+1},$$

which has been proved at P 3.12.

The equality holds for $a_1=a_2=\cdots=a_n=1$.

□

P 3.15. *Let $a_1,a_2,\ldots,a_n$ be nonnegative real numbers so that $a_1+a_2+\cdots+a_n\le n$. If $k\ge1-\frac{1}{n}$, then*

$$\frac{1-a_1}{ka_1^2+a_2+\cdots+a_n}+\frac{1-a_2}{a_1+ka_2^2+\cdots+a_n}+\cdots+\frac{1-a_n}{a_1+a_2+\cdots+ka_n^2}\ge0.$$

(Vasile C., 2006)

Solution. Let
$$s = \frac{a_1 + a_2 + \cdots + a_n}{n}, \qquad s \le 1.$$
We have three cases to consider.

Case 1: $s \le \frac{1}{n}$. The inequality is trivial because
$$a_i \le a_1 + a_2 + \cdots + a_n = ns \le 1$$
for $i = 1, 2, \ldots, n$.

Case 2: $\frac{1}{n} < s < 1$. Without loss of generality, assume that
$$a_1 \le \cdots \le a_j < 1 \le a_{j+1} \cdots \le a_n, \qquad j \in \{1, 2, \ldots, n\}.$$
Clearly, there are $b_1, b_2, \ldots, b_n$ so that $b_1 + b_2 + \cdots + b_n = n$ and
$$a_1 \le b_1 \le 1,\ \ldots,\ a_j \le b_j \le 1,\ b_{j+1} = a_{j+1},\ \ldots,\ b_n = a_n.$$
Write the desired inequality as
$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge 0,$$
where
$$f(u) = \frac{1-u}{ku^2 - u + ns}, \qquad u \in [0, ns].$$
For $u \in [0, 1]$, we have
$$f'(u) = \frac{k[(1-u)^2 - 1] + (1 - ns)}{(ku^2 - u + ns)^2} < 0,$$
hence f is strictly decreasing on $[0, 1]$ and
$$f(b_1) \le f(a_1),\ \ldots,\ f(b_j) \le f(a_j),\ f(b_{j+1}) = f(a_{j+1}),\ \ldots,\ f(b_n) = f(a_n).$$
Since
$$f(b_1) + f(b_2) + \cdots + f(b_n) \le f(a_1) + f(a_2) + \cdots + f(a_n),$$
it suffices to show that $f(b_1) + f(b_2) + \cdots + f(b_n) \ge 0$ for $b_1 + b_2 + \cdots + b_n = n$. This inequality is proved at Case 3.

Case 3: $s = 1$. Write the inequality as
$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \qquad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$
where
$$f(u) = \frac{1-u}{ku^2 - u + n}, \quad u \in [0, n].$$

From
$$f'(u) = \frac{k[(u-1)^2-1]-(n-1)}{(ku^2-u+n)^2},$$
it follows that f is decreasing on $[0, s_0]$ and increasing on $[s_0, n]$, where
$$s_0 = 1 + \sqrt{1+\frac{n-1}{k}} > 1 = s, \quad s_0 < n.$$
We will show that f is convex on $[1, s_0]$. We have
$$f''(u) = \frac{2g(u)}{(ku^2-u+n)^3},$$
where
$$g(u) = -k^2u^3 + 3k^2u^2 + 3k(n-1)u - kn - n + 1, \quad g'(u) = 3k(-ku^2 + 2ku + n - 1).$$
For $u \in [1, s_0]$, we have $g'(u) \ge 0$, g is increasing, therefore
$$\begin{aligned} g(u) \ge g(1) &= 2k^2 + (2n-3)k - n + 1 \\ &\ge \frac{2(n-1)^2}{n^2} + \frac{(2n-3)(n-1)}{n} - n + 1 \\ &= \frac{(n^2-1)(n-2)}{n^2} \ge 0, \end{aligned}$$
$f''(u) \ge 0$, $f(u)$ is convex for $u \in [s, s_0]$. By the RPCF-Theorem, it suffices to show that
$$\frac{1-x}{kx^2-x+n} + \frac{(n-1)(1-y)}{ky^2-y+n} \ge 0$$
for $0 \le x \le 1 \le y$ and $x + (n-1)y = n$. Since $(n-1)(1-y) = x - 1$, we have
$$\begin{aligned} \frac{1-x}{kx^2-x+n} + \frac{(n-1)(1-y)}{ky^2-y+n} &= (x-1)\left(-\frac{1}{kx^2-x+n} + \frac{1}{ky^2-y+n}\right) \\ &= \frac{(x-1)(x-y)(kx+ky-1)}{(kx^2-x+n)(ky^2-y+n)} \\ &= \frac{n(x-1)^2(kx+ky-1)}{(n-1)(kx^2-x+n)(ky^2-y+n)} \ge 0, \end{aligned}$$
because
$$k(x+y) - 1 \ge \frac{n-1}{n}(x+y) - 1 = \frac{(n-2)x}{n} \ge 0.$$
The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $k = 1 - \frac{1}{n}$, then the equality holds also for
$$a_1 = 0, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n-1}$$

(or any cyclic permutation).

Remark. For $k=1$, we get the following statement:

- *If $a_1,a_2,\dots,a_n$ are nonnegative real numbers so that $a_1+a_2+\cdots+a_n\le n$, then*

$$\frac{1-a_1}{a_1^2+a_2+\cdots+a_n}+\frac{1-a_2}{a_1+a_2^2+\cdots+a_n}+\cdots+\frac{1-a_n}{a_1+a_2+\cdots+a_n^2}\ge 0.$$

with equality for $a_1=a_2=\cdots=a_n=1$.

□

P 3.16. *Let $a_1,a_2,\dots,a_n$ be nonnegative real numbers so that $a_1+a_2+\cdots+a_n\le n$. If $k\ge 1-\dfrac{1}{n}$, then*

$$\frac{1-a_1}{1-a_1+ka_1^2}+\frac{1-a_2}{1-a_2+ka_2^2}+\cdots+\frac{1-a_n}{1-a_n+ka_n^2}\ge 0.$$

(Vasile C., 2006)

Solution. The proof is similar to the one of the preceding P 3.15. For the case 3, we need to show that

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf(s),\qquad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=\frac{1-u}{1-u+ku^2},\qquad u\in[0,n].$$

From

$$f'(u)=\frac{ku(u-2)}{(1-u+ku^2)^2},$$

it follows that f is decreasing on $[0,s_0]$ and increasing on $[s_0,n]$, where

$$s_0=2>s.$$

We will show that f is convex on $[1,s_0]$. For $u\in[1,s_0]$, we have

$$f''(u)=\frac{2kg(u)}{(1-u+ku^2)^3},\qquad g(u)=-ku^3+3ku^2-1.$$

Since

$$g'(u)=3ku(2-u)\ge 0,$$

g is increasing, $g(u)\ge g(1)=2k-1\ge 0$, hence $f''(u)\ge 0$ for $u\in[1,s_0]$. By the RPCF-Theorem, it suffices to show that

$$\frac{1-x}{1-x+kx^2}+\frac{(n-1)(1-y)}{1-y+ky^2}\ge 0$$

for $0 \le x \le 1 \le y$ and $x+(n-1)y = n$. Since $(n-1)(1-y) = x-1$, we have

$$\begin{aligned}\frac{1-x}{1-x+kx^2}+\frac{(n-1)(1-y)}{1-y+ky^2} &= (1-x)\left(\frac{1}{1-x+kx^2}-\frac{1}{1-y+ky^2}\right)\\ &= \frac{(1-x)(y-x)(kx+ky-1)}{(1-x+kx^2)(1-y+ky^2)}\\ &= \frac{n(x-1)^2(kx+ky-1)}{(n-1)(1-x+kx^2)(1-y+ky^2)}.\end{aligned}$$

Since

$$k(x+y)-1 \ge \frac{n-1}{n}(x+y)-1 = \frac{(n-2)x}{n} \ge 0,$$

the conclusion follows. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $k = 1-\frac{1}{n}$, then the equality holds also for

$$a_1 = 0, \qquad a_2 = a_3 = \cdots = a_n = \frac{n}{n-1}$$

(or any cyclic permutation).

□

P 3.17. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that $a_1+a_2+\cdots+a_n = n$. If $0 < k \le \frac{n}{n-1}$, then*

$$a_1^{k/a_1}+a_2^{k/a_2}+\cdots+a_n^{k/a_n} \le n.$$

(Vasile C., 2006)

Solution. According to the power mean inequality, we have

$$\left(\frac{a_1^{p/a_1}+a_2^{p/a_2}+\cdots+a_n^{p/a_n}}{n}\right)^{1/p} \ge \left(\frac{a_1^{q/a_1}+a_2^{q/a_2}+\cdots+a_n^{q/a_n}}{n}\right)^{1/q}$$

for all $p \ge q > 0$. Thus, it suffices to prove the desired inequality for

$$k = \frac{n}{n-1}, \qquad 1 < k \le 2.$$

Rewrite the desired inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n) \ge nf(s), \qquad s = \frac{a_1+a_2+\cdots+a_n}{n} = 1,$$

where

$$f(u) = -u^{k/u}, \qquad u \in \mathbb{I} = (0,n).$$

We have

$$f'(u) = ku^{\frac{k}{u}-2}(\ln u - 1),$$

$$f''(u) = ku^{\frac{k}{u}-4}[u + (1-\ln u)(2u - k + k\ln u)].$$

For $n = 2$, when $k = 2$ and $\mathbb{I} = (0,2)$, f is convex on $[1,2)$ because

$$1 - \ln u > 0, \quad 2u - k + k\ln u = 2u - 2 + 2\ln u \geq 2u - 2 \geq 0.$$

Therefore, we may apply the RHCF-Theorem. Consider now that $n \geq 3$. From the expression of f', it follows that f is decreasing on $(0, s_0]$ and increasing on $[s_0, n)$, where

$$s_0 = e > 1 = s.$$

In addition, we claim that f is convex on $[1, s_0]$. Indeed, since

$$1 - \ln u \geq 0, \quad 2u - k + k\ln u \geq 2 - k > 0,$$

we have $f'' > 0$ for $u \in [1, s_0]$. Therefore, by the RHCF-Theorem (for $n = 2$) and the RPCF-Theorem (for $n \geq 3$), we only need to show that

$$x^{k/x} + (n-1)y^{k/y} \leq n$$

for

$$0 < x \leq 1 \leq y, \quad x + (n-1)y = n.$$

We have

$$\frac{k}{x} \geq k > 1.$$

Also, from

$$\frac{k}{y} = \frac{n}{(n-1)y} > \frac{n}{x + (n-1)y} = 1, \qquad \frac{k}{y} = \frac{n}{(n-1)y} \leq \frac{2}{y} \leq 2,$$

we get

$$0 < \frac{k}{y} - 1 \leq 1.$$

Therefore, by Bernoulli's inequality, we have

$$\begin{aligned}
x^{k/x} + (n-1)y^{k/y} - n &= \frac{1}{\left(\frac{1}{x}\right)^{k/x}} + (n-1)y \cdot y^{k/y-1} - n \\
&\leq \frac{1}{1 + \frac{k}{x}\left(\frac{1}{x} - 1\right)} + (n-1)y\left[1 + \left(\frac{k}{y} - 1\right)(y-1)\right] - n \\
&= \frac{x^2}{x^2 - kx + k} - (k-1)x^2 - (2-k)x \\
&= \frac{-x(x-1)^2[(k-1)x + k(2-k)]}{x^2 - kx + k} \leq 0.
\end{aligned}$$

The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 3.18. *If a,b,c,d,e are nonzero real numbers so that $a+b+c+d+e=5$, then*

$$\left(7-\frac{5}{a}\right)^2+\left(7-\frac{5}{b}\right)^2+\left(7-\frac{5}{c}\right)^2+\left(7-\frac{5}{d}\right)^2+\left(7-\frac{5}{e}\right)^2 \ge 20.$$

(Vasile C., 2012)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d)+f(e)\ge 5f(s), \qquad s=\frac{a+b+c+d+e}{5}=1,$$

where

$$f(u)=\left(7-\frac{5}{u}\right)^2, \qquad u\in\mathbb{I}=\mathbb{R}\setminus\{0\}.$$

From

$$f'(u)=\frac{10(7u-5)}{u^3},$$

it follows that f is increasing on $(-\infty,0)\cup[s_0,\infty)$ and decreasing on $(0,s_0]$, where

$$s_0=\frac{5}{7}<1=s.$$

Since

$$\lim_{u\to-\infty} f(u)=49$$

and $f(s_0)=0$, we have

$$\min_{u\in\mathbb{I}} f(u)=f(s_0).$$

Also, f is convex on $[s_0,s]=[5/7,1]$ because

$$f''(u)=\frac{10(15-14u)}{u^4}>0.$$

According to the LPCF-Theorem and Note 4, we only need to show that

$$f(x)+4f(y)\ge 5f(1)$$

for all nonzero real x,y so that $x+4y=5$. Using Note 1, it suffices to prove that $h(x,y)\ge 0$, where

$$h(x,y)=\frac{g(x)-g(y)}{x-y}, \quad g(u)=\frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u)=5\left(\frac{9}{u}-\frac{5}{u^2}\right),$$

$$h(x,y)=\frac{5(5x+5y-9xy)}{x^2y^2}=\frac{5(6y-5)^2}{x^2y^2}\ge 0.$$

In accordance with Note 3, the equality holds for $a = b = c = d = e = 1$, and also for

$$a = \frac{5}{3}, \qquad b = c = d = e = \frac{5}{6}$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- *Let $a_1, a_2, \ldots, a_n$ be nonzero real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If $k = \dfrac{n}{n+\sqrt{n-1}}$, then*

$$\left(1-\frac{k}{a_1}\right)^2 + \left(1-\frac{k}{a_2}\right)^2 + \cdots + \left(1-\frac{k}{a_n}\right)^2 \ge n(1-k)^2,$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = \frac{n}{1+\sqrt{n-1}}, \qquad a_2 = a_3 = \cdots = a_n = \frac{n}{n-1+\sqrt{n-1}}$$

(or any cyclic permutation).

□

P 3.19. *If $a_1, a_2, \ldots, a_7$ are real numbers so that $a_1 + a_2 + \cdots + a_7 = 7$, then*

$$(a_1^2+2)(a_2^2+2)\cdots(a_7^2+2) \ge 3^7.$$

(Vasile C., 2007)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_7) \ge 7f(s), \qquad s = \frac{a_1 + a_2 + \cdots + a_7}{7} = 1,$$

where

$$f(u) = \ln(u^2+2), \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{2u}{u^2+2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty]$, where

$$s_0 = 0.$$

From

$$f''(u) = \frac{2(2-u^2)}{(u^2+2)^2},$$

it follows that $f''(u) > 0$ for $u \in [0,1]$, therefore f is convex on $[s_0, s]$. By the LPCF-Theorem, it suffices to prove that

$$f(x) + 6f(y) \ge 7f(1)$$

for $x, y \in \mathbb{R}$ so that $x + 6y = 7$. The inequality can be written as $g(y) \ge 0$, where

$$g(y) = \ln[(7-6y)^2 + 2] + 6\ln(y^2+2) - 7\ln 3, \qquad y \in \mathbb{R}.$$

From

$$\begin{aligned} g'(y) &= \frac{4(6y-7)}{12y^2 - 28y + 17} + \frac{12y}{y^2+2} \\ &= \frac{28(6y^3 - 13y^2 + 9y - 2)}{(12y^2 - 28y + 17)(y^2+2)} \\ &= \frac{28(2y-1)(3y-2)(y-1)}{(12y^2 - 28y + 17)(y^2+2)}, \end{aligned}$$

it follows that g is decreasing on $\left(-\infty, \frac{1}{2}\right] \cup \left[\frac{2}{3}, 1\right]$ and increasing on $\left[\frac{1}{2}, \frac{2}{3}\right] \cup [1, \infty)$; therefore,

$$g \ge \min\{g(1/2), g(1)\}.$$

Since $g(1) = 0$, we only need to show that $g(1/2) \ge 0$; that is, to show that $x = 4$ and $y = 1/2$ involve

$$(x^2+2)(y^2+2)^6 \ge 3^7.$$

Indeed, we have

$$(x^2+2)(y^2+2)^6 - 3^7 = 3^7\left(\frac{3^7}{2^{11}} - 1\right) = \frac{139 \cdot 3^7}{2^{11}} > 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_7 = 1$.

□

P 3.20. *Let $a_1, a_2, \ldots, a_n$ be real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If $k \ge \frac{n^2}{4(n-1)}$, then*

$$(a_1^2 + k)(a_2^2 + k)\cdots(a_n^2 + k) \ge (1+k)^n.$$

(Vasile C., 2007)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \qquad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \ln(u^2 + k), \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{2u}{u^2 + k},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty]$, where

$$s_0 = 0.$$

From

$$f''(u) = \frac{2(k - u^2)}{(u^2 + k)^2},$$

it follows that $f''(u) \ge 0$ for $u \in [0, 1]$, therefore f is convex on $[s_0, s]$. By the LPCF-Theorem and Note 2, it suffices to prove that $H(x, y) \ge 0$ for $x, y \in \mathbb{R}$ so that $x + (n-1)y = n$, where

$$H(x, y) = \frac{f'(x) - f'(y)}{x - y}.$$

We have

$$\begin{aligned}
\frac{1}{2}H(x, y) &= \frac{k - xy}{(x^2 + k)(y^2 + k)} \\
&\ge \frac{n^2 - 4(n-1)xy}{4(n-1)(x^2 + k)(y^2 + k)}, \\
&= \frac{[x + (n-1)y]^2 - 4(n-1)xy}{4(n-1)(x^2 + k)(y^2 + k)} \\
&= \frac{[x - (n-1)y)]^2}{4(n-1)(x^2 + k)(y^2 + k)} \ge 0.
\end{aligned}$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 3.21. *Let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 + a_2 + \cdots + a_n = n$. If $n \le 10$, then*

$$(a_1^2 - a_1 + 1)(a_2^2 - a_2 + 1) \cdots (a_n^2 - a_n + 1) \ge 1.$$

(Vasile C., 2007)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1$$

where

$$f(u) = \ln(u^2 - u + 1), \quad u \in \mathbb{R}.$$

From
$$f'(u) = \frac{2u-1}{u^2-u+1},$$
it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where
$$s_0 = \frac{1}{2} < 1 = s.$$
In addition, from
$$f''(u) = \frac{1+2u(1-u)}{(u^2-u+1)^2},$$
it follows that $f''(u) > 0$ for $u \in [s_0, 1]$, hence f is convex on $[s_0, s]$. According to LPCF-Theorem, we only need to show that
$$f(x) + (n-1)f(y) \ge nf(1)$$
for all real x, y so that $x + (n-1)y = n$. Write this inequality as $g(x) \ge 0$, where
$$g(x) = \ln(x^2 - x + 1) + (n-1)\ln(y^2 - y + 1), \qquad y = \frac{n-x}{n-1}.$$
Since $y'(x) = \dfrac{-1}{n-1}$, we have
$$g'(x) = \frac{2x-1}{x^2-x+1} + (n-1)y'\frac{2y-1}{y^2-y+1} = \frac{2x-1}{x^2-x+1} - \frac{2y-1}{y^2-y+1}$$
$$= \frac{(x-y)(1+x+y-2xy)}{(x^2-x+1)(y^2-y+1)} = \frac{(x-1)[2x^2-(n+2)x+2n-1]}{(n-1)^2(x^2-x+1)(y^2-y+1)}.$$
Because $2x^2-(n+2)x+2n-1 > 0$ for $n \le 10$, we have $g'(x) \le 0$ for $x \in (-\infty, 1]$ and $g'(x) \ge 0$ for $x \in [1, \infty)$. Therefore, since $g(x)$ is decreasing on $(-\infty, 1]$ and increasing on $[1, \infty)$, we have
$$g(x) \ge g(1) = 0.$$
The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$.

Remark 1. The inequality holds also for $n = 11$, $n = 12$ and $n = 13$, when the equation
$$2x^2 - (n+2)x + 2n - 1 = 0$$
has two positive roots, namely
$$x_1 = \frac{n+2-\sqrt{n^2-12(n-1)}}{4}, \qquad x_2 = \frac{n+2+\sqrt{n^2-12(n-1)}}{4},$$
satisfying $1 < x_1 < x_2$. Thus, $g(x)$ is decreasing on $(-\infty, 1] \cup [x_1, x_2]$ and increasing on $[1, x_1] \cup [x_2, \infty)$. Therefore, it suffices to show that
$$\min\{g(1), g(x_2)\} \ge 0.$$

We have $g(1) = 0$. For $n = 13$, we have

$$x_2 = 5, \qquad y_2 = \frac{13 - x_2}{12} = \frac{2}{3},$$

hence

$$g(x_2) = \ln(x_2^2 - x_2 + 1) + (n-1)\ln(y_2^2 - y_2 + 1) = \ln 21 + 12 \cdot \ln\frac{7}{9} = \ln\frac{7^{13}}{3^{23}} > 0.$$

For $n = 14$, the inequality does not hold.

Remark 2. By replacing $a_1, a_2, \ldots, a_n$ respectively with $1-a_1, 1-a_2, \ldots, 1-a_n$, we get the following statement:

- *Let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 + a_2 + \cdots + a_n = 0$. If $n \le 13$, then*

$$(1 - a_1 + a_1^2)(1 - a_2 + a_2^2)\cdots(1 - a_n + a_n^2) \ge 1,$$

with equality for $a_1 = a_2 = \cdots = a_n = 0$.

$\square$

P 3.22. *Let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 + a_2 + \cdots + a_n = n$. If $n \le 26$, then*

$$(a_1^2 - a_1 + 2)(a_2^2 - a_2 + 2)\cdots(a_n^2 - a_n + 2) \ge 2^n.$$

(Vasile C., 2007)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \qquad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \ln(u^2 - u + 2), \qquad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{2u - 1}{u^2 - u + 2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \frac{1}{2} < 1 = s.$$

In addition, from

$$f''(u) = \frac{3 + 2u(1-u)}{(u^2 - u + 2)^2},$$

it follows that $f''(u) > 0$ for $u \in [s_0, 1]$, hence f is convex on $[s_0, s]$. According to LPCF-Theorem, we only need to show that

$$f(x) + (n-1)f(y) \ge nf(1)$$

for all real x, y so that $x+(n-1)y = n$. Write this inequality as $g(x) \geq 0$, where

$$g(x) = \ln(x^2 - x + 2) + (n-1)\ln(y^2 - y + 2), \qquad y = \frac{n-x}{n-1}.$$

Since $y'(x) = \dfrac{-1}{n-1}$, we have

$$g'(x) = \frac{2x-1}{x^2-x+2} + (n-1)y'\frac{2y-1}{y^2-y+2} = \frac{2x-1}{x^2-x+2} - \frac{2y-1}{y^2-y+2}$$

$$= \frac{(x-y)(3+x+y-2xy)}{(x^2-x+2)(y^2-y+2)} = \frac{(x-1)[2x^2-(n+2)x+4n-3]}{(n-1)^2(x^2-x+1)(y^2-y+1)}.$$

Because $2x^2-(n+2)x+4n-3 > 0$ for $n \leq 26$, we have $g'(x) \leq 0$ for $x \in (-\infty, 1]$ and $g'(x) \geq 0$ for $x \in [1, \infty)$. Therefore, since $g(x)$ is decreasing on $(-\infty, 1]$ and increasing on $[1, \infty)$, we have

$$g(x) \geq g(1) = 0.$$

The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$.

Remark 1. The inequality holds also for $27 \leq n \leq 38$, when the equation

$$2x^2 - (n+2)x + 4n - 3 = 0$$

has two positive roots, namely

$$x_1 = \frac{n+2-\sqrt{n^2-28(n-1)}}{4}, \qquad x_2 = \frac{n+2+\sqrt{n^2-28(n-1)}}{4},$$

satisfying $1 < x_1 < x_2$. Thus, $g(x)$ is decreasing on $(-\infty, 1] \cup [x_1, x_2]$ and increasing on $[1, x_1] \cup [x_2, \infty)$. Therefore, it suffices to show that

$$\min\{g(1), g(x_2)\} \geq 0.$$

We have $g(1) = 0$ and $g(x_2) > 0$ for $27 \leq n \leq 38$. For $n = 39$, the inequality does not hold.

Remark 2. By replacing $a_1, a_2, \ldots, a_n$ respectively with $1-a_1, 1-a_2, \ldots, 1-a_n$, we get the following statement:

- *Let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 + a_2 + \cdots + a_n = 0$. If $n \leq 38$, then*

$$(2-a_1+a_1^2)(2-a_2+a_2^2)\cdots(2-a_n+a_n^2) \geq 2^n,$$

with equality for $a_1 = a_2 = \cdots = a_n = 0$.

□

P 3.23. *If a, b, c are nonnegative real numbers so that $a+b+c=3$, then*

$$(1-a+a^4)(1-b+b^4)(1-c+c^4) \geq 1.$$

Solution. Write the inequality as

$$f(a)+f(b)+f(c) \geq 3f(s), \qquad s=\frac{a+b+c}{3}=1,$$

where

$$f(u)=\ln(1-u+u^4), \quad u \in [0,3].$$

From

$$f'(u)=\frac{4u^3-1}{1-u+u^4},$$

it follows that f is decreasing on $[0, s_0]$ and increasing on $[s_0, 3]$, where

$$s_0=\frac{1}{\sqrt[3]{4}}<1=s.$$

Also, f is convex on $[s_0, 1]$ because

$$f''(u)=\frac{-4u^6-4u^3+12u^2-1}{(1-u+u^4)^2} \geq \frac{-4u^2-4u^2+12u^2-1}{(1-u+u^4)^2}=\frac{4u^2-1}{(1-u+u^4)^2}>0.$$

According to the LPCF-Theorem, we only need to show that

$$f(x)+2f(y) \geq 3f(1)$$

for all $x, y \geq 0$ so that $x+2y=3$. Using Note 2, it suffices to prove that $H(x, y) \geq 0$, where

$$H(x,y)=\frac{f'(x)-f'(y)}{x-y}.$$

We have

$$\begin{aligned}
H(x,y) &= \frac{(x+y)(x-y)^2-1+4(x^2+y^2+xy)-2xy(x+y)-4x^3y^3}{(1-x+x^4)(1-y+y^4)} \\
&\geq \frac{-1+4(x^2+y^2+xy)-2xy(x+y)-4x^3y^3}{(1-x+x^4)(1-y+y^4)} \\
&= \frac{h(x,y)}{(1-x+x^4)(1-y+y^4)},
\end{aligned}$$

where

$$h(x,y)=-1+2(x+y)[2(x+y)-xy]-4xy-4x^3y^3.$$

From $3=x+2y \geq 2\sqrt{2xy}$ and $(1-x)(1-y) \leq 0$, we get

$$xy \leq \frac{9}{8}, \qquad x+y \geq 1+xy.$$

Therefore,

$$\begin{aligned}
h(x,y) &\ge -1+2(1+xy)[2(1+xy)-xy]-4xy-4x^3y^3 \\
&= 3+2xy+2x^2y^2-4x^3y^3 \ge 3+2xy+2x^2y^2-5x^2y^2 \\
&= 3+2xy-3x^2y^2 \ge 3+2xy-4xy = 3-2xy > 0.
\end{aligned}$$

The proof is completed. The equality holds for $a=b=c=1$.

□

P 3.24. *If a,b,c,d are nonnegative real numbers so that $a+b+c+d=4$, then*

$$(1-a+a^3)(1-b+b^3)(1-c+c^3)(1-d+d^3) \ge 1.$$

(Vasile C., 2012)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d) \ge 4f(s), \qquad s=\frac{a+b+c+d}{4}=1,$$

where

$$f(u)=\ln(1-u+u^3), \qquad u\in[0,4].$$

From

$$f'(u)=\frac{3u^2-1}{1-u+u^3},$$

it follows that f is decreasing on $[0,s_0]$ and increasing on $[s_0,4]$, where

$$s_0=\frac{1}{\sqrt{3}}<1=s.$$

In addition, f is convex on $[s_0,1]$ because

$$f''(u)=\frac{-3u^4+6u-1}{(1-u+u^3)^2} \ge \frac{-3u+6u-1}{(1-u+u^3)^2} = \frac{3u-1}{(1-u+u^3)^2} > 0.$$

According to the LPCF-Theorem, we only need to show that

$$f(x)+3f(y) \ge 4f(1)$$

for all $x,y\ge 0$ so that $x+3y=4$. Using Note 2, it suffices to prove that $H(x,y)\ge 0$, where

$$H(x,y)=\frac{f'(x)-f'(y)}{x-y}.$$

We have

$$H(x,y)=\frac{(x-y)^2+3(x+y)-1-3x^2y^2}{(1-x+x^3)(1-y+y^3)} \ge \frac{3(x+y)-1-3x^2y^2}{(1-x+x^3)(1-y+y^3)}.$$

From $4 = x + 3y \geq 2\sqrt{3xy}$ and $(1-x)(1-y) \leq 0$, we get

$$xy \leq \frac{4}{3}, \quad x + y \geq 1 + xy.$$

Therefore,

$$\begin{aligned} 3(x+y) - 1 - 3x^2y^2 &\geq 3(1+xy) - 1 - 3x^2y^2 \\ &\geq 3(1+xy) - 1 - 4xy = 2 - xy > 0, \end{aligned}$$

hence $H(x,y) > 0$. The equality holds for $a = b = c = d = 1$.

□

P 3.25. *If a, b, c, d, e are nonzero real numbers so that $a + b + c + d + e = 5$, then*

$$5\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{e^2}\right) + 45 \geq 14\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}\right).$$

(Vasile C., 2013)

Solution. Write the desired inequality as

$$f(a) + f(b) + f(c) + f(d) + f(e) \geq 5f(s), \quad s = \frac{a+b+c+d+e}{5} = 1,$$

where

$$f(u) = \frac{5}{u^2} - \frac{14}{u} + 9, \quad u \in \mathbb{I} = \mathbb{R} \setminus \{0\}.$$

From

$$f'(u) = \frac{2(7u-5)}{u^3},$$

it follows that f is increasing on $(-\infty, 0) \cup [s_0, \infty)$ and decreasing on $(0, s_0]$, where

$$s_0 = \frac{5}{7} < 1 = s.$$

Since

$$\lim_{u \to -\infty} f(u) = 9$$

and $f(s_0) < f(1) = 0$, we have

$$\min_{u \in \mathbb{I}} f(u) = f(s_0).$$

From

$$f''(u) = \frac{2(15-14u)}{u^4},$$

it follows that f is convex on $[s_0,1]$. By the LPCF-Theorem, Note 4 and Note 1, it suffices to show that $h(x,y)\geq 0$ for all $x,y\in\mathbb{I}$ which satisfy $x+4y=5$, where

$$h(x,y)=\frac{g(x)-g(y)}{x-y},\quad g(u)=\frac{f(u)-f(1)}{u-1}.$$

Indeed, we have

$$g(u)=\frac{9}{u}-\frac{5}{u^2},$$

$$h(x,y)=\frac{5x+5y-9xy}{x^2y^2}=\frac{(6y-5)^2}{x^2y^2}\geq 0.$$

In accordance with Note 3, the equality holds for $a=b=c=d=e=1$, and also for

$$a=\frac{5}{3},\quad b=c=d=e=\frac{5}{6}$$

(or any cyclic permutation).

$\square$

P 3.26. *If a,b,c are positive real numbers so that $abc=1$, then*

$$\frac{7-6a}{2+a^2}+\frac{7-6b}{2+b^2}+\frac{7-6c}{2+c^2}\geq 1.$$

(Vasile C., 2008)

Solution. Using the substitution

$$a=e^x,\ \ b=e^y,\ \ c=e^z,$$

we need to show that

$$f(x)+f(y)+f(z)\geq 3f(s),$$

where

$$s=\frac{x+y+z}{3}=0$$

and

$$f(u)=\frac{7-6e^u}{2+e^{2u}},\quad u\in\mathbb{R}.$$

From

$$f'(u)=\frac{2(3e^u+2)(e^u-3)}{(2+e^{2u})^2},$$

it follows that f is decreasing on $(-\infty,s_0]$ and increasing on $[s_0,\infty)$, where

$$s_0=\ln 3>s.$$

We have

$$f''(u) = \frac{2t \cdot h(t)}{(2+t^2)^3}, \qquad h(t) = -3t^4 + 14t^3 + 36t^2 - 28t - 12, \qquad t = e^u.$$

We will show that $h(t) > 0$ for $t \in [1,3]$, hence f is convex on $[0, s_0]$. We have

$$\begin{aligned} h(t) &= 3(t^2-1)(9-t^2) + 14t^3 + 6t^2 - 28t + 15 \\ &\geq 14t^3 + 6t^2 - 28t + 15 \\ &= 14t^2(t-1) + 14(t-1)^2 + 6t^2 + 1 > 0. \end{aligned}$$

By the RPCF-Theorem, we only need to prove that

$$f(x) + 2f(y) \geq 3f(0)$$

for all real x, y so that $x + 2y = 0$. That is, to show that the original inequality holds for $b = c$ and $a = 1/c^2$. Write this inequality as

$$\frac{c^2(7c^2-6)}{2c^4+1} + \frac{2(7-6c)}{2+c^2} \geq 1,$$

$$(c-1)^2(c-2)^2(5c^2+6c+3) \geq 0.$$

By Note 3, the equality holds for $a = b = c = 1$, and also for

$$a = \frac{1}{4}, \qquad b = c = 2$$

(or any cyclic permutation).

□

P 3.27. *If a, b, c are positive real numbers so that $abc = 1$, then*

$$\frac{1}{a+5bc} + \frac{1}{b+5ca} + \frac{1}{c+5ab} \leq \frac{1}{2}.$$

(Vasile C., 2008)

Solution. Write the inequality as

$$\frac{a}{a^2+5} + \frac{b}{b^2+5} + \frac{c}{c^2+5} \leq \frac{1}{2}.$$

Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z,$$

we need to show that

$$f(x) + f(y) + f(z) \geq 3f(s),$$

where

$$s = \frac{x+y+z}{3} = 0$$

and

$$f(u) = \frac{-e^u}{e^{2u}+5}, \qquad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{e^u(e^{2u}-5)}{(e^{2u}+5)^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \frac{\ln 5}{2} > 0 = s.$$

Also, from

$$f''(u) = \frac{e^u(-e^{4u}+30e^{2u}-25)}{(e^{2u}+5)^3},$$

it follows that f is convex on $[s, s_0]$, because $u \in [0, s_0]$ involves $e^u \in [1, \sqrt{5}\,]$ and $e^{2u} \in [1, 5]$, hence

$$-e^{4u} + 30e^{2u} - 25 = e^{2u}(5 - e^{2u}) + 25(e^{2u} - 1) > 0.$$

By the RPCF-Theorem, we only need to prove the original inequality for $b = c$ and $a = 1/c^2$. Write this inequality as

$$\frac{c^2}{5c^4+1} + \frac{2c}{c^2+5} \le \frac{1}{2},$$

$$(c-1)^2(5c^4 - 10c^3 - 2c^2 + 6c + 5) \ge 0,$$

$$(c-1)^2[5(c-1)^4 + 2c(5c^2 - 16c + 13)] \ge 0.$$

The equality holds for $a = b = c = 1$.

□

P 3.28. *If a, b, c are positive real numbers so that $abc = 1$, then*

$$\frac{1}{4-3a+4a^2} + \frac{1}{4-3b+4b^2} + \frac{1}{4-3c+4c^2} \le \frac{3}{5}.$$

(Vasile Cirtoaje, 2008)

Solution. Let

$$a = e^x, \quad b = e^y, \quad c = e^z.$$

We need to show that

$$f(x) + f(y) + f(z) \ge 3f(s),$$

where

$$s = \frac{x+y+z}{3} = 0$$

and

$$f(u) = \frac{-1}{4-3e^u+4e^{2u}}, \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{e^u(8e^u-3)}{(4-3e^u+4e^{2u})^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \ln\frac{3}{8} < 0 = s.$$

We claim that f is convex on $[s_0, 0]$. Since

$$f''(u) = \frac{e^u(-64e^{3u}+36e^{2u}+55e^u-12)}{(4-3e^u+4e^{2u})^3},$$

we need to show that

$$-64t^3+36t^2+55t-12 \ge 0,$$

where

$$t = e^u \in \left[\frac{3}{8}, 1\right].$$

Indeed, we have

$$\begin{aligned} -64t^3+36t^2+55t-12 &> -72t^3+36t^2+48t-12 \\ &= 12(1-t)(6t^2+3t-1) \ge 0. \end{aligned}$$

By the LPCF-Theorem, we only need to prove the original inequality for $b = c$ and $a = 1/c^2$. Write this inequality as follows:

$$\frac{c^4}{4c^4-3c^2+4} + \frac{2}{4-3c+4c^2} \le \frac{3}{5},$$

$$28c^6-21c^5-48c^4+27c^3+42c^2-36c+8 \ge 0,$$

$$(c-1)^2(28c^4+35c^3-6c^2-20c+8) \ge 0.$$

It suffices to show that

$$7(4c^4+5c^3-c^2-3c+1) \ge 0.$$

Indeed,

$$4c^4+5c^3-c^2-3c+1 = c^2(2c-1)^2+9c^3-2c^2-3c+1$$

and

$$9c^3-2c^2-3c+1 = c(3c-1)^2+(2c-1)^2 > 0.$$

The equality holds for $a = b = c = 1$.

Remark. Since

$$\frac{1}{4-3a+4a^2} \geq \frac{1}{4-3a+4a^2+(1-a)^2} = \frac{1}{5(1-a+a^2)},$$

we get the following known inequality

$$\frac{1}{1-a+a^2} + \frac{1}{1-b+b^2} + \frac{1}{1-c+c^2} \leq 3.$$

□

P 3.29. *If a, b, c are positive real numbers so that $abc = 1$, then*

$$\frac{1}{(3a+1)(3a^2-5a+3)} + \frac{1}{(3b+1)(3b^2-5b+3)} + \frac{1}{(3c+1)(3c^2-5c+3)} \leq \frac{3}{4}.$$

Solution. Let

$$a = e^x, \quad b = e^y, \quad c = e^z.$$

We need to show that

$$f(x) + f(y) + f(z) \geq 3f(s),$$

where

$$s = \frac{x+y+z}{3} = 0$$

and

$$f(u) = \frac{-1}{(3e^u+1)(3e^{2u}-5e^u+3)}, \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{(3e^u-2)(9e^u-2)}{(3e^u+1)^2(3e^{2u}-5e^u+3)^2},$$

it follows that f is increasing on $(-\infty, s_1] \cup [s_0, \infty)$ and decreasing on $[s_1, s_0]$, where

$$s_1 = \ln 2 - \ln 9, \quad s_0 = \ln 2 - \ln 3, \quad s_1 < s_0 < 0 = s.$$

Since

$$\lim_{u \to -\infty} f(u) = f(s_0) = \frac{-1}{3},$$

we get

$$\min_{u \in \mathbb{R}} f(u) = f(s_0).$$

We claim that f is convex on $[s_0, 0]$. We have

$$f''(u) = \frac{t \cdot h(t)}{(3t+1)^3(3t^2-5t+3)^3},$$

where

$$t = e^u \in \left[\frac{2}{3}, 1\right], \quad h(t) = -729t^5 + 1188t^4 - 648t^3 + 387t^2 - 160t + 12.$$

Since the polynomial $h(t)$ has the real roots

$$t_1 \approx 0.0933, \quad t_2 \approx 0.5072, \quad t_3 \approx 1.11008,$$

it follows that $h(t) > 0$ for $t \in [2/3, 1] \subset [t_2, t_3]$, hence f is convex on $[s_0, 0]$. By the LPCF-Theorem, we only need to prove the original inequality for $b = c \leq 1$ and $a = 1/c^2$. Write this inequality as follows:

$$\frac{c^6}{(c^2+3)(3c^4-5c^2+3)} + \frac{2}{(3c+1)(3c^2-5c+3)} \leq \frac{3}{4}.$$

Since

$$c^2 + 3 \geq 2(c+1)$$

and

$$3c^4 - 5c^2 + 3 \geq c(3c^2 - 5c + 3),$$

it suffices to prove that

$$\frac{c^5}{2(c+1)(3c^2-5c+3)} + \frac{2}{(3c+1)(3c^2-5c+3)} \leq \frac{3}{4}.$$

This is equivalent to the obvious inequality

$$(1-c)^2(1 + 15c + 5c^2 - 14c^3 - 6c^4) \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 3.30. *Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be positive real numbers so that $a_1 a_2 \cdots a_n = 1$. If $p, q \geq 0$ so that $p + 4q \geq n - 1$, then*

$$\frac{1-a_1}{1+pa_1+qa_1^2} + \frac{1-a_2}{1+pa_2+qa_2^2} + \cdots + \frac{1-a_n}{1+pa_n+qa_n^2} \geq 0.$$

(Vasile C., 2008)

Solution. For $q = 0$, we get a known inequality (see Remark 2 from the proof of P 1.63). Consider further that $q > 0$. Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \ldots, n$, we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s),$$

where

$$s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0$$

and

$$f(u) = \frac{1 - e^u}{1 + pe^u + qe^{2u}}, \quad u \in \mathbb{R}.$$

From

$$f'(t) = \frac{e^u(qe^{2u} - 2qe^u - p - 1)}{(1 + pe^u + qe^{2u})^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \ln r_0 > 0 = s, \quad r_0 = 1 + \sqrt{1 + \frac{p+1}{q}}.$$

Also, we have

$$f''(u) = \frac{t \cdot h(t)}{(1 + pt + qt^2)^3},$$

where

$$h(t) = -q^2t^4 + q(p + 4q)t^3 + 3q(p + 2)t^2 + (p - 4q + p^2)t - p - 1, \quad t = e^u.$$

We will show that $h(t) \ge 0$ for $t \in [1, r_0]$, hence f is convex on $[0, s_0]$. We have

$$h'(t) = -4q^2t^3 + 3q(p + 4q)t^2 + 6q(p + 2)t + p - 4q + p^2,$$

$$h''(t) = 6q[-2qt^2 + (p + 4q)t + p + 2].$$

Since

$$h''(t) = 6q[2(-qt^2 + 2qt + p + 1) + p(t - 1)] \ge 12q(-qt^2 + 2qt + p + 1) \ge 0,$$

$h'(t)$ is increasing,

$$h'(t) \ge h'(1) = p^2 + 9pq + 8q^2 + p + 8q > 0,$$

h is increasing, hence

$$\begin{aligned} h(t) \ge h(1) &= p^2 + 4pq + 3q^2 + 2q - 1 = (p + 2q)^2 - (q - 1)^2 \\ &= (p + q + 1)(p + 3q - 1). \end{aligned}$$

Since

$$p + 3q - 1 \ge p + 3q - \frac{p + 4q}{n - 1} = \frac{p + 2q}{2} > 0,$$

$f''(u) > 0$ for $u \in [0, s_0]$, therefore f is convex on $[s, s_0]$. By the RPCF-Theorem, we only need to prove the original inequality for

$$a_2 = \cdots = a_n := t, \quad a_1 = 1/t^{n-1}, \quad t \ge 1.$$

Write this inequality as

$$\frac{t^{n-1}(t^{n-1}-1)}{t^{2n-2}+pt^{n-1}+q}+\frac{(n-1)(1-t)}{1+pt+qt^2}\geq 0,$$

or

$$pA+qB\geq C,$$

where

$$\begin{aligned}A&=t^{n-1}(t^n-nt+n-1),\\ B&=t^{2n}-t^{n+1}-(n-1)(t-1),\\ C&=t^{n-1}[(n-1)t^n+1-nt^{n-1}].\end{aligned}$$

Since $p+4q\geq n-1$ and $C\geq 0$ (by the AM-GM inequality applied to n positive numbers), it suffices to show that

$$pA+qB\geq\frac{(p+4q)C}{n-1},$$

which is equivalent to

$$p[(n-1)A-C]+q[(n-1)B-4C]\geq 0.$$

This is true if

$$(n-1)A-C\geq 0$$

and

$$(n-1)B-4C\geq 0$$

for $t\geq 1$. By the AM-GM inequality, we have

$$(n-1)A-C=nt^{n-1}[t^{n-1}+n-2-(n-1)t]\geq 0.$$

For $n=3$, we have

$$B=(t-1)^2(t^4+2t^3+2t^2+2t+2),$$

$$C=t^2(t-1)^2(2t+1),$$

$$\begin{aligned}B-2C&=(t-1)^2(t^4-2t^3+2t+2)\\ &=(t-1)^2[(t-1)^2(t^2-1)+3]\geq 0.\end{aligned}$$

Consider further that

$$n\geq 4.$$

Since

$$t-1\leq t^{n-1}(t-1),$$

we have

$$\begin{aligned} B &\geq t^{2n} - t^{n+1} - (n-1)t^{n-1}(t-1) \\ &= t^{n-1}[t^{n+1} - t^2 - (n-1)t + n - 1]. \end{aligned}$$

Thus, the inequality $(n-1)B - 4C \geq 0$ is true if

$$(n-1)[t^{n+1} - t^2 - (n-1)t + n - 1] - 4(n-1)t^n - 4 - 4nt^{n-1} \geq 0,$$

which is equivalent to $g(t) \geq 0$, where

$$g(t) = (n-1)t^{n+1} - 4(n-1)t^n + 4nt^{n-1} - (n-1)t^2 - (n-1)^2 t + n^2 - 2n - 3.$$

We have

$$g'(t) = (n-1)g_1(t), \quad g_1(t) = (n+1)t^n - 4nt^{n-1} + 4nt^{n-2} - 2t - n + 1,$$

$$g_1'(t) = n(n+1)t^{n-1} - 4n(n-1)t^{n-2} + 4n(n-2)t^{n-3} - 2.$$

Since

$$n(n+1)t^{n-1} + 4n(n-2)t^{n-3} \geq 4n\sqrt{(n+1)(n-2)}\,t^{n-2},$$

we get

$$\begin{aligned} g_1'(t) &\geq 4n\left[\sqrt{(n+1)(n-2)} - n + 1\right]t^{n-2} - 2 \\ &\geq 4n\left[\sqrt{(n+1)(n-2)} - n + 1\right] - 2 \\ &= \frac{4n(n-3)}{\sqrt{(n+1)(n-2)} + n - 1} - 2 \\ &> \frac{4n(n-3)}{(n+1) + n - 1} - 2 = 2(n-4) \geq 0. \end{aligned}$$

Therefore, $g_1(t)$ is increasing for $t \geq 1$, $g_1(t) \geq g_1(1) = 0$, $g(t)$ is increasing for $t \geq 1$, hence

$$g(t) \geq g(1) = 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark. For $p = 0$ and $q = 1$, we get the inequality (*Vasile C.*, 2006)

$$\frac{1-a}{1+a^2} + \frac{1-b}{1+b^2} + \frac{1-c}{1+c^2} + \frac{1-d}{1+d^2} + \frac{1-e}{1+e^2} \geq 0,$$

where a, b, c, d, e are positive real numbers so that $abcde = 1$. Replacing a, b, c, d, e by $1/a, 1/b, 1/c, 1/d, 1/e$, we get

$$\frac{1+a}{1+a^2} + \frac{1+b}{1+b^2} + \frac{1+c}{1+c^2} + \frac{1+d}{1+d^2} + \frac{1+e}{1+e^2} \leq 5,$$

where a, b, c, d, e are positive real numbers so that $abcde = 1$.

Notice that the inequality

$$\frac{1-a_1}{1+a_1^2} + \frac{1-a_2}{1+a_2^2} + \frac{1-a_3}{1+a_3^2} + \frac{1-a_4}{1+a_4^2} + \frac{1-a_5}{1+a_5^2} + \frac{1-a_6}{1+a_6^2} \geq 0$$

is not true for all positive numbers $a_1, a_2, a_3, a_4, a_5, a_6$ satisfying $a_1a_2a_3a_4a_5a_6 = 1$. Indeed, for $a_2 = a_3 = a_4 = a_5 = a_6 = 2$, the inequality becomes

$$\frac{1-a_1}{1+a_1^2} - 1 \geq 0,$$

which is false for $a_1 > 0$.

□

P 3.31. *If a, b, c are positive real numbers so that $abc = 1$, then*

$$\frac{1-a}{17+4a+6a^2} + \frac{1-b}{17+4b+6b^2} + \frac{1-c}{17+4c+6c^2} \geq 0.$$

(Vasile C., 2008)

Solution. Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z,$$

we need to show that

$$f(x) + g(y) + g(z) \geq 3f(s),$$

where

$$s = \frac{x+y+z}{3} = 0$$

and

$$f(u) = \frac{1-e^u}{1+pe^u+qe^{2u}}, \qquad u \in \mathbb{R},$$

with

$$p = \frac{4}{17}, \qquad q = \frac{6}{17}.$$

As we have shown in the proof of the preceding P 3.30, f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \ln r_0 > 0 = s, \qquad r_0 = 1 + \sqrt{1 + \frac{p+1}{q}} = 1 + \sqrt{\frac{9}{2}}.$$

In addition, since $p+3q-1=\dfrac{5}{17}>0$ (see the proof of P 3.30), f is convex on $[0,s_0]$. By the RPCF-Theorem, we only need to prove the original inequality for $b=c\ge 1$ and $a=1/c^2$. Write this inequality as follows:

$$\frac{c^2(c^2-1)}{c^4+pc^2+q}+\frac{2(1-c)}{1+pc+qc^2}\ge 0,$$

$$pA+qB\ge C,$$

where

$$A=c^2(c-1)^2(c+2),$$
$$B=(c-1)^2(c^4+2c^3+2c^2+2c+2),$$
$$C=c^2(c-1)^2(2c+1).$$

Indeed, we have

$$pA+qB-C=\frac{3(c-1)^2(c-2)^2(2c^2+2c+1)}{17}\ge 0.$$

In accordance with Note 3, the equality holds for $a=b=c=1$, and also for

$$a=\frac{1}{4},\qquad b=c=2$$

(or any cyclic permutation).

□

P 3.32. *If $a_1,a_2,\ldots,a_8$ are positive real numbers so that $a_1a_2\cdots a_8=1$, then*

$$\frac{1-a_1}{(1+a_1)^2}+\frac{1-a_2}{(1+a_2)^2}+\cdots+\frac{1-a_8}{(1+a_8)^2}\ge 0.$$

(Vasile C., 2006)

Solution. Using the substitutions $a_i=e^{x_i}$ for $i=1,2,\ldots,8$, we need to show that

$$f(x_1)+f(x_2)+\cdots+f(x_8)\ge 8f(s),$$

where

$$s=\frac{x_1+x_2+\cdots+x_8}{8}=0$$

and

$$f(u)=\frac{1-e^u}{(1+e^u)^2},\quad u\in\mathbb{R}.$$

From

$$f'(t)=\frac{e^u(e^u-3)}{(1+e^u)^3},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \ln 3 > 1 = s.$$

We have

$$f''(u) = \frac{e^u(8e^u - e^{2u} - 3)}{(1+e^u)^4}.$$

For $u \in [0, \ln 3]$, that is $e^u \in [1, 3]$, we have

$$8e^u - e^{2u} - 3 > 8e^u - 3e^u - 7 = (e^u - 1)(7 - e^u) \geq 0;$$

therefore, f is convex on $[s, s_0]$. By the RPCF-Theorem, we only need to prove the original inequality for $a_2 = \cdots = a_8 := t$ and $a_1 = 1/t^7$, where $t \geq 1$. For the nontrivial case $t > 1$, write this inequality as follows:

$$\frac{t^7(t^7-1)}{(t^7+1)^2} \geq \frac{7(t-1)}{(t+1)^2}.$$

$$\frac{t^7(t^7-1)(t+1)^2}{(t-1)(t^7+1)^2} \geq 7,$$

$$\frac{t^7(t^6+t^5+t^4+t^3+t^2+t+1)}{(t^6-t^5+t^4-t^3+t^2-t+1)^2} \geq 7.$$

Since

$$t^6 - t^5 + t^4 - t^3 + t^2 - t + 1 = t^4(t^2 - t + 1) - (t-1)(t^2+1) < t^4(t^2 - t + 1),$$

it suffices to show that

$$\frac{t^6+t^5+t^4+t^3+t^2+t+1}{t(t^2-t+1)^2} \geq 7,$$

which is equivalent to the obvious inequality

$$(t-1)^6 \geq 0.$$

Thus, the proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_8 = 1$.

Remark. The inequality

$$\frac{1-a_1}{(1+a_1)^2} + \frac{1-a_2}{(1+a_2)^2} + \cdots + \frac{1-a_9}{(1+a_9)^2} \geq 0$$

is not true for all positive numbers $a_1, a_2, \ldots, a_9$ satisfying $a_1 a_2 \cdots a_9 = 1$. Indeed, for $a_2 = a_3 = \cdots = a_9 = 3$, the inequality becomes

$$\frac{1-a_1}{(1+a_1)^2} - 1 \geq 0,$$

which is false for $a_1 > 0$.

□

P 3.33. *Let a,b,c be positive real numbers so that $abc=1$. If $k\in\left[\frac{-13}{3\sqrt{3}},\frac{13}{3\sqrt{3}}\right]$, then*

$$\frac{a+k}{a^2+1}+\frac{b+k}{b^2+1}+\frac{c+k}{c^2+1}\le\frac{3(1+k)}{2}.$$

(Vasile C., 2012)

Solution. The inequality is equivalent to

$$k\left(\sum\frac{1}{a^2+1}-\frac{3}{2}\right)\le\sum\left(\frac{1}{2}-\frac{a}{a^2+1}\right),$$

$$\sum\frac{(a-1)^2}{a^2+1}\ge k\left(\sum\frac{2}{a^2+1}-3\right). \qquad (*)$$

Thus, it suffices to prove it for $|k|=\frac{13}{3\sqrt{3}}$. On the other hand, replacing a,b,c by $1/a,1/b,1/c$, the inequality becomes

$$\sum\frac{(a-1)^2}{a^2+1}\ge k\left(3-\sum\frac{2}{a^2+1}\right). \qquad (**)$$

Based on $(*)$ and $(**)$, we only need to prove the desired inequality for

$$k=\frac{13}{3\sqrt{3}}.$$

Using the substitution

$$a=e^x,\quad b=e^y,\quad c=e^z,$$

we need to show that

$$f(x)+g(y)+g(z)\ge 3f(s),$$

where

$$s=\frac{x+y+z}{3}=0$$

and

$$f(u)=\frac{-e^u-k}{e^{2u}+1},\quad u\in\mathbb{R}.$$

From

$$f'(t)=\frac{e^{2u}+2ke^u-1}{(e^{2u}+1)^2},$$

it follows that f is decreasing on $(-\infty,s_0]$ and increasing on $[s_0,\infty)$, where

$$s_0=\ln r_0<0=s,\quad r_0=-k+\sqrt{k^2+1}=\frac{1}{3\sqrt{3}}.$$

Also, we have

$$f''(u)=\frac{t\cdot h(t)}{(1+t^2)^3},$$

where
$$h(t) = -t^4 - 4kt^3 + 6t^2 + 4kt - 1, \quad t = e^u.$$
We will show that $h(t) > 0$ for $t \in [r_0, 1]$, hence f is convex on $[s_0, s]$. Indeed, since
$$4kt = \frac{52t}{3\sqrt{3}} \geq \frac{52}{27} > 1,$$
we have
$$h(t) = -t^4 + 6t^2 - 1 + 4kt(1-t^2) \geq -t^4 + 6t^2 - 1 + (1-t^2) = t^2(5-t^2) > 0.$$
By the LPCF-Theorem, we only need to prove the original inequality for $b = c := t$ and $a = 1/t^2$, where $t > 0$. Write this inequality as
$$\frac{t^2(kt^2+1)}{t^4+1} + \frac{2(t+k)}{t^2+1} \leq \frac{3(1+k)}{2},$$
$$3t^6 - 4t^5 + t^4 + t^2 - 4t + 3 - k(1-t^2)^3 \geq 0,$$
$$(t-1)^2[(3+k)t^4 + 2(1+k)t^3 + 2t^2 + 2(1-k)t + 3 - k] \geq 0,$$
$$(t-1)^2\left(t-2+\sqrt{3}\right)^2\left[(27+13\sqrt{3})t^2 + 24(2+\sqrt{3})t + 33 + 17\sqrt{3}\right] \geq 0.$$
The equality holds for $a = b = c = 1$. If $k = \dfrac{13}{3\sqrt{3}}$, then the equality holds also for
$$a = 7 + 4\sqrt{3}, \quad b = c = 2 - \sqrt{3}$$
(or any cyclic permutation). If $k = \dfrac{-13}{3\sqrt{3}}$, then the equality holds also for
$$a = 7 - 4\sqrt{3}, \quad b = c = 2 + \sqrt{3}$$
(or any cyclic permutation).

□

P 3.34. *If a, b, c are positive real numbers and $0 < k \leq 2 + 2\sqrt{2}$, then*
$$\frac{a^3}{ka^2+bc} + \frac{b^3}{kb^2+ca} + \frac{c^3}{kc^2+ab} \geq \frac{a+b+c}{k+1}.$$

(Vasile C., 2011)

Solution. Due to homogeneity, we may assume that $abc = 1$. On this hypothesis, we write the inequality as follows:
$$\frac{a^4}{ka^3+1} + \frac{b^4}{kb^3+1} + \frac{b^4}{kb^3+1} \geq \frac{a}{k+1} + \frac{b}{k+1} + \frac{c}{k+1},$$

$$\frac{a^4-a}{ka^3+1}+\frac{b^4-b}{kb^3+1}+\frac{c^4-c}{kc^3+1}\ge 0.$$

Using the substitution

$$a=e^x,\quad b=e^y,\quad c=e^z,$$

we need to show that

$$f(x)+g(y)+g(z)\ge 3f(s),$$

where

$$s=\frac{x+y+z}{3}=0$$

and

$$f(u)=\frac{e^{4u}-e^u}{ke^{3u}+1},\quad u\in\mathbb{R}.$$

From

$$f'(t)=\frac{ke^{6u}+2(k+2)e^{3u}-1}{(ke^{3u}+1)^2},$$

it follows that f is decreasing on $(-\infty,s_0]$ and increasing on $[s_0,\infty)$, where

$$s_0=\ln r_0<0,\quad r_0=\sqrt[3]{\frac{-k-2+\sqrt{(k+1)(k+4)}}{k}}\in(0,1).$$

Also, we have

$$f''(u)=\frac{t\cdot h(t)}{(kt^3+1)^3},$$

where

$$h(t)=k^2t^9-k(4k+1)t^6+(13k+16)t^3-1,\quad t=e^u.$$

If $h(t)>0$ for $t\in[r_0,1]$, then f is convex on $[s_0,0]$. We will prove this only for $k=2+2\sqrt{2}$, when $r_0\approx 0.415$ and $h(t)\ge 0$ for $t\in[t_1,t_2]$, where $t_1\approx 0.2345$ and $t_2\approx 1.02$. Since $[r_0,1]\subset[t_1,t_2]$, the conclusion follows. By the LPCF-Theorem, we only need to prove the original inequality for $b=c$. Due to homogeneity, we may consider that $b=c=1$. Thus, we need to show that

$$\frac{a^3}{ka^2+1}+\frac{2}{a+k}\ge\frac{a+2}{k+1},$$

which is equivalent to the obvious inequality

$$(a-1)^2[a^2-(k-2)a+2]\ge 0.$$

For $k=2+2\sqrt{2}$, this inequality has the form

$$(a-1)^2(a-\sqrt{2})^2\ge 0.$$

The equality holds for $a=b=c$. If $k=2+2\sqrt{2}$, then the equality holds also for

$$\frac{a}{\sqrt{2}}=b=c$$

(or any cyclic permutation).

□

P 3.35. *If a,b,c,d,e are positive real numbers so that $abcde=1$, then*

$$2\left(\frac{1}{a+1}+\frac{1}{b+1}+\cdots+\frac{1}{e+1}\right)\geq 3\left(\frac{1}{a+2}+\frac{1}{b+2}+\cdots+\frac{1}{e+2}\right).$$

(Vasile C., 2012)

Solution. Write the inequality as

$$\frac{1-a}{(a+1)(a+2)}+\frac{1-b}{(b+1)(b+2)}+\frac{1-c}{(c+1)(c+2)}+\frac{1-d}{(d+1)(d+2)}+\frac{1-e}{(e+1)(e+2)}\geq 0.$$

Using the substitution

$$a=e^x,\quad b=e^y,\quad c=e^z,\quad d=e^t,\quad e=e^w,$$

we need to show that

$$f(x)+f(y)+f(z)+f(t)+f(w)\geq 5f(s),$$

where

$$s=\frac{x+y+z+t+w}{5}=0$$

and

$$f(u)=\frac{1-e^u}{(e^u+1)(e^u+2)},\quad u\in\mathbb{R}.$$

From

$$f'(u)=\frac{e^u(e^{2u}-2e^u-5)}{(e^u+1)^2(e^u+2)^2},$$

it follows that f is decreasing on $(-\infty,s_0]$ and increasing on $[s_0,\infty)$, where

$$s_0=\ln(1+\sqrt{6})<2,\qquad s<s_0.$$

Also, we have

$$f''(u)=\frac{t\cdot h(t)}{(t+1)^3(t+2)^3},\qquad t=e^u,$$

where

$$h(t)=-t^4+7t^3+21t^2+7t-10.$$

We will show that $h(t)>0$ for $t\in[1,2]$, hence f is convex on $[0,s_0]$. We have

$$h(t)\geq -2t^3+7t^3+21t^2+7t-10=5t^3+21t^2+7t-10>0.$$

By the RPCF-Theorem, we only need to prove the original inequality for

$$b=c=d=e:=t,\qquad a=1/t^4,\qquad t\geq 1.$$

Write this inequality as

$$\frac{t^4(t^4-1)}{(t^4+1)(2t^4+1)} \ge \frac{4(t-1)}{(t+1)(t+2)},$$

which is true if

$$t^4(t+1)(t+2)(t^3+t^2+t+1) \ge 4(t^4+1)(2t^4+1).$$

Since

$$(t^4+1)(2t^4+1) = 2t^8+3t^4+1 \le 2t^4(t^4+2),$$

it suffices to show that

$$(t+1)(t+2)(t^3+t^2+t+1) \ge 8(t^4+2).$$

This inequality is equivalent to

$$t^5-4t^4+6t^3+6t^2+5t-14 \ge 0,$$

$$t(t-1)^4+10(t^2-1)+4(t-1) \ge 0.$$

The equality holds for $a=b=c=d=e=1$.

□

P 3.36. *If $a_1, a_2, \ldots, a_{14}$ are positive real numbers so that $a_1a_2\cdots a_{14}=1$, then*

$$3\left(\frac{1}{2a_1+1}+\frac{1}{2a_2+1}+\cdots+\frac{1}{2a_{14}+1}\right) \ge 2\left(\frac{1}{a_1+1}+\frac{1}{a_2+1}+\cdots+\frac{1}{a_{14}+1}\right).$$

(Vasile C., 2012)

Solution. Write the inequality as

$$\frac{1-a_1}{(a_1+1)(2a_1+1)}+\frac{1-a_2}{(a_2+1)(2a_2+1)}+\cdots+\frac{1-a_{14}}{(a_{14}+1)(2a_{14}+1)} \ge 0.$$

Using the substitutions $a_i = e^{x_i}$ for $i=1,2,\ldots,14$, we need to show that

$$f(x_1)+f(x_2)+\cdots+f(x_{14}) \ge 14f(s),$$

where

$$s = \frac{x_1+x_2+\cdots+x_{14}}{14} = 0$$

and

$$f(u) = \frac{1-e^u}{(e^u+1)(2e^u+1)}, \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{2e^u(e^{2u} - 2e^u - 2)}{(e^u + 1)^2(2e^u + 1)^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \ln(1 + \sqrt{3}) < 2, \quad s < s_0.$$

Also, we have

$$f''(u) = \frac{2t \cdot h(t)}{(t+1)^3(2t+1)^3}, \quad t = e^u,$$

where

$$h(t) = -2t^4 + 11t^3 + 15t^2 + 2t - 2.$$

We will show that $h(t) > 0$ for $t \in [1, 2]$, hence f is convex on $[0, s_0]$. We have

$$h(t) \geq -4t^3 + 11t^3 + 15t^2 + 2t - 2 = 7t^3 + 15t^2 + 2t - 2 > 0.$$

By the RPCF-Theorem, we only need to prove the original inequality for

$$a_2 = a_3 = \cdots = a_{14} := t, \quad a_1 = 1/t^{13}, \quad t \geq 1.$$

Write this inequality as

$$\frac{t^{13}(t^{13} - 1)}{(t^{13} + 1)(t^{13} + 2)} \geq \frac{13(t-1)}{(t+1)(2t+1)}.$$

Since

$$(t^{13} + 1)(t^{13} + 2) = t^{26} + 3t^{13} + 2 \leq t^{13}(t^{13} + 5),$$

it suffices to show that

$$\frac{t^{13} - 1}{t^{13} + 5} \geq \frac{13(t-1)}{(t+1)(2t+1)},$$

which is equivalent to

$$t^{13}(t^2 - 5t + 7) - t^2 - 34t + 32 \geq 0.$$

Substituting

$$t = 1 + x, \quad x \geq 0,$$

the inequality becomes

$$(1 + x)^{13}(x^2 - 3x + 3) - x^2 - 36x - 3 \geq 0.$$

Since

$$(1 + x)^{13} \geq 1 + 13x + 78x^2,$$

it suffices to show that

$$(78x^2 + 13x + 1)(x^2 - 3x + 3) - x^2 - 36x - 3 \geq 0.$$

This inequality, equivalent to

$$x^2(78x^2-221x+196)\geq 0,$$

is true since

$$78x^2-221x+196\geq 64x^2-224x+196=4(4x-7)^2\geq 0.$$

The equality holds for $a_1=a_2=\cdots=a_{14}=1$.

□

P 3.37. *Let $a_1,a_2,\ldots,a_8$ be positive real numbers so that $a_1a_2\cdots a_8=1$. If $k>1$, then*

$$(k+1)\left(\frac{1}{ka_1+1}+\frac{1}{ka_2+1}+\cdots+\frac{1}{ka_8+1}\right)\geq 2\left(\frac{1}{a_1+1}+\frac{1}{a_2+1}+\cdots+\frac{1}{a_8+1}\right).$$

(Vasile C., 2012)

Solution. Write the inequality as

$$\frac{1-a_1}{(a_1+1)(ka_1+1)}+\frac{1-a_2}{(a_2+1)(ka_2+1)}+\cdots+\frac{1-a_8}{(a_8+1)(ka_8+1)}\geq 0.$$

Using the substitutions $a_i=e^{x_i}$ for $i=1,2,\ldots,8$, we need to show that

$$f(x_1)+f(x_2)+\cdots+f(x_8)\geq 8f(s),$$

where

$$s=\frac{x_1+x_2+\cdots+x_8}{8}=0$$

and

$$f(u)=\frac{1-e^u}{(e^u+1)(ke^u+1)},\quad u\in\mathbb{R}.$$

From

$$f'(u)=\frac{e^u(ke^{2u}-2ke^u-k-2)}{(e^u+1)^2(ke^u+1)^2},$$

it follows that f is decreasing on $(-\infty,s_0]$ and increasing on $[s_0,\infty)$, where

$$s_0=\ln\left(1+\sqrt{2+\frac{2}{k}}\right)<2,\qquad s<s_0.$$

Also, we have

$$f''(u)=\frac{t\cdot h(t)}{(t+1)^3(kt+1)^3},\qquad t=e^u,$$

where

$$h(t) = -k^2t^4 + k(5k+1)t^3 + 3k(k+3)t^2 + (k^2-k+2)t - k - 2.$$

We will show that $h(t) > 0$ for $t \in [1,2]$, hence f is convex on $[0, s_0]$. We have

$$\begin{aligned} h(t) &> -2k^2t^3 + k(5k+1)t^3 + 3k(k+3)t^2 + (k^2-k+2)t - k - 2 \\ &= k(3k+1)t^3 + 3k(k+3)t^2 + (k^2-k+2)t - k - 2 \\ &> 3k(k+3) + (k^2-k+2) - k - 2 > 0. \end{aligned}$$

By the RPCF-Theorem, we only need to prove the original inequality for

$$a_2 = a_3 = \cdots = a_8 := t, \qquad a_1 = 1/t^7, \qquad t \ge 1.$$

Write this inequality as

$$\frac{t^7(t^7-1)}{(t^7+1)(t^7+k)} \ge \frac{7(t-1)}{(t+1)(kt+1)}.$$

Since

$$(t^7+1)(t^7+k) = t^{14} + (k+1)t^7 + k \le t^7(t^7+2k+1),$$

it suffices to show that

$$\frac{t^7-1}{t^7+2k+1} \ge \frac{7(t-1)}{(t+1)(kt+1)},$$

which is equivalent to

$$k(t-1)P(t) + Q(t) \ge 0,$$

where

$$P(t) = t(t+1)(t^6+t^5+t^4+t^3+t^2+t+1) - 14,$$

$$Q(t) = (t+1)(t^7-1) - 7(t-1)(t^7+1).$$

Since $(t-1)P(t) \ge 0$ for $t \ge 1$, it suffices to consider the case $k = 1$. So, we need to show that

$$\frac{t^7-1}{t^7+3} \ge \frac{7(t-1)}{(t+1)^2},$$

which is equivalent to

$$t^7(t^2-5t+8) - t^2 - 23t + 20 \ge 0.$$

Substituting

$$t = 1 + x, \qquad x \ge 0,$$

the inequality becomes

$$(1+x)^7(x^2-3x+4) - x^2 - 25x - 4 \ge 0.$$

Since

$$(1+x)^7 \ge 1+7x+21x^2,$$

it suffices to show that

$$(21x^2+7x+1)(x^2-3x+4)-x^2-25x-4 \ge 0.$$

This inequality, equivalent to

$$x^2(21x^2-56x+63) \ge 0.$$

is true since

$$21x^2-56x+63 > 16x^2-56x+49 = (4x-7)^2 \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_8 = 1$.

□

P 3.38. *If $a_1, a_2, \ldots, a_9$ are positive real numbers so that $a_1 a_2 \cdots a_9 = 1$, then*

$$\frac{1}{2a_1+1}+\frac{1}{2a_2+1}+\cdots+\frac{1}{2a_9+1} \ge \frac{1}{a_1+2}+\frac{1}{a_2+2}+\cdots+\frac{1}{a_9+2}.$$

(Vasile C., 2012)

Solution. Write the inequality as

$$\frac{1-a_1}{(2a_1+1)(a_1+2)}+\frac{1-a_2}{(2a_2+1)(a_2+2)}+\cdots+\frac{1-a_9}{(2a_9+1)(a_9+2)} \ge 0.$$

Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \ldots, 9$, we need to show that

$$f(x_1)+f(x_2)+\cdots+f(x_9) \ge 9f(s),$$

where

$$s = \frac{x_1+x_2+\cdots+x_9}{9} = 0$$

and

$$f(u) = \frac{1-e^u}{(2e^u+1)(e^u+2)}, \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{e^u(2e^{2u}-4e^u-7)}{(2e^u+1)^2(e^u+2)^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \ln\left(1+\frac{3\sqrt{2}}{2}\right) < 2, \qquad s < s_0.$$

Also, we have

$$f''(u) = \frac{t \cdot h(t)}{(2t+1)^3(t+2)^3}, \quad t = e^u,$$

where

$$h(t) = -4t^4 + 26t^3 + 54t^2 + 19t - 14.$$

We will show that $h(t) > 0$ for $t \in [1,2]$, hence f is convex on $[0, s_0]$. We have

$$h(t) \geq -8t^3 + 26t^3 + 54t^2 + 19t - 14 = 18t^3 + 54t^2 + 19t - 14 > 0.$$

By the RPCF-Theorem, we only need to prove the original inequality for

$$a_2 = a_3 = \cdots = a_9 := t, \quad a_1 = 1/t^8, \quad t \geq 1.$$

Write this inequality as

$$\frac{t^8(t^8-1)}{(t^8+2)(2t^8+1)} \geq \frac{8(t-1)}{(2t+1)(t+2)}.$$

Since

$$(t^8+2)(2t^8+1) = 2t^{16} + 5t^8 + 2 \leq t^8(2t^8+7),$$

it suffices to show that

$$\frac{t^8-1}{2t^8+7} \geq \frac{8(t-1)}{(2t+1)(t+2)},$$

which is equivalent to

$$t^8(2t^2 - 11t + 18) - 2t^2 - 61t + 54 \geq 0.$$

Substituting

$$t = 1 + x, \quad x \geq 0,$$

the inequality becomes

$$(1+x)^8(2x^2 - 7x + 9) - 2x^2 - 65x - 9 \geq 0.$$

Since

$$(1+x)^8 \geq 1 + 8x + 28x^2,$$

it suffices to show that

$$(28x^2 + 8x + 1)(2x^2 - 7x + 9) - 2x^2 - 65x - 9 \geq 0.$$

This inequality, equivalent to

$$x^2(56x^2 - 180x + 196) \geq 0.$$

is true since

$$56x^2 - 180x + 196 \geq 49x^2 - 196x + 196 = 49(x-2)^2 \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_9 = 1$.

$\square$

P 3.39. *If $a_1, a_2, \ldots, a_n$ are real numbers so that*

$$a_1, a_2, \ldots, a_n \le \pi, \qquad a_1 + a_2 + \cdots + a_n = \pi,$$

then

$$\cos a_1 + \cos a_2 + \cdots + \cos a_n \le n \cos \frac{\pi}{n}.$$

(Vasile C., 2000

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge n f(s), \qquad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{\pi}{n},$$

where

$$f(u) = -\cos u, \qquad u \in \mathbb{I} = [-(n-2)\pi, \pi].$$

Let

$$s_0 = 0 < s.$$

We see that f is increasing on $[s_0, \pi] = \mathbb{I}_{\ge s_0}$ and $f(u) \ge f(s_0) = -1$ for $u \in \mathbb{I}$. In addition, f is convex on $[s_0, s]$. Thus, by the LPCF-Theorem, we only need to prove that $g(x) \le 0$, where

$$g(x) = \cos x + (n-1)\cos y - n\cos s, \qquad x + (n-1)y = \pi, \qquad \pi \ge x \ge s \ge y \ge 0.$$

Since $y' = \dfrac{-1}{n-1}$, we get

$$g'(x) = -\sin x + \sin y = -2\sin\frac{x-y}{2}\cos\frac{x+y}{2}.$$

We have $g'(x) \le 0$ because

$$0 < \frac{x+y}{2} \le \frac{x+(n-1)y}{2} = \frac{\pi}{2}$$

and

$$0 \le \frac{x-y}{2} < \frac{\pi}{2}.$$

From $g' \le 0$, it follows that g is decreasing, hence $g(x) \le g(s) = 0$.

The equality holds for $a_1 = a_2 = \cdots = a_n = \dfrac{\pi}{n}$. If $n = 2$, then the inequality is an identity.

Remark. In the same manner, we can prove the following generalization:

- *If $a_1, a_2, \ldots, a_n$ are real numbers so that*

$$a_1, a_2, \ldots, a_n \le \pi, \qquad \frac{a_1 + a_2 + \cdots + a_n}{n} = s, \qquad 0 < s \le \frac{\pi}{4},$$

then

$$\cos a_1 + \cos a_2 + \cdots + \cos a_n \le n \cos s,$$

with equality for $a_1 = a_2 = \cdots = a_n = s$.

$\square$

P 3.40. *If $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ are real numbers so that*

$$a_1, a_2, \ldots, a_n \geq \frac{-1}{n-2}, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$\frac{a_1^2}{a_1^2 - a_1 + 1} + \frac{a_2^2}{a_2^2 - a_2 + 1} + \cdots + \frac{a_n^2}{a_n^2 - a_n + 1} \leq n.$$

(Vasile Cirtoaje, 2012)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \qquad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{1-u}{u^2 - u + 1}, \qquad u \in \mathbb{I} = \left[\frac{-1}{n-2}, \frac{n^2 - n - 1}{n-2}\right].$$

Let $s_0 = 2$. We have $s < s_0$ and

$$\min_{u \in \mathbb{I}} f(u) = f(s_0)$$

because

$$f(u) - f(2) = \frac{1-u}{u^2 - u + 1} + \frac{1}{3} = \frac{(u-2)^2}{3(u^2 - u + 1)} \geq 0.$$

From

$$f'(u) = \frac{u(u-2)}{(u^2 - u + 1)^2},$$

$$f''(u) = \frac{2(3u^2 - u^3 - 1)}{(u^2 - u + 1)^3} = \frac{2u^2(2-u) + 2(u^2 - 1)}{(u^2 - u + 1)^3},$$

it follows that f is convex on $[1, s_0]$. However, we can't apply the RPCF-Theorem in its original form because f is not decreasing on $\mathbb{I}_{\leq s_0}$. According to Theorem 1, we may replace this condition with $ns - (n-1)s_0 \leq \inf \mathbb{I}$. Indeed, we have

$$ns - (n-1)s_0 = n - 2(n-1) = -n + 2 \leq \frac{-1}{n-2} = \inf \mathbb{I}.$$

So, it suffices to show that $f(x) + (n-1)f(y) \geq nf(1)$ for all $x, y \in \mathbb{I}$ so that

$$x + (n-1)y = n.$$

According to Note 1, we only need to show that $h(x, y) \geq 0$, where

$$g(u) = \frac{f(u) - f(1)}{u - 1}, \quad h(x, y) = \frac{g(x) - g(y)}{x - y}.$$

We have

$$g(u)=\frac{-1}{u^2-u+1},$$

$$h(x,y)=\frac{x+y-1}{(x^2-x+1)(y^2-y+1)}=\frac{(n-2)x+1}{(n-1)(x^2-x+1)(y^2-y+1)}\geq 0.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$, and also for

$$a_1=\frac{-1}{n-2},\qquad a_2=a_3=\cdots=a_n=\frac{n-1}{n-2}$$

(or any cyclic permutation).

□

P 3.41. *If $a_1,a_2,\ldots,a_n$ $(n\geq 3)$ are nonzero real numbers so that*

$$a_1,a_2,\ldots,a_n\geq\frac{-n}{n-2},\qquad a_1+a_2+\cdots+a_n=n,$$

then

$$\frac{1}{a_1^2}+\frac{1}{a_2^2}+\cdots+\frac{1}{a_n^2}\geq\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}.$$

(Vasile Cirtoaje, 2012)

Solution. According to P 2.25-(a) in Volume 1, the inequality is true for $n=3$. Assume further that $n\geq 4$ and write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\geq nf(s),\qquad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=\frac{1}{u^2}-\frac{1}{u},\qquad u\in\mathbb{I}=\left[\frac{-n}{n-2},\frac{n(2n-3)}{n-2}\right]\setminus\{0\}.$$

Let

$$s_0=2,\quad s<s_0.$$

From

$$f(u)-f(2)=\frac{1}{u^2}-\frac{1}{u}+\frac{1}{4}=\frac{(u-2)^2}{4u^2}\geq 0,$$

it follows that

$$\min_{u\in\mathbb{I}}f(u)=f(s_0),$$

while from

$$f'(u)=\frac{u-2}{u^3},\qquad f''(u)=\frac{2(3-u)}{u^4},$$

it follows that f is convex on $[s, s_0]$. However, we can't apply the RPCF-Theorem because f is not decreasing on $\mathbb{I}_{\le s_0}$. According to Theorem 1 and Note 6, we may replace this condition with $ns-(n-1)s_0 \le \inf \mathbb{I}$. For $n \ge 4$, we have

$$ns-(n-1)s_0 = n-2(n-1) = -n+2 \le \frac{-n}{n-2} = \inf \mathbb{I}.$$

So, according to Note 1, it suffices to show that $h(x,y) \ge 0$ for all $x, y \in \mathbb{I}$ so that $x+(n-1)y = n$. We have

$$g(u) = \frac{f(u)-f(1)}{u-1} = \frac{-1}{u^2},$$

$$h(x,y) = \frac{g(x)-g(y)}{x-y} = \frac{x+y}{x^2y^2} = \frac{(n-2)x+n}{(n-1)x^2y^2} \ge 0.$$

The proof is completed. By Note 3, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = \frac{-n}{n-2}, \qquad a_2 = a_3 = \cdots = a_n = \frac{n}{n-2}$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- *Let $a_1, a_2, \ldots, a_n \ge \frac{-n}{n-2}$ so that $a_1+a_2+\cdots+a_n = n$. If $n \ge 3$ and $k \ge 0$, then*

$$\frac{1-a_1}{k+a_1^2} + \frac{1-a_2}{k+a_2^2} + \cdots + \frac{1-a_n}{k+a_n^2} \ge 0,$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = \frac{-n}{n-2}, \qquad a_2 = a_3 = \cdots = a_n = \frac{n}{n-2}$$

(or any cyclic permutation).

□

P 3.42. *If $a_1, a_2, \ldots, a_n \ge -1$ so that $a_1+a_2+\cdots+a_n = n$, then*

$$(n+1)\left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2}\right) \ge 2n + (n-1)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

(Vasile C., 2013)

Solution. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\geq nf(s),\qquad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=\frac{n+1}{u^2}-\frac{n-1}{u},\quad u\in\mathbb{I}=[-1,2n-1]\setminus\{0\}.$$

Let

$$s_0=\frac{2(n+1)}{n-1}\in\mathbb{I},\qquad s<s_0.$$

Since

$$f(u)-f(s_0)=\frac{[(n-1)u-2(n+1)]^2}{4(n+1)u^2}\geq 0,$$

we have

$$\min_{u\in\mathbb{I}}f(u)=f(s_0).$$

From

$$f'(u)=\frac{(n-1)u-2(n+1)}{u^3},\qquad f''(u)=\frac{6(n+1)-2(n-1)u}{u^4},$$

it follows that f is convex on $[1,s_0]$. Since f is not decreasing on $\mathbb{I}_{\leq s_0}$, according to Theorem 1 and Note 6, we may replace this condition in RPCF-Theorem with $ns-(n-1)s_0\leq\inf\mathbb{I}$. We have

$$ns-(n-1)s_0=n-2(n+1)=-n-2<-1=\inf\mathbb{I}.$$

According to Note 1, we only need to show that $h(x,y)\geq 0$ for $-1\leq x\leq 1\leq y$ and $x+(n-1)y=n$. We have

$$g(u)=\frac{f(u)-f(1)}{u-1}=-\frac{2}{u}-\frac{n+1}{u^2}$$

and

$$h(x,y)=\frac{g(x)-g(y)}{x-y}=\frac{2xy+(n+1)(x+y)}{x^2y^2}=\frac{(x+1)(n^2+n-2x)}{(n-1)x^2y^2}\geq 0.$$

According to Note 4, the equality holds for $a_1=a_2=\cdots=a_n=1$, and also for

$$a_1=-1,\qquad a_2=\cdots=a_n=\frac{n+1}{n-1}$$

(or any cyclic permutation).

$\square$

P 3.43. *If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are real numbers so that*

$$a_1, a_2, \ldots, a_n \ge \frac{-(3n-2)}{n-2}, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$\frac{1-a_1}{(1+a_1)^2} + \frac{1-a_2}{(1+a_2)^2} + \cdots + \frac{1-a_n}{(1+a_n)^2} \ge 0.$$

(Vasile C., 2014)

Solution. According to P 2.25-(b) in Volume 1, the inequality is true for $n = 3$. Assume further that $n \ge 4$ and write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{1-u}{(1+u)^2}, \qquad u \in \mathbb{I} = \left[\frac{-(3n-2)}{n-2}, \frac{4n^2-7n+2}{n-2}\right] \setminus \{-1\}.$$

Let

$$s_0 = 3, \qquad s < s_0.$$

From

$$f(u) - f(3) = \frac{1-u}{(1+u)^2} + \frac{1}{8} = \frac{(u-3)^2}{8(u+1)^2} \ge 0,$$

it follows that

$$\min_{u \in \mathbb{I}} f(u) = f(s_0).$$

From

$$f'(u) = \frac{u-3}{(u+1)^3}, \qquad f''(u) = \frac{2(5-u)}{(u+1)^4},$$

it follows that f is convex on $[1, s_0]$. We can't apply the RPCF-Theorem in its original form because f is not decreasing on $\mathbb{I}_{\le s_0}$. However, according to Theorem 1 and Note 6, we may replace this condition with $ns - (n-1)s_0 \le \inf \mathbb{I}$. Indeed, for $n \ge 4$, we have

$$ns - (n-1)s_0 = n - 3(n-1) = -2n + 3 \le \frac{-(3n-2)}{n-2} = \inf \mathbb{I}.$$

According to Note 1, it suffices to show that $h(x, y) \ge 0$ for all $x, y \in \mathbb{I}$ so that $x \le 1 \le y$ and $x + (n-1)y = n$. We have

$$g(u) = \frac{f(u) - f(1)}{u-1} = \frac{-1}{(u+1)^2},$$

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{x + y + 2}{(x+1)^2(y+1)^2} = \frac{(n-2)x + 3n - 2}{(n-1)(x+1)^2(y+1)^2} \ge 0.$$

In accordance with Note 3, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for
$$a_1 = \frac{-(3n-2)}{n-2}, \qquad a_2 = a_3 = \cdots = a_n = \frac{n+2}{n-2}$$
(or any cyclic permutation).

□

P 3.44. *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If $n \ge 3$ and $k \ge 2 - \frac{2}{n}$, then*
$$\frac{1-a_1}{(1-ka_1)^2} + \frac{1-a_2}{(1-ka_2)^2} + \cdots + \frac{1-a_n}{(1-ka_n)^2} \ge 0.$$

(Vasile C., 2012)

Solution. According to P 3.99 in Volume 1, the inequality is true for $n = 3$. Assume further that $n \ge 4$ and write the inequality as
$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \qquad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$
where
$$f(u) = \frac{1-u}{(1-ku)^2}, \qquad u \in \mathbb{I} = [0, n] \setminus \{1/k\}.$$
Let
$$s_0 = 2 - 1/k, \qquad 1 = s < s_0.$$
Since
$$f(u) - f(s_0) = \frac{1-u}{(1-ku)^2} + \frac{1}{4k(k-1)} = \frac{(ku-2k+1)^2}{4k(k-1)(1-ku)^2} \ge 0,$$
we have
$$\min_{u \in \mathbb{I}} f(u) = f(s_0).$$
From
$$f'(u) = \frac{ku-2k+1}{(ku-1)^3}, \qquad f''(u) = \frac{2k(-ku+3k-2)}{(1-ku)^4},$$
it follows that f is convex on $[1, s_0]$. We can't apply the RPCF-Theorem because f is not decreasing on $\mathbb{I}_{\le s_0}$. According to Theorem 1 and Note 6, we may replace this condition with $ns - (n-1)s_0 \le \inf \mathbb{I}$. Indeed, we have
$$ns - (n-1)s_0 \le n - (n-1) \cdot \frac{3n-4}{2(n-1)} = \frac{4-n}{2} \le 0 = \inf \mathbb{I}.$$

So, it suffices to show that $f(x)+(n-1)f(y) \ge nf(1)$ for all $x, y \in \mathbb{I}$ so that $x \le 1 \le y$ and $x+(n-1)y = n$. According to Note 1, we only need to show that $h(x, y) \ge 0$, where

$$g(u) = \frac{f(u)-f(1)}{u-1}, \quad h(x,y) = \frac{g(x)-g(y)}{x-y}.$$

Since

$$g(u) = \frac{-1}{(1-ku)^2}, \qquad h(x,y) = \frac{k[k(x+y)-2]}{(1-kx)^2(1-ky)^2},$$

we need to show that $k(x+y)-2 \ge 0$. Indeed, we have

$$\frac{k(x+y)-2}{2} \ge \frac{(n-1)(x+y)}{n} - 1 = \frac{(n-1)(x+y)}{n} - \frac{x+(n-1)y}{n} = \frac{(n-2)x}{n} \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $k = 2 - \dfrac{2}{n}$, then the equality also holds for

$$a_1 = 0, \qquad a_2 = a_3 = \cdots = a_n = \frac{n}{n-1}$$

(or any cyclic permutation).

□

Chapter 4

Partially Convex Function Method for Ordered Variables

4.1 Theoretical Basis

The following statement is known as Right Partially Convex Function Theorem for Ordered Variables (RPCF-OV Theorem).

RPCF-OV Theorem (*Vasile Cirtoaje*, 2014). *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $[s, s_0]$, where $s, s_0 \in \mathbb{I}$, $s < s_0$. In addition, f is decreasing on $\mathbb{I}_{\le s_0}$ and $f(u) \ge f(s_0)$ for $u \in \mathbb{I}$. The inequality*

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying

$$a_1 + a_2 + \cdots + a_n = ns$$

and

$$a_1 \le a_2 \le \cdots \le a_m \le s, \quad m \in \{1, 2, \ldots, n-1\},$$

if and only if

$$f(x) + (n-m)f(y) \ge (1+n-m)f(s)$$

for all $x, y \in \mathbb{I}$ so that $x \le s \le y$ and $x + (n-m)y = (1+n-m)s$.

Proof. For

$$a_1 = x, \quad a_2 = \cdots = a_m = s, \quad a_{m+1} = \cdots = a_n = y,$$

the inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s)$$

becomes

$$f(x) + (n-m)f(y) \ge (1+n-m)f(s);$$

therefore, the necessity is obvious. By Lemma from Chapter 3, to prove the sufficiency, it suffices to consider that $a_1, a_2, \ldots, a_n \in \mathbb{J}$, where

$$\mathbb{J} = \mathbb{I}_{\le s_0}.$$

Because f is convex on $\mathbb{J}_{\ge s}$, the desired inequality follows from HCF-OV Theorem applied to the interval $\mathbb{J}$.

Similarly, we can prove Left Partially Convex Function Theorem for Ordered Variables (LPCF-OV Theorem).

LPCF-OV Theorem. *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $[s_0, s]$, where $s_0, s \in \mathbb{I}$, $s_0 < s$. In addition, f is increasing on $\mathbb{I}_{\ge s_0}$ and $f(u) \ge f(s_0)$ for $u \in \mathbb{I}$. The inequality*

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying

$$a_1 + a_2 + \cdots + a_n = ns$$

and

$$a_1 \ge a_2 \ge \cdots \ge a_m \ge s, \quad m \in \{1, 2, \ldots, n-1\},$$

if and only if

$$f(x) + (n-m)f(y) \ge (1+n-m)f(s)$$

for all $x, y \in \mathbb{I}$ so that $x \ge s \ge y$ and $x + (n-m)y = (1+n-m)s$.

The RPCF-OV Theorem and the LPCF-OV Theorems are respectively generalizations of the RPCF Theorem and LPCF Theorem, because the last theorems can be obtained from the first theorems for $m = 1$.

Note 1. Let us denote

$$g(u) = \frac{f(u) - f(s)}{u - s}, \quad h(x, y) = \frac{g(x) - g(y)}{x - y}.$$

We may replace the hypothesis condition in the RPCF-OV Theorem and the LPCF-OV Theorem, namely

$$f(x) + mf(y) \ge (1+m)f(s),$$

by the condition

$$h(x, y) \ge 0 \ \text{ for all } \ x, y \in \mathbb{I} \ \text{ so that } \ x + my = (1+m)s.$$

Note 2. Assume that f is differentiable on $\mathbb{I}$, and let

$$H(x, y) = \frac{f'(x) - f'(y)}{x - y}.$$

The desired inequality of Jensen's type in the RPCF-OV Theorem and the LPCF-OV Theorem holds true by replacing the hypothesis

$$f(x)+mf(y) \geq (1+m)f(s)$$

with the more restrictive condition

$$H(x,y) \geq 0 \;\; \textit{for all} \;\; x,y \in \mathbb{I} \;\; \textit{so that} \;\; x+my=(1+m)s.$$

Note 3. The desired inequalities in the RPCF-OV Theorem and the LPCF-OV Theorem become equalities for

$$a_1 = a_2 = \cdots = a_n = s.$$

In addition, if there exist $x, y \in \mathbb{I}$ so that

$$x+(n-m)y=(1+n-m)s, \quad f(x)+(n-m)f(y)=(1+n-m)f(s), \quad x \neq y,$$

then the equality holds also for

$$a_1 = x, \quad a_2 = \cdots = a_m = s, \quad a_{m+1} = \cdots = a_n = y$$

(or any cyclic permutation). Notice that these equality conditions are equivalent to

$$x+(n-m)y=(1+n-m)s, \quad h(x,y)=0$$

($x < y$ for RHCF-OV Theorem, and $x > y$ for LHCF-OV Theorem).

Note 4. The RPCF-OV Theorem is also valid in the case where f is defined on $\mathbb{I} \setminus \{u_0\}$, where u_0 is an interior point of $\mathbb{I}$ so that $u_0 > s0$. Similarly, LPCF Theorem is also valid in the case in which f is defined on $\mathbb{I} \setminus \{u_0\}$, where u_0 is an interior point of $\mathbb{I}$ so that $u_0 < s0$.

Note 5. The RPCF-Theorem holds true by replacing the condition

$$f \textit{ is decreasing on } \mathbb{I}_{\leq s_0}$$

with

$$ns-(n-1)s_0 \leq \inf \mathbb{I}.$$

More precisely, the following theorem holds:

Theorem 1. *Let f be a function defined on a real interval $\mathbb{I}$, convex on $[s, s_0]$ and satisfying*

$$\min_{u \in \mathbb{I}_{\geq s}} f(u) = f(s_0),$$

where

$$s, s_0 \in \mathbb{I}, \;\; s < s_0, \;\; (1+n-m)s-(n-m)s_0 \leq \inf \mathbb{I}.$$

The inequality

$$f(a_1)+f(a_2)+\cdots+f(a_n) \geq nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ *satisfying*

$$a_1 + a_2 + \cdots + a_n = ns$$

and

$$a_1 \le a_2 \le \cdots \le a_m \le s, \quad m \in \{1, 2, \ldots, n-1\},$$

if and only if

$$f(x) + (n-m)f(y) \ge (1+n-m)f(s)$$

for all $x, y \in \mathbb{I}$ *so that* $x \le s \le y$ *and* $x + (n-m)y = (1+n-m)s$.

The proof of this theorem is similar to the one of Theorem 1 from chapter 3.

Similarly, the LPCF-Theorem holds true by replacing the condition

$$f \text{ is increasing on } \mathbb{I}_{\ge s_0}$$

with

$$ns - (n-1)s_0 \ge \sup \mathbb{I}.$$

More precisely, the following theorem holds:

Theorem 2. *Let* f *be a function defined on a real interval* $\mathbb{I}$*, convex on* $[s_0, s]$ *and satisfying*

$$\min_{u \in \mathbb{I}_{\le s}} f(u) = f(s_0),$$

where

$$s, s_0 \in \mathbb{I}, \quad s > s_0, \quad (1+n-m)s - (n-m)s_0 \ge \sup \mathbb{I}.$$

The inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ *satisfying*

$$a_1 + a_2 + \cdots + a_n = ns$$

and

$$a_1 \ge a_2 \ge \cdots \ge a_m \ge s, \quad m \in \{1, 2, \ldots, n-1\},$$

if and only if

$$f(x) + (n-m)f(y) \ge (1+n-m)f(s)$$

for all $x, y \in \mathbb{I}$ *so that* $x \ge s \ge y$ *and* $x + (n-m)y = (1+n-m)s$.

Note 6. Theorem 1 is also valid in the case in which f is defined on $\mathbb{I} \setminus \{u_0\}$, where u_0 is an interior point of $\mathbb{I}$ so that $u_0 \notin [s, s_0]$. Similarly, Theorem 2 is also valid in the case in which f is defined on $\mathbb{I} \setminus \{u_0\}$, where u_0 is an interior point of $\mathbb{I}$ so that $u_0 \notin [s_0, s]$.

Note 7. We can extend *weighted* Jensen's inequality to right and left partially convex functions with ordered variables establishing the WRPCF-OV Theorem and the WLPCF-OV Theorem (*Vasile Cirtoaje*, 2014).

WRPCF-OV Theorem. *Let $p_1, p_2, \ldots, p_n$ be positive real numbers so that*

$$p_1 + p_2 + \cdots + p_n = 1,$$

and let f be a real function defined on an interval $\mathbb{I}$ and convex on $[s, s_0]$, where $s, s_0 \in int(\mathbb{I})$, $s < s_0$. In addition, f is decreasing on $\mathbb{I}_{\le s_0}$ and $f(u) \ge f(s_0)$ for $u \in \mathbb{I}$. The inequality

$$p_1 f(x_1) + p_2 f(x_2) + \cdots + p_n f(x_n) \ge f(p_1 x_1 + p_2 x_2 + \cdots + p_n x_n)$$

holds for all $x_1, x_2, \ldots, x_n \in \mathbb{I}$ so that $p_1 x_1 + p_2 x_2 + \cdots + p_n x_n = s$ and

$$x_1 \le x_2 \le \cdots \le x_n, \quad x_m \le s, \quad m \in \{1, 2, \ldots, n-1\},$$

if and only if

$$f(x) + k f(y) \ge (1+k) f(s)$$

for all $x, y \in \mathbb{I}$ satisfying

$$x \le s \le y, \quad x + ky = (1+k)s,$$

where

$$k = \frac{p_{m+1} + p_{m+2} + \cdots + p_n}{p_1}.$$

WLPCF-OV Theorem. *Let $p_1, p_2, \ldots, p_n$ be positive real numbers so that*

$$p_1 + p_2 + \cdots + p_n = 1,$$

and let f be a real function defined on an interval $\mathbb{I}$ and convex on $[s_0, s]$, where $s_0, s \in \mathbb{I}$, $s_0 < s$. In addition, f is increasing on $\mathbb{I}_{\ge s_0}$ and $f(u) \ge f(s_0)$ for $u \in \mathbb{I}$. The inequality

$$p_1 f(x_1) + p_2 f(x_2) + \cdots + p_n f(x_n) \ge f(p_1 x_1 + p_2 x_2 + \cdots + p_n x_n)$$

holds for all $x_1, x_2, \ldots, x_n \in \mathbb{I}$ so that $p_1 x_1 + p_2 x_2 + \cdots + p_n x_n = s$ and

$$x_1 \ge x_2 \ge \cdots \ge x_n, \quad x_m \ge s, \quad m \in \{1, 2, \ldots, n-1\},$$

if and only if

$$f(x) + k f(y) \ge (1+k) f(s)$$

for all $x, y \in \mathbb{I}$ satisfying

$$x \ge s \ge y, \quad x + ky = (1+k)s,$$

where

$$k = \frac{p_{m+1} + p_{m+2} + \cdots + p_n}{p_1}.$$

For the most commonly used case

$$p_1 = p_2 = \cdots = p_n = \frac{1}{n},$$

the WRPCF-OV Theorem and the WLPCF-OV Theorem yield the RPCF-OV Theorem and the LPCF-OV Theorem, respectively.

4.2 Applications

4.1. If a,b,c,d are real numbers so that

$$a \le 1 \le b \le c \le d, \qquad a+b+c+d=4,$$

then

$$\frac{a}{3a^2+1}+\frac{b}{3b^2+1}+\frac{c}{3c^2+1}+\frac{d}{3d^2+1} \le 1.$$

4.2. If a,b,c,d are real numbers so that

$$a \ge b \ge 1 \ge c \ge d, \qquad a+b+c+d=4,$$

then

$$\frac{16a-5}{32a^2+1}+\frac{16b-5}{32b^2+1}+\frac{16c-5}{32c^2+1}+\frac{16d-5}{32d^2+1} \le \frac{4}{3}.$$

4.3. If a,b,c,d,e are real numbers so that

$$a \ge b \ge 1 \ge c \ge d \ge e, \qquad a+b+c+d+e=5,$$

then

$$\frac{18a-5}{12a^2+1}+\frac{18b-5}{12b^2+1}+\frac{18c-5}{12c^2+1}+\frac{18d-5}{12d^2+1}+\frac{18e-5}{12e^2+1} \le 5.$$

4.4. If a,b,c,d,e are real numbers so that

$$a \ge b \ge 1 \ge c \ge d \ge e, \qquad a+b+c+d+e=5,$$

then

$$\frac{a(a-1)}{3a^2+4}+\frac{b(b-1)}{3b^2+4}+\frac{c(c-1)}{3c^2+4}+\frac{d(d-1)}{3d^2+4}+\frac{e(e-1)}{3e^2+4} \ge 0.$$

4.5. Let $a_1,a_2,\ldots,a_{2n} \ne -k$ be real numbers so that

$$a_1 \ge \cdots \ge a_n \ge 1 \ge a_{n+1} \ge \cdots \ge a_{2n}, \qquad a_1+a_2+\cdots+a_{2n}=2n.$$

If $k \ge \dfrac{n+1}{2\sqrt{n}}$, then

$$\frac{a_1(a_1-1)}{(a_1+k)^2}+\frac{a_2(a_2-1)}{(a_2+k)^2}+\cdots+\frac{a_{2n}(a_{2n}-1)}{(a_{2n}+k)^2} \ge 0.$$

4.6. Let $a_1, a_2, \dots, a_{2n} \neq -k$ be real numbers so that

$$a_1 \ge \cdots \ge a_n \ge 1 \ge a_{n+1} \ge \cdots \ge a_{2n}, \qquad a_1 + a_2 + \cdots + a_{2n} = 2n.$$

If $k \ge 1 + \dfrac{n+1}{\sqrt{n}}$, then

$$\frac{a_1^2 - 1}{(a_1 + k)^2} + \frac{a_2^2 - 1}{(a_2 + k)^2} + \cdots + \frac{a_{2n}^2 - 1}{(a_{2n} + k)^2} \ge 0.$$

4.7. If $a_1, a_2, \dots, a_n$ are positive real numbers so that

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$a_1^{3/a_1} + a_2^{3/a_2} + \cdots + a_n^{3/a_n} \le n.$$

4.8. If $a_1, a_2, \dots, a_{11}$ are real numbers so that

$$a_1 \ge a_2 \ge 1 \ge a_3 \ge \cdots \ge a_{11}, \qquad a_1 + a_2 + \cdots + a_{11} = 11,$$

then

$$(1 - a_1 + a_1^2)(1 - a_2 + a_2^2) \cdots (1 - a_{11} + a_{11}^2) \ge 1.$$

4.9. If $a_1, a_2, \dots, a_8$ are nonzero real numbers so that

$$a_1 \ge a_2 \ge a_3 \ge a_4 \ge 1 \ge a_5 \ge a_6 \ge a_7 \ge a_8, \qquad a_1 + a_2 + \cdots + a_8 = 8,$$

then

$$5\left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_8^2}\right) + 72 \ge 14\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_8}\right).$$

4.10. If a, b, c, d are positive real numbers so that

$$a \le b \le 1 \le c \le d, \qquad abcd = 1,$$

then

$$\frac{7 - 6a}{2 + a^2} + \frac{7 - 6b}{2 + b^2} + \frac{7 - 6c}{2 + c^2} + \frac{7 - 6d}{2 + d^2} \ge \frac{4}{3}.$$

4.11. If a, b, c are positive real numbers so that

$$a \le b \le 1 \le c, \qquad abc = 1,$$

then

$$\frac{7-4a}{2+a^2} + \frac{7-4b}{2+b^2} + \frac{7-4c}{2+c^2} \ge 3.$$

4.12. If a, b, c are positive real numbers so that

$$a \ge 1 \ge b \ge c, \qquad abc = 1,$$

then

$$\frac{23-8a}{3+2a^2} + \frac{23-8b}{3+2b^2} + \frac{23-8c}{3+2c^2} \ge 9.$$

4.13. Let $a_1, a_2, \dots, a_n$ be positive real numbers so that

$$a_1 \le \cdots \le a_{n-1} \le 1 \le a_n, \qquad a_1 a_2 \cdots a_n = 1.$$

If $p, q \ge 0$ so that $p + 3q \ge 1$, then

$$\frac{1-a_1}{1+pa_1+qa_1^2} + \frac{1-a_2}{1+pa_2+qa_2^2} + \cdots + \frac{1-a_n}{1+pa_n+qa_n^2} \ge 0.$$

4.14. If a, b, c, d, e are real numbers so that

$$-2 \le a \le b \le 1 \le c \le d \le e, \qquad a+b+c+d+e = 5,$$

then

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{e^2} \ge \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}.$$

4.3 Solutions

P 4.1. *If a,b,c,d are real numbers so that*

$$a \le 1 \le b \le c \le d, \quad a+b+c+d=4,$$

then

$$\frac{a}{3a^2+1}+\frac{b}{3b^2+1}+\frac{c}{3c^2+1}+\frac{d}{3d^2+1} \le 1.$$

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d) \ge 4f(s), \quad s=\frac{a+b+c+d}{4}=1,$$

where

$$f(u)=\frac{-u}{3u^2+1}, \quad u \in \mathbb{R}.$$

From

$$f'(u)=\frac{3u^2-1}{(3u^2+1)^2},$$

it follows that f is increasing on $(-\infty,-s_0]\cup[s_0,\infty)$ and decreasing on $[-s_0,s_0]$, where $s_0=1/\sqrt{3}$. Since

$$\lim_{u\to-\infty} f(u)=0$$

and $f(s_0)<0$, it follows that

$$\min_{u\in\mathbb{R}} f(u)=f(s_0).$$

From

$$f''(u)=\frac{18u(1-u^2)}{(3u^2+1)^3},$$

it follows that f is convex on $[0,1]$, hence on $[s_0,1]$. Therefore, we may apply the LPCF-OV Theorem for $n=4$ and $m=1$. We only need to show that $f(x)+f(y)\ge 2f(1)$ for all real x,y so that $x+y=2$. Using Note 1, it suffices to prove that $h(x,y)\ge 0$, where

$$h(x,y)=\frac{g(x)-g(y)}{x-y}, \quad g(u)=\frac{f(u)-f(1)}{u-1}.$$

Indeed, we have

$$g(u)=\frac{3u-1}{4(3u^2+1)},$$

$$h(x,y)=\frac{3(1+x+y-3xy)}{4(3x^2+1)(3y^2+1)}=\frac{9(1-xy)}{4(3x^2+1)(3y^2+1)}\ge 0,$$

since

$$4(1-xy)=(x+y)^2-4xy=(x-y)^2\geq 0.$$

Thus, the proof is completed. The equality holds for $a=b=c=d=1$.

Remark. Similarly, we can prove the following generalization:

- *If $a_1,a_2,\ldots,a_n$ are real numbers so that*

$$a_1\leq 1\leq a_2\leq\cdots\leq a_n,\qquad a_1+a_2+\cdots+a_n=n,$$

then

$$\frac{a_1}{3a_1^2+1}+\frac{a_2}{3a_2^2+1}+\cdots+\frac{a_n}{3a_n^2+1}\leq\frac{n}{4},$$

with equality for $a_1=a_2=\cdots=a_n=1$.

□

P 4.2. *If a,b,c,d are real numbers so that*

$$a\geq b\geq 1\geq c\geq d,\qquad a+b+c+d=4,$$

then

$$\frac{16a-5}{32a^2+1}+\frac{16b-5}{32b^2+1}+\frac{16c-5}{32c^2+1}+\frac{16d-5}{32d^2+1}\leq\frac{4}{3}.$$

(Vasile C., 2012)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d)\geq 4f(s),\qquad s=\frac{a+b+c+d}{4}=1,$$

where

$$f(u)=\frac{5-16u}{32u^2+1},\qquad u\in\mathbb{R}.$$

As shown in the proof of P 3.1, f is convex on $[s_0,1]$, increasing for $u\geq s_0$ and

$$\min_{u\in\mathbb{R}}f(u)=f(s_0),$$

where

$$s_0=\frac{5+\sqrt{33}}{16}\approx 0.6715.$$

Therefore, we may apply the LPCF-OV Theorem for $n=4$ and $m=2$. We only need to show that $f(x)+2f(y)\geq 3f(1)$ for all real x,y so that $x+2y=3$. Using Note 1, it suffices to prove that $h(x,y)\geq 0$, where

$$h(x,y)=\frac{g(x)-g(y)}{x-y},\quad g(u)=\frac{f(u)-f(1)}{u-1}.$$

Indeed, we have

$$g(u)=\frac{32(2u-1)}{3(32u^2+1)},$$

$$h(x,y)=\frac{64(1+16x+16y-32xy)}{3(32x^2+1)(32y^2+1)}=\frac{64(4x-5)^2}{3(32x^2+1)(32y^2+1)}\geq 0.$$

From $x+2y=3$ and $h(x,y)=0$, we get $x=5/4$ and $y=7/8$. Therefore, in accordance with Note 3, the equality holds for $a=b=c=d=1$, and also for

$$a=\frac{5}{4},\qquad b=1,\qquad c=d=\frac{7}{8}.$$

Remark. Similarly, we can prove the following generalization:

- *If $a_1,a_2,\dots,a_n$ $(n\geq 3)$ are real numbers so that*

$$a_1\geq\cdots\geq a_{n-2}\geq 1\geq a_{n-1}\geq a_n,\qquad a_1+a_2+\cdots+a_n=n,$$

then

$$\frac{16a_1-5}{32a_1^2+1}+\frac{16a_2-5}{32a_2^2+1}+\cdots+\frac{16a_n-5}{32a_n^2+1}\leq\frac{n}{3},$$

with equality for $a_1=a_2=\cdots=a_n=1$, and also for

$$a_1=\frac{5}{4},\qquad a_2=\cdots=a_{n-2}=1,\qquad a_{n-1}=a_n=\frac{7}{8}.$$

□

P 4.3. *If a,b,c,d,e are real numbers so that*

$$a\geq b\geq 1\geq c\geq d\geq e,\qquad a+b+c+d+e=5,$$

then

$$\frac{18a-5}{12a^2+1}+\frac{18b-5}{12b^2+1}+\frac{18c-5}{12c^2+1}+\frac{18d-5}{12d^2+1}+\frac{18e-5}{12e^2+1}\leq 5.$$

(Vasile C., 2012)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d)+f(e)\geq 5f(s),\qquad s=\frac{a+b+c+d+e}{5}=1,$$

where

$$f(u)=\frac{5-18u}{12u^2+1},\qquad u\in\mathbb{R}.$$

As shown in the proof of P 3.2, f is convex on $[s_0,1]$, increasing for $u\geq s_0$ and

$$\min_{u\in\mathbb{R}}f(u)=f(s_0),$$

where

$$s_0 = \frac{5+\sqrt{52}}{18} \approx 0.678.$$

Therefore, applying the LPCF-OV Theorem for $n = 5$ and $m = 3$, we only need to show that $f(x)+3f(y) \ge 4f(1)$ for all real x, y so that $x+3y = 4$. Using Note 1, it suffices to prove that $h(x, y) \ge 0$, where

$$h(x,y) = \frac{g(x)-g(y)}{x-y}, \quad g(u) = \frac{f(u)-f(1)}{u-1}.$$

Indeed, we have

$$g(u) = \frac{6(2u-1)}{12u^2+1},$$

$$h(x,y) = \frac{12(1+6x+6y-12xy)}{(12x^2+1)(12y^2+1)} = \frac{12(2x-3)^2}{(12x^2+1)(12y^2+1)} \ge 0.$$

From $x+3y = 4$ and $h(x, y) = 0$, we get $x = 3/2$ and $y = 5/6$. Therefore, in accordance with Note 3, the equality holds for $a = b = c = d = e = 1$, and also for

$$a = \frac{3}{2}, \quad b = 1, \quad c = d = e = \frac{5}{6}.$$

Remark. Similarly, we can prove the following generalization:

- *If $a_1, a_2, \ldots, a_n$ $(n \ge 4)$ are real numbers so that*

$$a_1 \ge \cdots \ge a_{n-3} \ge 1 \ge a_{n-2} \ge a_{n-1} \ge a_n, \quad a_1 + a_2 + \cdots + a_n = n,$$

then

$$\frac{18a_1-5}{12a_1^2+1} + \frac{18a_2-5}{12a_2^2+1} + \cdots + \frac{18a_n-5}{12a_n^2+1} \le n,$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = \frac{3}{2}, \quad a_2 = \cdots = a_{n-3} = 1, \quad a_{n-2} = a_{n-1} = a_n = \frac{5}{6}.$$

□

P 4.4. *If a, b, c, d, e are real numbers so that*

$$a \ge b \ge 1 \ge c \ge d \ge e, \quad a+b+c+d+e = 5,$$

then

$$\frac{a(a-1)}{3a^2+4} + \frac{b(b-1)}{3b^2+4} + \frac{c(c-1)}{3c^2+4} + \frac{d(d-1)}{3d^2+4} + \frac{e(e-1)}{3e^2+4} \ge 0.$$

(Vasile C., 2012)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d)+f(e) \ge 5f(s), \quad s=\frac{a+b+c+d+e}{5}=1,$$

where

$$f(u)=\frac{u^2-u}{3u^2+4}, \quad u\in\mathbb{R}.$$

As shown in the proof of P 3.5, f is convex on $[s_0,1]$, increasing for $u\ge s_0$ and

$$\min_{u\in\mathbb{R}} f(u)=f(s_0),$$

where

$$s_0=\frac{-4+2\sqrt{7}}{3}\approx 0.43.$$

Therefore, we may apply the LPCF-OV Theorem for $n=5$ and $m=2$. We only need to show that $f(x)+3f(y)\ge 4f(1)$ for all real x,y so that $x+3y=4$. Using Note 1, it suffices to prove that $h(x,y)\ge 0$. Indeed, we have

$$g(u)=\frac{f(u)-f(1)}{u-1}=\frac{u}{3u^2+4},$$

$$h(x,y)=\frac{g(x)-g(y)}{x-y}=\frac{4-3xy}{(3x^2+4)(3y^2+4)}=\frac{(x-2)^2}{(12x^2+1)(12y^2+1)}\ge 0.$$

From $x+3y=4$ and $h(x,y)=0$, we get $x=2$ and $y=2/3$. Therefore, in accordance with Note 3, the equality holds for

$$a=b=c=d=e=1,$$

and also for

$$a=2,\ b=1,\ c=d=e=\frac{2}{3}.$$

Remark. Similarly, we can prove the following generalizations:

- *If $a_1,a_2,\dots,a_n$ $(n\ge 4)$ are real numbers so that*

$$a_1\ge\cdots\ge a_{n-3}\ge 1\ge a_{n-2}\ge a_{n-1}\ge a_n, \quad a_1+a_2+\cdots+a_n=n,$$

then

$$\frac{a_1(a_1-1)}{3a_1^2+4}+\frac{a_2(a_2-1)}{3a_2^2+4}+\cdots+\frac{a_n(a_n-1)}{3a_n^2+4}\ge 0,$$

with equality for $a_1=a_2=\cdots=a_n=1$, and also for

$$a_1=2, \quad a_2=\cdots=a_{n-3}=1, \quad a_{n-2}=a_{n-1}=a_n=\frac{2}{3}.$$

- *If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are real numbers so that*

$$a_1 \ge a_2 \ge 1 \ge a_3 \ge \cdots \ge a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$\frac{a_1(a_1-1)}{4(n-2)a_1^2+(n-1)^2} + \frac{a_2(a_2-1)}{4(n-2)a_2^2+(n-1)^2} + \cdots + \frac{a_n(a_n-1)}{4(n-2)a_n^2+(n-1)^2} \ge 0,$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = \frac{n-1}{2}, \qquad a_2 = 1, \qquad a_3 = \cdots = a_n = \frac{n-1}{2(n-2)}.$$

□

P 4.5. *Let $a_1, a_2, \ldots, a_{2n} \ne -k$ be real numbers so that*

$$a_1 \ge \cdots \ge a_n \ge 1 \ge a_{n+1} \ge \cdots \ge a_{2n}, \qquad a_1 + a_2 + \cdots + a_{2n} = 2n.$$

If $k \ge \dfrac{n+1}{2\sqrt{n}}$, then

$$\frac{a_1(a_1-1)}{(a_1+k)^2} + \frac{a_2(a_2-1)}{(a_2+k)^2} + \cdots + \frac{a_{2n}(a_{2n}-1)}{(a_{2n}+k)^2} \ge 0.$$

(Vasile C., 2012)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_{2n}) \ge 2nf(s), \qquad s = \frac{a_1 + a_2 + \cdots + a_{2n}}{2n} = 1,$$

where

$$f(u) = \frac{u(u-1)}{(u+k)^2}, \quad u \in \mathbb{I} = \mathbb{R} \setminus \{-k\}.$$

As shown in the proof of P 3.8, f is convex on $[s_0, 1]$, increasing for $u \ge s_0$ and

$$\min_{u \in \mathbb{I}} f(u) = f(s_0),$$

where

$$s_0 = \frac{k}{2k+1} < 1.$$

Having in view Note 4, we may apply the LPCF-OV Theorem for $2n$ real numbers and $m = n$. We only need to show that $f(x) + nf(y) \ge (n+1)f(1)$ for $x, y \in \mathbb{I}$ so that $x + ny = n+1$. Using Note 1, it suffices to prove that $h(x,y) \ge 0$. We have

$$g(u) = \frac{f(u) - f(1)}{u-1} = \frac{u}{(u+k)^2},$$

$$h(x,y)=\frac{g(x)-g(y)}{x-y}=\frac{k^2-xy}{(x+k)^2(y+k)^2}\geq 0,$$

because

$$k^2-xy\geq \frac{(n+1)^2}{4n}-xy=\frac{(x+ny)^2}{4n}-xy=\frac{(x-ny)^2}{4n}\geq 0.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$. If $k=\dfrac{n+1}{2\sqrt{n}}$, then the equality holds also for

$$a_1=\frac{n+1}{2},\qquad a_2=\cdots=a_n=1,\qquad a_{n+1}=\cdots=a_{2n}=\frac{n+1}{2n}.$$

□

P 4.6. *Let $a_1,a_2,\ldots,a_{2n}\neq -k$ be real numbers so that*

$$a_1\geq\cdots\geq a_n\geq 1\geq a_{n+1}\geq\cdots\geq a_{2n},\qquad a_1+a_2+\cdots+a_{2n}=2n.$$

If $k\geq 1+\dfrac{n+1}{\sqrt{n}}$, then

$$\frac{a_1^2-1}{(a_1+k)^2}+\frac{a_2^2-1}{(a_2+k)^2}+\cdots+\frac{a_{2n}^2-1}{(a_{2n}+k)^2}\geq 0.$$

(Vasile C., 2012)

Solution. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_{2n})\geq 2nf(s),\qquad s=\frac{a_1+a_2+\cdots+a_{2n}}{2n}=1,$$

where

$$f(u)=\frac{u^2-1}{(u+k)^2},\quad u\in\mathbb{I}=\mathbb{R}\setminus\{-k\}.$$

As shown in the proof of P 3.9, f is convex on $[s_0,1]$, increasing for $u\geq s_0$ and

$$\min_{u\in\mathbb{I}} f(u)=f(s_0),$$

where

$$s_0=\frac{-1}{k}\in(-1,0).$$

According to Note 4, we may apply the LPCF-OV Theorem for $2n$ real numbers and $m=n$. Thus, we only need to show that $f(x)+nf(y)\geq (n+1)f(1)$ for $x,y\in\mathbb{I}$ so that $x+ny=n+1$. Using Note 1, it suffices to prove that $h(x,y)\geq 0$. We have

$$g(u)=\frac{f(u)-f(1)}{u-1}=\frac{u+1}{(u+k)^2},$$

$$h(x,y)=\frac{g(x)-g(y)}{x-y}=\frac{(k-1)^2-1-x-y-xy}{(x+k)^2(y+k)^2}\ge 0,$$

because

$$(k-1)^2-1-x-y-xy\ge\frac{(n+1)^2}{n}-1-x-y-xy=\frac{(ny-1)^2}{n}\ge 0.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$. If $k=1+\dfrac{n+1}{\sqrt{n}}$, then the equality holds also for

$$a_1=n,\qquad a_2=\cdots=a_n=1,\qquad a_{n+1}=\cdots=a_{2n}=\frac{1}{n}.$$

□

P 4.7. *If $a_1,a_2,\ldots,a_n$ are positive real numbers so that*

$$a_1\ge 1\ge a_2\ge\cdots\ge a_n,\qquad a_1+a_2+\cdots+a_n=n,$$

then

$$a_1^{3/a_1}+a_2^{3/a_2}+\cdots+a_n^{3/a_n}\le n.$$

(Vasile C., 2012)

Solution. Rewrite the desired inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf(s),\qquad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=-u^{3/u},\qquad u\in\mathbb{I}=(0,n).$$

We have

$$f'(u)=3u^{\frac{3}{u}-2}(\ln u-1),$$

$$f''(u)=3u^{\frac{3}{u}-4}g(t),\qquad g(t)=u+(1-\ln u)(2u-3+3\ln u).$$

From the expression of f', it follows that f is decreasing on $(0,s_0]$ and increasing on $[s_0,n)$, where

$$s_0=e.$$

In addition, we claim that $f''(u)\ge$ for $u\in[1,e]$. If $u\in[3/2,e]$, then

$$g(t)>(1-\ln u)(2u-3)\ge 0.$$

Also,for $u\in[1,3/2]$, we have

$$g(t)=3(u-1)+(6-2u-3\ln u)\ln u\ge(6-2u-3\ln u)\ln u\ge 3\left(1-\ln\frac{3}{2}\right)\ln u>0.$$

Since f is convex on $[1, s_0]$, we may apply the RPCF-OV Theorem for $m = n-1$. We only need to show that $f(x)+f(y) \geq 2f(1)$ for all $x, y > 0$ so that $x + y = 2$. The inequality $f(x)+f(y) \geq 2f(1)$ is equivalent to

$$x^{3/x} + y^{3/y} \leq 2,$$

which is just the inequality in P 3.32 from Volume 2. The equality holds for

$$a_1 = a_2 = \cdots = a_n = 1.$$

□

P 4.8. *If $a_1, a_2, \ldots, a_{11}$ are real numbers so that*

$$a_1 \geq a_2 \geq 1 \geq a_3 \geq \cdots \geq a_{11}, \qquad a_1 + a_2 + \cdots + a_{11} = 11,$$

then

$$(1-a_1+a_1^2)(1-a_2+a_2^2)\cdots(1-a_{11}+a_{11}^2) \geq 1.$$

(Vasile C., 2012)

Solution. Rewrite the desired inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_{11}) \geq 11f(s), \quad s = \frac{a_1+a_2+\cdots+a_{11}}{11} = 1,$$

where

$$f(u) = \ln(1-u+u^2), \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{2u-1}{1-u+u^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = 1/2.$$

Also, from

$$f''(u) = \frac{1+2u(1-u)}{(1-u+u^2)^2},$$

it follows that $f''(u) > 0$ for $u \in [s_0, 1]$, hence f is convex on $[s_0, 1]$. Therefore, applying the LPCF-OV Theorem for $n = 11$ and $m = 2$, we only need to show that $f(x)+9f(y) \geq 9f(1)$ for all real x, y so that $x+9y = 10$. Using Note 2, it suffices to prove that $H(x, y) \geq 0$, where

$$H(x, y) = \frac{f'(x)-f'(y)}{x-y} = \frac{1+x+y-2xy}{(1-x+x^2)(1-y+y^2)}.$$

Since

$$1+x+y-2xy=18y^2-8y+1=2y^2+(4y-1)^2>0,$$

the conclusion follows. The equality holds for $a_1=a_2=\cdots=a_{11}=1$.

Remark. By replacing $a_1,a_2,\ldots,a_{11}$ respectively with $1-a_1,1-a_2,\ldots,1-a_{11}$, we get the following statement.

- *If $a_1,a_2,\ldots,a_{11}$ are real numbers so that*

$$a_1\le a_2\le 0\le a_3\le\cdots\le a_{11},\qquad a_1+a_2+\cdots+a_{11}=0,$$

then

$$(1-a_1+a_1^2)(1-a_2+a_2^2)\cdots(1-a_{11}+a_{11}^2)\ge 1,$$

with equality for $a_1=a_2=\cdots=a_n=0$.

□

P 4.9. *If $a_1,a_2,\ldots,a_8$ are nonzero real numbers so that*

$$a_1\ge a_2\ge a_3\ge a_4\ge 1\ge a_5\ge a_6\ge a_7\ge a_8,\qquad a_1+a_2+\cdots+a_8=8,$$

then

$$5\left(\frac{1}{a_1^2}+\frac{1}{a_2^2}+\cdots+\frac{1}{a_8^2}\right)+72\ge 14\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_8}\right).$$

(Vasile C., 2012)

Solution. Write the desired inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_8)\ge 8f(s),\quad s=\frac{a_1+a_2+\cdots+a_8}{8}=1,$$

where

$$f(u)=\frac{5}{u^2}-\frac{14}{u}+9,\quad u\in\mathbb{I}=\mathbb{R}\setminus\{0\}.$$

As shown in the proof of P 3.25, f is convex on $[s_0,1]$, increasing for $u\ge s_0$ and

$$\min_{u\in\mathbb{I}}f(u)=f(s_0),$$

where

$$s_0=\frac{5}{7}.$$

Taking into account Note 4, we may apply the LPCF-OV Theorem for $n=8$ and $m=4$. We only need to show that $f(x)+4f(y)\ge 5f(1)$ for $x,y\in\mathbb{I}$ so that $x+4y=5$. Using Note 1, it suffices to prove that $h(x,y)\ge 0$. Indeed, we have

$$g(u)=\frac{f(u)-f(1)}{u-1}=\frac{9}{u}-\frac{5}{u^2},$$

$$h(x,y)=\frac{g(x)-g(y)}{x-y}=\frac{5(x+y)-9xy}{x^2y^2}$$
$$=\frac{(x+4y)(x+y)-9xy}{x^2y^2}=\frac{(x-2y)^2}{x^2y^2}\geq 0.$$

In accordance with Note 3, the equality holds for $a_1=a_2=\cdots=a_8=1$, and also for

$$a_1=\frac{5}{3},\qquad a_2=a_3=a_4=1,\qquad a_5=a_6=a_7=a_8=\frac{5}{6}.$$

□

P 4.10. *If a,b,c,d are positive real numbers so that*

$$a\leq b\leq 1\leq c\leq d,\qquad abcd=1,$$

then

$$\frac{7-6a}{2+a^2}+\frac{7-6b}{2+b^2}+\frac{7-6c}{2+c^2}+\frac{7-6d}{2+d^2}\geq\frac{4}{3}.$$

(Vasile C., 2012)

Solution. Using the substitution

$$a=e^x,\ b=e^y,\ c=e^z,\ d=e^w,$$

we need to show that

$$f(x)+f(y)+f(z)+f(w)\geq 4f(s),$$

where

$$x\leq y\leq 0\leq z\leq w,\qquad s=\frac{x+y+z+w}{4}=0,$$
$$f(u)=\frac{7-6e^u}{2+e^{2u}},\quad u\in\mathbb{R}.$$

As shown in the proof of P 3.26, f is convex on $[0,s_0]$, is decreasing on $(-\infty,s_0]$ and increasing on $[s_0,\infty)$, where

$$s_0=\ln 3.$$

Therefore, we may apply the RPCF-OV Theorem for $n=4$ and $m=2$. We only need to show that $f(x)+2f(y)\geq 3f(0)$ for all real x,y so that $x+2y=0$; that is, to prove that

$$\frac{7-6a}{2+a^2}+\frac{2(7-6d)}{2+d^2}\geq 1$$

for $a,d>0$ so that $ad^2=1$. This is equivalent to

$$(d-1)^2(d-2)^2(5d^2+6d+3)\geq 0,$$

which is clearly true. In accordance with Note 3, the equality holds for $a = b = c = d = 1$, and also for

$$a = \frac{1}{4}, \quad b = 1, \quad c = d = 2.$$

□

P 4.11. *If a, b, c are positive real numbers so that*

$$a \le b \le 1 \le c, \quad abc = 1,$$

then

$$\frac{7-4a}{2+a^2} + \frac{7-4b}{2+b^2} + \frac{7-4c}{2+c^2} \ge 3.$$

(Vasile C., 2012)

Solution. Using the substitution

$$a = e^x, \ b = e^y, \ c = e^z,$$

we need to show that

$$f(x) + f(y) + f(z) \ge 3f(s),$$

where

$$x \le y \le 0 \le z, \quad s = \frac{x+y+z}{3} = 0,$$

$$f(u) = \frac{7-4e^u}{2+e^{2u}}, \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{2e^u(2e^u+1)(e^u-4)}{(2+e^{2u})^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \ln 4.$$

Also, we have

$$f''(u) = \frac{4t \cdot h(t)}{(2+t^2)^3}, \quad t = e^u,$$

where

$$h(t) = -t^4 + 7t^3 + 12t^2 - 14t - 4.$$

We will show that $h(t) \ge 0$ for $t \in [1, 4]$, hence f is convex on $[0, s_0]$. Indeed,

$$h(t) = (t-1)[t^2(-t+6) + 18t + 4] \ge 0.$$

Therefore, we may apply the RPCF-OV Theorem for $n = 3$ and $m = 2$. We only need to show that $f(x) + f(y) \geq 2f(0)$ for all real x, y so that $x + y = 0$. That is, to prove that

$$\frac{7-4a}{2+a^2} + \frac{7-4b}{2+b^2} \geq 2$$

for all $a, b > 0$ so that $ab = 1$. This is equivalent to

$$(a-1)^4 \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 4.12. *If a, b, c are positive real numbers so that*

$$a \geq 1 \geq b \geq c, \qquad abc = 1,$$

then

$$\frac{23-8a}{3+2a^2} + \frac{23-8b}{3+2b^2} + \frac{23-8c}{3+2c^2} \geq 9.$$

(Vasile C., 2012)

***Solution*.** Using the substitution

$$a = e^x, \; b = e^y, \; c = e^z,$$

we need to show that

$$f(x) + f(y) + f(z) \geq 3f(s),$$

where

$$x \geq 1 \geq y \geq z, \qquad s = \frac{x+y+z}{3} = 0,$$

$$f(u) = \frac{23-8e^u}{3+2e^{2u}}, \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{4e^u(4e^u+1)(e^u-6)}{(3+2e^{2u})^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where $s_0 = \ln 6$. Also, we have

$$f''(u) = \frac{8t \cdot h(t)}{(3+2t^2)^3}, \qquad t = e^u,$$

where

$$h(t) = -4t^4 + 46t^3 + 36t^2 - 69t - 9.$$

We will show that $h(t) \geq 0$ for $t \in [1, 6]$, hence f is convex on $[0, s_0]$. Indeed,

$$h(t) = (t-1)(2t+3)[2t(-t+12)+3] \geq 0.$$

Therefore, we may apply the RPCF-OV Theorem for $n = 3$ and $m = 2$. We only need to show that $f(x)+f(y) \geq 2f(0)$ for all real x, y so that $x+y=0$. That is, to prove that

$$\frac{23-8a}{3+2a^2}+\frac{23-8b}{3+2b^2} \geq 6.$$

for all $a, b > 0$ so that $ab = 1$. This is equivalent to

$$(a-1)^4 \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 4.13. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that*

$$a_1 \leq \cdots \leq a_{n-1} \leq 1 \leq a_n, \qquad a_1a_2\cdots a_n = 1.$$

If $p, q \geq 0$ so that $p+3q \geq 1$, then

$$\frac{1-a_1}{1+pa_1+qa_1^2}+\frac{1-a_2}{1+pa_2+qa_2^2}+\cdots+\frac{1-a_n}{1+pa_n+qa_n^2} \geq 0.$$

(Vasile C., 2012)

Solution. For $q = 0$, we need to show that $p \geq 1$ involves

$$\frac{1-a_1}{1+pa_1}+\frac{1-a_2}{1+pa_2}+\cdots+\frac{1-a_n}{1+pa_n} \geq 0.$$

This is just the inequality from P 2.24. Consider next that $q > 0$. Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \ldots, n$, we need to show that

$$f(x_1)+f(x_2)+\cdots+f(x_n) \geq nf(s),$$

where

$$x_1 \leq \cdots \leq x_{n-1} \leq 0 \leq x_n, \qquad s = \frac{x_1+x_2+\cdots+x_n}{n} = 0,$$

$$f(u) = \frac{1-e^u}{1+pe^u+qe^{2u}}, \quad u \in \mathbb{R}.$$

As shown in the proof of P 3.30, if $p+3q-1 \geq 0$, then f is convex on $[0, s_0]$, where

$$s_0 = \ln r_0 > 0, \qquad r_0 = 1+\sqrt{1+\frac{p+1}{q}}.$$

In addition, f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$. Therefore, we may apply the RPCF-OV Theorem for $m = n-1$. We only need to show that $f(x)+f(y) \geq 2f(0)$ for all real x, y so that $x+y=0$; that is, to prove that

$$\frac{1-a}{1+pa+qa^2}+\frac{1-b}{1+pb+qb^2} \geq 0$$

for $a, b > 0$ so that $ab = 1$. This is equivalent to

$$(a-1)^2[(p-1)a + q(a^2+a+1)] \geq 0,$$

which is true because

$$(p-1)a + q(a^2+a+1) \geq (p-1)a + q(3a) = (p+3q-1)a \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 4.14. *If a, b, c, d, e are real numbers so that*

$$-2 \leq a \leq b \leq 1 \leq c \leq d \leq e, \qquad a+b+c+d+e = 5,$$

then

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{e^2} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}.$$

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) + f(e) \geq 5f(s), \qquad s = \frac{a+b+c+d+e}{5} = 1,$$

where

$$f(u) = \frac{1}{u^2} - \frac{1}{u}, \qquad u \in \mathbb{I} = [-2, 7] \setminus \{0\}.$$

Let

$$s_0 = 2, \quad s < s_0.$$

From

$$f(u) - f(2) = \frac{1}{u^2} - \frac{1}{u} + \frac{1}{4} = \frac{(u-2)^2}{4u^2} \geq 0,$$

it follows that

$$\min_{u \in \mathbb{I}} f(u) = f(s_0),$$

while from

$$f'(u) = \frac{u-2}{u^3}, \qquad f''(u) = \frac{2(3-u)}{u^4},$$

it follows that f is convex on $[s, s_0]$. We can't apply the the RPCF-OV Theorem because f is not decreasing on $\mathbb{I}_{\leq s_0}$. According to Theorem 1 (applied for $n = 5$ and $m = 2$) and Note 6, we may replace this condition with $(1+n-m)s-(n-m)s_0 \leq \inf \mathbb{I}$. Indeed, we have

$$(1+n-m)s - (n-m)s_0 = 4 - 6 = -2 = \inf \mathbb{I}.$$

So, according to Note 1, it suffices to show that $h(x,y) \geq 0$ for all $x, y \in \mathbb{I}$ so that $x + 3y = 4$. We have

$$g(u) = \frac{f(u) - f(1)}{u-1} = \frac{-1}{u^2},$$

$$h(x,y) = \frac{g(x) - g(y)}{x - y} = \frac{x+y}{x^2y^2} = \frac{2(x+2)}{3x^2y^2} \geq 0.$$

The proof is completed. By Note 3, the equality holds for $a = b = c = d = e = 1$, and also for

$$a = -2, \quad b = 1, \quad c = d = e = 2.$$

□

Chapter 5

EV Method for Nonnegative Variables

5.1 Theoretical Basis

The Equal Variables Method is an effective tool for solving some difficult symmetric inequalities.

EV-Theorem (*Vasile Cirtoaje*, 2005). *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be fixed nonnegative real numbers, and let*

$$0 \le x_1 \le x_2 \le \cdots \le x_n$$

so that

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n, \qquad x_1^k + x_2^k + \cdots + x_n^k = a_1^k + a_2^k + \cdots + a_n^k,$$

where k is a nonnegative real number $(k \ne 1)$; $k = 0$ means $x_1 x_2 \cdots x_n = a_1 a_2 \cdots a_n$. Let f be a real-valued function, continuous on $[0, \infty)$ and differentiable on $(0, \infty)$, so that the joined function

$$g(x) = f'\left(x^{\frac{1}{k-1}}\right)$$

is strictly convex on $(0, \infty)$. Then, the sum

$$S_n = f(x_1) + f(x_2) + \cdots + f(x_n)$$

is maximal only for

$$x_1 = x_2 = \cdots = x_{n-1} \le x_n,$$

and minimal only for $x_1 = 0$ or $0 < x_1 \le x_2 = x_3 = \cdots = x_n$.

To prove the EV-Theorem, we need the EV-Lemma and the EV-Proposition below.

EV-Lemma. *Let a, b, c be fixed nonnegative real numbers, not all equal and, for $k \ge 0$, at most one of them equal to zero, and let $x \le y \le z$ be nonnegative real numbers so that*

$$x + y + z = a + b + c, \qquad x^k + y^k + z^k = a^k + b^k + c^k,$$

where k is a real number ($k \neq 1$); for $k = 0$, the second equation is $xyz = abc$. Then, the range of y is an interval $[m, M]$ with $m < M$; in addition,

(1) $y = m$ *if and only if* $x = y < z$;

(2) $y = M$ *if and only if* $0 = x < y \le z$ *or* $0 < x \le y = z$.

Proof. We show first, by the contradiction method, that $x < z$. Indeed, if $x = z$, then

$$x = z \ \Rightarrow\ x = y = z \ \Rightarrow\ x^k + y^k + z^k = 3\left(\frac{x+y+z}{3}\right)^k$$
$$\Rightarrow\ a^k + b^k + c^k = 3\left(\frac{a+b+c}{3}\right)^k \ \Rightarrow\ a = b = c,$$

which is false. Notice that the last implication follows from Jensen's inequalities

$$a^k + b^k + c^k \ge 3\left(\frac{a+b+c}{3}\right)^k, \qquad k \in (-\infty, 0) \cup (1, \infty),$$

$$a^k + b^k + c^k \le 3\left(\frac{a+b+c}{3}\right)^k, \qquad k \in (0, 1),$$

$$abc \le \left(\frac{a+b+c}{3}\right)^3, \qquad k = 0,$$

where the equality holds if and only if $a = b = c$.

According to the relations

$$x + z = a + b + c - y, \qquad x^k + z^k = a^k + b^k + c^k - y^k,$$

we may consider x and z as functions of y. From

$$x' + z' = -1, \qquad x^{k-1}x' + z^{k-1}z' = -y^{k-1},$$

we get

$$x' = \frac{y^{k-1} - z^{k-1}}{z^{k-1} - x^{k-1}} \le 0, \qquad z' = \frac{y^{k-1} - x^{k-1}}{x^{k-1} - z^{k-1}} \le 0. \tag{*}$$

Let us define the nonnegative functions

$$f_1(y) = y - x(y), \qquad f_2(y) = z(y) - y. \qquad f_3(y) = x(y).$$

Since

$$f_1'(y) = 1 - x'(y) > 0, \qquad f_2'(y) = z'(y) - 1 < 0, \qquad f_3'(y) = x'(y) \le 0,$$

these functions are strictly increasing, decreasing and decreasing, respectively. Thus, the inequality $f_1(y) \ge 0$ (with f_1 increasing) involves $y \ge m$, where m is a root of the equation $x(y) = y$, and the inequality $f_2(y) \ge 0$ (with f_2 decreasing) involves involves $y \le y_2$, where y_2 is a root of the equation $z(y) = y$. If $x(y_2) \ge 0$, then

y_2 is the maximal value of y. Otherwise, the maximal value of y is given by the inequality $f_3(y) \ge 0$ (with f_3 decreasing), which involves $y \le y_3$, where y_3 is a root of the equation $x(y) = 0$. Therefore, $y \in [m, M]$, with $y = m$ for $x = y$, and $y = M$ for either $y = z$ or $x = 0$.

EV-Proposition. *Let a, b, c be fixed nonnegative real numbers, and let $0 \le x \le y \le z$ so that*

$$x + y + z = a + b + c, \qquad x^k + y^k + z^k = a^k + b^k + c^k,$$

where k is a real number ($k \neq 1$); $k = 0$ means $xyz = abc$. Let f be a real-valued function, continuous on $[0, \infty)$ and differentiable on $(0, \infty)$, so that the joined function

$$g(x) = f'\left(x^{\frac{1}{k-1}}\right)$$

is strictly convex on $(0, \infty)$. Then, the sum

$$S_3 = f(x) + f(y) + f(z)$$

is maximal only when $0 \le x = y \le z$, and minimal only when $x = 0$ or $0 < x \le y = z$.

Proof. If $a = b = c$, then

$$a^k + b^k + c^k = 3\left(\frac{a+b+c}{3}\right)^k,$$

hence

$$x^k + y^k + z^k = 3\left(\frac{x+y+z}{3}\right)^k,$$

which involves $x = y = z$. If $k > 0$ and two of a, b, c are equal to zero, then

$$a^k + b^k + c^k = (a + b + c)^k,$$

hence

$$x^k + y^k + z^k = (x + y + z)^k,$$

which involves $x = y = 0$. In both cases, the extremum conditions in the statement ($x = y$ and either $x = 0$ or $y = z$) are satisfied. Consider further that a, b, c are not all equal and at most one of them is equal to zero. As shown in the proof of the EV-Lemma, we have $x < z$. According to the relations

$$x + z = a + b + c - y, \quad x^k + z^k = a^k + b^k + c^k - y^k,$$

we may consider x and z as functions of y. Thus, we have

$$S_3 = f(x(y)) + f(y) + f(z(y)) := F(y).$$

According to the EV-Lemma, it suffices to show that F is maximal for $y = m$ and is minimal for $y = M$. Using (*), we have

$$\begin{aligned} F'(y) &= x'f'(x) + f'(y) + z'f'(z) \\ &= \frac{y^{k-1} - z^{k-1}}{z^{k-1} - x^{k-1}} g(x^{k-1}) + g(y^{k-1}) + \frac{y^{k-1} - x^{k-1}}{x^{k-1} - z^{k-1}} g(z^{k-1}), \end{aligned}$$

which, for $x < y < z$, is equivalent to

$$\frac{F'(y)}{(y^{k-1} - x^{k-1})(y^{k-1} - z^{k-1})} = \frac{g(x^{k-1})}{(x^{k-1} - y^{k-1})(x^{k-1} - z^{k-1})}$$

$$+ \frac{g(y^{k-1})}{(y^{k-1} - z^{k-1})(y^{k-1} - x^{k-1})} + \frac{g(z^{k-1})}{(z^{k-1} - x^{k-1})(z^{k-1} - y^{k-1})}.$$

Since g is strictly convex, the right hand side is positive. Moreover, since

$$(y^{k-1} - x^{k-1})(y^{k-1} - z^{k-1}) < 0,$$

we have $F'(y) < 0$ for $y \in (m, M)$ (see the EV-Lemma), hence F is strictly decreasing on $[m, M]$. Therefore, F is maximal for $y = m$ (when $0 \le x = y \le z$) and is minimal for $y = M$ (when $x = 0$ or $0 < x \le y = z$.

Proof of the EV-Theorem. Since $X = \{x_1, x_2, \ldots, x_n\}$ is defined as a compact set in $\mathbb{R}^+_{\ltimes}$, S_n attains its minimum and maximum. For $n = 3$, the EV-Theorem follows immediately from the EV-Proposition. To prove the theorem for $n \ge 4$, we use the contradiction method.

(a) For the sake of contradiction, assume that S_n is maximal at $(b_1, b_2, \ldots, b_n)$, where $b_1 \le b_2 \le \cdots \le b_n$ and $b_1 < b_{n-1}$. Let x_1, x_{n-1} and x_n be real numbers so that $x_1 \le x_{n-1} \le x_n$ and

$$x_1 + x_{n-1} + x_n = b_1 + b_{n-1} + b_n, \quad x_1^k + x_{n-1}^k + x_n^k = b_1^k + b_{n-1}^k + b_n^k.$$

According to the EV-Proposition, the sum $f(x_1) + f(x_{n-1}) + f(x_n)$ is maximal for $x_1 = x_{n-1}$, when

$$f(x_1) + f(x_{n-1}) + f(x_n) > f(b_1) + f(b_{n-1}) + f(b_n).$$

This result contradicts the assumption that S_n attains its maximum at $(b_1, b_2, \ldots, b_n)$ with $b_1 < b_{n-1}$.

(b) Similarly, we can prove that S_n is minimal for $n \ge 4$ when either $x_1 = 0$ or

$$0 < x_1 \le x_2 = \cdots = x_n.$$

Corollary 1. *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be fixed nonnegative real numbers, and let*

$$0 \le x_1 \le x_2 \le \cdots \le x_n$$

so that

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n,$$
$$x_1^2 + x_2^2 + \cdots + x_n^2 = a_1^2 + a_2^2 + \cdots + a_n^2.$$

Let f be a real-valued function, continuous on $[0, \infty)$ and differentiable on $(0, \infty)$, so that the joined function

$$g(x) = f'(x)$$

is strictly convex on $(0, \infty)$. The sum

$$S_n = f(x_1) + f(x_2) + \cdots + f(x_n)$$

is maximal only when

$$x_1 = x_2 = \cdots = x_{n-1} \le x_n \ ,$$

and is minimal only when either $x_1 = 0$ *or* $0 < x_1 \le x_2 = x_3 = \cdots = x_n$.

Corollary 2. *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be fixed positive real numbers, and let*

$$0 < x_1 \le x_2 \le \cdots \le x_n$$

so that

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n,$$
$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}.$$

Let f be a real-valued function, continuous and differentiable on $(0, \infty)$, so that the joined function

$$g(x) = f'\left(\frac{1}{\sqrt{x}}\right)$$

is strictly convex on $(0, \infty)$. The sum

$$S_n = f(x_1) + f(x_2) + \cdots + f(x_n)$$

is maximal only when

$$x_1 = x_2 = \cdots = x_{n-1} \le x_n \ ,$$

and is minimal only when

$$x_1 \le x_2 = x_3 = \cdots = x_n.$$

Corollary 3. *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be fixed nonnegative real numbers, and let*

$$0 \le x_1 \le x_2 \le \cdots \le x_n$$

so that

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n, \qquad x_1 x_2 \cdots x_n = a_1 a_2 \cdots a_n.$$

Let f be a real-valued function, continuous on $[0,\infty)$ and differentiable on $(0,\infty)$, so that the joined function

$$g(x) = f'(1/x)$$

is strictly convex on $(0,\infty)$. The sum

$$S_n = f(x_1) + f(x_2) + \cdots + f(x_n)$$

is maximal only when

$$x_1 = x_2 = \cdots = x_{n-1} \le x_n \ ,$$

and is minimal only when either $x_1 = 0$ *or* $0 < x_1 \le x_2 = x_3 = \cdots = x_n$.

Corollary 4. *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be fixed nonnegative real numbers, and let*

$$0 \le x_1 \le x_2 \le \cdots \le x_n$$

so that

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n, \qquad x_1^k + x_2^k + \cdots + x_n^k = a_1^k + a_2^k + \cdots + a_n^k,$$

where k is a real number $(k \ne 0,\ k \ne 1)$.

(1) *For $k < 0$, the product $P_n = x_1 x_2 \cdots x_n$ is maximal when*

$$0 < x_1 \le x_2 = x_3 = \cdots = x_n,$$

and is minimal only when

$$0 < x_1 = x_2 = \cdots = x_{n-1} \le x_n;$$

(2) *For $k > 0$, the product $P_n = x_1 x_2 \cdots x_n$ is maximal when*

$$x_1 = x_2 = \cdots = x_{n-1} \le x_n,$$

and is minimal only when either $x_1 = 0$ *or* $0 < x_1 \le x_2 = x_3 = \cdots = x_n$.

Note 1. The EV-Theorem, Corollary 1 and Corollary 3 are also valid for the cases when $x_1, x_2, \ldots, x_n > 0$, f is continuous and differentiable on $(0,\infty)$, $f(0+) = \pm\infty$ and the sum S_n has a global maximum (minimum).

From the EV-Theorem and Note 1, we can obtain some interesting particular results, which are useful in many applications.

Corollary 5. *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be fixed nonnegative real numbers, and let*

$$0 \le x_1 \le x_2 \le \cdots \le x_n$$

so that

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n, \qquad x_1^k + x_2^k + \cdots + x_n^k = a_1^k + a_2^k + \cdots + a_n^k.$$

Let us denote

$$S_n = x_1^m + x_2^m + \cdots + x_n^m.$$

Case $1:\ k < 0.$

(a) *If* $m \in (k,0) \cup (1,\infty)$, *then* S_n *is maximal only for*

$$0 < x_1 = x_2 = \cdots = x_{n-1} \le x_n,$$

and is minimal only for

$$0 < x_1 \le x_2 = x_3 = \cdots = x_n.$$

(b) *If* $m \in (-\infty,k) \cup (0,1)$, *then* S_n *is minimal only for*

$$0 < x_1 = x_2 = \cdots = x_{n-1} \le x_n,$$

and is maximal only for

$$0 < x_1 \le x_2 = x_3 = \cdots = x_n.$$

Case $2:\ 0 \le k < 1$ ($k = 0$ *means* $x_1x_2\cdots x_n = a_1a_2\cdots a_n$).

(a) *If* $m \in (0,k) \cup (1,\infty)$, *then* S_n *is maximal only for*

$$0 \le x_1 = x_2 = \cdots = x_{n-1} \le x_n,$$

and is minimal only for either $x_1 = 0$ *or* $0 < x_1 \le x_2 = x_3 = \cdots = x_n$.

(b) *If* $m \in (-\infty,0)$, *then* S_n *is minimal only for*

$$0 < x_1 = x_2 = \cdots = x_{n-1} \le x_n,$$

and is maximal (if it has a global maximum) only for

$$0 < x_1 \le x_2 = x_3 = \cdots = x_n.$$

(c) *If* $m \in (k,1)$, *then* S_n *is minimal only for*

$$0 \le x_1 = x_2 = \cdots = x_{n-1} \le x_n,$$

and is maximal only for either $x_1 = 0$ *or* $0 < x_1 \le x_2 = x_3 = \cdots = x_n$.

Case $3:\ k > 1.$

(a) *If* $m \in (0,1) \cup (k,\infty)$, *then* S_n *is maximal only for*

$$0 \le x_1 = x_2 = \cdots = x_{n-1} \le x_n,$$

and is minimal only for either $x_1 = 0$ *or* $0 < x_1 \le x_2 = x_3 = \cdots = x_n$.

(b) *If* $m \in (-\infty, 0)$*, then* S_n *is minimal only for*

$$0 < x_1 = x_2 = \cdots = x_{n-1} \le x_n,$$

and is maximal (if it has a global maximum) only for

$$0 < x_1 \le x_2 = x_3 = \cdots = x_n.$$

(c) *If* $m \in (1, k)$*, then* S_n *is minimal only for*

$$0 \le x_1 = x_2 = \cdots = x_{n-1} \le x_n,$$

and is maximal only for either $x_1 = 0$ *or* $0 < x_1 \le x_2 = x_3 = \cdots = x_n$.

Proof. We apply the EV-Theorem and Note 1 to the function

$$f(u) = m(m-1)(m-k)u^m.$$

We have

$$f'(u) = m^2(m-2)(m-k)u^{m-1}$$

and

$$g(x) = m^2(m-1)(m-k)x^{\frac{m-1}{k-1}}, \qquad g''(x) = \frac{m^2(m-1)^2(m-k)^2}{(k-1)^2}x^{\frac{1+m-2k}{k-1}}.$$

Since $g''(x) > 0$ for $x > 0$, g is strictly convex on $(0, \infty)$.

Corollary 6. *Let* $a_1, a_2, \dots, a_n$ $(n \ge 3)$ *be fixed nonnegative real numbers, and let*

$$0 \le x_1 \le x_2 \le \cdots \le x_n$$

so that

$$x_1^p + x_2^p + \cdots + x_n^p = a_1^p + a_2^p + \cdots + a_n^p, \qquad x_1^q + x_2^q + \cdots + x_n^q = a_1^q + a_2^q + \cdots + a_n^q,$$

where

$$p, q \in \{1, 2, 3\}, \qquad p \ne q.$$

The symmetric sum

$$S_n = \sum_{1 \le i_1 < i_2 < i_3 \le n} x_{i_1} x_{i_2} x_{i_3}$$

is maximal only for

$$0 \le x_1 = x_2 = \cdots = x_{n-1} \le x_n,$$

and is minimal only for either $x_1 = 0$ *or* $0 < x_1 \le x_2 = x_3 = \cdots = x_n$.

Proof. Taking into account that

$$6\sum_{1\le i_1<i_2<i_3\le n} x_{i_1}x_{i_2}x_{i_3} = \left(\sum x_1\right)^3 - 3\left(\sum x_1\right)\left(\sum x_1^2\right) + 2\sum x_1^3,$$

Corollary 6 is a consequence of Corollary 5. For $p = 2$ and $q = 3$, according to this identity, the sum $\sum_{1\le i_1<i_2<i_3\le n} x_{i_1}x_{i_2}x_{i_3}$ is maximal/minimal when $\sum x_1$ is maximal/minimal. Therefore, we need to show that if

$$x_1^2 + x_2^2 + \cdots + x_n^2 = constant, \qquad x_1^3 + x_2^3 + \cdots + x_n^3 = constant,$$

then the sum $\sum x_1$ is maximal for

$$0 \le x_1 = x_2 = \cdots = x_{n-1} \le x_n,$$

and is minimal for either $x_1 = 0$ or $0 < x_1 \le x_2 = x_3 = \cdots = x_n$. This follows by replacing $x_1, x_2, \ldots, x_n$ with $x_1^2, x_2^2, \ldots, x_n^2$ in Corollary 5, case $k = 3/2$ and $m = 1/2$.

Note 2. The EV-Theorem and Corollaries 1-3 can be extended to the cases where:

(a) $x_1, x_2, \ldots, x_n \ge m \ge 0$, f is continuous on $[m, \infty)$ and differentiable on (m, ∞), and $g(x)$ is strictly convex for $x^{\frac{1}{k-1}} > m$; so, the sum

$$S_n = f(x_1) + f(x_2) + \cdots + f(x_n)$$

is maximal for $x_1 = x_2 = \cdots = x_{n-1} \le x_n$, and is minimal for either $x_1 = m$ or $m < x_1 \le x_2 = x_3 = \cdots = x_n$;

(b) $0 \le x_1, x_2, \ldots, x_n \le M$, f is continuous on $[0, M]$ and differentiable on $(0, M)$, and $g(x)$ is strictly convex for $x^{\frac{1}{k-1}} < M$; so, the sum

$$S_n = f(x_1) + f(x_2) + \cdots + f(x_n)$$

is maximal for either $x_n = M$ or $x_1 = x_2 = \cdots = x_{n-1} \le x_n$, and is minimal $x_1 \le x_2 = x_3 = \cdots = x_n$;

Note 3. The EV-Theorem and Corollaries 1-3 can be extended to the cases where:

(a) $x_1, x_2, \ldots, x_n > m \ge 0$, f is continuous and differentiable on (m, ∞), $f(m+) = \pm\infty$, $g(x)$ is strictly convex for $x^{\frac{1}{k-1}} > m$ and the sum S_n has a global maximum (minimum);

(b) $0 \le x_1, x_2, \ldots, x_n < M$, f is continuous and differentiable on $[0, M)$, $f(M-) = \pm\infty$, $g(x)$ is strictly convex for $x^{\frac{1}{k-1}} < M$ and the sum S_n has a global maximum (minimum).

5.2 Applications

5.1. If a,b,c,d are nonnegative real numbers so that

$$a+b+c+d=a^3+b^3+c^3+d^3=2,$$

then

$$\frac{7}{4}\le a^2+b^2+c^2+d^2\le 2.$$

5.2. If $a_1,a_2,\dots,a_9$ are nonnegative real numbers so that

$$a_1+a_2+\cdots+a_9=a_1^2+a_2^2+\cdots+a_9^2=3,$$

then

$$3\le a_1^3+a_2^3+\cdots+a_9^3\le \frac{14}{3}.$$

5.3. If a,b,c,d are nonnegative real numbers so that

$$a+b+c+d=a^2+b^2+c^2+d^2=\frac{27}{7},$$

then

$$\frac{5427}{1372}\le a^3+b^3+c^3+d^3\le \frac{1377}{343}.$$

5.4. If a,b,c are positive real numbers so that $abc=1$, then

$$a^5+b^5+c^5\ge\sqrt{3(a^7+b^7+c^7)}.$$

5.5. If a,b,c,d are positive real numbers so that $abcd=1$, then

$$a^3+b^3+c^3+d^3\ge\sqrt{4(a^4+b^4+c^4+d^4)}.$$

5.6. If a,b,c,d are nonnegative real numbers so that $a+b+c+d=4$, then

$$\frac{bcd}{11a+16}+\frac{cda}{11b+16}+\frac{dab}{11c+16}+\frac{abc}{11d+16}\le\frac{4}{27}.$$

5.7. If a, b, c are real numbers, then

$$\frac{bc}{3a^2+b^2+c^2}+\frac{ca}{3b^2+c^2+a^2}+\frac{ab}{3c^2+a^2+b^2}\le\frac{3}{5}.$$

5.8. If a, b, c are nonnegative real numbers so that $a+b+c=3$, then

(a) $$\frac{bc}{a^2+2}+\frac{ca}{b^2+2}+\frac{ab}{c^2+2}\le\frac{9}{8};$$

(b) $$\frac{bc}{a^2+3}+\frac{ca}{b^2+3}+\frac{ab}{c^2+3}\le\frac{11\sqrt{33}-45}{24};$$

(c) $$\frac{bc}{a^2+4}+\frac{ca}{b^2+4}+\frac{ab}{c^2+4}\le\frac{3}{5}.$$

5.9. If a, b, c, d are nonnegative real numbers so that

$$(3a+1)(3b+1)(3c+1)(3d+1)=64,$$

then

$$abc+bcd+cda+dab\le 1.$$

5.10. If $a_1, a_2, \ldots, a_n$ and p, q are nonnegative real numbers so that

$$a_1+a_2+\cdots+a_n=p+q, \qquad a_1^3+a_2^3+\cdots+a_n^3=p^3+q^3,$$

then

$$a_1^2+a_2^2+\cdots+a_n^2\le p^2+q^2.$$

5.11. If a, b, c are nonnegative real numbers, then

$$a\sqrt{a^2+4b^2+4c^2}+b\sqrt{b^2+4c^2+4a^2}+c\sqrt{c^2+4a^2+4b^2}\ge(a+b+c)^2.$$

5.12. If a, b, c are nonnegative real numbers so that $ab+bc+ca=3$, then

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\le\frac{3}{2(a+b+c)}+\frac{a+b+c}{3}.$$

5.13. If a, b, c are nonnegative real numbers so that $ab + bc + ca = 3$, then

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{3}{a+b+c} + \frac{a+b+c}{6}.$$

5.14. Let a, b, c be nonnegative real numbers, no two of which are zero. If

$$a^2 + b^2 + c^2 = 3,$$

then

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} + \frac{a+b+c}{9} \geq \frac{11}{2(a+b+c)}.$$

5.15. Let a, b, c be nonnegative real numbers, no two of which are zero. If

$$a + b + c = 4,$$

then

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{15}{8 + ab + bc + ca}.$$

5.16. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{1}{a+b+c} + \frac{2}{\sqrt{ab+bc+ca}}.$$

5.17. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{3-\sqrt{3}}{a+b+c} + \frac{2+\sqrt{3}}{2\sqrt{ab+bc+ca}}.$$

5.18. Let a, b, c be nonnegative real numbers, no two of which are zero, so that

$$ab + bc + ca = 3.$$

If

$$0 \leq k \leq \frac{9+5\sqrt{3}}{6} \approx 2.943,$$

then

$$\frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{c+a} \geq \frac{9(1+k)}{a+b+c+3k}.$$

5.19. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \geq \frac{20}{a+b+c+6\sqrt{ab+bc+ca}}.$$

5.20. If a, b, c are positive real numbers so that

$$7(a^2+b^2+c^2)=11(ab+bc+ca),$$

then

$$\frac{51}{28} \leq \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \leq 2.$$

5.21. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that

$$\frac{a_1^2+a_2^2+\cdots+a_n^2}{n+3}=\left(\frac{a_1+a_2+\cdots+a_n}{n+1}\right)^2,$$

then

$$\frac{(n+1)(2n-1)}{2} \leq (a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right) \leq \frac{3n^2(n+1)}{2(n+2)}.$$

5.22. If a, b, c, d are nonnegative real numbers so that $a+b+c+d=3$, then

$$abc+bcd+cda+dab \leq 1+\frac{176}{81}\,abcd.$$

5.23. If a, b, c, d are nonnegative real numbers so that $a+b+c+d=3$, then

$$a^2b^2c^2+b^2c^2d^2+c^2d^2a^2+d^2a^2b^2+\frac{3}{4}abcd \leq 1.$$

5.24. If a, b, c, d are nonnegative real numbers so that $a+b+c+d=3$, then

$$a^2b^2c^2+b^2c^2d^2+c^2d^2a^2+d^2a^2b^2+\frac{4}{3}(abcd)^{3/2} \leq 1.$$

5.25. If a, b, c, d are nonnegative real numbers so that $a+b+c+d=4$, then

$$a^2b^2c^2+b^2c^2d^2+c^2d^2a^2+d^2a^2b^2+2(abcd)^{3/2} \leq 6.$$

5.26. If a, b, c are nonnegative real numbers so that $a+b+c=3$, then

$$11(ab+bc+ca)+4(a^2b^2+b^2c^2+c^2a^2) \le 45.$$

5.27. If a, b, c are nonnegative real numbers so that $a+b+c=3$, then

$$a^2b^2+b^2c^2+c^2a^2+a^3b^3+b^3c^3+c^3a^3 \ge 6abc.$$

5.28. If a, b, c are nonnegative real numbers so that $a+b+c=3$, then

$$2(a^2+b^2+c^2)+5\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right) \ge 21.$$

5.29. If a, b, c are nonnegative real numbers so that $ab+bc+ca=3$, then

$$\sqrt{\frac{1+2a}{3}}+\sqrt{\frac{1+2b}{3}}+\sqrt{\frac{1+2c}{3}} \ge 3.$$

5.30. Let a, b, c be nonnegative real numbers, no two of which are zero. If

$$0 \le k \le 15,$$

then

$$\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}+\frac{k}{(a+b+c)^2} \ge \frac{9+k}{4(ab+bc+ca)}.$$

5.31. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}+\frac{24}{(a+b+c)^2} \ge \frac{8}{ab+bc+ca}.$$

5.32. If a, b, c are nonnegative real numbers, no two of which are zero, so that

$$k(a^2+b^2+c^2)+(2k+3)(ab+bc+ca)=9(k+1), \qquad 0 \le k \le 6,$$

then

$$\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}+\frac{9k}{(a+b+c)^2} \ge \frac{3}{4}+k.$$

5.33. If a, b, c are nonnegative real numbers, no two of which are zero, then

(a) $$\frac{2}{a^2+b^2}+\frac{2}{b^2+c^2}+\frac{2}{c^2+a^2}\ge\frac{8}{a^2+b^2+c^2}+\frac{1}{ab+bc+ca};$$

(b) $$\frac{2}{a^2+b^2}+\frac{2}{b^2+c^2}+\frac{2}{c^2+a^2}\ge\frac{7}{a^2+b^2+c^2}+\frac{6}{(a+b+c)^2};$$

(c) $$\frac{2}{a^2+b^2}+\frac{2}{b^2+c^2}+\frac{2}{c^2+a^2}\ge\frac{45}{4(a^2+b^2+c^2)+ab+bc+ca}.$$

5.34. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}+\frac{3}{a^2+b^2+c^2}\ge\frac{4}{ab+bc+ca}.$$

5.35. If a, b, c are nonnegative real numbers, no two of which are zero, then

(a) $$\frac{3}{a^2+ab+b^2}+\frac{3}{b^2+bc+c^2}+\frac{3}{c^2+ca+a^2}\ge\frac{5}{ab+bc+ca}+\frac{4}{a^2+b^2+c^2};$$

(b) $$\frac{3}{a^2+ab+b^2}+\frac{3}{b^2+bc+c^2}+\frac{3}{c^2+ca+a^2}\ge\frac{1}{ab+bc+ca}+\frac{24}{(a+b+c)^2};$$

(c) $$\frac{1}{a^2+ab+b^2}+\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2}\ge\frac{21}{2(a^2+b^2+c^2)+5(ab+bc+ca)}.$$

5.36. Let f be a real-valued function, continuous on $[0,\infty)$ and differentiable on $(0,\infty)$, so that $f'''(u)\ge 0$ for $u\in(0,\infty)$. If $a, b, c\ge 0$, then

$$f(a^2+2bc)+f(b^2+2ca)+f(c^2+2ab)\le f(a^2+b^2+c^2)+2f(ab+bc+ca).$$

5.37. If a, b, c are the lengths of the side of a triangle, then

$$\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\le\frac{85}{36(ab+bc+ca)}.$$

5.38. If a, b, c are the lengths of the side of a triangle so that $a+b+c=3$, then

$$\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\le\frac{3(a^2+b^2+c^2)}{4(ab+bc+ca)}.$$

5.39. Let $a, b, c \geq \dfrac{2}{5}$ so that $a + b + c = 3$. Then,

$$\frac{1}{3+2(a^2+b^2)} + \frac{1}{3+2b^2+c^2)} + \frac{1}{3+2(c^2+a^2)} \leq \frac{3}{7}.$$

5.40. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$\frac{2}{2+a^2+b^2} + \frac{2}{2+b^2+c^2} + \frac{2}{2+c^2+a^2} \leq \frac{99}{63+a^2+b^2+c^2}.$$

5.41. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$\frac{1}{3+a^2+b^2} + \frac{1}{3+b^2+c^2} + \frac{1}{3+c^2+a^2} \leq \frac{18}{27+a^2+b^2+c^2}.$$

5.42. If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then

$$\frac{5}{3+a^2+b^2} + \frac{5}{3+b^2+c^2} + \frac{5}{3+c^2+a^2} \geq \frac{27}{6+a^2+b^2+c^2}.$$

5.43. If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 4$, then

$$\sum \frac{3}{3+2(a^2+b^2+c^2)} \leq \frac{296}{218+a^2+b^2+c^2+d^2}.$$

5.44. If a, b, c are nonnegative real numbers so that $ab + bc + ca = 3$, then

$$\frac{4}{2+a^2+b^2} + \frac{4}{2+b^2+c^2} + \frac{4}{2+c^2+a^2} \geq \frac{21}{4+a^2+b^2+c^2}.$$

5.45. If a, b, c are nonnegative real numbers so that $a^2 + b^2 + c^2 = 3$, then

$$\frac{1}{10-(a+b)^2} + \frac{1}{10-(b+c)^2} + \frac{1}{10-(c+a)^2} \leq \frac{1}{2}.$$

5.46. If a, b, c are nonnegative real numbers, no two of which are zero, so that $a^4 + b^4 + c^4 = 3$, then

$$\frac{1}{a^5+b^5} + \frac{1}{b^5+c^5} + \frac{1}{c^5+a^5} \geq \frac{3}{2}.$$

5.47. If $a_1,a_2,\ldots,a_n$ are nonnegative real numbers so that $a_1+a_2+\cdots+a_n=n$, then

$$\sqrt{a_1^2+1}+\sqrt{a_2^2+1}+\cdots+\sqrt{a_n^2+1}\ge\sqrt{2\left(1-\frac{1}{n}\right)(a_1^2+a_2^2+\cdots+a_n^2)+2(n^2-n+1)}.$$

5.48. If $a_1,a_2,\ldots,a_n$ are nonnegative real numbers so that $a_1+a_2+\cdots+a_n=n$, then

$$\sum\sqrt{(3n-4)a_1^2+n}\ge\sqrt{(3n-4)(a_1^2+a_2^2+\cdots+a_n^2)+n(4n^2-7n+4)}.$$

5.49. If a,b,c are nonnegative real numbers so that $a+b+c=3$, then

$$\sqrt{a^2+4}+\sqrt{b^2+4}+\sqrt{c^2+4}\le\sqrt{\frac{8}{3}(a^2+b^2+c^2)+37}.$$

5.50. If a,b,c are nonnegative real numbers so that $a+b+c=3$, then

$$\sqrt{32a^2+3}+\sqrt{32b^2+3}+\sqrt{32c^2+3}\le\sqrt{32(a^2+b^2+c^2)+219}.$$

5.51. If $a_1,a_2,\ldots,a_n$ are positive real numbers so that $a_1+a_2+\cdots+a_n=n$, then

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}+\frac{2n\sqrt{n-1}}{a_1^2+a_2^2+\cdots+a_n^2}\ge n+2\sqrt{n-1}.$$

5.52. If $a,b,c\in[0,1]$, then

$$(1+3a^2)(1+3b^2)(1+3c^2)\ge(1+ab+bc+ca)^3.$$

5.53. If a,b,c are nonnegative real numbers so that $a+b+c=ab+bc+ca$, then

$$\frac{1}{4+5a^2}+\frac{1}{4+5b^2}+\frac{1}{4+5c^2}\ge\frac{1}{3}.$$

5.54. If a,b,c,d are positive real numbers so that $a+b+c+d=4abcd$, then

$$\frac{1}{1+3a}+\frac{1}{1+3b}+\frac{1}{1+3c}+\frac{1}{1+3d}\ge 1.$$

5.55. If $a_1, a_2, \ldots, a_n$ are positive real numbers so that

$$a_1 + a_2 + \cdots + a_n = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n},$$

then

$$\frac{1}{1+(n-1)a_1} + \frac{1}{1+(n-1)a_2} + \cdots + \frac{1}{1+(n-1)a_n} \ge 1.$$

5.56. If a, b, c, d, e are nonnegative real numbers so that $a^4 + b^4 + c^4 + d^4 + e^4 = 5$, then

$$7(a^2 + b^2 + c^2 + d^2 + e^2) \ge (a + b + c + d + e)^2 + 10.$$

5.57. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then

$$(a_1^2 + a_2^2 + \cdots + a_n^2)^2 - n^2 \ge \frac{n(n-1)}{n^2 - n + 1}\left(a_1^4 + a_2^4 + \cdots + a_n^4 - n\right).$$

5.58. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that $a_1^2 + a_2^2 + \cdots + a_n^2 = n$, then

$$a_1^3 + a_2^3 + \cdots + a_n^3 \ge \sqrt{n^2 - n + 1 + \left(1 - \frac{1}{n}\right)(a_1^6 + a_2^6 + \cdots + a_n^6)}.$$

5.59. If a, b, c are positive real numbers so that $abc = 1$, then

$$4\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + \frac{50}{a + b + c} \ge 27.$$

5.60. If a, b, c are positive real numbers so that $abc = 1$, then

$$a^3 + b^3 + c^3 + 15 \ge 6\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

5.61. Let $a_1, a_2, \ldots, a_n$ be positive numbers so that $a_1 a_2 \cdots a_n = 1$. If $k \ge n - 1$, then

$$a_1^k + a_2^k + \cdots + a_n^k + (2k - n)n \ge (2k - n + 1)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

5.62. Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be nonnegative numbers so that $a_1+a_2+\cdots+a_n = n$, and let k be an integer satisfying $2 \le k \le n+2$. If

$$r = \left(\frac{n}{n-1}\right)^{k-1} - 1,$$

then

$$a_1^k + a_2^k + \cdots + a_n^k - n \ge nr(1 - a_1 a_2 \cdots a_n).$$

5.63. If a, b, c are positive real numbers so that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 3$, then

$$4(a^2 + b^2 + c^2) + 9 \ge 21abc.$$

5.64. If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = n$, then,

$$a_1 + a_2 + \cdots + a_n - n \le e_{n-1}(a_1 a_2 \cdots a_n - 1),$$

where

$$e_{n-1} = \left(1 + \frac{1}{n-1}\right)^{n-1}.$$

5.65. If $a_1, a_2, \ldots, a_n$ are positive real numbers, then

$$\frac{a_1^n + a_2^n + \cdots + a_n^n}{a_1 a_2 \cdots a_n} + n(n-1) \ge (a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

5.66. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers, then

$$(n-1)(a_1^n + a_2^n + \cdots + a_n^n) + n a_1 a_2 \cdots a_n \ge (a_1 + a_2 + \cdots + a_n)(a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1}).$$

5.67. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers, then

$$(n-1)(a_1^{n+1} + a_2^{n+1} + \cdots + a_n^{n+1}) \ge (a_1 + a_2 + \cdots + a_n)(a_1^n + a_2^n + \cdots + a_n^n - a_1 a_2 \cdots a_n).$$

5.68. If $a_1, a_2, \ldots, a_n$ are positive real numbers, then

$$(a_1 + a_2 + \cdots + a_n - n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n\right) + a_1 a_2 \cdots a_n + \frac{1}{a_1 a_2 \cdots a_n} \ge 2.$$

5.69. If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1 a_2 \cdots a_n = 1$, then

$$\left| \frac{1}{\sqrt{a_1 + a_2 + \cdots + a_n - n}} - \frac{1}{\sqrt{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n}} \right| < 1.$$

5.70. If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1 a_2 \cdots a_n = 1$, then

$$a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} + \frac{n^2(n-2)}{a_1 + a_2 + \cdots + a_n} \ge (n-1)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

5.71. If a, b, c are nonnegative real numbers, then

$$(a + b + c - 3)^2 \ge \frac{abc - 1}{abc + 1}(a^2 + b^2 + c^2 - 3).$$

5.72. If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then

$$(a_1 a_2 \cdots a_n)^{\frac{1}{\sqrt{n-1}}}(a_1^2 + a_2^2 + \cdots + a_n^2) \le n.$$

5.73. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 + a_2 + \cdots + a_n = n - 1$, then

$$\sqrt[n]{\frac{n-1}{a_1 a_2 \cdots a_n}} \ \ge\ 4\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n(n-1)}}.$$

5.74. If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1^3 + a_2^3 + \cdots + a_n^3 = n$, then

$$a_1 + a_2 + \cdots + a_n \ge n \sqrt[n+1]{a_1 a_2 \cdots a_n}.$$

5.75. Let a, b, c be nonnegative real numbers so that $ab + bc + ca = 3$. If

$$k \ge 2 - \frac{\ln 4}{\ln 3} \approx 0.738,$$

then

$$a^k + b^k + c^k \ge 3.$$

5.76. Let a,b,c be nonnegative real numbers so that $a+b+c=3$. If

$$k \ge \frac{\ln 9-\ln 8}{\ln 3-\ln 2} \approx 0.29,$$

then

$$a^k+b^k+c^k \ge ab+bc+ca.$$

5.77. If $a_1,a_2,\ldots,a_n$ $(n\ge 4)$ are nonnegative numbers so that $a_1+a_2+\cdots+a_n=n$, then

$$\frac{1}{n+1-a_2a_3\cdots a_n}+\frac{1}{n+1-a_3a_4\cdots a_1}+\cdots+\frac{1}{n+1-a_1a_2\cdots a_{n-1}} \le 1.$$

5.78. If a,b,c are nonnegative real numbers so that

$$a+b+c\ge 2, \qquad ab+bc+ca\ge 1,$$

then

$$\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c} \ge 2.$$

5.79. If a,b,c,d are positive real numbers so that $abcd=1$, then

$$(a+b+c+d)^4 \ge 36\sqrt{3}\,(a^2+b^2+c^2+d^2).$$

5.80. If a,b,c are nonnegative real numbers so that $ab+bc+ca=1$, then

$$\sqrt{33a^2+16}+\sqrt{33b^2+16}+\sqrt{33c^2+16} \le 9(a+b+c).$$

5.81. If a,b,c are positive real numbers so that $a+b+c=3$, then

$$a^2b^2+b^2c^2+c^2a^2 \le \frac{3}{\sqrt[3]{abc}}.$$

5.82. If $a_1,a_2,\ldots,a_n$ $(n\le 81)$ are nonnegative real numbers so that

$$a_1^2+a_2^2+\cdots+a_n^2=a_1^5+a_2^5+\cdots+a_n^5,$$

then

$$a_1^6+a_2^6+\cdots+a_n^6 \le n.$$

5.83. If a, b, c are nonnegative real numbers so that $a+b+c=3$, then

$$1+\sqrt{1+a^3+b^3+c^3} \ge \sqrt{3(a^2+b^2+c^2)}.$$

5.84. If a, b, c are nonnegative real numbers so that $a+b+c=3$, then

$$\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a} \le \sqrt{16+\frac{2}{3}(ab+bc+ca)}.$$

5.85. If $a, b, c \in [0,4]$ and $ab+bc+ca=4$, then

$$\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a} \le 3+\sqrt{5}.$$

5.86. If a, b, c are positive real numbers so that $abc=1$, then

(a) $$\frac{a+b+c}{3} \ge \sqrt[3]{\frac{2+a^2+b^2+c^2}{5}};$$

(b) $$a^3+b^3+c^3 \ge \sqrt{3(a^4+b^4+c^4)}.$$

5.87. If a, b, c, d are nonnegative real numbers so that $a+b+c+d=4$, then

$$(a^2+b^2+c^2+d^2-4)(a^2+b^2+c^2+d^2+18) \le 10(a^3+b^3+c^3+d^3-4).$$

5.88. If a, b, c, d are nonnegative real numbers such that

$$a+b+c+d=4,$$

then

$$(a^4+b^4+c^4+d^4)^2 \ge (a^2+b^2+c^2+d^2)(a^5+b^5+c^5+d^5).$$

5.89. If a, b, c, d are nonnegative real numbers such that

$$a+b+c+d=4,$$

then

$$13(a^2+b^2+c^2+d^2)^2 \ge 12(a^4+b^4+c^4+d^4)+160.$$

5.90. If $a_1, a_2, \ldots, a_8$ are nonnegative real numbers, then

$$19(a_1^2 + a_2^2 + \cdots + a_8^2)^2 \ge 12(a_1 + a_2 + \cdots + a_8)(a_1^3 + a_2^3 + \cdots + a_8^3).$$

5.91. If a, b, c are nonnegative real numbers so that

$$5(a^2 + b^2 + c^2) = 17(ab + bc + ca),$$

then

$$3\sqrt{\frac{3}{5}} \le \sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} \le \frac{1+\sqrt{7}}{\sqrt{2}}.$$

5.92. If a, b, c are nonnegative real numbers so that

$$8(a^2 + b^2 + c^2) = 9(ab + bc + ca),$$

then

$$\frac{19}{12} \le \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \le \frac{141}{88}.$$

5.93. If $a, b, c \in (0, 2]$ such that $a + b + c = 3$, then

$$\sqrt{\frac{2(b+c)}{a} - 1} + \sqrt{\frac{2(c+a)}{b} - 1} + \sqrt{\frac{2(a+b)}{c} - 1} \ge \frac{9}{\sqrt{ab+bc+ca}}.$$

5.94. Let a, b, c and x, y, z be nonnegative real numbers such that

$$x^3 + y^3 + z^3 = a^3 + b^3 + c^3.$$

Then,

$$\frac{(a+b+c)(x+y+z)}{ab+bc+ca+xy+yz+zx} \ge \sqrt[3]{3}.$$

5.95. If a, b, c, d are positive numbers such that

$$a + b + c + d = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d},$$

then

$$ab + ac + ad + bc + bd + cd + 3abcd \ge 9.$$

5.96. If a_1, a_2, a_3, a_4, a_5 are nonnegative real numbers, then

$$\frac{(a_1^3 + a_2^3 + a_3^3 + a_4^3 + a_5^3)^2}{a_1^4 + a_2^4 + a_3^4 + a_4^4 + a_5^4} \ge \frac{1}{2} \sum_{i<j} a_i a_j.$$

5.97. If $a_1, a_2, \ldots, a_n \ge 0$ such that

$$a_1 + a_2 + \cdots + a_n = n,$$

then

$$\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n} \le \sqrt{2n - 1 + 2\left(1 - \frac{1}{n}\right) \sum_{i<j} a_i a_j}.$$

5.98. If $a_1, a_2, \ldots, a_n \ge 0$ such that

$$a_1 + a_2 + \cdots + a_n = \sum_{i<j} a_i a_j > 0,$$

then

$$\frac{(n-1)(n-2)}{2}(a_1 + a_2 + \cdots + a_n) + \sum_{i<j} \sqrt{a_i a_j} \ge n(n-1).$$

5.99. Let

$$F(a_1, a_2, \ldots, a_n) = n(a_1^2 + a_2^2 + \cdots + a_n^2) - (a_1 + a_2 + \cdots + a_n)^2,$$

where $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 \le a_2 \le \cdots \le a_n$ and

$$a_1^2(a_2^2 + a_3^2 + \cdots + a_n^2) \ge n - 1.$$

Then,

$$F(a_1, a_2, \ldots, a_n) \ge F\left(\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}\right).$$

5.100. Let

$$F(a_1, a_2, \ldots, a_n) = a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1 a_2 \cdots a_n},$$

where $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 \le a_2 \le \cdots \le a_n$ and

$$a_1(a_2 + a_3 + \cdots + a_n) \ge n - 1.$$

Then,

$$F(a_1, a_2, \ldots, a_n) \ge F\left(\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}\right).$$

5.101. Let

$$F(a_1,a_2,\dots,a_n)=\sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}}-\frac{a_1+a_2+\cdots+a_n}{n}\ ,$$

where $a_1,a_2,\dots,a_n$ are positive real numbers such that $a_1\le a_2\le\cdots\le a_n$ and

$$a_1^{n-1}(a_2+a_3+\cdots+a_n)\ge n-1.$$

Then,

$$F(a_1,a_2,\dots,a_n)\ge F\left(\frac{1}{a_1},\frac{1}{a_2},\dots,\frac{1}{a_n}\right).$$

5.102. If $a_1,a_2,\dots,a_n$ $(n\ge 4)$ are positive real numbers such that

$$a_1+a_2+\cdots+a_n=n,\qquad a_n=\max\{a_1,a_2,\dots,a_n\},$$

then

$$n\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_{n-1}}\right)\ge 4(a_1^2+a_2^2+\cdots+a_n^2)+n(n-5).$$

5.103. If a,b,c are nonnegative real numbers so that $ab+bc+ca=3$, then

$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1}\le 1.$$

5.3 Solutions

P 5.1. *If a, b, c, d are nonnegative real numbers so that*

$$a+b+c+d = a^3+b^3+c^3+d^3 = 2,$$

then

$$\frac{7}{4} \le a^2+b^2+c^2+d^2 \le 2.$$

(Vasile C., 2010)

Solution. The right inequality follows from the Cauchy-Schwarz inequality

$$(a^2+b^2+c^2+d^2)^2 \le (a+b+c+d)(a^3+b^3+c^3+d^3).$$

The equality holds for $a = b = 0$ and $c = d = 1$ (or any permutation).

To prove the left inequality, assume that $a \le b \le c \le d$, then apply Corollary 5 for $k = 3$ and $m = 2$:

- *If a, b, c, d are nonnegative real numbers so that*

$$a+b+c+d = 2\,, \qquad a^3+b^3+c^3+d^3 = 2, \qquad a \le b \le c \le d,$$

then

$$S_4 = a^2+b^2+c^2+d^2$$

is minimal for $a = b = c$.

So, we only need to prove that the equations

$$3a+d = 3a^3+d^3 = 2, \quad a, d \ge 0,$$

imply

$$\frac{7}{4} \le 3a^2+d^2.$$

Indeed, from $3a+d = 3a^3+d^3 = 2$, we get $a = 1/4$ and $d = 5/4$, when

$$3a^2+d^2 = \frac{7}{4}.$$

The left inequality is an equality for

$$a = b = c = \frac{1}{4}, \qquad d = \frac{5}{4}$$

(or any cyclic permutation).

□

P 5.2. *If $a_1, a_2, \ldots, a_9$ are nonnegative real numbers so that*

$$a_1 + a_2 + \cdots + a_9 = a_1^2 + a_2^2 + \cdots + a_9^2 = 3,$$

then

$$3 \le a_1^3 + a_2^3 + \cdots + a_9^3 \le \frac{14}{3}.$$

(Vasile C., 2010)

Solution. The left inequality follows from the Cauchy-Schwarz inequality

$$(a_1 + a_2 + \cdots + a_9)(a_1^3 + a_2^3 + \cdots + a_9^3) \ge (a_1^2 + a_2^2 + \cdots + a_9^2)^2.$$

The equality holds for $a_1 = a_2 = \cdots = a_6 = 0$ and $a_7 = a_8 = a_9 = 1$ (or any permutation).

To prove the right inequality, assume that

$$a_1 \le a_2 \le \cdots \le a_9,$$

then apply Corollary 5 for $k = 2$ and $m = 3$:

- *If $a_1, a_2, \ldots, a_9$ are nonnegative real numbers so that*

$$a_1 + a_2 + \cdots + a_9 = 3\ , \qquad a_1^2 + a_2^2 + \cdots + a_9^2 = 3, \qquad a_1 \le a_2 \le \cdots \le a_9,$$

then

$$S_9 = a_1^3 + a_2^3 + \cdots + a_9^3$$

is maximal for $a_1 = a_2 = \cdots = a_8 \le a_9$.

Thus, we only need to prove that the equations

$$8a + b = 3, \qquad 8a^2 + b^2 = 3, \qquad a, b \ge 0,$$

involve

$$8a^3 + b^3 \le \frac{14}{3}.$$

Indeed, from the equations above, we get $a = 1/6$ and $b = 5/3$, when

$$8a^3 + b^3 = \frac{1}{27} + \frac{125}{27} = \frac{14}{3}.$$

The equality holds for

$$a_1 = a_2 = \cdots = a_8 = \frac{1}{6}, \qquad a_9 = \frac{5}{3}$$

(or any cyclic permutation).

□

P 5.3. *If a,b,c,d are nonnegative real numbers so that*

$$a+b+c+d=a^2+b^2+c^2+d^2=\frac{27}{7},$$

then

$$\frac{5427}{1372}\le a^3+b^3+c^3+d^3\le\frac{1377}{343}.$$

(Vasile C., 2014)

Solution. Assume that $a\le b\le c\le d$.

(a) To prove the right inequality, we apply Corollary 5 for $k=2$ and $m=3$:

- *If a,b,c,d are nonnegative real numbers so that*

$$a+b+c+d=\frac{27}{7}, \qquad a^2+b^2+c^2+d^2=\frac{27}{7}, \qquad a\le b\le c\le d,$$

then

$$S_4=a^3+b^3+c^3+d^3$$

is maximal for $a=b=c\le d$

Thus, we only need to prove that the equations

$$3a+d=\frac{27}{7}, \qquad 3a^2+d^2=\frac{27}{7}, \qquad a,d\ge 0,$$

involve

$$3a^3+d^3\le\frac{1377}{343}.$$

Indeed, from the equations above, we get $a=6/7$ and $d=9/7$, when

$$3a^3+d^3=3\left(\frac{6}{7}\right)^3+\left(\frac{9}{7}\right)^3=\frac{1377}{343}.$$

The equality holds for

$$a=b=c=\frac{6}{7}, \qquad d=\frac{9}{7}$$

(or any cyclic permutation).

(b) To prove the left inequality, we apply Corollary 5 for $k=2$ and $m=3$:

- *If a,b,c,d are nonnegative real numbers so that*

$$a+b+c+d=\frac{27}{7}, \qquad a^2+b^2+c^2+d^2=\frac{27}{7}, \qquad a\le b\le c\le d,$$

then

$$S_4=a^3+b^3+c^3+d^3$$

is minimal for either $a = 0$ or $a \le b = c = d$.

The case $a = 0$ is not possible because from

$$b + c + d = \frac{27}{7}, \qquad b^2 + c^2 + d^2 = \frac{27}{7},$$

we get

$$3(b^2 + c^2 + d^2) - (b + c + d)^2 = \frac{27}{7}\left(3 - \frac{27}{7}\right) < 0,$$

which contradicts the known inequality

$$3(b^2 + c^2 + d^2) \ge b + c + d)^2.$$

For $a \le b = c = d$, we need to prove that the equations

$$a + 3d = \frac{27}{7}, \qquad a^2 + 3d^2 = \frac{27}{7}, \qquad a, d \ge 0,$$

involve

$$a^3 + 3d^3 \ge \frac{5427}{1372}.$$

Indeed, from the equations above, we get $a = 9/14$ and $d = 15/14$, when

$$a^3 + 3d^3 = \left(\frac{9}{14}\right)^3 + 3\left(\frac{15}{14}\right)^3 = \frac{5427}{1372}.$$

The equality holds for

$$a = \frac{9}{14}, \qquad b = c = d = \frac{15}{14}$$

(or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

• *Let k be a positive real number ($k > 2$), and let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that*

$$a_1 + a_2 + \cdots + a_n = a_1^2 + a_2^2 + \cdots + a_n^2 = \frac{(n-1)^3}{n^2 - 3n + 3}.$$

The sum

$$S_n = a_1^k + a_2^k + \cdots + a_n^k$$

is maximal for

$$a_1 = \cdots = a_{n-1} = \frac{(n-1)(n-2)}{n^2 - 3n + 3}, \qquad a_n = \frac{(n-1)^2}{n^2 - 3n + 3},$$

and is minimal for

$$a_1 = \frac{(n-1)^2(n-2)}{n(n^2 - 3n + 3)}, \qquad a_2 = \cdots = a_n = \frac{(n-1)(n^2 - 2n + 2)}{n(n^2 - 3n + 3)}.$$

$\square$

P 5.4. *If a,b,c are positive real numbers so that $abc=1$, then*

$$a^5+b^5+c^5 \ge \sqrt{3(a^7+b^7+c^7)}.$$

(Vasile C., 2014)

Solution. Substituting

$$a=x^{1/5},\ b=y^{1/5},\ c=z^{1/5},$$

we need to show that $xyz=1$ involves

$$x+y+z \ge \sqrt{3(x^{7/5}+y^{7/5}+z^{7/5})}.$$

Assume that $x\le y\le z$, then apply Corollary 5 for $k=0$ and $m=7/5$:

- *If x,y,z are positive real numbers so that*

$$x+y+z=constant\ , \quad xyz=1, \quad x\le y\le z,$$

then

$$S_3=x^{7/5}+y^{7/5}+z^{7/5}$$

is maximal for $x=y$.

So, it suffices to prove the original inequality for $a=b$. Write this inequality in the homogeneous form

$$(a^5+b^5+c^5)^2 \ge 3abc(a^7+b^7+c^7).$$

We only need to prove this inequality for $a=b=1$; that is, to show that $f(c)\ge 0$, where

$$f(c)=(c^5+2)^2-3c(c^7+2), \quad c>0.$$

We have

$$f'(c)=10c^4(c^5+2)-24c^7-6,$$

$$f''(c)=2c^3g(t), \quad g(t)=45c^5-84c^3+40.$$

By the AM-GM inequality, we get

$$\begin{aligned}g(t)&=15c^5+15c^5+15c^5+20+20-84c^3\ge 5\sqrt[5]{(15c^5)^3\cdot 20^2}-84c^3\\&=\sqrt[5]{27\cdot 16}\left(25-14\sqrt[5]{18}\right)c^3>0,\end{aligned}$$

hence $f''(c)>0$, $f'(c)$ is increasing. Since $f'(0)=1$, it follows that $f'(c)\le 0$ for $c\le 1$, $f'(c)\ge 0$ for $c\ge 1$, therefore f is decreasing on $(0,1]$ and increasing on $[1,\infty)$; consequently, $f(c)\ge f(1)=0$. The equality occurs for $a=b=c=1$.

□

P 5.5. *If a,b,c,d are positive real numbers so that $abcd=1$, then*

$$a^3+b^3+c^3+d^3 \ge \sqrt{4(a^4+b^4+c^4+d^4)}.$$

(Vasile C., 2014)

Solution. Substituting

$$a=x^{1/3},\ b=y^{1/3},\ c=z^{1/3},\ d=t^{1/3},$$

we need to show that $xyzt=1$ involves

$$x+y+z+t \ge \sqrt{4(x^{4/3}+y^{4/3}+z^{4/3}+t^{4/3})}.$$

Apply Corollary 5, case $k=0$ and $m=4/3$:

- *If x,y,z,t are positive real numbers so that*

$$x+y+z+t = constant\ , \qquad xyzt=1, \qquad x\le y\le z\le t,$$

then

$$S_4 = x^{4/3}+y^{4/3}+z^{4/3}+t^{4/3}$$

is maximal for $x=y=z$.

Therefore, it suffices to prove the original inequality for $a=b=c$. Write the original inequality in the homogeneous form

$$(a^3+b^3+c^3+d^3)^2 \ge 4\sqrt{abcd}\ (a^4+b^4+c^4+d^4).$$

We only need to prove this inequality for $a=b=c=1$; that is, to show that

$$(d^3+3)^2 \ge 4\sqrt{d}\ (d^4+3).$$

Putting $u=\sqrt{d}$, we have

$$\begin{aligned}(d^3+3)^2-4\sqrt{d}\ (d^4+3) &= (u^6+3)^2-4u(u^8+3)\\ &= (u^3-1)^4+4(u+2)(u-1)^2 \ge 0.\end{aligned}$$

The equality holds for $a=b=c=d=1$.

□

P 5.6. *If a,b,c,d are nonnegative real numbers so that $a+b+c+d=4$, then*

$$\frac{bcd}{11a+16}+\frac{cda}{11b+16}+\frac{dab}{11c+16}+\frac{abc}{11d+16} \le \frac{4}{27}.$$

(Vasile C., 2008)

Solution. For $a=0$, the inequality becomes

$$bcd \le \frac{64}{27},$$

where $b,c,d \ge 0$, $b+c+d=4$. By the AM-GM inequality, we have

$$bcd \le \left(\frac{b+c+d}{3}\right)^3 = \left(\frac{4}{3}\right)^3 = \frac{64}{27}.$$

For $abcd \ne 0$, we write the inequality in the form

$$f(a)+f(b)+f(c)+f(d)+\frac{4}{(1+k)abcd} \ge 0,$$

where

$$f(u)=\frac{-1}{u(u+k)}, \qquad k=\frac{16}{11}, \qquad u>0.$$

We have $f(0+)=-\infty$ and

$$f'(u)=\frac{2u+k}{(u^2+ku)^2},$$

$$g(x)=f'(1/x)=\frac{kx^4+2x^3}{(kx+1)^2},$$

$$g''(x)=\frac{2x(k^3x^3+4k^2x^2+6kx+6)}{(kx+1)^4}.$$

Since $g''(x)>0$ for $x>0$, g is strictly convex on $(0,\infty)$. By Corollary 3 and Note 1, *if*

$$a+b+c+d=4, \qquad abcd=constant, \qquad 0<a\le b\le c\le d,$$

then the sum

$$S_4=f(a)+f(b)+f(c)+f(d)$$

is minimal for $b=c=d$. Thus, we only need to prove that

$$\frac{b^3}{11a+16}+\frac{3ab^2}{11b+16} \le \frac{4}{27}$$

for $a+3b=4$. The inequality is equivalent to

$$\frac{b^3}{3(20-11b)}+\frac{3b^2(4-3b)}{11b+16} \le \frac{4}{21},$$

$$(b-1)^2(4-3b)(231b+80) \ge 0,$$

$$(b-1)^2a(231b+80) \ge 0.$$

The equality holds for $a=b=c=d=1$, and also for

$$a=0, \qquad b=c=d=\frac{4}{3}$$

(or any cyclic permutation).

□

P 5.7. *If a, b, c are real numbers, then*

$$\frac{bc}{3a^2+b^2+c^2}+\frac{ca}{3b^2+c^2+a^2}+\frac{ab}{3c^2+a^2+b^2}\le\frac{3}{5}.$$

(Vasile Cirtoaje and Pham Kim Hung, 2005)

Solution. For $a=0$, the inequality is true because

$$\frac{bc}{b^2+c^2}\le\frac{1}{2}<\frac{3}{5}.$$

Consider further that a, b, c are different from zero. The inequality remains unchanged by replacing a, b, c with $-a, -b, -c$, respectively. Thus, we only need to consider the case $a<0$, $b, c>0$, and the case $a, b, c>0$. In the first case, it suffices to show that

$$\frac{bc}{3a^2+b^2+c^2}\le\frac{3}{5}.$$

Indeed, we have

$$\frac{bc}{3a^2+b^2+c^2}<\frac{bc}{b^2+c^2}\le\frac{1}{2}<\frac{3}{5}.$$

Consider now the case $a, b, c>0$. Replacing a, b, c with $\sqrt{a}, \sqrt{b}, \sqrt{c}$, the inequality becomes

$$\frac{1}{\sqrt{a}(3a+b+c)}+\frac{1}{\sqrt{b}(3b+c+a)}+\frac{1}{\sqrt{c}(3c+a+b)}\le\frac{3}{5\sqrt{abc}}.$$

Due to homogeneity, we may consider that $a+b+c=2$. So, we need to show that

$$f(a)+f(b+f(c)+\frac{6}{5\sqrt{abc}}\ge 0,$$

where

$$f(u)=\frac{-1}{\sqrt{u}(u+1)},\qquad u>0.$$

We have $f(0+)=-\infty$ and

$$f'(u)=\frac{3u+1}{2u\sqrt{u}(u+1)^2},$$

$$g(x)=f'(1/x)=\frac{x^2\sqrt{x}(x+3)}{2(x+1)^2},$$

$$g''(x)=\frac{\sqrt{x}(3x^3+11x^2+5x+45)}{8(x+1)^4}.$$

Since $g''(x)>0$ for $x>0$, g is strictly convex on $(0,\infty)$. By Corollary 3 and Note 1, *if*

$$a+b+c=2,\qquad abc=constant,\qquad 0<a\le b\le c,$$

then the sum

$$S_3 = f(a) + f(b) + f(c)$$

is minimal for $b = c$. Thus, we only need to prove the original homogeneous inequality for $b = c = 1$; that is,

$$\frac{1}{3a^2+2} + \frac{2a}{a^2+4} \le \frac{3}{5},$$

$$9a^4 - 30a^3 + 37a^2 - 20a + 4 \ge 0,$$

$$(a-1)^2(3a-2)^2 \ge 0.$$

The equality holds for $a = b = c$, and also for

$$3a = 2b = 2c$$

(or any cyclic permutation).

□

P 5.8. *If a, b, c are nonnegative real numbers so that $a + b + c = 3$, then*

(a)
$$\frac{bc}{a^2+2} + \frac{ca}{b^2+2} + \frac{ab}{c^2+2} \le \frac{9}{8};$$

(b)
$$\frac{bc}{a^2+3} + \frac{ca}{b^2+3} + \frac{ab}{c^2+3} \le \frac{11\sqrt{33}-45}{24};$$

(c)
$$\frac{bc}{a^2+4} + \frac{ca}{b^2+4} + \frac{ab}{c^2+4} \le \frac{3}{5}.$$

(Vasile C., 2008)

Solution. For the nontrivial case $abc \ne 0$, we can write the desired inequalities in the form

$$f(a) + f(b) + f(c) + \frac{m}{abc} \ge 0,$$

where

$$f(u) = \frac{-1}{u(u^2+k)}, \qquad k \in \{2,3,4\}, \qquad u > 0.$$

We have $f(0+) = -\infty$ and

$$f'(u) = \frac{3u^2+k}{u^2(u^2+k)^2},$$

$$g(x) = f'(1/x) = \frac{kx^6+3x^4}{(kx^2+1)^2},$$

$$g''(x) = \frac{2x^2(k^3x^6+4k^2x^4-3kx^2+18)}{(kx^2+1)^4}.$$

Since
$$k^3x^6+4k^2x^4-3kx^2+18>4k^2x^4-3kx^2+18>0,$$
we have $g''(x)>0$, hence g is strictly convex on $(0,\infty)$. According to Corollary 3 and Note 1, *if*
$$a+b+c=3, \quad abc=constant, \quad 0<a\le b\le c,$$
then the sum
$$S_3=f(a)+f(b)+f(c)$$
is minimal for $b=c$. Thus, we only need to prove the original inequalities for $b=c$.

(a) We only need to prove the homogeneous inequality
$$\frac{bc}{9a^2+2(a+b+c)^2}+\frac{ca}{9b^2+2(a+b+c)^2}+\frac{ab}{9c^2+2(a+b+c)^2}\le\frac{1}{8}$$
for $b=c=1$, that is
$$\frac{1}{11a^2+8a+8}+\frac{2a}{2a^2+8a+17}\le\frac{1}{8},$$
$$\frac{2a}{2a^2+8a+17}\le\frac{a(11a+8)}{8(11a^2+8a+8)},$$
$$a(22a^3-72a^2+123a+8)\ge 0.$$
Since
$$22a^3-72a^2+123a+8>20a^3-80a^2+80a=20a(a-2)^2\ge 0,$$
the conclusion follows. The equality holds for $a=0$ and $b=c=3/2$ (or any cyclic permutation).

(b) Let
$$m=\frac{11\sqrt{33}-45}{72}\approx 0.253, \quad r=\frac{\sqrt{33}-5}{4}\approx 0.186.$$
We only need to prove the homogeneous inequality
$$\frac{bc}{3a^2+(a+b+c)^2}+\frac{ca}{3b^2+(a+b+c)^2}+\frac{ab}{3c^2+(a+b+c)^2}\le m$$
for $b=c=1$; that is, to show that $f(a)\le m$, where
$$f(a)=\frac{1}{4(a^2+a+1)}+\frac{2a}{a^2+4a+7}.$$

We have

$$f'(a) = \frac{-8a^6 - 18a^5 + 15a^4 + 28a^3 + 18a^2 - 42a + 7}{4(a^2 + a + 1)^2(a^2 + 4a + 7)^2}$$
$$= \frac{(1-a)^2(7 + 7a + 4a^2)(1 - 5a - 2a^2)}{4(a^2 + a + 1)^2(a^2 + 4a + 7)^2}.$$

Since $f'(a) \geq 0$ for $a \in [0, r]$, and $f'(a) \leq 0$ for $a \in [r, \infty)$, f is increasing on $[0, r]$ and decreasing on $[r, \infty)$; therefore,

$$f(a) \geq f(r) = m.$$

The equality holds for

$$a/r = b = c$$

(or any cyclic permutation).

(c) We only need to prove the homogeneous inequality

$$\frac{bc}{9a^2 + 4(a+b+c)^2} + \frac{ca}{9b^2 + 4(a+b+c)^2} + \frac{ab}{9c^2 + 4(a+b+c)^2} \leq \frac{1}{15}$$

for $b = c = 1$, that is

$$\frac{1}{13a^2 + 16a + 16} + \frac{2a}{4a^2 + 16a + 25} \leq \frac{1}{15},$$

$$52a^4 - 118a^3 + 105a^2 - 64a + 25 \geq 0,$$

$$(a-1)^2(52a^2 - 14a + 25) \geq 0.$$

Since

$$52a^2 - 14a + 25 > 7a^2 - 14a + 7 = 7(a-1)^2 \geq 0,$$

the conclusion follows. The equality holds for $a = b = c = 1$.

□

P 5.9. *If a, b, c, d are nonnegative real numbers so that*

$$(3a+1)(3b+1)(3c+1)(3d+1) = 64,$$

then

$$abc + bcd + cda + dab \leq 1.$$

(Vasile C., 2014)

Solution. For $d = 0$, we need to show that

$$(3a+1)(3b+1)(3c+1) = 64$$

involves $abc \le 1$. Indeed, by the AM-GM inequality, we have

$$64 = (3a+1)(3b+1)(3c+1) \ge \left(4\sqrt[4]{a^3}\right)\left(4\sqrt[4]{b^3}\right)\left(4\sqrt[4]{c^3}\right) = 64\sqrt[4]{(abc)^3},$$

hence $abc \le 1$. Consider further that $a,b,c,d > 0$ and use the contradiction method. Assume that

$$abc + bcd + cda + dab > 1,$$

and prove that

$$(3a+1)(3b+1)(3c+1) > 64.$$

It suffices to show that

$$abc + bcd + cda + dab \ge 1$$

involves

$$(3a+1)(3b+1)(3c+1) \ge 64.$$

Replacing a,b,c,d by $1/a, 1/b, 1/c, 1/d$, we need to show that

$$a + b + c + d = abcd$$

involves

$$\left(\frac{3}{a}+1\right)\left(\frac{3}{b}+1\right)\left(\frac{3}{c}+1\right)\left(\frac{3}{d}+1\right) \ge 64,$$

which is equivalent to

$$f(a) + f(b) + f(c) + f(d) \le -6\ln 2,$$

where

$$f(u) = -\ln\left(\frac{3}{u}+1\right), \qquad u > 0.$$

We have $f(0+) = -\infty$ and

$$g(x) = f'(1/x) = \frac{3x^2}{3x+1}, \qquad g''(x) = \frac{6}{(3x+1)^3} > 0,$$

hence g is strictly convex on $(0,\infty)$. By Corollary 3 and Note 1, *if a,b,c,d are positive real numbers so that*

$$a + b + c + d = constant\ , \qquad abcd = constant\ , \qquad a \le b \le c \le d,$$

then

$$S_4 = f(a) + f(b) + f(c) + f(d)$$

is maximal for $a = b = c$.

Thus, we only need to prove that

$$\left(\frac{3}{a}+1\right)^3\left(\frac{3}{d}+1\right) \ge 64$$

for $3a+d=a^3d$, that is

$$\frac{3}{d}=\frac{a^3-1}{a}, \quad 1<a\le d.$$

Write this inequality as

$$(3+a)^3(3+d)\ge 64a^3d,$$

$$(3+a)^4(3+d)\ge 64a^3d(3+a),$$

$$4\left(1+\frac{a-1}{4}\right)^4(3+d)\ge a^3d(3+a).$$

By Bernoulli's inequality, we have

$$\left(1+\frac{a-1}{4}\right)^4 \ge 1+4\cdot\frac{a-1}{4}=a.$$

Thus, it suffices to show that

$$4(3+d)\ge a^2d(3+a),$$

which is equivalent to

$$\frac{12}{d}\ge a^3+3a^2-4,$$

$$\frac{4(a^3-1)}{a}\ge a^3+3a^2-4,$$

$$a^4-a^3-4a+4\le 0,$$

$$(a-1)(a^3-4)\le 0.$$

This is true if $a^3\le 4$. Indeed, we have

$$0\le\frac{3}{a}-\frac{3}{d}=\frac{3}{a}-\frac{a^3-1}{a}=\frac{4-a^3}{a}.$$

The proof is completed. The original inequality is an equality for

$$a=b=c=1, \quad d=0$$

(or any cyclic permutation).

□

P 5.10. *If $a_1, a_2, \ldots, a_n$ and p, q are nonnegative real numbers so that*

$$a_1 + a_2 + \cdots + a_n = p + q, \qquad a_1^3 + a_2^3 + \cdots + a_n^3 = p^3 + q^3,$$

then

$$a_1^2 + a_2^2 + \cdots + a_n^2 \le p^2 + q^2.$$

(Vasile C., 2013)

Solution. For $n = 2$, the inequality is an equality. Consider now that $n \ge 3$ and $a_1 \le a_2 \le \cdots \le a_n$. We will apply Corollary 5 for $k = 3$ and $m = 2$:

- *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that $a_1 \le a_2 \le \cdots \le a_n$ and*

$$a_1 + a_2 + \cdots + a_n = p + q, \qquad a_1^3 + a_2^3 + \cdots + a_n^3 = p^3 + q^3,$$

then

$$S_n = a_1^2 + a_2^2 + \cdots + a_n^2$$

is maximal for either $a_1 = 0$ or $a_2 = a_3 = \cdots = a_n$.

In the first case $a_1 = 0$, the conclusion follows by induction method. In the second case, for

$$a_1 = a, \qquad a_2 = a_3 = \cdots = a_n = b,$$

we need to show that

$$a^2 + (n-1)b^2 \le p^2 + q^2$$

for

$$a + (n-1)b = p + q, \qquad a^3 + (n-1)b^3 = p^3 + q^3.$$

Since

$$3(p^2 + q^2) = (p+q)^2 + \frac{2(p^3 + q^3)}{p+q},$$

the inequality can be written as

$$3a^2 + 3(n-1)b^2 \le [a + (n-1)b]^2 + \frac{2[a^3 + (n-1)b^3]}{a + (n-1)b},$$

which is equivalent to

$$(n-1)(n-2)b^2[3a + (n-3)b] \ge 0.$$

The equality holds when $n-2$ of $a_1, a_2, \ldots, a_n$ are equal to zero.

□

P 5.11. *If a, b, c are nonnegative real numbers, then*

$$a\sqrt{a^2 + 4b^2 + 4c^2} + b\sqrt{b^2 + 4c^2 + 4a^2} + c\sqrt{c^2 + 4a^2 + 4b^2} \ge (a + b + c)^2.$$

(Vasile C., 2010)

Solution. Due to homogeneity and symmetry, we may assume that

$$a^2+b^2+c^2=3, \qquad 0 \le a \le b \le c \le \sqrt{3}.$$

Under this assumption, we write the desired inequality as

$$f(a)+f(b)+f(c)+\frac{1}{\sqrt{3}}(a+b+c)^2 \le 0,$$

where

$$f(u)=-u\sqrt{4-u^2}, \qquad 0 \le u \le \sqrt{3}.$$

We have

$$g(x)=f'(x)=\frac{2(x^2-2)}{\sqrt{4-x^2}},$$

$$g''(x)=\frac{48}{(4-x^2)^{5/2}}.$$

Since $g''(x)>0$ for $x \in (0,2)$, g is strictly convex on $[0,\sqrt{3}]$. According to Corollary 1, *if*

$$a+b+c=constant\ , \qquad a^2+b^2+c^2=3\ , \qquad 0 \le a \le b \le c,$$

then the sum

$$S_3=f(a)+f(b)+f(c)$$

is maximal for $a=b\le c$. Thus, we only need to prove the original inequality for $a=b$. Since the inequality is an identity for $a=b=0$, we may consider $a=b=1$ and $c \ge 1$. We need to prove that

$$2\sqrt{4c^2+5}+c\sqrt{c^2+8} \ge (c+2)^2.$$

By squaring, the inequality becomes

$$c\sqrt{(4c^2+5)(c^2+8)} \ge 2c^3+8c-1.$$

This is true if

$$c^2(4c^2+5)(c^2+8) \ge (2c^3+8c-1)^2,$$

which is equivalent to

$$5c^4+4c^3-24c^2+16c-1 \ge 0,$$

$$(c-1)^2(5c^2+14c-1) \ge 0.$$

The equality holds for $a=b=c$, and also for $a=b=0$ (or any cyclic permutation).

□

P 5.12. *If a,b,c are nonnegative real numbers so that $ab+bc+ca=3$, then*

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\le\frac{3}{2(a+b+c)}+\frac{a+b+c}{3}.$$

(Vasile C., 2010)

Solution. Write the inequality in the homogeneous form

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\le\frac{3}{2(a+b+c)}+\frac{a+b+c}{ab+bc+ca}.$$

Due to homogeneity and symmetry, we may assume that

$$a+b+c=1,\qquad 0\le a\le b\le c,\qquad ab+bc+ca>0.$$

Under this assumption, we write the desired inequality as

$$f(a)+f(b)+f(c)\le\frac{3}{2}+\frac{1}{ab+bc+ca},$$

where

$$f(u)=\frac{1}{1-u},\qquad 0\le u<1.$$

We will apply Corollary 1 to the function f, which satisfies $f(1-)=\infty$ and

$$g(x)=f'(x)=\frac{1}{(1-x)^2},$$

$$g''(x)=\frac{6}{(1-x)^4}.$$

Since $g''(x)>0$, g is strictly convex on $[0,1)$. According to Corollary 1 and Note 3, *if*

$$a+b+c=1\ ,\qquad ab+bc+ca=constant\ ,\qquad 0\le a\le b\le c,$$

then the sum

$$S_3=f(a)+f(b)+f(c)$$

is maximal for $a=b\le c$. Thus, we only need to prove the homogeneous inequality for $a=b=1$ and $c\ge 1$; that is,

$$1+\frac{4}{c+1}\le\frac{3}{c+2}+\frac{2(c+2)}{2c+1},$$

which reduces to

$$(c-1)^2\ge 0.$$

The original inequality is an equality for $a=b=c=1$.

□

P 5.13. *If a,b,c are nonnegative real numbers so that $ab+bc+ca=3$, then*

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq\frac{3}{a+b+c}+\frac{a+b+c}{6}.$$

(Vasile C., 2010)

Solution. Proceeding in the same manner as in the proof of the preceding P 5.12, we only need to prove the homogeneous inequality

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq\frac{3}{a+b+c}+\frac{a+b+c}{2(ab+bc+ca)}$$

for $a=0$ and for $a\leq b=c=1$.

Case 1: $a=0$. The homogeneous inequality reduces to

$$\frac{1}{b}+\frac{1}{c}\geq\frac{2}{b+c}+\frac{b+c}{2bc},$$

which is equivalent to

$$(b-c)^2\geq 0.$$

Case 2: $a\leq b=c=1$. The homogeneous inequality becomes

$$\frac{1}{2}+\frac{2}{a+1}\geq\frac{3}{a+2}+\frac{a+2}{2(2a+1)},$$

$$\frac{1}{2}-\frac{a+2}{2(2a+1)}\geq\frac{3}{a+2}-\frac{2}{a+1},$$

$$\frac{a-1}{2(2a+1)}\geq\frac{a-1}{(a+1)(a+2)},$$

$$a(a-1)^2\geq 0.$$

The equality holds for $a=b=c=1$, and also for

$$a=0,\qquad b=c=\sqrt{3}$$

(or any cyclic permutation).

□

P 5.14. *Let a,b,c be nonnegative real numbers, no two of which are zero. If*

$$a^2+b^2+c^2=3,$$

then

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}+\frac{a+b+c}{9}\geq\frac{11}{2(a+b+c)}.$$

(Vasile C., 2010)

Solution. Using the same method as in the proof of P 5.12, we only need to prove the homogeneous inequality

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}+\frac{a+b+c}{3(a^2+b^2+c^2)}\ge\frac{11}{2(a+b+c)}$$

for $a=0$ and for $a\le b=c=1$.

Case 1: $a=0$. The homogeneous inequality reduces to

$$\frac{1}{b}+\frac{1}{c}+\frac{1}{b+c}+\frac{b+c}{3(b^2+c^2)}\ge\frac{11}{2(b+c)},$$

$$\frac{b+c}{bc}+\frac{b+c}{3(b^2+c^2)}\ge\frac{9}{2(b+c)},$$

$$(b+c)^2\left[\frac{1}{bc}+\frac{1}{3(b^2+c^2)}\right]\ge\frac{9}{2}.$$

Using the substitution

$$x=\frac{b^2+c^2}{bc},\qquad x\ge 2,$$

the inequality becomes

$$(x+2)\left(1+\frac{1}{3x}\right)\ge\frac{9}{2},$$

which is equivalent to

$$6x^2-13x+4\ge 0,$$

$$x+2(x-2)(3x-1)\ge 0.$$

Case 2: $a\le 1=b=c$. The homogeneous inequality becomes

$$\frac{1}{2}+\frac{2}{a+1}+\frac{a+2}{3(a^2+2)}\ge\frac{11}{2(a+2)},$$

$$\frac{a+2}{3(a^2+2)}+\frac{a^2-4a-1}{2(a+1)(a+2)}\ge 0,$$

$$3a^4-10a^3+13a^2-8a+2\ge 0,$$

$$(a-1)^2(3a^2-4a+2)\ge 0,$$

$$(a-1)^2[a^2+2(a-1)^2]\ge 0.$$

The equality holds for $a=b=c=1$.

$\square$

P 5.15. *Let a, b, c be nonnegative real numbers, no two of which are zero. If*

$$a+b+c=4,$$

then

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \geq \frac{15}{8+ab+bc+ca}.$$

(Vasile C., 2010)

Solution. Using the same method as in P 5.12, we only need to prove the homogeneous inequality

$$\frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a} \geq \frac{15(a+b+c)}{(a+b+c)^2+2(ab+bc+ca)}$$

for $a=0$ and for $a \leq b=c=1$.

Case 1: $a=0$. The homogeneous inequality reduces to

$$\frac{2(b+c)}{bc}+\frac{2}{b+c} \geq \frac{15(b+c)}{(b+c)^2+2bc},$$

$$\frac{2(b+c)^2}{bc}+2 \geq \frac{15(b+c)^2}{(b+c)^2+2bc}.$$

Using the substitution

$$x=\frac{(b+c)^2}{bc}, \qquad x \geq 4,$$

the inequality becomes

$$2x+2 \geq \frac{15x}{x+2},$$

which is equivalent to

$$2x^2-9x+4 \geq 0,$$

$$(x-4)(2x-1) \geq 0.$$

Case 2: $a \leq 1,\ b=c=1.$ The homogeneous inequality becomes

$$1+\frac{4}{a+1} \geq \frac{15(a+2)}{(a+2)^2+2(2a+1)},$$

$$\frac{a+5}{a+1} \geq \frac{15(a+2)}{a^2+8a+6},$$

$$a(a-1)^2 \geq 0.$$

The equality holds for $a=b=c=4/3$, and also for

$$a=0, \qquad b=c=2$$

(or any cyclic permutation).

□

P 5.16. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \ge \frac{1}{a+b+c}+\frac{2}{\sqrt{ab+bc+ca}}.$$

(Vasile C., 2010)

Solution. Using the same method as in P 5.12, we only need to prove the desired homogeneous inequality for $a=0$ and for $0<a\le b=c=1$.

Case 1: $a=0$. The inequality reduces to the obvious form

$$\frac{1}{b}+\frac{1}{c} \ge \frac{2}{\sqrt{bc}}.$$

Case 2: $0<a\le 1=b=c$. The inequality becomes

$$\frac{1}{2}+\frac{2}{a+1} \ge \frac{1}{a+2}+\frac{2}{\sqrt{2a+1}},$$

$$\frac{1}{2}-\frac{1}{a+2} \ge \frac{2}{\sqrt{2a+1}}-\frac{2}{a+1},$$

$$\frac{a}{2(a+2)} \ge \frac{2(a+1-\sqrt{2a+1})}{(a+1)\sqrt{2a+1}},$$

$$\frac{a}{2(a+2)} \ge \frac{2a^2}{(a+1)\sqrt{2a+1}\,(a+1+\sqrt{2a+1})}.$$

Since

$$\sqrt{2a+1}\,(a+1+\sqrt{2a+1}) \ge \sqrt{2a+1}(\sqrt{2a+1}+\sqrt{2a+1}) = 2(2a+1),$$

it suffices to show that

$$\frac{a}{2(a+2)} \ge \frac{a^2}{(a+1)(2a+1)},$$

which is equivalent to

$$a(1-a)\ge 0.$$

The equality holds for

$$a=0, \qquad b=c$$

(or any cyclic permutation).

□

P 5.17. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \ge \frac{3-\sqrt{3}}{a+b+c}+\frac{2+\sqrt{3}}{2\sqrt{ab+bc+ca}}.$$

(Vasile C., 2010)

Solution. As shown in the proof of P 5.12, it suffices to consider the cases $a = 0$ and $a \le b = c = 1$.

Case 1: $a = 0$. The inequality reduces to

$$\frac{1}{b} + \frac{1}{c} \ge \frac{2-\sqrt{3}}{b+c} + \frac{2+\sqrt{3}}{2\sqrt{bc}}.$$

It suffices to show that

$$\frac{1}{b} + \frac{1}{c} \ge \frac{2-\sqrt{3}}{2\sqrt{bc}} + \frac{2+\sqrt{3}}{2\sqrt{bc}},$$

which is equivalent to the obvious inequality

$$\frac{1}{b} + \frac{1}{c} \ge \frac{2}{\sqrt{bc}}.$$

Case 2: $a \le 1 = b = c$. The inequality reduces to

$$\frac{1}{2} + \frac{2}{a+1} \ge \frac{3-\sqrt{3}}{a+2} + \frac{2+\sqrt{3}}{2\sqrt{2a+1}}.$$

Using the substitution

$$2a + 1 = 3x^2, \qquad x \ge \frac{\sqrt{3}}{3},$$

the inequality becomes

$$\frac{1}{2} + \frac{4}{3x^2+1} \ge \frac{6-2\sqrt{3}}{3(x^2+1)} + \frac{2+\sqrt{3}}{2\sqrt{3}\,x},$$

$$\frac{1}{2} + \frac{4}{3x^2+1} - \frac{2}{x^2+1} - \frac{1}{2x} \ge \frac{1}{\sqrt{3}\,x} - \frac{2}{\sqrt{3}\,(x^2+1)},$$

$$\frac{3x^5 - 3x^4 - 4x^2 + 5x - 1}{2x(x^2+1)(3x^2+1)} \ge \frac{1}{\sqrt{3}}\left(\frac{1}{x} - \frac{2}{x^2+1}\right),$$

$$\frac{(x-1)^2(3x^3+3x^2+3x-1)}{2x(x^2+1)(3x^2+1)} \ge \frac{(x-1)^2}{\sqrt{3}\,x(x^2+1)}.$$

This is true if

$$\frac{3x^3+3x^2+3x-1}{2(3x^2+1)} \ge \frac{\sqrt{3}}{3},$$

which is equivalent to

$$9x^3 + 3(3-2\sqrt{3})x^2 + 9x - 3 - 2\sqrt{3} \ge 0,$$

$$(3x - \sqrt{3}\,)[3x^2 + (3-\sqrt{3})x + 2 + \sqrt{3}] \ge 0.$$

The equality holds for $a = b = c$, and also for

$$a = 0, \qquad b = c$$

(or any cyclic permutation).

□

P 5.18. *Let a, b, c be nonnegative real numbers, no two of which are zero, so that*

$$ab + bc + ca = 3.$$

If

$$0 \le k \le \frac{9 + 5\sqrt{3}}{6} \approx 2.943,$$

then

$$\frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{c+a} \ge \frac{9(1+k)}{a+b+c+3k}.$$

(Vasile Cirtoaje and Lorian Saceanu, 2014)

Solution. From

$$(a+b+c)^2 \ge 3(ab+bc+ca),$$

we get

$$a + b + c \ge 3.$$

Let

$$m = \frac{9+5\sqrt{3}}{6}, \qquad m \ge k.$$

We claim that

$$\frac{1+m}{a+b+c+3m} \ge \frac{1+k}{a+b+c+3k}.$$

Indeed, this inequality is equivalent to the obvious inequality

$$(m-k)(a+b+c-3) \ge 0.$$

Thus, we only need to show that

$$\frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{c+a} \ge \frac{9(1+m)}{a+b+c+3m},$$

which can be rewritten in the homogeneous form

$$\frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{c+a} \ge \frac{9(1+m)}{a+b+c+m\sqrt{3(ab+bc+ca)}}.$$

As shown in the proof of P 5.12, it suffices to prove this homogeneous inequality for $a = 0$ and for $a \le b = c = 1$.

Case 1: $a = 0$. The inequality reduces to

$$\frac{2}{b} + \frac{2}{c} + \frac{2}{b+c} \ge \frac{9(1+m)}{b+c+m\sqrt{3bc}}.$$

Substituting

$$x = \frac{b+c}{\sqrt{bc}}, \quad x \ge 2,$$

the inequality becomes

$$2x + \frac{2}{x} \ge \frac{9(1+m)}{x + m\sqrt{3}},$$

$$2x^3 + 2\sqrt{3}\, mx^2 - (7+9m)x + 2\sqrt{3}\, m \ge 0,$$

$$(x-2)[2x^2 + 2(\sqrt{3}\, m + 2)x - \sqrt{3}\, m] \ge 0.$$

Case 2: $a \le 1 = b = c$. The inequality has the form

$$1 + \frac{4}{a+1} \ge \frac{9(1+m)}{a + 2 + m\sqrt{3(2a+1)}}.$$

Using the substitution

$$2a + 1 = 3x^2, \qquad x \ge \frac{\sqrt{3}}{3},$$

the inequality becomes

$$\frac{3x^2+9}{3x^2+1} \ge \frac{6(1+m)}{x^2 + 2mx + 1},$$

$$x^4 + 2mx^3 - 2(3m+1)x^2 + 6mx + 1 - 2m \ge 0,$$

$$(x-1)^2[x^2 + 2(m+1)x + 1 - 2m] \ge 0,$$

which is true since

$$\begin{aligned} x^2 + 2(m+1)x + 1 - 2m &\ge \frac{1}{3} + \frac{2(m+1)\sqrt{3}}{3} + 1 - 2m \\ &= \frac{2[2+\sqrt{3} - (3-\sqrt{3})m]}{3} = 0. \end{aligned}$$

The equality holds for $a = b = c = 1$. If $k = \dfrac{9+5\sqrt{3}}{6}$, then the equality holds also for

$$a = 0, \qquad b = c = \sqrt{3}$$

(or any cyclic permutation).

□

P 5.19. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \ge \frac{20}{a + b + c + 6\sqrt{ab + bc + ca}}.$$

(Vasile C., 2010)

Solution. The proof is similar to the one of P 5.12. Finally, we only need to prove the inequality for $a = 0$ and for $a \le b = c = 1$.

Case 1: $a = 0$. The inequality reduces to

$$\frac{1}{b} + \frac{1}{c} + \frac{1}{b+c} \ge \frac{20}{b+c+6\sqrt{bc}}.$$

Substituting

$$x = \frac{b+c}{\sqrt{bc}}, \quad x \ge 2,$$

the inequality becomes

$$x + \frac{1}{x} \ge \frac{20}{x+6},$$

$$x^3 + 6x^2 - 19x + 6 \ge 0,$$

$$(x-2)(x^2 + 8x - 3) \ge 0.$$

Case 2: $a \le 1 = b = c$. We need to show that

$$\frac{1}{2} + \frac{2}{a+1} \ge \frac{20}{a+2+6\sqrt{2a+1}}.$$

Using the substitution

$$2a + 1 = x^2, \quad x \ge 1,$$

the inequality becomes

$$\frac{x^2+9}{2(x^2+1)} \ge \frac{40}{x^2+12x+3},$$

$$x^4 + 12x^3 - 68x^2 + 108x - 53 \ge 0,$$

$$(x-1)(x^3 + 13x^2 - 55x + 53) \ge 0.$$

It is true since

$$\begin{aligned} x^3 + 13x^2 - 55x + 53 &= (x-1)^3 + 16x^2 - 58x + 54 \\ &= (x-1)^3 + 16\left(x - \frac{29}{16}\right)^2 + \frac{23}{16} > 0. \end{aligned}$$

The equality holds for

$$a = 0, \quad b = c$$

(or any cyclic permutation).

$\square$

P 5.20. *If a, b, c are positive real numbers so that*

$$7(a^2+b^2+c^2) = 11(ab+bc+ca),$$

then

$$\frac{51}{28} \le \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \le 2.$$

(Vasile C., 2008)

Solution. Due to homogeneity and symmetry, we may consider that

$$a+b+c=1, \quad 0 < a \le b \le c < 1.$$

Thus, we need to show that

$$a+b+c=1, \quad a^2+b^2+c^2 = \frac{11}{25}, \quad 0 < a \le b \le c < 1$$

involves

$$\frac{51}{28} \le \frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} \le 2.$$

We apply Corollary 1 to the function

$$f(u) = \frac{u}{1-u}, \quad 0 \le u < 1.$$

We have $f(1-) = \infty$ and

$$g(x) = f'(x) = \frac{1}{(1-x)^2}, \quad g''(x) = \frac{6}{(1-x)^4}.$$

Since $g''(x) > 0$, g is strictly convex on $[0,1)$. According to Corollary 1 and Note 3, *if*

$$a+b+c=1\ , \quad a^2+b^2+c^2 = \frac{11}{25}\ , \quad 0 \le a \le b \le c < 1,$$

then the sum

$$S_3 = f(a) + f(b) + f(c)$$

is maximal for $a = b \le c$, and is minimal for either $a = 0$ or $0 < a \le b = c$. Note that the case $a = 0$ is not possible because it involves $7(b^2+c^2) = 11bc$, which is false.

(1) To prove the right original inequality for $a = b \le c$, let us denote

$$t = \frac{c}{a}, \quad t \ge 1.$$

The hypothesis $7(a^2+b^2+c^2) = 11(ab+bc+ca)$ involves $t = 3$, hence

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{2a}{a+c} + \frac{c}{2a} = \frac{2}{1+t} + \frac{t}{2} = 2.$$

The right inequality is an equality for $a = b = \frac{c}{3}$ (or any cyclic permutation).

(2) To prove the left original inequality for $0 < a \le b = c$, let us denote

$$t = \frac{a}{b}, \quad 0 < t \le 1.$$

The hypothesis $7(a^2 + b^2 + c^2) = 11(ab + bc + ca)$ involves $t = \frac{1}{7}$, hence

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{a}{2b} + \frac{2b}{a+b} = \frac{t}{2} + \frac{2}{t+1} = \frac{51}{28}.$$

The left inequality is an equality for $7a = b = c$ (or any cyclic permutation).

□

P 5.21. *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that*

$$\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n+3} = \left(\frac{a_1 + a_2 + \cdots + a_n}{n+1}\right)^2,$$

then

$$\frac{(n+1)(2n-1)}{2} \le (a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \le \frac{3n^2(n+1)}{2(n+2)}.$$

(Vasile C., 2008)

Solution. For $n = 2$, both inequalities are identities. For $n \ge 3$, assume that

$$a_1 \le a_2 \le \cdots \le a_n.$$

The case $a_1 = 0$ is not possible because the hypothesis involves

$$\frac{a_2^2 + \cdots + a_n^2}{(a_2 + \cdots + a_n)^2} = \frac{n+3}{(n+1)^2} < \frac{1}{n-1},$$

which contradicts the Cauchy-Schwarz inequality

$$\frac{a_2^2 + \cdots + a_n^2}{(a_2 + \cdots + a_n)^2} \ge \frac{1}{n-1}.$$

Due to homogeneity and symmetry, we may consider that

$$a_1 + a_2 + \cdots + a_n = n + 1,$$

which implies

$$a_1^2 + a_2^2 + \cdots + a_n^2 = n + 3.$$

Thus, we need to show that

$$a_1 + a_2 + \cdots + a_n = n+1, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n+3, \qquad 0 < a_1 \le a_2 \le \cdots \le a_n$$

involves

$$\frac{2n-1}{2} \le \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \le \frac{3n^2}{2(n+2)}.$$

We apply Corollary 5 for $k = 2$ and $m = -1$:

- *If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $0 < a_1 \le a_2 \le \cdots \le a_n$ and*

$$a_1 + a_2 + \cdots + a_n = n+1, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n+3,$$

then

$$S_n = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

is minimal for

$$0 < a_1 = a_2 = \cdots = a_{n-1} \le a_n,$$

and is maximal for

$$a_1 \le a_2 = a_3 = \cdots = a_n.$$

(1) To prove the left original inequality, we only need to consider the case

$$a_1 = a_2 = \cdots = a_{n-1} \le a_n.$$

The hypothesis

$$\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n+3} = \left(\frac{a_1 + a_2 + \cdots + a_n}{n+1}\right)^2$$

implies

$$\frac{(n-1)a_1^2 + a_n^2}{n+3} = \left[\frac{(n-1)a_1 + a_n}{n+1}\right]^2,$$

$$(2a_1 - a_n)[2a_1 - (n+2)a_n] = 0,$$

$$a_1 = \frac{a_n}{2},$$

hence

$$\begin{aligned}(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) &= [(n-1)a_1 + a_n]\left(\frac{n-1}{a_1} + \frac{1}{a_n}\right) \\ &= (n-1)^2 + 1 + (n-1)\left(\frac{a_1}{a_n} + \frac{a_n}{a_1}\right) \\ &= \frac{(n+1)(2n-1)}{2}.\end{aligned}$$

The equality holds for

$$a_1 = a_2 = \cdots = a_{n-1} = \frac{a_n}{2}$$

(or any cyclic permutation).

(2) To prove the right original inequality, we only need to consider the case

$$a_1 \le a_2 = a_3 = \cdots = a_n.$$

The hypothesis involves

$$(a_1 - 2a_n)[(n+2)a_1 - 2a_n] = 0,$$

$$a_1 = \frac{2a_n}{n+2},$$

hence

$$\begin{aligned}(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) &= [(n-1)a_1 + a_n]\left(\frac{n-1}{a_1} + \frac{1}{a_n}\right) \\ &= (n-1)^2 + 1 + (n-1)\left(\frac{a_1}{a_n} + \frac{a_n}{a_1}\right) \\ &= \frac{3n^2(n+1)}{2(n+2)}.\end{aligned}$$

The equality holds for

$$a_1 = a_2 = \cdots = a_{n-1} = \frac{2a_n}{n+2}$$

(or any cyclic permutation).

□

P 5.22. *If a, b, c, d are nonnegative real numbers so that $a + b + c + d = 3$, then*

$$abc + bcd + cda + dab \le 1 + \frac{176}{81}\, abcd.$$

(Vasile C., 2005)

Solution. Assume that

$$a \le b \le c \le d.$$

For $a = 0$, we need to show that $b + c + d = 3$ implies

$$bcd \le 1,$$

which follows immediately from the AM-GM inequality:

$$bcd \le \left(\frac{b+c+d}{3}\right)^3 = 1.$$

For $a > 0$, rewrite the inequality in the form

$$abcd\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \le 1+\frac{176}{81}\,abcd$$

and apply Corollary 5 for $k = 0$ and $m = -1$:

- *If*

$$a+b+c+d=3, \qquad abcd=constant, \qquad 0<a\le b\le c\le d,$$

then

$$S_4=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$$

is maximal for

$$a \le b = c = d.$$

Thus, we only need to prove the homogeneous inequality

$$27(a+b+c+d)(abc+bcd+cda+dab) \le (a+b+c+d)^4+176abcd$$

for $a \le b = c = d = 1.$ The inequality becomes

$$27(a+3)(3a+1) \le (a+3)^4+176a,$$

$$a^4+12a^3-27a^2+14a \ge 0,$$

$$a(a-1)^2(a+14) \ge 0.$$

The equality holds for $a = b = c = d = 3/4$, and also for

$$a=0, \qquad b=c=d=1$$

(or any cyclic permutation).

□

P 5.23. *If a,b,c,d are nonnegative real numbers so that $a+b+c+d=3$, then*

$$a^2b^2c^2+b^2c^2d^2+c^2d^2a^2+d^2a^2b^2+\frac{3}{4}abcd \le 1.$$

(Gabriel Dospinescu and Vasile Cirtoaje, 2005)

Solution. Assume that

$$a \le b \le c \le d.$$

For $a = 0$, we need to show that

$$b^2c^2d^2 \le 1,$$

which follows immediately from the AM-GM inequality:

$$bcd \le \left(\frac{b+c+d}{3}\right)^3 = 1.$$

For $a > 0$, rewrite the inequality in the form

$$a^2b^2c^2d^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right)+\frac{3}{4}abcd \le 1,$$

and apply Corollary 5 for $k = 0$ and $m = -2$:

- *If*

$$a+b+c+d = 3, \qquad abcd = constant, \qquad 0 < a \le b \le c \le d,$$

then

$$S_4 = \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}$$

is maximal for $a \le b = c = d$.

Thus, we only need to prove the homogeneous inequality

$$\left(\frac{a+b+c+d}{3}\right)^6 \ge a^2b^2c^2+b^2c^2d^2+c^2d^2a^2+d^2a^2b^2+\frac{1}{12}abcd(a+b+c+d)^2$$

for $a \le b = c = d = 1$; that is, to show that $0 < a \le 1$ implies

$$\left(1+\frac{a}{3}\right)^6 \ge 1+3a^2+\frac{1}{12}a(a+3)^2.$$

Since

$$\left(1+\frac{a}{3}\right)^3 = 1+a+\frac{a^2}{3}+\frac{a^3}{27} > 1+a+\frac{a^2}{3},$$

it suffices to show that

$$\left(1+a+\frac{a^2}{3}\right)^2 \ge 1+3a^2+\frac{1}{12}a(a+3)^2,$$

which is equivalent to the obvious inequality

$$4a^4+3a(1-a)(15-7a) \ge 0.$$

The equality holds for

$$a = 0, \qquad b = c = d = 1$$

(or any cyclic permutation).

□

P 5.24. *If a,b,c,d are nonnegative real numbers so that $a+b+c+d=3$, then*

$$a^2b^2c^2+b^2c^2d^2+c^2d^2a^2+d^2a^2b^2+\frac{4}{3}(abcd)^{3/2}\le 1.$$

(Vasile C., 2005)

Solution. The proof is similar to the one of the preceding P 5.23. We need to prove that

$$\left(1+\frac{a}{3}\right)^6\ge 1+3a^2+\frac{4}{3}a^{3/2}$$

for $0\le a\le 1$. Since

$$2a^{3/2}\le a^2+a,$$

it suffices to show that

$$\left(1+\frac{a}{3}\right)^6\ge 1+\frac{2}{3}a+\frac{11}{3}a^2.$$

Since

$$\left(1+\frac{a}{3}\right)^3=1+a+\frac{a^2}{3}+\frac{a^3}{27}\ge 1+a+\frac{a^2}{3}$$

and

$$\begin{aligned}\left(1+a+\frac{a^2}{3}\right)^2&=1+2a+\frac{5}{3}a^2+\frac{2}{3}a^3+\frac{1}{9}a^4\\&\ge 1+2a+\frac{5}{3}a^2+\frac{2}{3}a^3,\end{aligned}$$

it suffices to show that

$$1+2a+\frac{5}{3}a^2+\frac{2}{3}a^3\ge 1+\frac{2}{3}a+\frac{11}{3}a^2,$$

which is equivalent to the obvious inequality

$$a(1-a)(2-a)\ge 0.$$

The equality holds for

$$a=0,\qquad b=c=d=1$$

(or any cyclic permutation).

□

P 5.25. *If a,b,c,d are nonnegative real numbers so that $a+b+c+d=4$, then*

$$a^2b^2c^2+b^2c^2d^2+c^2d^2a^2+d^2a^2b^2+2(abcd)^{3/2}\le 6.$$

(Vasile C., 2005)

Solution. The proof is similar to the one of P 5.23. We need to prove that

$$6\left(\frac{a+3}{4}\right)^6 \ge 1+3a^2+2a^{3/2}$$

for $0 \le a \le 1$. Since

$$2a^{3/2} \le a^2 + a,$$

it suffices to show that

$$6\left(\frac{a+3}{4}\right)^6 \ge 1+a+4a^2.$$

Using the substitution

$$x = \frac{1-a}{4}, \qquad 0 \le x \le \frac{1}{4},$$

the inequality becomes

$$3(1-x)^6 \ge 3-18x+32x^2,$$

$$x^2(13-60x+45x^2-18x^3+3x^4) \ge 0.$$

It is true since

$$\begin{aligned} 2(13-60x+45x^2-18x^3+3x^4) &> 25-120x+90x^2-40x^3 \\ &= 5(1-4x)(5-4x+2x^2) \ge 0. \end{aligned}$$

The equality holds for $a=b=c=d=1$.

□

P 5.26. *If a,b,c are nonnegative real numbers so that $a+b+c=3$, then*

$$11(ab+bc+ca)+4(a^2b^2+b^2c^2+c^2a^2) \le 45.$$

(Vasile C., 2005)

Solution. Assume that $a \le b \le c$. For $a=0$, we need to show that $b+c=3$ involves

$$11bc+4b^2c^2 \le 45.$$

We have

$$bc \le \left(\frac{b+c}{2}\right)^2 = \frac{9}{4},$$

hence

$$11bc+4b^2c^2 \le \frac{99}{4}+\frac{81}{4} = 45.$$

For $a>0$, rewrite the desired inequality in the form

$$11abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+4a^2b^2c^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \le 45.$$

According to Corollary 5 (case $k=2$ and $m<0$), *if*

$$a+b+c=3, \qquad abc=constant, \qquad 0<a\le b\le c,$$

then the sums $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ *and* $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$ *are maximal for* $0<a\le b=c$.
Therefore, we only need to prove that $a+2b=3$ involves

$$11(2ab+b^2)+4(2a^2b^2+b^4)\le 45,$$

which is equivalent to

$$15-22b-13b^2+32b^3-12b^4\ge 0,$$

$$(3-2b)(1-b)^2(5+6b)\ge 0,$$
$$a(1-b)^2(5+6b)\ge 0.$$

The equality holds for $a=b=c=1$, and also for

$$a=0, \qquad b=c=\frac{3}{2}$$

(or any cyclic permutation).

Remark. In the same manner, we can prove the following statement:

- *If a,b,c,d are nonnegative real numbers so that $a+b+c+d=4$, then*

$$abc+bcd+cda+dab+a^2b^2c^2+b^2c^2d^2+c^2d^2a^2+d^2a^2b^2\le 8,$$

with equality for $a=b=c=d=1$.

□

P 5.27. *If a,b,c are nonnegative real numbers so that $a+b+c=3$, then*

$$a^2b^2+b^2c^2+c^2a^2+a^3b^3+b^3c^3+c^3a^3\ge 6abc.$$

(Vasile C., 2005)

Solution. Assume that $a\le b\le c$. For $a=0$, the inequality is trivial. For $a>0$, rewrite the desired inequality in the form

$$abc\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+a^2b^2c^2\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)\ge 6.$$

According to Corollary 5 (case $k=0$ and $m<0$), *if*

$$a+b+c=3, \qquad abc=constant, \qquad 0<a\le b\le c,$$

then the sums $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$ *and* $\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}$ *are maximal for* $0<a\le b=c$.

Thus, we only need to prove that

$$2a^2b^2+b^4+2a^3b^3+b^6\ge 6ab^2$$

for

$$a+2b=3, \qquad 1\le b<3/2.$$

The inequality is equivalent to

$$b^3(14-33b+24b^2-5b^3)\ge 0,$$

$$b^3(1-b)^2(14-5b)\ge 0.$$

The equality holds for $a=b=c=1$, and also for

$$a=b=0, \qquad c=3$$

(or any cyclic permutation).

□

P 5.28. *If a,b,c are nonnegative real numbers so that $a+b+c=3$, then*

$$2(a^2+b^2+c^2)+5\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\ge 21.$$

(Vasile C., 2008)

Solution. Apply Corollary 5 for $k=2$ and $m=1/2$:

- *If*

$$a+b+c=3, \qquad a^2+b^2+c^2=constant, \qquad 0\le a\le b\le c,$$

then

$$S_3=\sqrt{a}+\sqrt{b}+\sqrt{c}$$

is minimal for either $a=0$ or $0<a\le b=c$.

Case 1: $a=0$. We need to show that $b+c=3$ involves

$$2(b^2+c^2)+5\left(\sqrt{b}+\sqrt{c}\right)\ge 21,$$

which is equivalent to

$$5\sqrt{3+2\sqrt{bc}}\ge 3+4bc.$$

Substituting

$$x=\sqrt{bc}, \quad 0\le x\le \frac{b+c}{2}=\frac{3}{2},$$

the inequality becomes

$$5\sqrt{3+2x} \ge 3+4x^2,$$

$$25(3+2x) \ge (3+4x^2)^2.$$

This inequality is equivalent to $f(x) \ge 0$, where

$$f(x) = \frac{66}{x} + 50 - 24x - 16x^3, \qquad 0 < x \le 3/2.$$

Since f is decreasing, we have

$$f(x) \ge f(3/2) = 4 > 0.$$

Case 2: $0 < a \le b = c$. We need to show that

$$2(a^2+2b^2) + 5\left(\sqrt{a} + 2\sqrt{b}\,\right) \ge 21$$

for

$$a+2b = 3, \quad 1 \le b < \frac{3}{2}.$$

Write the inequality as

$$5\sqrt{3-2b} + 10\sqrt{b} \ge 3 + 24b - 12b^2.$$

Substituting

$$x = \sqrt{b}, \quad 1 \le x < \sqrt{\frac{3}{2}},$$

the inequality becomes

$$5\sqrt{3-2x^2} \ge 3 - 10x + 24x^2 - 12x^4,$$

$$12(x^2-1)^2 \ge 5\left(3 - 2x - \sqrt{3-2x^2}\right),$$

$$12(x^2-1)^2 \ge \frac{30(x-1)^2}{3-2x+\sqrt{3-2x^2}},$$

which is true if

$$2(x+1)^2 \ge \frac{5}{3-2x+\sqrt{3-2x^2}}.$$

It suffices to show that

$$2(x+1)^2 \ge \frac{5}{3-2x},$$

which is equivalent to

$$1 + 8x - 2x^2 - 4x^3 \ge 0,$$

$$x(5-4x)\left(\frac{7}{4}+x\right) + \frac{4-3x}{4} \ge 0.$$

Since

$$x < \sqrt{\frac{3}{2}} < \frac{5}{4} < \frac{4}{3},$$

the conclusion follows.

The equality holds for $a = b = c = 1$.

$\square$

P 5.29. *If a, b, c are nonnegative real numbers so that $ab + bc + ca = 3$, then*

$$\sqrt{\frac{1+2a}{3}} + \sqrt{\frac{1+2b}{3}} + \sqrt{\frac{1+2c}{3}} \geq 3.$$

(Vasile C., 2008)

Solution. Write the hypothesis $ab + bc + ca = 3$ as

$$(a+b+c)^2 = 6 + a^2 + b^2 + c^2,$$

and apply Corollary 1 to

$$f(u) = \sqrt{\frac{1+2u}{3}}, \qquad u \geq 0.$$

We have

$$g(x) = f'(x) = \frac{1}{\sqrt{3(1+2x)}},$$

$$g''(x) = \frac{\sqrt{3}}{(1+2x)^{5/2}}.$$

Since $g''(x) > 0$ for $x \geq 0$, g is strictly convex on $[0, \infty)$. According to Corollary 1, *if*

$$a+b+c = constant, \qquad a^2+b^2+c^2 = constant, \qquad 0 \leq a \leq b \leq c,$$

then the sum

$$S_3 = f(a) + f(b) + f(c)$$

is minimal for either $a = 0$ or $0 < a \leq b = c$.

Case 1: $a = 0$. We need to show that $bc = 3$ involves

$$\sqrt{1+2b} + \sqrt{1+2c} \geq 3\sqrt{3} - 1.$$

By squaring, the inequality becomes

$$b + c + \sqrt{13 + 2(b+c)} \geq 13 - 3\sqrt{3}.$$

We have $b+c \ge 2\sqrt{bc} = 2\sqrt{3}$, hence

$$b+c+\sqrt{13+2(b+c)} \ge 2\sqrt{3}+\sqrt{13+4\sqrt{3}} = 4\sqrt{3}+1 > 13-3\sqrt{3}.$$

Case 2: $0 < a \le b = c$. From $ab+bc+ca=3$, it follows that

$$a = \frac{3-b^2}{2b}, \qquad 0 < b < \sqrt{3}.$$

Thus, the inequality can be written as

$$\sqrt{1+\frac{3-b^2}{b}} + 2\sqrt{1+2b} \ge 3\sqrt{3}.$$

Substituting

$$t = \sqrt{\frac{1+2b}{3}}, \qquad \frac{1}{\sqrt{3}} < t < \sqrt{\frac{1+2\sqrt{3}}{3}} < \frac{5}{4},$$

the inequality turns into

$$\sqrt{\frac{3+4t^2-3t^4}{2(3t^2-1)}} \ge 3-2t.$$

By squaring, we need to show that

$$7-8t-14t^2+24t^3-9t^4 \ge 0,$$

which is equivalent to

$$(1-t)^2(7+6t-9t^2) \ge 0.$$

This is true since

$$7+6t-9t^2 = 8-(3t-1)^2 > 8-\left(\frac{15}{4}-1\right)^2 = \frac{7}{16} > 0.$$

The equality holds for $a=b=c=1$.

□

P 5.30. *Let a,b,c be nonnegative real numbers, no two of which are zero. If*

$$0 \le k \le 15,$$

then

$$\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}+\frac{k}{(a+b+c)^2} \ge \frac{9+k}{4(ab+bc+ca)}.$$

(Vasile C., 2007)

Solution. Due to homogeneity and symmetry, we may consider that

$$a+b+c=1, \quad 0 \le a \le b \le c.$$

On this assumption, the inequality becomes

$$\frac{1}{(1-a)^2}+\frac{1}{(1-b)^2}+\frac{1}{(1-c)^2}+k \ge \frac{9+k}{2(1-a^2-b^2-c^2)}.$$

To prove it, we apply Corollary 1 to the function

$$f(u)=\frac{1}{(1-u)^2}, \quad 0 \le u < 1.$$

We have $f(1-)=\infty$ and

$$g(x)=f'(x)=\frac{2}{(1-x)^3}, \quad g''(x)=\frac{24}{(1-x)^5}.$$

Since $g''(x)>0$, g is strictly convex on $[0,1)$. According to Corollary 1 and Note 3, *if*

$$a+b+c=1, \quad a^2+b^2+c^2=constant, \quad 0 \le a \le b \le c,$$

then the sum

$$S_3=f(a)+f(b)+f(c)$$

is minimal for either $a=0$ or $0<a \le b=c$.

Case 1: $a=0$. For

$$x=\frac{b}{c}+\frac{c}{b}, \quad x \ge 2,$$

the original inequality becomes

$$\frac{1}{b^2}+\frac{1}{c^2}+\frac{1+k}{(b+c)^2} \ge \frac{9+k}{4bc},$$

$$x+\frac{1+k}{x+2} \ge \frac{9+k}{4},$$

$$(x-2)(4x+7-k) \ge 0.$$

This is true since

$$4x+7-k \ge 15-k \ge 0.$$

Case 2: $0<a \le b=c$. The original inequality becomes

$$\frac{2}{(a+b)^2}+\frac{1}{4b^2}+\frac{k}{(a+2b)^2} \ge \frac{9+k}{4b(2a+b)},$$

$$\frac{a(a-b)^2}{2b^2(a+b)^2(2a+b)}+\frac{ka(4b-a)}{4b(a+2b)^2(2a+b)} \ge 0.$$

The equality holds for

$$a = 0, \qquad b = c$$

(or any cyclic permutation). If $k = 0$ (Iran 1996 inequality), then the equality holds also for $a = b = c$.

□

P 5.31. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

$$\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} + \frac{24}{(a+b+c)^2} \ge \frac{8}{ab+bc+ca}.$$

(Vasile C., 2007)

Solution. As shown in the proof of the preceding P 5.30, it suffices to prove the inequality for $a = 0$, and for $0 < a \le b = c$.

Case 1: $a = 0$. For

$$x = \frac{b}{c} + \frac{c}{b}, \qquad x \ge 2,$$

the original inequality becomes

$$\frac{1}{b^2} + \frac{1}{c^2} + \frac{25}{(b+c)^2} \ge \frac{8}{bc},$$

$$x + \frac{25}{x+2} \ge 8,$$

$$(x-3)^2 \ge 0.$$

Case 2: $0 < a \le b = c$. Due to homogeneity, we only need to prove the homogeneous inequality for $0 < a \le b = c = 1$; that is,

$$\frac{2}{(a+1)^2} + \frac{1}{4} + \frac{24}{(a+2)^2} \ge \frac{8}{2a+1}.$$

It suffices to show that

$$\frac{2}{(a+1)^2} \ge \frac{8}{2a+1} - \frac{24}{(a+2)^2},$$

which is equivalent to

$$\frac{1}{(1+a)^2} \ge \frac{4(1-a)^2}{(2a+1)(a+2)^2},$$

$$a(2a^2 + 9a + 12) \ge 4a^2(a^2 - 2).$$

This is true since

$$a(2a^2 + 9a + 12) \ge 0 \ge 4a^2(a^2 - 2).$$

The equality holds for

$$a = 0, \qquad \frac{b}{c} + \frac{c}{b} = 3$$

(or any cyclic permutation).

Remark. Actually, the following generalization holds:

- *Let a, b, c be nonnegative real numbers, no two of which are zero. If $k \ge 15$, then*

$$\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} + \frac{k}{(a+b+c)^2} \ge \frac{2(\sqrt{k+1}-1)}{ab+bc+ca},$$

with equality for

$$a = 0, \qquad \frac{b}{c} + \frac{c}{b} = \sqrt{k+1} - 2$$

(or any cyclic permutation).

□

P 5.32. *If a, b, c are nonnegative real numbers, no two of which are zero, so that*

$$k(a^2+b^2+c^2) + (2k+3)(ab+bc+ca) = 9(k+1), \qquad 0 \le k \le 6,$$

then

$$\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} + \frac{9k}{(a+b+c)^2} \ge \frac{3}{4} + k.$$

(Vasile C., 2007)

Solution. Write the inequality in the homogeneous form

$$\frac{4}{(a+b)^2} + \frac{4}{(b+c)^2} + \frac{4}{(c+a)^2} + \frac{36k}{(a+b+c)^2} \ge \frac{9(k+1)(4k+3)}{k(a^2+b^2+c^2)+(2k+3)(ab+bc+ca)}.$$

As shown in the proof of P 5.30, it suffices to prove this inequality for $a = 0$, and for $0 < a \le b = c$.

Case 1: $a = 0$. Let

$$x = \frac{b}{c} + \frac{c}{b}, \qquad x \ge 2.$$

The homogeneous inequality becomes

$$4\left(\frac{1}{b^2} + \frac{1}{c^2}\right) + \frac{36k+4}{(b+c)^2} \ge \frac{9(k+1)(4k+3)}{k(b^2+c^2)+(2k+3)bc},$$

$$4x + \frac{36k+4}{x+2} \ge \frac{9(k+1)(4k+3)}{kx+2k+3},$$

$$4kx^3 + 4(4k+3)x^2 - (43k+3)x - 2(5k+21) \ge 0,$$

$$(x-2)[4kx^2+4(6k+3)x+5k+21] \geq 0.$$

Case 2: $0 < a \leq b = c$. We only need to prove the homogeneous inequality for $b = c = 1$. The inequality becomes

$$\frac{8}{(a+1)^2}+1+\frac{36k}{(a+2)^2} \geq \frac{9(k+1)(4k+3)}{ka^2+(4k+6)a+4k+3},$$

$$ka^6+(10k+6)a^5-(14k-12)a^4-(10k+18)a^3+(17k-24)a^2+(24-4k)a \geq 0,$$

$$a(a-1)^2[ka^3+6(2k+1)a^2+3(3k+8)a+4(6-k)] \geq 0.$$

Clearly, the last inequality is true for $0 \leq k \leq 6$.

The equality holds for $a = b = c$, and also for

$$a = 0, \qquad b = c$$

(or any cyclic permutation).

□

P 5.33. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

(a) $$\frac{2}{a^2+b^2}+\frac{2}{b^2+c^2}+\frac{2}{c^2+a^2} \geq \frac{8}{a^2+b^2+c^2}+\frac{1}{ab+bc+ca};$$

(b) $$\frac{2}{a^2+b^2}+\frac{2}{b^2+c^2}+\frac{2}{c^2+a^2} \geq \frac{7}{a^2+b^2+c^2}+\frac{6}{(a+b+c)^2};$$

(c) $$\frac{2}{a^2+b^2}+\frac{2}{b^2+c^2}+\frac{2}{c^2+a^2} \geq \frac{45}{4(a^2+b^2+c^2)+ab+bc+ca}.$$

(Vasile C., 2007)

Solution. (a) Due to homogeneity and symmetry, we may consider that

$$a^2+b^2+c^2 = 1, \qquad 0 \leq a \leq b \leq c.$$

On this assumption, the inequality can be written as

$$\frac{2}{1-a^2}+\frac{2}{1-b^2}+\frac{2}{1-c^2} \geq 8+\frac{2}{(a+b+c)^2-1}.$$

To prove it, we apply Corollary 1 to the function

$$f(u) = \frac{1}{1-u^2}, \qquad 0 \leq u < 1.$$

We have $f(1-) = \infty$ and

$$g(x) = f'(x) = \frac{2x}{(1-x^2)^2}, \qquad g''(x) = \frac{24x(1+x^2)}{(1-x^2)^4}.$$

Since $g''(x) > 0$ for $x \in (0,1)$, g is strictly convex on $[0,1)$. According to Corollary 1 and Note 3, *if*

$$a+b+c = constant, \qquad a^2+b^2+c^2 = 1, \qquad 0 \le a \le b \le c,$$

then the sum

$$S_3 = f(a)+f(b)+f(c)$$

is minimal for either $a = 0$ *or* $0 < a \le b = c$.

Case 1: $a = 0$. For

$$x = \frac{b}{c}+\frac{c}{b}, \qquad x \ge 2,$$

the original inequality becomes

$$\frac{2}{b^2}+\frac{2}{c^2} \ge \frac{6}{b^2+c^2}+\frac{1}{bc},$$

$$2x \ge \frac{6}{x}+1,$$

$$(x-2)(2x+3) \ge 0.$$

Case 2: $0 < a \le b = c$. Due to homogeneity, it suffices to prove the original inequality for $b = c = 1$. Thus, we need to show that

$$1+\frac{4}{a^2+1} \ge \frac{8}{a^2+2}+\frac{1}{2a+1},$$

which is equivalent to

$$\frac{2a}{2a+1} \ge \frac{4a^2}{(a^2+1)(a^2+2)},$$

$$a(a^4-a^2-2a+2) \ge 0,$$

$$a(a-1)^2(a^2+2a+2) \ge 0.$$

The equality holds for $a = b = c$, and also for $a = 0, b = c$ (or any cyclic permutation).

(b) The proof is similar to the one of the inequality in (a). For $a = 0$ and

$$x = \frac{b}{c}+\frac{c}{b}, \qquad x \ge 2,$$

the original inequality becomes

$$\frac{2}{b^2}+\frac{2}{c^2} \ge \frac{5}{b^2+c^2}+\frac{6}{(b+c)^2},$$

$$2x \ge \frac{5}{x}+\frac{6}{x+2},$$

$$(x-2)(2x^2+8x+5) \ge 0.$$

For $b=c=1$, the original inequality is

$$1+\frac{4}{a^2+1} \ge \frac{7}{a^2+2}+\frac{6}{(a+2)^2},$$

$$a(a^5+4a^4-2a^3-15a+12) \ge 0,$$

$$a(a-1)^2(a^3+6a^2+9a+12) \ge 0.$$

The equality holds for $a=b=c$, and also for $a=0, b=c$ (or any cyclic permutation).

(c) The proof is also similar to the one of the inequality in (a). For $a=0$ and

$$x=\frac{b}{c}+\frac{c}{b}, \qquad x \ge 2,$$

the original inequality becomes

$$2\left(\frac{1}{b^2}+\frac{1}{c^2}\right)+\frac{2}{b^2+c^2} \ge \frac{45}{4(b^2+c^2)+bc},$$

$$2x+\frac{2}{x} \ge \frac{45}{4x+1},$$

$$(x-2)(8x^2+18x-1) \ge 0.$$

For $b=c=1$, the original inequality can be written as

$$1+\frac{4}{a^2+1} \ge \frac{45}{4a^2+2a+9},$$

$$a(2a^3+a^2-8a+5) \ge 0,$$

$$a(a-1)^2(2a+5) \ge 0.$$

The equality holds for $a=b=c$, and also for $a=0, b=c$ (or any cyclic permutation).

□

P 5.34. *If a,b,c are nonnegative real numbers, no two of which are zero, then*

$$\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}+\frac{3}{a^2+b^2+c^2} \ge \frac{4}{ab+bc+ca}.$$

(Vasile C., 2007)

Solution. As shown in the proof of the preceding P 5.33, it suffices to prove the inequality for $a=0$, and for $0<a\le b=c$.

Case 1: $a=0$. For

$$x=\frac{b}{c}+\frac{c}{b}, \qquad x\ge 2,$$

the original inequality becomes

$$\frac{1}{b^2}+\frac{1}{c^2}+\frac{4}{b^2+c^2}\ge\frac{4}{bc},$$

$$x+\frac{4}{x}\ge 4,$$

$$(x-2)^2\ge 0.$$

Case 2: $0<a\le b=c$. Due to homogeneity, it suffices to prove the original inequality for $0<a\le b=c=1$. Thus, we need to show that

$$\frac{1}{2}+\frac{2}{a^2+1}+\frac{3}{a^2+2}\ge\frac{4}{2a+1}.$$

It suffices to show that

$$\frac{2}{a+1}+\frac{3}{a+2}\ge\frac{4}{2a+1}-\frac{1}{2},$$

which is equivalent to

$$\frac{5a+7}{a^2+3a+2}\ge\frac{7-2a}{4a+2},$$

$$a(2a^2+19a+21)\ge 0,$$

The equality holds for

$$a=0, \qquad b=c$$

(or any cyclic permutation).

Remark. Actually, the following generalization holds:

- *Let a,b,c be nonnegative real numbers, no two of which are zero.*
  (a) *If $-4\le k\le 3$, then*

$$\frac{2}{a^2+b^2}+\frac{2}{b^2+c^2}+\frac{2}{c^2+a^2}+\frac{2k}{a^2+b^2+c^2}\ge\frac{k+5}{ab+bc+ca},$$

with equality for

$$a=0, \qquad b=c$$

(or any cyclic permutation).

(b) *If $k\ge 3$, then*

$$\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}+\frac{k}{a^2+b^2+c^2}\ge\frac{2\sqrt{k+1}}{ab+bc+ca},$$

with equality for

$$a = 0, \qquad \frac{b}{c} + \frac{c}{b} = \sqrt{k+1}$$

(or any cyclic permutation).

□

P 5.35. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

(a) $$\frac{3}{a^2+ab+b^2} + \frac{3}{b^2+bc+c^2} + \frac{3}{c^2+ca+a^2} \ge \frac{5}{ab+bc+ca} + \frac{4}{a^2+b^2+c^2};$$

(b) $$\frac{3}{a^2+ab+b^2} + \frac{3}{b^2+bc+c^2} + \frac{3}{c^2+ca+a^2} \ge \frac{1}{ab+bc+ca} + \frac{24}{(a+b+c)^2};$$

(c) $$\frac{1}{a^2+ab+b^2} + \frac{1}{b^2+bc+c^2} + \frac{1}{c^2+ca+a^2} \ge \frac{21}{2(a^2+b^2+c^2)+5(ab+bc+ca)}.$$

(Vasile C., 2007)

Solution. (a) Due to homogeneity and symmetry, we may consider that

$$a+b+c = 1, \qquad 0 \le a \le b \le c.$$

Let

$$p = \frac{1+a^2+b^2+c^2}{2}.$$

Since

$$\frac{1}{2(b^2+bc+c^2)} = \frac{1}{(a+b+c)^2+a^2+b^2+c^2-2a(a+b+c)} = \frac{1}{2(p-a)},$$

the inequality can be written as

$$\frac{3}{p-a} + \frac{3}{p-b} + \frac{3}{p-c} \ge \frac{5}{1-p} + \frac{4}{2p-1}.$$

To prove it, we apply Corollary 1 to the function

$$f(u) = \frac{3}{p-u}, \qquad 0 \le u < p.$$

We have $f(p-) = \infty$ and

$$g(x) = f'(x) = \frac{3}{(p-x)^2}, \qquad g''(x) = \frac{18}{(p-x)^4}.$$

Since $g''(x) > 0$, g is strictly convex on $[0, p)$. According to Corollary 1 and Note 3, *if*

$$a+b+c = 1, \qquad a^2+b^2+c^2 = 2p-1 = constant, \qquad 0 \le a \le b \le c,$$

then the sum

$$S_3 = f(a) + f(b) + f(c)$$

is minimal for either $a = 0$ *or* $0 < a \le b = c$.

Case 1: $a = 0$. For

$$x = \frac{b}{c} + \frac{c}{b}, \qquad x \ge 2,$$

the original inequality becomes

$$3\left(\frac{1}{b^2} + \frac{1}{c^2}\right) + \frac{3}{b^2 + bc + c^2} \ge \frac{5}{bc} + \frac{4}{b^2 + c^2},$$

which is equivalent to

$$3x + \frac{3}{x+1} \ge 5 + \frac{4}{x},$$

$$(x-2)(3x^2 + 4x + 2) \ge 0.$$

Case 2: $0 < a \le b = c$. Due to homogeneity, it suffices to prove the original inequality for $b = c = 1$. Thus, we need to show that

$$\frac{6}{a^2 + a + 1} + 1 \ge \frac{5}{2a + 1} + \frac{4}{a^2 + 2},$$

which is equivalent to

$$a(a^4 - a^3 + 3a^2 - 7a + 4) \ge 0,$$

$$a(a-1)^2(a^2 + a + 4) \ge 0.$$

The equality holds for $a = b = c$, and also for $a = 0, b = c$ (or any cyclic permutation).

(b) The proof is similar to the one of the inequality in (a). For $a = 0$, the original inequality becomes

$$3\left(\frac{1}{b^2} + \frac{1}{c^2}\right) + \frac{3}{b^2 + bc + c^2} \ge \frac{1}{bc} + \frac{24}{(b+c)^2},$$

which is equivalent to

$$3x + \frac{3}{x+1} \ge 1 + \frac{24}{x+2}, \qquad x = \frac{b}{c} + \frac{c}{b},$$

$$(x-2)(3x^2 + 14x + 10) \ge 0.$$

For $b = c = 1$, the original inequality becomes

$$\frac{6}{a^2 + a + 1} + 1 \ge \frac{1}{2a + 1} + \frac{24}{a^2 + 2},$$

which is equivalent to

$$a(a^4+5a^3-9a^2-a+4)\geq 0,$$

$$a(a-1)^2(a^2+7a+4)\geq 0.$$

The equality holds for $a=b=c$, and also for $a=0, b=c$ (or any cyclic permutation).

(c) The proof is similar to the one of the inequality in (a). For $a=0$, the original inequality becomes

$$\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{b^2+bc+c^2}\geq\frac{21}{2(b^2+c^2)+5bc},$$

which is equivalent to

$$x+\frac{1}{x+1}\geq\frac{21}{2x+5},\qquad x=\frac{b}{c}+\frac{c}{b},$$

$$(x-2)(2x^2+11x+8)\geq 0.$$

For $b=c=1$, the original inequality becomes

$$\frac{2}{a^2+a+1}+\frac{1}{3}\geq\frac{21}{2a^2+10a+9},$$

which is equivalent to

$$a(a^3+6a^2-15a+8)\geq 0,$$

$$a(a-1)^2(a+8)\geq 0.$$

The equality holds for $a=b=c$, and also for $a=0, b=c$ (or any cyclic permutation).

□

P 5.36. *Let f be a real-valued function, continuous on $[0,\infty)$ and differentiable on $(0,\infty)$, so that $f'''(u)\geq 0$ for $u\in(0,\infty)$. If $a,b,c\geq 0$, then*

$$f(a^2+2bc)+f(b^2+2ca)+f(c^2+2ab)\leq f(a^2+b^2+c^2)+2f(ab+bc+ca).$$

Solution. Denoting

$$x=a^2+2bc,\quad y=b^2+2ca,\quad z=c^2+2ab,$$

the inequality becomes

$$f(x)+f(y)+f(z)\leq f(a^2+b^2+c^2)+2f(ab+bc+ca).$$

Assume that

$$a+b+c = constant, \qquad a^2+b^2+c^2 = constant,$$

which involve

$$2(ab+bc+ca) = (a+b+c)^2 - (a^2+b^2+c^2) = constant.$$

We have

$$x+y+z = (a+b+c)^2 = constant,$$

$$x^2+y^2+z^2 = (a^2+b^2+c^2)^2 + 2(ab+bc+ca)^2 = constant.$$

According to the EV-Theorem (Corollary 1), since $f'''(u) \ge 0$ for $u \in (0,\infty)$, the sum $f(x)+f(y)+f(z)$ is maximal for $x = y \le z$, that is

$$a^2+2bc = b^2+2ca \le c^2+2ab.$$

From $a^2+2bc = b^2+2ca$, we get $a = b$ or $a+b = 2c$. If $a+b = 2c$, the inequality $b^2+2ca \le c^2+2ab$ is equivalent to $(b-c)^2 \le 0$, which involves $b = c$. Thus it suffices to prove the required inequality for two equal variables, when the inequality is an identity.
The equality holds for $a = b$ or $b = c$ or $c = a$.

Remark 1. The inequality is also true for a real-valued function f, continuous on $(0,\infty)$ and differentiable on $(0,\infty)$, so that $f'''(u) \ge 0$ for $u \in (0,\infty)$ and $\lim_{u\to 0} f(u) = \pm\infty$.

Remark 2. The following inequalities hold:

$$\frac{1}{a^2+2bc} + \frac{1}{b^2+2ca} + \frac{1}{c^2+2ab} \ge \frac{1}{a^2+b^2+c^2} + \frac{2}{ab+bc+ca},$$

$$\sqrt{a^2+2bc} + \sqrt{b^2+2ca} + \sqrt{c^2+2ab} \le \sqrt{a^2+b^2+c^2} + 2\sqrt{ab+bc+ca},$$

$$\frac{1}{\sqrt{a^2+2bc}} + \frac{1}{\sqrt{b^2+2ca}} + \frac{1}{\sqrt{c^2+2ab}} \ge \frac{1}{\sqrt{a^2+b^2+c^2}} + \frac{2}{\sqrt{ab+bc+ca}},$$

$$(a^2+2bc)(b^2+2ca)(c^2+2ab) \le (a^2+b^2+c^2)(ab+bc+ca)^2.$$

□

P 5.37. *If a, b, c are the lengths of the side of a triangle, then*

$$\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} \le \frac{85}{36(ab+bc+ca)}.$$

(Vasile C., 2007)

Solution. Use the substitution

$$a = y + z, \qquad b = z + x, \qquad c = x + y,$$

where x, y, z are nonnegative real numbers. Due to homogeneity and symmetry, we may consider that

$$x + y + z = 2, \qquad 0 \le x \le y \le z.$$

We need to show that

$$\frac{1}{(x+2)^2} + \frac{1}{(y+2)^2} + \frac{1}{(z+2)^2} \le \frac{85}{18(12 - x^2 - y^2 - z^2)},$$

which can be written as

$$f(x) + f(y) + f(z) + \frac{85}{18(12 - x^2 - y^2 - z^2)} \ge 0,$$

where

$$f(u) = \frac{-1}{(u+2)^2}, \qquad u \ge 0.$$

We have

$$g(x) = f'(x) = \frac{2}{(x+2)^3}, \qquad g''(x) = \frac{24}{(x+2)^5}.$$

Since $g''(x) > 0$ for $x \ge 0$, g is strictly convex on $[0, \infty)$. According to Corollary 1, *if*

$$x + y + z = 2, \qquad x^2 + y^2 + z^2 = constant, \qquad 0 \le x \le y \le z,$$

then the sum

$$S_3 = f(x) + f(y) + f(z)$$

is minimal for either $x = 0$ *or* $0 < x \le y = z$.

Case 1: $x = 0$. This implies $a = b + c$. Since

$$\frac{1}{(a+b)^2} + \frac{1}{(c+a)^2} = \frac{5(b^2 + c^2) + 8bc}{(2b^2 + 2c^2 + 5bc)^2}$$

and

$$ab + bc + ca = a(b + c) + bc = (b + c)^2 + bc = b^2 + c^2 + 3bc,$$

we need to show that

$$\frac{5(b^2 + c^2) + 8bc}{(2b^2 + 2c^2 + 5bc)^2} + \frac{1}{(b+c)^2} \le \frac{85}{36(b^2 + c^2 + 3bc)}.$$

For $bc = 0$, the inequality is true. For $bc \ne 0$, substituting

$$t = \frac{b}{c} + \frac{c}{b}, \qquad t \ge 2,$$

the inequality becomes

$$\frac{5t+8}{(2t+5)^2}+\frac{1}{t+2}\le\frac{85}{36(t+3)},$$

$$\frac{5t+8}{(2t+5)^2}\le\frac{49t+62}{36(t+2)(t+3)}.$$

It suffices to show that

$$\frac{5t+8}{(2t+5)^2}\le\frac{48t+64}{36(t+2)(t+3)},$$

which is equivalent to

$$\frac{5t+8}{(2t+5)^2}\le\frac{12t+16}{9(t+2)(t+3)},$$

$$3t^3+7t^2-10t-32\ge 0,$$

$$(t-2)(3t^2+13t+16)\ge 0.$$

Case 2: $0<x\le y=z$. This involves $b=c$. Since the original inequality is homogeneous, we may consider $b=c=1$ and $0\le a\le b+c=2$. Thus, we only need to show that

$$\frac{1}{4}+\frac{2}{(a+1)^2}\le\frac{85}{36(2a+1)},$$

which is equivalent to

$$(a-2)(9a^2-2a+1)\le 0.$$

The equality holds for a degenerated isosceles triangle with $a=b+c$, $b=c$ (or any cyclic permutation).

□

P 5.38. *If a,b,c are the lengths of the side of a triangle so that $a+b+c=3$, then*

$$\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\le\frac{3(a^2+b^2+c^2)}{4(ab+bc+ca)}.$$

(Vasile C., 2007)

Solution. Write the inequality in the homogeneous form

$$\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\le\frac{27(a^2+b^2+c^2)}{4(a+b+c)^2(ab+bc+ca)}.$$

As shown in the proof of the preceding P 5.37, it suffices to prove this inequality for $a=b+c$ and for $b=c=1$.

Case 1: $a = b + c$. Since

$$\frac{1}{(a+b)^2} + \frac{1}{(c+a)^2} = \frac{5(b^2+c^2)+8bc}{(2b^2+2c^2+5bc)^2}$$

and

$$\frac{27(a^2+b^2+c^2)}{4(a+b+c)^2(ab+bc+ca)} = \frac{27(b^2+c^2+bc)}{8(b+c)^2(b^2+c^2+3bc)},$$

we need to show that

$$\frac{5(b^2+c^2)+8bc}{(2b^2+2c^2+5bc)^2} + \frac{1}{(b+c)^2} \le \frac{27(b^2+c^2+bc)}{8(b+c)^2(b^2+c^2+3bc)}.$$

For $bc = 0$, the inequality is true. For $bc \neq 0$, substituting

$$t = \frac{b}{c} + \frac{c}{b}, \qquad t \ge 2,$$

the inequality becomes

$$\frac{5t+8}{(2t+5)^2} + \frac{1}{t+2} \le \frac{27(t+1)}{8(t+2)(t+3)},$$

$$\frac{9t^2+38t+41}{(2t+5)^2} \le \frac{27(t+1)}{8(t+3)}.$$

It suffices to show that

$$\frac{9t^2+45t+27}{(2t+5)^2} \le \frac{27(t+1)}{8(t+3)},$$

which is equivalent to

$$\frac{t^2+5t+3}{(2t+5)^2} \le \frac{3(t+1)}{8(t+3)},$$

$$4t^3 + t(8t-9) + 3 \ge 0.$$

Case 2: $b = c = 1,\ a \le b + c = 2$. The homogeneous inequality becomes

$$\frac{2}{(a+1)^2} + \frac{1}{4} \le \frac{27(a^2+2)}{4(2a+1)(a+2)^2}.$$

Since

$$4(2a+1)(a+2) \le 9(a+1)^2,$$

it suffices to show that

$$\frac{2}{(a+1)^2} + \frac{1}{4} \le \frac{3(a^2+2)}{(a+1)^2(a+2)},$$

which is equivalent to

$$(a-6)(a-1)^2 \le 0.$$

The equality holds for a an equilateral triangle.

□

P 5.39. *Let $a, b, c \ge \dfrac{2}{5}$ so that $a+b+c=3$. Then,*

$$\frac{1}{3+2(a^2+b^2)}+\frac{1}{3+2(b^2+c^2)}+\frac{1}{3+2(c^2+a^2)} \le \frac{3}{7}.$$

(Vasile C., 2006)

Solution. For $a \le b \le c$, we have

$$\frac{2}{5} \le a \le b \le c \le \frac{11}{5}.$$

Indeed,

$$c = 3-a-b \le 3-\frac{2}{5}-\frac{2}{5}=\frac{11}{5}.$$

Using the substitution

$$m=\frac{3}{2}+a^2+b^2+c^2, \qquad m \ge \frac{3}{2}+\frac{1}{3}(a+b+c)^2=\frac{9}{2}\ ,$$

we have to show that

$$f(a)+f(b)+f(c) \le \frac{6}{7}$$

for

$$a+b+c=3, \qquad a^2+b^2+c^2=m-\frac{3}{2}, \qquad \frac{2}{5} \le a \le b \le c \le \frac{11}{5},$$

$$f(u)=\frac{1}{m-u^2}, \qquad \frac{2}{5} \le u \le \frac{11}{5}.$$

From

$$g(x)=f'(x)=\frac{2x}{(m-x^2)^2}, \qquad g''(x)=\frac{24x(m+x^2)}{(m-x^2)^4},$$

it follows that $g''(x)>0$, hence g is strictly convex. By Corollary 1 and Note 2, *if*

$$a+b+c=3, \qquad a^2+b^2+c^2=constant, \qquad \frac{2}{5} \le a \le b \le c \le \frac{11}{5},$$

then the sum

$$S_3=f(a)+f(b)+f(c)$$

is maximal for either $c=11/5$ or $a=b\le c$. The case $c=11/5$ leads to $a=b=2/5$, when the inequality is an equality. In the second case, we need to prove that

$$\frac{1}{3+4a^2}+\frac{2}{3+2(a^2+c^2)} \le \frac{3}{7}$$

for $2a+c=3$, $\dfrac{2}{5} \le a \le c$. Write the inequality as follows

$$\frac{1}{3+4a^2}+\frac{2}{21-24a+10a^2} \le \frac{3}{7},$$

$$\frac{1}{3+4a^2} \le \frac{49-72a+30a^2}{7(21-24a+10a^2)},$$

$$a(a-1)^2(5a-2) \ge 0.$$

The equality holds for $a=b=c=1$, and also for

$$a=b=\frac{2}{5}, \qquad c=\frac{11}{5}$$

(or any cyclic permutation).

Remark In the same manner, we can prove the following statement:

- *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1+a_2+\cdots+a_n=n$. If $k \ge \dfrac{n^2-1}{n^2-n-1}$, then*

$$\sum \frac{1}{k+a_2^2+\cdots+a_n^2} \le \frac{n}{k+n-1},$$

with equality for $a_1=a_2=\cdots=a_n=1$. If $k=\dfrac{n^2-1}{n^2-n-1}$, then the equality holds also for

$$a_1=\cdots=a_{n-1}=\frac{1}{n^2-n-1}, \qquad a_n=n-\frac{n-1}{n^2-n-1}$$

(or any cyclic permutation).

□

P 5.40. *If a,b,c are nonnegative real numbers so that $a+b+c=3$, then*

$$\frac{2}{2+a^2+b^2}+\frac{2}{2+b^2+c^2}+\frac{2}{2+c^2+a^2} \le \frac{99}{63+a^2+b^2+c^2}.$$

(Vasile C., 2009)

Solution. The proof is similar to the one of P 5.39. Thus, we only need to prove the inequality for $0 \le a=b \le c$; that is, to show that $2a+c=3$ involves

$$\frac{1}{1+a^2}+\frac{4}{2+a^2+c^2} \le \frac{99}{63+2a^2+c^2}.$$

Write this inequality as follows

$$\frac{1}{a^2+1}+\frac{4}{5a^2-12a+11} \le \frac{33}{2(a^2-2a+12)},$$

$$49a^4-112a^3+78a^2-16a+1 \ge 0,$$

$$(a-1)^2(7a-1)^2 \geq 0.$$

The equality holds for $a=b=c=1$, and also for

$$a=b=\frac{1}{7}, \qquad c=\frac{19}{7}$$

(or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

- *Let a,b,c be nonnegative real numbers so that $a+b+c=3$. If $\frac{8}{5} \leq k \leq 3$, then*

$$\frac{k+2}{k+a^2+b^2}+\frac{k+2}{k+b^2+c^2}+\frac{k+2}{k+c^2+a^2} \leq \frac{9(3k^2+11k+10)}{9(k^2+2k+6)+(5k-8)(a^2+b^2+c^2)},$$

with equality for $a=b=c=1$, and also for

$$a=b=\frac{3-k}{7}, \qquad c=\frac{2k+15}{7}$$

(or any cyclic permutation).

□

P 5.41. *If a,b,c are nonnegative real numbers so that $a+b+c=3$, then*

$$\frac{1}{3+a^2+b^2}+\frac{1}{3+b^2+c^2}+\frac{1}{3+c^2+a^2} \leq \frac{18}{27+a^2+b^2+c^2}.$$

(Vasile C., 2009)

Solution. The proof is similar to the one of P 5.39. Thus, we only need to prove the inequality for $0 \leq a=b \leq c$. Therefore, we only need to show that $2a+c=3$ involves

$$\frac{1}{3+2a^2}+\frac{2}{3+a^2+c^2} \leq \frac{18}{27+2a^2+c^2}.$$

Write this inequality as follows

$$\frac{1}{2a^2+3}+\frac{2}{5a^2-12a+12} \leq \frac{3}{a^2-2a+6},$$

$$a^2(a-1)^2 \geq 0.$$

The equality holds for $a=b=c=1$, and also for

$$a=b=0, \qquad c=3$$

(or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

- *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If $k \geq \frac{n}{n-2}$, then*

$$\sum \frac{1}{k + a_2^2 + \cdots + a_n^2} \leq \frac{n^2(n+k)}{n(n^2+kn+k^2)+(kn-n-k)(a_1^2+a_2^2+\cdots+a_n^2)},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = \cdots = a_{n-1} = 0, \qquad a_n = n$$

(or any cyclic permutation).

□

P 5.42. *If a, b, c are nonnegative real numbers so that $a+b+c=3$, then*

$$\frac{5}{3+a^2+b^2} + \frac{5}{3+b^2+c^2} + \frac{5}{3+c^2+a^2} \geq \frac{27}{6+a^2+b^2+c^2}.$$

(Vasile C., 2014)

Solution. Using the substitution

$$m = 3 + a^2 + b^2 + c^2,$$

we have to show that

$$f(a) + f(b) + f(c) \geq \frac{27}{24+m}$$

for

$$a+b+c=3, \qquad a^2+b^2+c^2 = m-3, \qquad 0 \leq a \leq b \leq c,$$

$$f(u) = \frac{5}{m-u^2}, \qquad 0 \leq u \leq \sqrt{m-3}.$$

From

$$g(x) = f'(x) = \frac{10x}{(m-x^2)^2}, \qquad g''(x) = \frac{120x(m+x^2)}{(m-x^2)^4},$$

it follows that $g''(x) \geq 0$ for $0 \leq x \leq \sqrt{m-3}$, hence g is strictly convex. By Corollary 1, *if*

$$a+b+c=3, \qquad a^2+b^2+c^2 = constant, \qquad 0 \leq a \leq b \leq c,$$

then the sum

$$S_3 = f(a) + f(b) + f(c)$$

is minimal for either $a=0$ *or* $0<a\le b=c$. Write the inequality in the homogeneous form

$$\sum \frac{5}{(a+b+c)^2+3(a^2+b^2)} \ge \frac{27}{2(a+b+c)^2+3(a^2+b^2+c^2)}.$$

Case 1: $a=0$. The homogeneous inequality becomes

$$\frac{5}{(b+c)^2+3b^2}+\frac{5}{(b+c)^2+3c^2}+\frac{5}{(b+c)^2+3(b^2+c^2)} \ge \frac{27}{2(b+c)^2+3(b^2+c^2)},$$

$$\frac{5[5(b^2+c^2)+4bc]}{4(b^2+c^2)^2+10bc(b^2+c^2)+13b^2c^2}+\frac{5}{4(b^2+c^2)+2bc} \ge \frac{27}{5(b^2+c^2)+4bc}.$$

For the nontrivial case $bc\ne 0$, substituting

$$\frac{b}{c}+\frac{c}{b}=t, \qquad t\ge 2,$$

we may write the inequality as

$$\frac{5(5t+4)}{4t^2+10t+13}+\frac{5}{4t+2} \ge \frac{27}{5t+4},$$

$$\frac{5(5t+4)}{4t^2+10t+13} \ge \frac{83t+34}{2(2t+1)(5t+4)}.$$

Since

$$83t+34\le 90t+20,$$

it suffices to show that

$$\frac{5t+4}{4t^2+10t+13} \ge \frac{9t+2}{(2t+1)(5t+4)},$$

which is equivalent to

$$14t^3+7t^2-65t-10\ge 0,$$

$$(t-2)(14t^2+35t+5)\ge 0.$$

Case 2: $0<a\le b=c$. We only need to prove the homogeneous inequality for $b=c=1$; that is,

$$\frac{10}{(a+2)^2+3(a^2+1)}+\frac{5}{(a+2)^2+6} \ge \frac{27}{2(a+2)^2+3(a^2+2)},$$

$$\frac{10}{4a^2+4a+7}+\frac{5}{a^2+4a+10} \ge \frac{27}{5a^2+8a+14},$$

$$a(a^3-3a+2)\ge 0,$$

$$a(a-1)^2(a+2)\ge 0.$$

The equality holds for $a = b = c = 1$, and also for

$$a = 0, \qquad b = c = \frac{3}{2}$$

(or any cyclic permutation).

Remark 1. Similarly, we can prove the following generalization:

- *Let a, b, c be nonnegative real numbers so that $a + b + c = 3$. If $k \ge 0$, then*

$$\frac{1}{k+a^2+b^2}+\frac{1}{k+b^2+c^2}+\frac{1}{k+c^2+a^2} \ge \frac{9(4k+15)}{3(4k^2+15k+9)+(8k+21)(a^2+b^2+c^2)}.$$

with equality for $a = b = c = 1$, and also for

$$a = 0, \qquad b = c = \frac{3}{2}$$

(or any cyclic permutation).

For $k = 0$, we get the inequality in P 1.171 from Volume 2:

$$\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2} \ge \frac{45}{(a+b+c)^2+7(a^2+b^2+c^2)}.$$

Remark 2. More general, the following statement holds:

- *Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If $k \ge 0$, then*

$$\sum \frac{1}{k+a_2^2+\cdots+a_n^2} \ge \frac{p}{q+a_1^2+a_2^2+\cdots+a_n^2},$$

where

$$p = \frac{n^2(n-1)^2k+n^3(n^2-n-1)}{(n-1)^3k+n(n^3-2n^2-n+1)}, \qquad q = \frac{n(n-1)^2k^2+n^2(n^2-n-1)k+n^3}{(n-1)^3k+n(n^3-2n^2-n+1)},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = 0, \qquad a_2 = \cdots = a_n = \frac{n}{n-1}$$

(or any cyclic permutation).

For $k = 0$ and $k = n$, we get the inequalities

$$\sum \frac{1}{a_2^2+\cdots+a_n^2} \ge \frac{n^2(n^2-n-1)}{n^2+(n^3-2n^2-n+1)(a_1^2+a_2^2+\cdots+a_n^2)},$$

$$\sum \frac{2n-1}{n+a_2^2+\cdots+a_n^2} \ge \frac{n^2(2n-3)}{n(n-1)+(n-2)(a_1^2+a_2^2+\cdots+a_n^2)}.$$

□

P 5.43. *If a, b, c, d are nonnegative real numbers so that $a+b+c+d=4$, then*

$$\sum \frac{3}{3+2(a^2+b^2+c^2)} \le \frac{296}{218+a^2+b^2+c^2+d^2}.$$

(Vasile C., 2009)

Solution. The proof is similar to the one of P 5.39. Thus, we only need to prove the inequality for $0 \le a = b = c \le d$, that is to show that $3a+d=4$ involves

$$\frac{1}{1+2a^2} + \frac{9}{3+4a^2+2d^2} \le \frac{296}{218+3a^2+d^2}.$$

Write this inequality as follows

$$\frac{1}{1+2a^2} + \frac{9}{35-48a+22a^2} \le \frac{148}{3(39-4a+2a^2)},$$

$$(a-1)^2(14a-1)^2 \ge 0.$$

The equality holds for $a=b=c=d=1$, and also for

$$a=b=c=\frac{1}{14}, \qquad d=\frac{53}{14}$$

(or any cyclic permutation).

□

P 5.44. *If a, b, c are nonnegative real numbers so that $ab+bc+ca=3$, then*

$$\frac{4}{2+a^2+b^2} + \frac{4}{2+b^2+c^2} + \frac{4}{2+c^2+a^2} \ge \frac{21}{4+a^2+b^2+c^2}.$$

(Vasile C., 2014)

Solution. The proof is similar to the one of P 5.42. Thus, we only need to prove the inequality for $a=0$ and for $0 < a \le b = c$.

Case 1: $a=0$. We need to show that $bc=3$ involves

$$\frac{1}{2+b^2} + \frac{1}{2+c^2} + \frac{1}{2+b^2+c^2} \ge \frac{21}{4(4+b^2+c^2)}.$$

Denote

$$x = b^2+c^2, \qquad x \ge 2bc = 6.$$

Since

$$\frac{1}{2+b^2} + \frac{1}{2+c^2} = \frac{4+b^2+c^2}{b^2c^2+2(b^2+c^2)+4} = \frac{x+4}{2x+13},$$

we only need to show that

$$\frac{x+4}{2x+13}+\frac{1}{x+2}\geq\frac{21}{4(x+4)}.$$

Since

$$\frac{x+4}{2x+13}+\frac{1}{x+2}=\frac{x^2+8x+21}{(2x+13)(x+2)}\geq\frac{7(2x+3)}{(2x+13)(x+2)},$$

it suffices to show that

$$\frac{2x+3}{(2x+13)(x+2)}\geq\frac{3}{4(x+4)}.$$

This inequality reduces to

$$(x-6)(2x+5)\geq 0.$$

Case 2: $0<a\leq b=c$. Let

$$q=ab+bc+ca.$$

We only need to prove the homogeneous inequality

$$\frac{4}{2q+3(a^2+b^2)}+\frac{4}{2q+3(b^2+c^2)}+\frac{4}{2q+3(c^2+a^2)}\geq\frac{21}{4q+3(a^2+b^2+c^2)}$$

for $b=c=1$. Thus, we need to show that

$$\frac{8}{2(2a+1)+3(a^2+1)}+\frac{4}{2(2a+1)+6}\geq\frac{21}{4(2a+1)+3(a^2+2)},$$

which is equivalent to

$$\frac{8}{3a^2+4a+5}+\frac{1}{a+2}\geq\frac{21}{3a^2+8a+10},$$

$$\frac{a^2+4a+7}{(3a^2+4a+5)(a+2)}\geq\frac{7}{3a^2+8a+10},$$

$$a(3a^3-a^2-7a+5)\geq 0,$$

$$a(a-1)^2(3a+5)\geq 0.$$

The equality holds for $a=b=c=1$, and also for

$$a=0,\qquad b=c=\sqrt{3}$$

(or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

- *Let a,b,c be nonnegative real numbers so that $ab+bc+ca=3$. If $k\geq 0$, then*

$$\frac{1}{k+a^2+b^2}+\frac{1}{k+b^2+c^2}+\frac{1}{k+c^2+a^2}\geq\frac{9(k+5)}{3(k^2+5k+2)+2(k+4)(a^2+b^2+c^2)}.$$

with equality for $a=b=c=1$, and also for

$$a=0, \qquad b=c=\sqrt{3}$$

(or any cyclic permutation).

For $k=0$, we get the inequality in P 1.171 from Volume 2:

$$\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2} \ge \frac{45}{2(ab+bc+ca)+8(a^2+b^2+c^2)}.$$

□

P 5.45. *If a,b,c are nonnegative real numbers so that $a^2+b^2+c^2=3$, then*

$$\frac{1}{10-(a+b)^2}+\frac{1}{10-(b+c)^2}+\frac{1}{10-(c+a)^2} \le \frac{1}{2}.$$

(Vasile C., 2006)

Solution. Let

$$s=a+b+c, \qquad s \le 3.$$

We need to show that

$$\frac{1}{10-(s-a)^2}+\frac{1}{10-(s-b)^2}+\frac{1}{10-(s-c)^2} \le \frac{1}{2}$$

for $a+b+c=s$ and $a^2+b^2+c^2=3$. Apply Corollary 1 to the function

$$f(u)=\frac{-1}{10-(s-u)^2}, \qquad 0 \le u \le s \le 3.$$

We have

$$g(x)=f'(x)=\frac{2(s-x)}{[10-(s-x)^2]^2},$$

$$g''(x)=\frac{24(s-x)[10+(s-x)^2]}{[10-(s-x)^2]^4}.$$

Since $g''(x)>0$ for $x \in [0,s)$, g is strictly convex on $[0,s]$. According to the Corollary 1, *if*

$$a+b+c=s, \qquad a^2+b^2+c^2=3, \qquad 0 \le a \le b \le c,$$

then

$$S_3=f(a)+f(b)+f(c)$$

is minimal for either $a=0$ *or* $0<a\le b=c$. Therefore, we only need to prove the homogeneous inequality

$$\sum \frac{1}{10(a^2+b^2+c^2)-3(b+c)^2} \le \frac{1}{2(a^2+b^2+c^2)}$$

for $a=0$ and for $b=c=1$.

Case 1: $a=0$. The homogeneous inequality becomes

$$\frac{1}{7(b^2+c^2)-6bc}+\frac{1}{10b^2+7c^2}+\frac{1}{7b^2+10c^2} \le \frac{1}{2(b^2+c^2)}.$$

This is true since

$$\frac{1}{7(b^2+c^2)-6bc} \le \frac{1}{4(b^2+c^2)}$$

and

$$\begin{aligned}\frac{1}{10b^2+7c^2}+\frac{1}{7b^2+10c^2} &= \frac{17(b^2+c^2)}{70(b^2+c^2)+149b^2c^2}\\ &\le \frac{17(b^2+c^2)}{70(b^2+c^2)+140b^2c^2}\\ &= \frac{17}{70(b^2+c^2)} < \frac{1}{4(b^2+c^2)}.\end{aligned}$$

Case 2: $b=c=1$. The homogeneous inequality turns into

$$\frac{1}{2(5a^2+4)}+\frac{2}{7a^2-6a+17} \le \frac{1}{2(a^2+2)},$$

$$\frac{2}{7a^2-6a+17} \le \frac{2a^2+1}{(5a^2+4)(a^2+2)},$$

$$4a^4-12a^3+13a^2-6a+1 \ge 0,$$

$$(a-1)^2(2a-1)^2 \ge 0.$$

The equality holds for $a=b=c=1$, and also for

$$2a=b=c=\frac{2}{\sqrt{3}}$$

(or any cyclic permutation).

□

P 5.46. *If* a,b,c *are nonnegative real numbers, no two of which are zero, so that* $a^4+b^4+c^4=3$*, then*

$$\frac{1}{a^5+b^5}+\frac{1}{b^5+c^5}+\frac{1}{c^5+a^5} \ge \frac{3}{2}.$$

(Vasile C., 2010)

Solution. Using the substitution

$$x = a^4, \quad y = b^4, \quad z = c^4, \qquad p = x^{5/4} + y^{5/4} + z^{5/4},$$

we need to show that $x + y + z = 3$ and $x^{5/4} + y^{5/4} + z^{5/4} = p$ involve

$$f(x) + f(y) + f(z) \ge \frac{3}{2},$$

where

$$f(u) = \frac{1}{p - u^{5/4}}, \qquad 0 \le u < p^{4/5}.$$

We will apply the EV-Theorem for $k = 5/4$. We have

$$f'(u) = \frac{5u^{1/4}}{4(p - u^{5/4})^2},$$

$$g(x) = f'\left(x^{\frac{1}{k-1}}\right) = f'(x^4) = \frac{5x}{4(p - x^5)^2},$$

$$g''(x) = \frac{75x^4(2p + 3x^5)}{2(p - x^5)^4}.$$

Since $g''(x) \ge 0$, g is strictly convex. According to the EV-Theorem and Note 3, *if*

$$x + y + z = 3, \qquad x^{5/4} + y^{5/4} + z^{5/4} = p = constant, \qquad 0 \le x \le y \le z,$$

then

$$S_3 = f(x) + f(y) + f(z)$$

is minimal for either $x = 0$ or $0 < x \le y = z$. Thus, we only need to prove the homogeneous inequality

$$\frac{1}{a^5 + b^5} + \frac{1}{b^5 + c^5} + \frac{1}{c^5 + a^5} \ge \frac{3}{2}\left(\frac{3}{a^4 + b^4 + c^4}\right)^{5/4}$$

for $a = 0$ and $0 < a \le b = c = 1$.

Case 1: $a = 0$. The homogeneous inequality becomes

$$\frac{1}{b^5} + \frac{1}{c^5} + \frac{1}{b^5 + c^5} \ge \frac{3}{2}\left(\frac{3}{b^4 + c^4}\right)^{5/4},$$

$$\left(\frac{b}{c}\right)^{5/2} + \left(\frac{c}{b}\right)^{5/2} + \frac{1}{\left(\frac{b}{c}\right)^{5/2} + \left(\frac{c}{b}\right)^{5/2}} \ge \left(\frac{3}{2}\right)^{9/4} \left[\frac{2}{\left(\frac{b}{c}\right)^2 + \left(\frac{c}{b}\right)^2}\right]^{5/4},$$

$$t^{5/2} + t^{-5/2} + \frac{1}{t^{5/2} + t^{-5/2}} \ge \left(\frac{3}{2}\right)^{9/4} \left(\frac{2}{t^2 + t^{-2}}\right)^{5/4},$$

$$2A^{5/2} + \frac{1}{2A^{5/2}} \geq \left(\frac{3}{2}\right)^{9/4} \cdot \frac{1}{B^{5/2}},$$

where

$$A = \left(\frac{t^{5/2} + t^{-5/2}}{2}\right)^{2/5}, \qquad B = \left(\frac{t^2 + t^{-2}}{2}\right)^{1/2}, \qquad t = \frac{b}{c}.$$

By power mean inequality, we have $A \geq B \geq 1$. Since

$$2A^{5/2} + \frac{1}{2A^{5/2}} - \left(2B^{5/2} + \frac{1}{2B^{5/2}}\right) = \left(A^{5/2} - B^{5/2}\right)\left(2 - \frac{1}{2A^{5/2}B^{5/2}}\right) \geq 0,$$

it suffices to show that

$$2B^{5/2} + \frac{1}{2B^{5/2}} \geq \left(\frac{3}{2}\right)^{9/4} \cdot \frac{1}{B^{5/2}},$$

$$4B^5 + 1 \geq \left(\frac{3^9}{2^5}\right)^{1/4},$$

which is true if

$$5 \geq \left(\frac{3^9}{2^5}\right)^{1/4},$$

$$32 \cdot 5^4 \geq 3^9.$$

This inequality follows by multiplying the inequalities

$$5^4 > 23 \cdot 3^3$$

and

$$32 \cdot 23 > 3^6.$$

Case 2: $0 < a \leq 1 = b = c$. The homogeneous inequality becomes

$$\frac{a^5 + 5}{a^5 + 1} \geq 3\left(\frac{3}{a^4 + 2}\right)^{5/4},$$

which is true if $g(a) \geq 0$, where

$$g(a) = \ln(a^5 + 5) - \ln(a^5 + 1) + \frac{5}{4}\ln(a^4 + 2) - \frac{9\ln 3}{4},$$

with

$$\frac{g'(a)}{5a^3} = \frac{a}{a^5 + 5} - \frac{a}{a^5 + 1} + \frac{1}{a^4 + 2} = \frac{a^{10} + 2a^5 - 8a + 5}{(a^4 + 5)(a^5 + 1)(a^4 + 2)}$$
$$= \frac{(a-1)(a^9 + a^8 + a^7 + a^6 + a^5 + 3a^4 + 3a^3 + 3a^2 + 3a - 5)}{(a^4 + 5)(a^5 + 1)(a^4 + 2)}.$$

There exists $d \in (0,1)$ so that $g'(d)=0$, $g'(a)>0$ for $a \in [0,d)$ and $g'(a)<0$ for $a \in (d,1)$. Therefore, g is increasing on $[0,d]$ and is decreasing on $[d,1]$. Since $g(1)=0$, we only need to show that $g(0)\ge 0$. Indeed,

$$g(0)=\frac{1}{4}\ln\frac{5^4\cdot 2^5}{3^9}>0.$$

The equality holds for $a=b=c=1$.

□

P 5.47. *If $a_1,a_2,\ldots,a_n$ are nonnegative real numbers so that $a_1+a_2+\cdots+a_n=n$, then*

$$\sqrt{a_1^2+1}+\sqrt{a_2^2+1}+\cdots+\sqrt{a_n^2+1}\ge\sqrt{2\left(1-\frac{1}{n}\right)(a_1^2+a_2^2+\cdots+a_n^2)+2(n^2-n+1)}.$$

(Vasile C., 2014)

Solution. For $n=2$, we need to show that $a_1+a_2=2$ involves

$$\sqrt{a_1^2+1}+\sqrt{a_2^2+1}\ge\sqrt{a_1^2+a_2^2+6}.$$

By squaring, the inequality becomes

$$\sqrt{(a_1^2+1)(a_2^2+1)}\ge 2,$$

which follows immediately from the Cauchy-Schwarz inequality:

$$(a_1^2+1)(a_2^2+1)=(a_1^2+1)(1+a_2^2)\ge(a_1+a_2)^2=4.$$

Assume further that $n\ge 3$ and $a_1\le a_2\le\cdots\le a_n$. We will apply Corollary 1 to the function

$$f(u)=-\sqrt{u^2+4},\quad u\ge 0.$$

We have

$$g(x)=f'(x)=\frac{-x}{\sqrt{x^2+4}},$$

$$g''(x)=\frac{12x}{(x^2+4)^{5/2}}.$$

Since $g''(x)>0$ for $x>0$, $g(x)$ is strictly convex for $x\ge 0$. By Corollary 1, *if $a_1\le a_2\le\cdots\le a_n$ and*

$$a_1+a_2+\cdots+a_n=n,\qquad a_1^2+a_2^2+\cdots+a_n^2=constant,$$

then the sum

$$S_n=f(a_1)+f(a_2)+\cdots+f(a_n)$$

is maximal for $a_1 = a_2 = \cdots = a_{n-1}$. Thus, we only need to show that

$$\sqrt{a^2+1}+(n-1)\sqrt{b^2+1} \ge \sqrt{2\left(1-\frac{1}{n}\right)[a^2+(n-1)b^2]+2(n^2-n+1)}.$$

for

$$a+(n-1)b=n.$$

By squaring, the inequality becomes

$$2n(n-1)\sqrt{(a^2+1)(b^2+1)} \ge (n-2)a^2-(n-2)(n-1)^2b^2+n^3,$$

which is equivalent to

$$\sqrt{(b^2+1)[(n-1)^2b^2-2n(n-1)b+n^2+1]} \ge n-(n-2)b.$$

This is true if

$$(b^2+1)[(n-1)^2b^2-2n(n-1)b+n^2+1] \ge [n-(n-2)b]^2,$$

which is equivalent o

$$(n-1)^2b^4-2n(n-1)b^3+(n^2+2n-2)b^2-2nb+1 \ge 0,$$

$$(b-1)^2[(n-1)b-1]^2 \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = a_2 = \cdots = a_{n-1} = \frac{1}{n-1}, \qquad a_n = n-1$$

(or any cyclic permutation).

□

P 5.48. *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then*

$$\sum \sqrt{(3n-4)a_1^2+n} \ge \sqrt{(3n-4)(a_1^2+a_2^2+\cdots+a_n^2)+n(4n^2-7n+4)}.$$

(Vasile C., 2009)

Solution. The proof is similar to the one of the preceding P 5.47. Thus, it suffices to prove the inequality for $a_1 = a_2 = \cdots = a_{n-1}$. Write the inequality in the homogeneous form

$$\sum \sqrt{n(3n-4)a_1^2+S^2} \ge \sqrt{n(3n-4)(a_1^2+a_2^2+\cdots+a_n^2)+(4n^2-7n+4)S^2},$$

where $S = a_1 + a_2 + \cdots + a_n$. We only need to prove this inequality for $a_1 = a_2 = \cdots = a_{n-1} = 1$, that is

$$(n-1)\sqrt{n(3n-4)+(n-1+a_n)^2} + \sqrt{n(3n-4)a_n^2+(n-1+a_n)^2} \geq$$

$$\geq \sqrt{n(3n-4)(n-1+a_n^2)+(4n^2-7n+4)(n-1+a_n)^2},$$

which is equivalent to

$$\sqrt{(n-1)[a_n^2+2(n-1)a_n+4n^2-6n+1]} + \sqrt{(3n-1)a_n^2+2a_n+n-1} \geq$$

$$\geq \sqrt{(7n-4)a_n^2+2(4n^2-7n+4)a_n+4n^3-8n^2+7n-4}.$$

By squaring, the inequality turns into

$$2\sqrt{(n-1)[(3n-1)a_n^2+2a_n+n-1][a_n^2+2(n-1)a_n+4n^2-6n+1]} \geq$$

$$(3n-2)a_n^2+2(n-1)(3n-2)a_n+2n^2-n-2.$$

Squaring again, we get

$$(a_n-1)^2(a_n-2n+3)^2 \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = a_2 = \cdots = a_{n-1} = \frac{a_n}{2n-3} = \frac{n}{3n-4}$$

(or any cyclic permutation).

Remark. For $n = 3$, we get the inequality

$$\sqrt{5a^2+3} + \sqrt{5b^2+3} + \sqrt{5c^2+3} \geq \sqrt{5(a^2+b^2+c^2)+57},$$

where a, b, c are nonnegative real numbers so that $a+b+c = 3$. By squaring, the inequality turns into

$$\sqrt{(5a^2+3)(5b^2+3)} + \sqrt{(5b^2+3)(5c^2+3)} + \sqrt{(5c^2+3)(5a^2+3)} \geq 24,$$

with equality for $a = b = c = 1$, and also for

$$a = b = \frac{c}{3} = \frac{3}{5}$$

(or any cyclic permutation).

□

P 5.49. *If a,b,c are nonnegative real numbers so that $a+b+c=3$, then*

$$\sqrt{a^2+4}+\sqrt{b^2+4}+\sqrt{c^2+4} \le \sqrt{\frac{8}{3}(a^2+b^2+c^2)+37}.$$

(Vasile C., 2009)

Solution. Assume that $a \le b \le c$, and apply Corollary 1 to the function a

$$f(u) = -\sqrt{u^2+4}, \quad u \ge 0.$$

We have

$$g(x) = f'(x) = \frac{-x}{\sqrt{x^2+4}},$$

$$g''(x) = \frac{12x}{(x^2+4)^{5/2}}.$$

Since $g''(x) > 0$ for $x > 0$, $g(x)$ is strictly convex for $x \ge 0$. By Corollary 1, *if*

$$a+b+c=3, \qquad a^2+b^2+c^2 = constant\ , \qquad a \le b \le c,$$

then the sum

$$S_3 = f(a)+f(b)+f(c)$$

is minimal for either $a=0$ or $0<a \le b=c$. Thus, we only need to prove the desired inequality for these cases.

Case 1: $a=0$. We need to prove that $b+c=3$ involves

$$\sqrt{b^2+4}+\sqrt{c^2+4} \le \sqrt{\frac{8}{3}(b^2+c^2)+37} - 2.$$

Substituting

$$b = \frac{3x}{2}, \qquad c = \frac{3y}{2},$$

we need to prove that $x+y=2$ involves

$$\sqrt{9x^2+16}+\sqrt{9y^2+16} \le 2\sqrt{6(x^2+y^2)+37} - 4.$$

By squaring, the inequality becomes

$$2\sqrt{(9x^2+16)(9y^2+16)} \le 15(x^2+y^2)+132-16\sqrt{6(x^2+y^2)+37}.$$

Denoting

$$p = xy, \qquad 0 \le p \le 1,$$

we have

$$x^2+y^2 = 4-2p, \quad (9x^2+16)(9y^2+16) = 81p^2-288p+832,$$

and the inequality becomes

$$\sqrt{81p^2-288p+832} \le -15p+96-8\sqrt{61-12p},$$

$$\sqrt{\frac{81}{4}p^2-72p+208} \le -\frac{15}{2}p+(48-4\sqrt{61-12p}),$$

By squaring again (the right hand side is positive), the inequality becomes

$$\frac{81}{4}p^2-72p+208 \le \frac{225}{4}p^2-15p(48-4\sqrt{61-12p})+(48-4\sqrt{61-12p})^2,$$

$$3p^2-70p+256 \ge (32-5p)\sqrt{61-12p}.$$

Since

$$2\sqrt{61-12p} \le 7+\frac{61-12p}{7} = \frac{2(55-6p)}{7},$$

it suffices to show that

$$7(3p^2-70p+256) \ge (32-5p)(55-6p),$$

which is equivalent to

$$(1-p)(32+9p) \ge 0.$$

Case 2: $b=c$. We need to prove that

$$a+2b=3$$

implies

$$\sqrt{a^2+4}+2\sqrt{b^2+4} \le \sqrt{\frac{8}{3}(a^2+2b^2)+37}.$$

By squaring, the inequality becomes

$$12\sqrt{(a^2+4)(b^2+4)} \le 5a^2+4b^2+51,$$

which is equivalent to

$$\sqrt{(4b^2-12b+13)(b^2+4)} \le 2b^2-5b+8.$$

By squaring again, the inequality becomes

$$2b^3-7b^2+8b-3 \le 0,$$

$$(b-1)^2(2b-3) \le 0,$$

$$(b-1)^2a \ge 0.$$

The equality holds for $a=b=c=1$, and also for

$$a=0, \qquad b=c=\frac{3}{2}$$

(or any cyclic permutation).

□

P 5.50. *If a,b,c are nonnegative real numbers so that $a+b+c=3$, then*

$$\sqrt{32a^2+3}+\sqrt{32b^2+3}+\sqrt{32c^2+3}\le\sqrt{32(a^2+b^2+c^2)+219}.$$

(Vasile C., 2009)

Solution. The proof is similar to the one of P 5.49. Thus, it suffices to prove the homogeneous inequality

$$\sum\sqrt{96a^2+(a+b+c)^2}\le\sqrt{96(a^2+b^2+c^2)+73(a+b+c)^2}$$

for $a=0$ and for $b=c=1$.

Case 1: $a=0$. We have to show that

$$b+c+\sqrt{97b^2+2bc+c^2}+\sqrt{b^2+2bc+97c^2}\le\sqrt{169(b^2+c^2)+146bc}.$$

Since $2bc\le b^2+c^2$, it suffices to prove that

$$b+c+\sqrt{98b^2+2c^2}+\sqrt{2b^2+98c^2}\le\sqrt{169(b^2+c^2)+146bc}.$$

By squaring, we get

$$(b+c)\left(\sqrt{98b^2+2c^2}+\sqrt{2b^2+98c^2}\right)+2\sqrt{(49b^2+c^2)(b^2+49c^2)}\le$$

$$\le 34(b^2+c^2)+72bc.$$

Since

$$\sqrt{98b^2+2c^2}+\sqrt{2b^2+98c^2}\le\sqrt{2(98b^2+2c^2+2b^2+98c^2)}=10\sqrt{2(b^2+c^2)}$$

and

$$10(b+c)\sqrt{2(b^2+c^2)}\le 20(b+c)^2,$$

it suffices to show that

$$\sqrt{(49b^2+c^2)(b^2+49c^2)}\le 7(b^2+c^2)+36bc.$$

Squaring again, the inequality becomes

$$bc(b-c)^2\ge 0.$$

Case 2: $b=c=1$. The homogeneous inequality turns into

$$\sqrt{97a^2+4a+4}+2\sqrt{a^2+4a+100}\le\sqrt{169a^2+292a+484}.$$

By squaring, we get

$$\sqrt{(97a^2+4a+4)(a^2+4a+100)}\le 17a^2+68a+20.$$

Squaring again, the inequality reduces to

$$a(a-1)^2(a+12) \geq 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

Remark. By squaring, we deduce the inequality

$$\sqrt{(32a^2+3)(32b^2+3)} + \sqrt{(32b^2+3)(32c^2+3)} + \sqrt{(32c^2+3)(32a^2+3)} \leq 105,$$

with equality for $a = b = c = 1$, and also for

$$a = 0, \qquad b = c = \frac{3}{2}$$

(or any cyclic permutation).

□

P 5.51. *If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then*

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} + \frac{2n\sqrt{n-1}}{a_1^2 + a_2^2 + \cdots + a_n^2} \geq n + 2\sqrt{n-1}.$$

(Vasile C., 2009)

Solution. For $n = 2$, the inequality reduces to

$$(a_1 a_2 - 1)^2 \geq 0.$$

Consider further that $n \geq 3$ and $a_1 \leq a_2 \leq \cdots \leq a_n$. By Corollary 5 (case $k = 2$ and $m = -1$), *if $a_1 \leq a_2 \leq \cdots \leq a_n$ and*

$$a_1 + a_2 + \cdots + a_n = n, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = constant,$$

then the sum

$$S_n = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

is minimal for $a_1 = \cdots = a_{n-1} \leq a_n$. Therefore, we only need to prove that

$$\frac{n-1}{a_1} + \frac{1}{a_n} + \frac{2n\sqrt{n-1}}{(n-1)a_1^2 + a_n^2} \geq n + 2\sqrt{n-1},$$

for $(n-1)a_1 + a_n = n$. The inequality is equivalent to

$$(a_1 - 1)^2 \left(a_1 - \frac{n}{n-1+\sqrt{n-1}} \right)^2 \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = a_2 = \cdots = a_{n-1} = \frac{a_n}{\sqrt{n-1}}$$

(or any cyclic permutation).

□

P 5.52. *If* $a, b, c \in [0, 1]$, *then*

$$(1+3a^2)(1+3b^2)(1+3c^2) \ge (1+ab+bc+ca)^3.$$

Solution. Since

$$2(ab+bc+ca) = (a+b+c)^2 - (a^2+b^2+c^2),$$

we may apply Corollary 1 to the function

$$f(u) = -\ln(1+3u^2), \qquad u \in [0, 1],$$

to prove the inequality

$$f(a) + f(b) + f(c) + 3\ln(1+ab+bc+ca) \le 0.$$

We have

$$g(x) = f'(x) = \frac{-6x}{1+3x^2},$$

$$g''(x) = \frac{108x(1-x^2)}{(1+3x^2)^3}.$$

Since $g''(x) > 0$ for $x \in (0, 1)$, g is strictly convex on $[0, 1]$. According to Corollary 1 and Note 2, *if*

$$a+b+c = constant, \qquad a^2+b^2+c^2 = constant, \qquad 0 \le a \le b \le c \le 1,$$

then

$$S_3 = f(a) + f(b) + f(c)$$

is maximal for $a = b \le c$. or for $c = 1$. Thus, we only need to prove the original inequality for these cases.

Case 1: $a = b \le c$. We need to show that

$$(1+3a^2)^2(1+3c^2) \ge (1+a^2+2ac)^3.$$

For $c = 0$, the inequality is an equality. For fixed c, $0 < c \le 1$, we need to show that $h(a) \ge 0$, where

$$h(a) = 2\ln(1+3a^2) + \ln(1+3c^2) - 3\ln(1+a^2+2ac), \qquad a \in [0, c].$$

From

$$h'(a) = \frac{12a}{1+3a^2} - \frac{6(a+c)}{1+a^2+2ac} = \frac{6(1-a^2)(a-c)}{(1+3a^2)(1+a^2+2ac)} \le 0,$$

it follows that h is decreasing on $[0,c]$, hence $h(a) \ge h(c) = 0$.

Case 2: $c = 1$. We need to show that

$$4(1+3a^2)(1+3b^2) \ge (1+a)^3(1+b)^3.$$

This is true because

$$2(1+3a^2) \ge (1+a)^3, \qquad 2(1+3b^2) \ge (1+b)^3.$$

The first inequality is equivalent to

$$(1-a)^3 \ge 0.$$

The proof is completed. The equality holds for $a = b = c$.

□

P 5.53. *If a,b,c are nonnegative real numbers so that $a+b+c = ab+bc+ca$, then*

$$\frac{1}{4+5a^2} + \frac{1}{4+5a^2} + \frac{1}{4+5a^2} \ge \frac{1}{3}.$$

(Vasile C., 2007)

Solution. By expanding, the inequality becomes

$$4(a^2+b^2+c^2) + 15 \ge 25a^2b^2c^2 + 5(a^2b^2+b^2c^2+c^2a^2).$$

Let $p = a+b+c$. Since

$$a^2+b^2+c^2 = p^2-2p, \qquad a^2b^2+b^2c^2+c^2a^2 = p^2-2pabc,$$

the inequality becomes

$$(2p-4)^2 \ge (p-5abc)^2,$$

$$(3p-4-5abc)(p+5abc-4) \ge 0.$$

We will show that $3p \ge 4+5abc$ and $p+5abc \ge 4$. According to Corollary 4 (case $n=3$, $k=2$) or P 3.57 in Volume 1, *if*

$$a+b+c = constant, \qquad ab+bc+ca = constant, \qquad 0 \le a \le b \le c \le d,$$

then the product abc is maximal for $a=b$, and is minimal for $a=0$ or $b=c$. Thus, we only need to prove that $3p \ge 4+5abc$ for $a=b$, and $p+5abc \ge 4$ for $a=0$ and for $b=c$.

For $a = b$, from $a + b + c = ab + bc + ca$ we get

$$c = \frac{a(2-a)}{2a-1}, \qquad \frac{1}{2} < a \le 2,$$

hence

$$3p - 4 - 5abc = (3 - 5a^2)c + 6a - 4 = \frac{(a-1)^2(5a^2+4)}{2a-1} \ge 0.$$

For $a = 0$, from $a + b + c = ab + bc + ca$ we get

$$c = \frac{b}{b-1}, \qquad b > 1,$$

hence

$$p + 5abc - 4 = b + c - 4 = \frac{(b-2)^2}{b-1} \ge 0.$$

For $b = c$, from $a + b + c = ab + bc + ca$ we get

$$a = \frac{b(2-b)}{2b-1}, \qquad \frac{1}{2} < b \le 2,$$

hence

$$p + 5abc - 4 = a(5b^2+1) + 2b - 4 = \frac{(2-b)(5b^3 - 3b + 2)}{2b-1}$$
$$= \frac{(2-b)[4b^3 + (b-1)^2(b+2)]}{2b-1} \ge 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 0$ and $b = c = 2$ (or any cyclic permutation).

□

P 5.54. *If a, b, c, d are positive real numbers so that $a + b + c + d = 4abcd$, then*

$$\frac{1}{1+3a} + \frac{1}{1+3b} + \frac{1}{1+3c} + \frac{1}{1+3d} \ge 1.$$

(Vasile C., 2007)

Solution. By expanding, the inequality becomes

$$1 + 3(ab + ac + ad + bc + bd + cd) \ge 19abcd,$$

$$2 + 3(a + b + c + d)^2 \ge 3(a^2 + b^2 + c^2 + d^2) + 38abcd.$$

According to Corollary 5 (case $n = 4$, $k = 0$, $m = 2$), *if*

$$a + b + c + d = constant, \qquad abcd = constant, \qquad 0 < a \le b \le c \le d,$$

then the sum

$$S_4 = a^2 + b^2 + c^2 + d^2$$

is maximal for $a = b = c \le d$. Thus, we only need to prove that

$$3a + d = 4a^3 d, \quad d = \frac{3a}{4a^3 - 1}, \quad a > \frac{1}{\sqrt[3]{4}},$$

involves

$$\frac{3}{3a+1} + \frac{1}{3d+1} \ge 1,$$

$$\frac{3}{3a+1} + \frac{4a^3 - 1}{4a^3 + 9a - 1} \ge 1,$$

$$4a^3 - 9a^2 + 6a - 1 \ge 0,$$

$$(a-1)^2(4a-1) \ge 0.$$

The equality holds for $a = b = c = d = 1$.

Open problem. *If $a_1, a_2, \ldots, a_n$ ($n \ge 3$) are positive real numbers so that*

$$a_1 + a_2 + \cdots + a_n = n a_1 a_2 \cdots a_n,$$

then

$$\frac{1}{1+(n-1)a_1} + \frac{1}{1+(n-1)a_2} + \cdots + \frac{1}{1+(n-1)a_n} \ge 1.$$

□

P 5.55. *If $a_1, a_2, \ldots, a_n$ are positive real numbers so that*

$$a_1 + a_2 + \cdots + a_n = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n},$$

then

$$\frac{1}{1+(n-1)a_1} + \frac{1}{1+(n-1)a_2} + \cdots + \frac{1}{1+(n-1)a_n} \ge 1.$$

(Vasile C., 1996)

Solution. For $n = 2$, the inequality is an identity. For $n \ge 3$, we consider

$$a_1 \le a_2 \le \cdots \le a_n,$$

and apply Corollary 2 to the function

$$f(u) = \frac{1}{1+(n-1)u}, \quad u > 0.$$

We have

$$f'(u) = \frac{-(n-1)}{[1+(n-1)u]^2},$$

$$g(x) = f'\left(\frac{1}{\sqrt{x}}\right) = \frac{-(n-1)x}{[\sqrt{x}+n-1]^2},$$

$$g''(x) = \frac{3(n-1)^2}{2\sqrt{x}(\sqrt{x}+n-1)^4}.$$

Since $g''(x) > 0$ for $x > 0$, g is strictly convex on $[0,\infty)$. By Corollary 2, *if* $0 < a_1 \le a_2 \le \cdots \le a_n$ *and*

$$a_1 + a_2 + \cdots + a_n = constant, \qquad \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = constant,$$

then the sum

$$S_n = f(a_1) + f(a_2) + \cdots + f(a_n)$$

is minimal for $a_2 = \cdots = a_n$. Therefore, we only need to show that

$$\frac{1}{1+(n-1)a} + \frac{n-1}{1+(n-1)b} \ge 1$$

for

$$a + (n-1)b = \frac{1}{a} + \frac{n-1}{b}, \qquad 0 < a \le b.$$

Write the hypothesis as

$$\frac{1}{a} - a = (n-1)\left(b - \frac{1}{b}\right),$$

which involves $a \le 1 \le b$ and

$$\frac{1}{a} - a \ge b - \frac{1}{b}, \qquad ab \le 1.$$

Write the desired inequality as

$$\frac{n-1}{1+(n-1)b} \ge 1 - \frac{1}{1+(n-1)a},$$

which is equivalent to

$$\frac{n-1}{1+(n-1)b} \ge \frac{(n-1)a}{1+(n-1)a},$$

$$1 - a \ge (n-1)a(b-1).$$

For the nontrivial case $b \ne 1$, we have

$$1 - a - (n-1)a(b-1) = 1 - a - \frac{b(1-a^2)}{a(b^2-1)}a(b-1) = \frac{(1-a)(1-ab)}{b+1} \ge 0.$$

If $n \ge 3$, then the equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 5.56. *If a,b,c,d,e are nonnegative real numbers so that $a^4+b^4+c^4+d^4+e^4=5$, then*

$$7(a^2+b^2+c^2+d^2+e^2) \ge (a+b+c+d+e)^2+10.$$

(Vasile C., 2008)

Solution. According to Corollary 5 (case $n=5$, $k=4$, $m=2$), *if*

$$a+b+c+d+e=constant, \quad a^4+b^4+c^4+d^4+e^4=5, \quad 0 \le a \le b \le c \le d \le e,$$

then the sum

$$S_4 = a^2+b^2+c^2+d^2+e^2$$

is minimal for $a=b=c=d \le e$. Thus, we only need to prove the homogeneous inequality

$$[7(a^2+b^2+c^2+d^2+e^2)-(a+b+c+d+e)^2]^2 \ge 20(a^4+b^4+c^4+d^4+e^4)$$

for $a=b=c=d=0$ and $a=b=c=d=1$. The first case is trivial. In the second case, the inequality becomes

$$[7(4+e^2)-(4+e)^2]^2 \ge 20(4+e^4),$$

$$(3e^2-4e+6)^2 \ge 5e^4+20,$$

$$e^4-6e^3+13e^2-12e+4 \ge 0,$$

$$(e-1)^2(e-2)^2 \ge 0.$$

The equality holds for $a=b=c=d=e=1$, and also for

$$a=b=c=d=\frac{e}{2}=\frac{1}{\sqrt{2}}$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- *If $a_1,a_2,\ldots,a_n$ are nonnegative real numbers so that*

$$a_1^4+a_2^4+\cdots+a_n^4=n,$$

then

$$(n+\sqrt{n-1})(a_1^2+a_2^2+\cdots+a_n^2-n) \ge (a_1+a_2+\cdots+a_n)^2-n^2,$$

with equality for $a_1=a_2=\cdots=a_n=1$, and also for

$$a_1=\cdots=a_{n-1}=\frac{a_n}{\sqrt{n-1}}=\frac{1}{\sqrt[4]{n-1}}$$

(or any cyclic permutation).

□

P 5.57. *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then*

$$(a_1^2 + a_2^2 + \cdots + a_n^2)^2 - n^2 \ge \frac{n(n-1)}{n^2 - n + 1}\left(a_1^4 + a_2^4 + \cdots + a_n^4 - n\right).$$

(Vasile C., 2008)

Solution. For $n = 2$, the inequality reduces to $(a_1a_2 - 1)^2 \ge 0$. For $n \ge 3$, we apply Corollary 5 for $k = 2$ and $m = 4$: *if* $0 \le a_1 \le a_2 \le \cdots \le a_n$ *and*

$$a_1 + a_2 + \cdots + a_n = n, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = constant,$$

then

$$S_n = a_1^4 + a_2^4 + \cdots + a_n^4$$

is maximal for $a_1 = \cdots = a_{n-1} \le a_n$. Thus, we only need to prove the homogeneous inequality

$$n^2(n^2 - n + 1)(a_1^2 + a_2^2 + \cdots + a_n^2)^2 \ge (n^2 - 2n + 2)(a_1 + a_2 + \cdots + a_n)^4 + n^3(n-1)S_n,$$

for $a_1 = \cdots = a_{n-1} = 0$ and for $a_1 = \cdots = a_{n-1} = 1$. For the nontrivial case $a_1 = \cdots = a_{n-1} = 1$, the inequality becomes

$$n^2(n^2 - n + 1)(n - 1 + a_n^2)^2 \ge (n^2 - 2n + 2)(n - 1 + a_n)^4 + n^3(n-1)(n - 1 + a_n^4),$$

$$(a_n - 1)^2[a_n - (n-1)^2]^2 \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = \cdots = a_{n-1} = \frac{1}{n-1}, \qquad a_n = n - 1$$

(or any cyclic permutation).

□

P 5.58. *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that $a_1^2 + a_2^2 + \cdots + a_n^2 = n$, then*

$$a_1^3 + a_2^3 + \cdots + a_n^3 \ge \sqrt{n^2 - n + 1 + \left(1 - \frac{1}{n}\right)(a_1^6 + a_2^6 + \cdots + a_n^6)}.$$

(Vasile C., 2008)

Solution. For $n = 2$, the inequality is equivalent to

$$a_1^6 + a_2^6 + 4a_1^3a_2^3 \ge 6,$$

$$(a_1^2 + a_2^2)^3 - 3a_1^2a_2^2(a_1^2 + a_2^2) + 4a_1^3a_2^3 \ge 6,$$

$$2a_1^3a_2^3 - 3a_1^2a_2^2 + 1 \ge 0,$$

$$(a_1a_2-1)^2(2a_1a_2+1)\geq 0.$$

For $n\geq 3$, we apply Corollary 5 for $k=3/2$ and $m=3$: *if* $0\leq x_1\leq x_2\leq\cdots\leq x_n$ *and*

$$x_1+x_2+\cdots+x_n=n,\qquad x_1^{3/2}+x_2^{3/2}+\cdots+x_n^{3/2}=constant,$$

then

$$S_n=x_1^3+x_2^3+\cdots+x_n^3$$

is maximal for $x_1=\cdots=x_{n-1}\leq x_n$. Thus, we only need to prove the homogeneous inequality

$$(a_1^3+a_2^3+\cdots+a_n^3)^2\geq\frac{n^2-n+1}{n^3}(a_1^2+a_2^2+\cdots+a_n^2)^3+\left(1-\frac{1}{n}\right)(a_1^6+a_2^6+\cdots+a_n^6)$$

for $a_1=\cdots=a_{n-1}=0$ and for $a_1=\cdots=a_{n-1}=1$. For the nontrivial case $a_1=\cdots=a_{n-1}=1$, the inequality becomes

$$n^3(n-1+a_n^3)^2\geq(n^2-n+1)(n-1+a_n^2)^3+n^2(n-1)(n-1+a_n^6),$$

$$(a_n-1)^2(a_n-n+1)^2(a_n^2+2na_n+n-1)\geq 0.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$, and also for

$$a_1=\cdots=a_{n-1}=\frac{a_n}{n-1}=\frac{1}{\sqrt{n-1}}$$

(or any cyclic permutation).

□

P 5.59. *If a,b,c are positive real numbers so that $abc=1$, then*

$$4\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\frac{50}{a+b+c}\geq 27.$$

(Vasile C., 2012)

Solution. According to Corollary 5 (case k=0 and $m=-1$, *if*

$$a+b+c=constant,\qquad abc=1,\qquad 0<a\leq b\leq c,$$

then

$$S_3=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

is minimal for $0<a=b\leq c$. Thus, we only need to prove that

$$4\left(\frac{2}{a}+\frac{1}{c}\right)+\frac{50}{2a+c}\geq 27$$

for

$$a^2c = 1, \qquad a \le 1.$$

The inequality is equivalent to

$$8a^6 - 54a^4 - 26a^3 - 27a + 8 \ge 0,$$

$$(2a-1)^2(2a^4 + 2a^3 - 12a^2 + 5a + 8) \ge 0.$$

It is true for $a \in (0,1]$ because

$$2a^4 + 2a^3 - 12a^2 + 5a + 8 > -12a^2 + 4a + 8 = 4(1-a)(2+3a) \ge 0.$$

The equality holds for

$$a = b = \frac{1}{2}, \qquad c = 4$$

(or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

- *If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1a_2\cdots a_n = 1$, then*

$$2^n\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) + \frac{(2^n+n-1)^2}{a_1+a_2+\cdots+a_n} \ge 2n(2^n+1),$$

with equality for

$$a_1 = \cdots = a_{n-1} = \frac{1}{2}, \qquad a_n = 2^{n-1}$$

(or any cyclic permutation).

For

$$a_1 = \cdots = a_{n-1} = a \le 1, \qquad a^{n-1}a_n = 1,$$

the inequality is equivalent to $f(a) \ge 0$, where

$$f(a) = 2^n\left(\frac{n-1}{a} + a^{n-1}\right) + \frac{(2^n+n-1)^2a^{n-1}}{(n-1)a^n+1} - 2n(2^n+1).$$

We have

$$\begin{aligned}\frac{f'(a)}{n-1} &= \frac{2^n(a^n-1)}{a^2} - \frac{(2^n+n-1)^2a^{n-2}(a^n-1)}{[(n-1)a^n+1]^2}\\ &= \frac{(a^n-1)(2^na^n-1)[(n-1)^2a^n-2^n]}{a^2[(n-1)a^n+1]^2}.\end{aligned}$$

Since

$$(n-1)^2a^n - 2^n \le (n-1)^2 - 2^n < 0,$$

it follows that $f'(a) < 0$ for $a \in \left(0, \frac{1}{2}\right)$, and $f'(a) > 0$ for $a \in \left(\frac{1}{2}, 1\right)$. Therefore, f is decreasing on $\left(0, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, 1\right]$, hence

$$f(a) \ge f\left(\frac{1}{2}\right) = 0.$$

□

P 5.60. *If a, b, c are positive real numbers so that $abc = 1$, then*

$$a^3 + b^3 + c^3 + 15 \ge 6\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

(Michael Rozenberg, 2006)

Solution. Replacing a, b, c by their reverses $1/a, 1/b, 1/c$, we need to show that $abc = 1$ involves

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + 15 \ge 6(a + b + c).$$

According to Corollary 5 (case k=0 and $m = -3$, *if*

$$a + b + c = constant, \qquad abc = 1, \qquad 0 < a \le b \le c,$$

then

$$S_3 = \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}$$

is minimal for $0 < a = b \le c$. Thus, we only need to prove that

$$\frac{2}{a^3} + \frac{1}{c^3} + 15 \ge 6(2a + c)$$

for

$$a^2c = 1, \qquad a \le 1.$$

The inequality is equivalent to

$$\frac{2}{a^3} + a^6 + 15 \ge 6\left(2a + \frac{1}{a^2}\right),$$

$$a^9 - 12a^4 + 15a^3 - 6a + 2 \ge 0,$$

$$(1 - a)^2(2 - 2a - 6a^2 + 5a^3 + 4a^4 + 3a^5 + 2a^6 + a^7) \ge 0.$$

It suffices to show that

$$2 - 2a - 6a^2 + 5a^3 + 3a^4 \ge 0.$$

Indeed, we have

$$2(2 - 2a - 6a^2 + 5a^3 + 3a^4) = (2 - 3a)^2\left(1 + 2a + \frac{3}{4}a^2\right) + a^3\left(1 - \frac{3}{4}a\right) \ge 0.$$

The equality holds for $a = b = c = 1$.

□

P 5.61. *Let $a_1, a_2, \ldots, a_n$ be positive numbers so that $a_1 a_2 \cdots a_n = 1$. If $k \ge n-1$, then*

$$a_1^k + a_2^k + \cdots + a_n^k + (2k-n)n \ge (2k-n+1)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

(Vasile C., 2008)

Solution. For $n = 2$ and $k = 1$, the inequality is an identity. For $n = 2$ and $k > 1$, we need to show that $f(a) \ge 0$ for $a > 0$, where

$$f(a) = a^k + a^{-k} + 4(k-1) - (2k-1)(a + a^{-1}).$$

We have

$$f'(a) = k(a^{k-1} - a^{-k-1}) - (2k-1)(1 - a^{-2}),$$

$$f''(a) = k[(k-1)a^{k-2} + (k+1)a^{-k-2}] - 2(2k-1)a^{-3}.$$

By the weighted AM-GM inequality, we get

$$(k-1)a^{k-2} + (k+1)a^{-k-2} \ge 2ka^{\frac{(k-1)(k-2)+(k+1)(-k-2)}{2k}} = 2ka^{-3},$$

hence

$$f''(a) \ge 2k^2 a^{-3} - 2(2k-1)a^{-3} = 2(k-1)^2 a^{-3} > 0,$$

f' is strictly increasing. Since $f'(1) = 0$, it follows that $f'(a) < 0$ for $a < 1$ and $f'(a) > 0$ for $a > 1$, f is decreasing on $(0, 1]$ and increasing on $[1, \infty)$, hence $f(a) \ge f(1) = 0$.

Consider further that $n \ge 3$. Replacing $a_1, a_2, \ldots, a_n$ by $1/a_1, 1/a_2, \ldots, 1/a_n$, we need to show that $a_1 a_2 \cdots a_n = 1$ involves

$$\frac{1}{a_1^k} + \frac{1}{a_2^k} + \cdots + \frac{1}{a_n^k} + (2k-n)n \ge (2k-n+1)(a_1 + a_2 + \cdots + a_n).$$

According to Corollary 5, *if* $0 < a_1 \le a_2 \le \cdots \le a_n$ *and*

$$a_1 + a_2 + \cdots + a_n = constant, \qquad a_1 a_2 \cdots a_n = 1,$$

then

$$S_n = \frac{1}{a_1^k} + \frac{1}{a_2^k} + \cdots + \frac{1}{a_n^k}$$

is minimal for $0 < a_1 = \cdots = a_{n-1} \le a_n$. Thus, we only need to prove the original inequality for $a_1 = \cdots = a_{n-1} \ge 1$; that is, to show that $t \ge 1$ involves $f(t) \ge 0$, where

$$f(t) = (n-1)t^k + \frac{1}{t^{k(n-1)}} + (2k-n)n - (2k-n+1)\left(\frac{n-1}{t} + t^{n-1}\right).$$

We have

$$f'(t) = \frac{(n-1)g(t)}{t^{kn-k+1}}, \qquad g(t) = k(t^{kn}-1) - (2k-n+1)t^{kn-k-1}(t^n-1),$$

$$g'(t) = t^{kn-k-2}h(t), \qquad h(t) = k^2nt^{k+1} - (2k-n+1)[(k+1)(n-1)t^n - kn + k + 1],$$

$$h'(t) = (k+1)nt^{n-1}[k^2t^{k-n+1} - (2k-n+1)(n-1)].$$

If $k = n-1$, then $h(t) = n(n-1)(n-2) > 0$. If $k > n-1$, then

$$k^2t^{k-n+1} - (2k-n+1)(n-1) \ge k^2 - (2k-n+1)(n-1) = (k-n+1)^2 > 0,$$

$h'(t) > 0$ for $t \ge 1$, h is strictly increasing on $[1, \infty)$, hence

$$h(t) \ge h(1) = n[(k-1)^2 + n - 2] > 0.$$

From $h > 0$, we get $g' > 0$, g is strictly increasing, $g(t) \ge g(1) = 0$ for $t \ge 1$, $f'(t) > 0$ for $t > 1$, f is strictly increasing, $f(t) \ge f(1) = 0$ for $t \ge 1$.

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $n = 2$ and $k = 1$, then the equality holds for $a_1a_2 = 1$.

□

P 5.62. *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be nonnegative numbers so that $a_1+a_2+\cdots+a_n = n$, and let k be an integer satisfying $2 \le k \le n+2$. If*

$$r = \left(\frac{n}{n-1}\right)^{k-1} - 1,$$

then

$$a_1^k + a_2^k + \cdots + a_n^k - n \ge nr(1 - a_1a_2\cdots a_n).$$

(Vasile C., 2005)

Solution. According to Corollary 4, *if* $0 \le a_1 \le a_2 \le \cdots \le a_n$ *and*

$$a_1 + a_2 + \cdots + a_n = n, \qquad a_1^k + a_2^k + \cdots + a_n^k = constant,$$

then the product

$$P = a_1a_2\cdots a_n$$

is minimal for either $a_1 = 0$ or $0 < a_1 \le a_2 = \cdots = a_n$.

Case 1: $a_1 = 0$. We need to show that

$$a_2^k + \cdots + a_n^k \ge \frac{n^k}{(n-1)^{k-1}}$$

for $a_2 + \cdots + a_n = n$. This follows by Jensen's inequality

$$a_2^k + \cdots + a_n^k \ge (n-1)\left(\frac{a_2 + \cdots + a_n}{n-1}\right)^k.$$

Case 2: $0 < a_1 \le a_2 = \cdots = a_n$. Denoting $a_1 = x$ and $a_2 = y$ $(x \le y)$, we only need to show that

$$f(x) \ge 0,$$

where

$$f(x) = x^k + (n-1)y^k + nrxy^{n-1} - n(r+1), \qquad y = \frac{n-x}{n-1}, \qquad 0 < x \le 1 \le y.$$

It is easy to check that

$$f(0) = f(1) = 0.$$

Since

$$y' = \frac{-1}{n-1},$$

we have

$$\begin{aligned} f'(x) &= k(x^{k-1} - y^{k-1}) + nry^{n-2}(y-x) \\ &= (y-x)[nry^{n-2} - k(y^{k-2} + y^{k-3}x + \cdots + x^{k-2})] \\ &= (y-x)y^{n-2}[nr - kg(x)], \end{aligned}$$

where

$$g(x) = \frac{1}{y^{n-k}} + \frac{x}{y^{n-k+1}} + \cdots + \frac{x^{k-2}}{y^{n-2}}.$$

We see that $f'(x)$ has the same sign as

$$h(x) = nr - kg(x).$$

Since the function

$$y(x) = \frac{n-x}{n-1}$$

is strictly decreasing, g is strictly increasing for $2 \le k \le n$. Also, g is strictly increasing for $k = n+1$, when

$$\begin{aligned} g(x) &= y + x + \frac{x^2}{y} + \cdots + \frac{x^{n-1}}{y^{n-2}} \\ &= \frac{(n-2)x + n}{n-1} + \frac{x^2}{y} + \cdots + \frac{x^{n-1}}{y^{n-2}}, \end{aligned}$$

and for $k = n+2$, when

$$\begin{aligned} g(x) &= y^2 + yx + x^2 + \frac{x^3}{y} + \cdots + \frac{x^n}{y^{n-2}} \\ &= \frac{(n^2-3n+3)x^2 + n(n-3)x + n^2}{(n-1)^2} + \frac{x^3}{y} + \cdots + \frac{x^n}{y^{n-2}}. \end{aligned}$$

Therefore, the function $h(x)$ is strictly decreasing for $x \in [0,1]$. Since $f(0) = f(1) = 0$, there exists $x_1 \in (0,1)$ so that $f(x)$ is increasing on $[0,x_1]$ and decreasing on $[x_1,1]$. As a consequence, $f(x) \ge 0$ for $x \in [0,1]$.

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = 0, \qquad a_2 = \cdots = a_n = \frac{n}{n-1}$$

(or any cyclic permutation).

Remark. For the particular case $k = n$, the inequality has been posted in 2004 on Art of Problem Solving website by *Gabriel Dospinescu* and *Calin Popa*.

□

P 5.63. *If a,b,c are positive real numbers so that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3$, then*

$$4(a^2+b^2+c^2)+9 \ge 21abc.$$

(Vasile C., 2006)

Solution. Replacing a,b,c by their reverses $1/a,1/b,1/c$, we need to show that $a+b+c=3$ involves

$$4\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+9 \ge \frac{21}{abc}.$$

According to Corollary 5 (case k=0 and $m=-2$), *if*

$$a+b+c=3, \qquad abc = constant, \qquad 0 < a \le b \le c,$$

then

$$S_3 = \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

is minimal for $0 < a = b \le c$. Thus, we only need to prove that

$$4\left(\frac{2}{a^2}+\frac{1}{c^2}\right)+9 \ge \frac{21}{a^2c}$$

for $2a+b=3$. The inequality is equivalent to

$$(9a^2+8)c^2 - 21c + 4a^2 \ge 0,$$

$$4a^4 - 12a^3 + 13a^2 - 6a + 1 \ge 0,$$

$$(a-1)^2(2a-1)^2 \ge 0.$$

The equality holds for $a = b = c = 1$, and also for

$$a = b = 2, \qquad c = \frac{1}{2}$$

(or any cyclic permutation).

□

P 5.64. *If* $a_1, a_2, \ldots, a_n$ *are positive real numbers so that* $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = n$, *then*

$$a_1 + a_2 + \cdots + a_n - n \le e_{n-1}(a_1 a_2 \cdots a_n - 1),$$

where

$$e_{n-1} = \left(1 + \frac{1}{n-1}\right)^{n-1}.$$

(Gabriel Dospinescu and Calin Popa, 2004)

Solution. For $n = 2$, the inequality is an identity. For $n \ge 3$, replacing $a_1, a_2, \ldots, a_n$ by $1/a_1, 1/a_2, \ldots, 1/a_n$, we need to show that $a_1 + a_2 + \cdots + a_n = n$ involves

$$a_1 a_2 \cdots a_n \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n + e_{n-1}\right) \le e_{n-1}.$$

According to Corollary 5 (case $k = 0$ and $m = -1$), *if* $0 < a_1 \le a_2 \le \cdots \le a_n$ *and*

$$a_1 + a_2 + \cdots + a_n = n, \qquad a_1 a_2 \cdots a_n = constant,$$

then

$$S_n = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

is maximal for $0 < a_1 \le a_2 = \cdots = a_n$. Using the notation $a_1 = x$ and $a_2 = y$, we only need to show that $f(x) \le 0$ for

$$x + (n-1)y = n, \qquad 0 < x \le 1,$$

where

$$\begin{aligned} f(x) &= xy^{n-1}\left(\frac{1}{x} + \frac{n-1}{y} - n + e_{n-1}\right) - e_{n-1} \\ &= y^{n-1} + (n-1)xy^{n-2} - (n - e_{n-1})xy^{n-1} - e_{n-1}. \end{aligned}$$

Since

$$y' = \frac{-1}{n-1},$$

we get

$$\frac{f'(x)}{y^{n-3}} = (y - x)h(x),$$

where

$$h(x) = n - 2 - (n - e_{n-1})y = n - 2 - (n - e_{n-1})\frac{n - x}{n-1}$$

is a linear increasing function. Since

$$h(0) = \frac{n}{n-1}\left(e_{n-1} - 3 + \frac{2}{n}\right) < 0$$

and

$$h(1)=e_{n-1}-2>0,$$

there exists $x_1\in(0,1)$ so that $h(x_1)=0$, $h(x)<0$ for $x\in[0,x_1)$, and $h(x)>0$ for $x\in(x_1,1]$. Consequently, f is strictly decreasing on $[0,x_1]$ and strictly increasing on $[x_1,1]$. From

$$f(0)=f(1)=0,$$

it follows that $f(x)\le 0$ for $x\in[0,1]$.

The equality holds for $a_1=a_2=\cdots=a_n=1$. If $n=2$, then the equality holds for $a_1+a_2=2a_1a_2$.

□

P 5.65. *If $a_1,a_2,\ldots,a_n$ are positive real numbers, then*

$$\frac{a_1^n+a_2^n+\cdots+a_n^n}{a_1a_2\cdots a_n}+n(n-1)\ge(a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right).$$

(Vasile C., 2004)

Solution. For $n=2$, the inequality is an identity. For $n\ge 3$, according to Corollary 5 (case $k=0$ and $m\in\{-1,n\}$), *if $0<a_1\le a_2\le\cdots\le a_n$ and*

$$a_1+a_2+\cdots+a_n=constant,\qquad a_1a_2\cdots a_n=constant,$$

then the sum $\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}$ is maximal and the sum $a_1^n+a_2^n+\cdots+a_n^n$ is minimal for

$$0<a_1\le a_2=\cdots=a_n.$$

Consequently, we only need to prove the desired homogeneous inequality for $a_2=\cdots=a_n=1$, when it becomes

$$a_1^n+(n-2)a_1\ge(n-1)a_1^2.$$

Indeed, by the AM-GM inequality, we have

$$a_1^n+(n-2)a_1\ge(n-1)\sqrt[n-1]{a_1^n\cdot a_1^{n-2}}=(n-1)a_1^2.$$

For $n\ge 3$, the equality holds when $a_1=a_2=\cdots=a_n$.

□

P 5.66. *If $a_1,a_2,\ldots,a_n$ are nonnegative real numbers, then*

$$(n-1)(a_1^n+a_2^n+\cdots+a_n^n)+na_1a_2\cdots a_n\ge(a_1+a_2+\cdots+a_n)(a_1^{n-1}+a_2^{n-1}+\cdots+a_n^{n-1}).$$

(Janos Suranyi, MSC-Hungary)

Solution. For $n=2$, the inequality is an identity. For $n \ge 3$, according to Corollary 5 (case $k=n$ and $m=n-1$), *if* $0 \le a_1 \le a_2 \le \cdots \le a_n$ *and*

$$a_1+a_2+\cdots+a_n = constant, \qquad a_1^n+a_2^n+\cdots+a_n^n = constant,$$

then the sum $a_1^{n-1}+a_2^{n-1}+\cdots+a_n^{n-1}$ *is maximal and the product* $a_1a_2\cdots a_n$ *is minimal for either* $a_1=0$ *or* $0<a_1 \le a_2=\cdots=a_n$. Consequently, we only need to consider these cases.

Case 1: $a_1=0$. The inequality reduces to

$$(n-1)(a_2^n+\cdots+a_n^n) \ge (a_2+\cdots+a_n)(a_2^{n-1}+\cdots+a_n^{n-1}),$$

which follows immediately from Chebyshev's inequality.

Case 2: $0<a_1 \le a_2=\cdots=a_n$. Due to homogeneity, we may set $a_2=\cdots=a_n=1$, when the inequality becomes

$$(n-2)a_1^n+a_1 \ge (n-1)a_1^{n-1}.$$

Indeed, by the AM-GM inequality, we have

$$(n-2)a_1^n+a_1 \ge (n-1)\sqrt[n-1]{a_1^{n(n-2)}\cdot a_1} = (n-1)a_1^{n-1}.$$

For $n \ge 3$, the equality holds when $a_1=a_2=\cdots=a_n$, and also when

$$a_1=0, \qquad a_2=\cdots=a_n$$

(or any cyclic permutation).

□

P 5.67. *If* $a_1,a_2,\ldots,a_n$ *are nonnegative real numbers, then*

$$(n-1)(a_1^{n+1}+a_2^{n+1}+\cdots+a_n^{n+1}) \ge (a_1+a_2+\cdots+a_n)(a_1^n+a_2^n+\cdots+a_n^n-a_1a_2\cdots a_n).$$

(Vasile C., 2006)

Solution. For $n=2$, the inequality is an identity. For $n \ge 3$, according to Corollary 5 (case $k=n+1$ and $m=n$), *if* $0 \le a_1 \le a_2 \le \cdots \le a_n$ *and*

$$a_1+a_2+\cdots+a_n = constant, \qquad a_1^{n+1}+a_2^{n+1}+\cdots+a_n^{n+1} = constant,$$

then the sum $a_1^n+a_2^n+\cdots+a_n^n$ *is maximal and the product* $a_1a_2\cdots a_n$ *is minimal for either* $a_1=0$ *or* $0<a_1 \le a_2=\cdots=a_n$. Consequently, we only need to consider these cases.

Case 1: $a_1=0$. The inequality reduces to

$$(n-1)(a_2^{n+1}+\cdots+a_n^{n+1}) \ge (a_2+\cdots+a_n)(a_2^n+\cdots+a_n^n),$$

which follows immediately from Chebyshev's inequality.

Case 2: $0 < a_1 \le a_2 = \cdots = a_n$. Due to homogeneity, we may set $a_2 = \cdots = a_n = 1$, when the inequality becomes

$$(n-2)a_1^{n+1} + a_1^2 \ge (n-1)a_1^n.$$

Indeed, by the AM-GM inequality, we have

$$(n-2)a_1^{n+1} + a_1^2 \ge (n-1)\sqrt[n-1]{a_1^{(n+1)(n-2)} \cdot a_1^2} = (n-1)a_1^n.$$

For $n \ge 3$, the equality holds when $a_1 = a_2 = \cdots = a_n$, and also when

$$a_1 = 0, \qquad a_2 = \cdots = a_n$$

(or any cyclic permutation).

□

P 5.68. *If* $a_1, a_2, \ldots, a_n$ *are positive real numbers, then*

$$(a_1 + a_2 + \cdots + a_n - n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n\right) + a_1a_2\cdots a_n + \frac{1}{a_1a_2\cdots a_n} \ge 2.$$

(Vasile C., 2006)

Solution. For $n = 2$, the inequality reduces to

$$(1-a_1)^2(1-a_2)^2 \ge 0.$$

Consider further that $n \ge 3$. Since the inequality remains unchanged by replacing each a_i with $1/a_i$, we may consider $a_1a_2\cdots a_n \ge 1$. By the AM-GM inequality, we get

$$a_1 + a_2 + \cdots + a_n \ge n\sqrt[n]{a_1a_2\cdots a_n} \ge n.$$

According to Corollary 5 (case $k = 0$ and $m = -1$), *if* $0 < a_1 \le a_2 \le \cdots \le a_n$ *and*

$$a_1 + a_2 + \cdots + a_n = constant, \qquad a_1a_2\cdots a_n = constant,$$

then the sum

$$S_n = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

is minimal for $0 < a_1 = a_2 = \cdots = a_{n-1} \le a_n$. Consequently, we only need to consider

$$a_1 = a_2 = \cdots = a_{n-1} = x, \qquad a_n = y, \qquad x \le y.$$

The inequality becomes

$$[(n-1)x + y - n]\left(\frac{n-1}{x} + \frac{1}{y} - n\right) + x^{n-1}y + \frac{1}{x^{n-1}y} \ge 2,$$

$$\left(x^{n-1}+\frac{n-1}{x}-n\right)y+\left[\frac{1}{x^{n-1}}+(n-1)x-n\right]\frac{1}{y}\geq\frac{n(n-1)(x-1)^2}{x}.$$

Since

$$\begin{aligned}x^{n-1}+\frac{n-1}{x}-n&=\frac{x-1}{x}\left[(x^{n-1}-1)+(x^{n-2}-1)+\cdots+(x-1)\right]\\&=\frac{(x-1)^2}{x}\left[x^{n-2}+2x^{n-3}+\cdots+(n-1)\right],\end{aligned}$$

and

$$\frac{1}{x^{n-1}}+(n-1)x-n=\frac{(x-1)^2}{x}\left[\frac{1}{x^{n-2}}+\frac{2}{x^{n-3}}+\cdots+(n-1)\right],$$

it is enough to prove the inequality

$$\left[x^{n-2}+2x^{n-3}+\cdots+(n-1)\right]y+\left[\frac{1}{x^{n-2}}+\frac{2}{x^{n-3}}+\cdots+(n-1)\right]\frac{1}{y}\geq n(n-1),$$

which is equivalent to

$$\left(x^{n-2}y+\frac{1}{x^{n-2}y}-2\right)+2\left(x^{n-3}y+\frac{1}{x^{n-3}y}-2\right)+\cdots+(n-1)\left(y+\frac{1}{y}-2\right)\geq 0,$$

$$\frac{(x^{n-2}y-1)^2}{x^{n-2}y}+\frac{2(x^{n-3}y-1)^2}{x^{n-3}y}+\cdots+\frac{(n-1)(y-1)^2}{y}\geq 0.$$

The equality holds if $n-1$ of the numbers a_i are equal to 1.

□

P 5.69. *If* $a_1,a_2,\ldots,a_n$ *are positive real numbers so that* $a_1a_2\cdots a_n=1$, *then*

$$\left|\frac{1}{\sqrt{a_1+a_2+\cdots+a_n-n}}-\frac{1}{\sqrt{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}-n}}\right|<1.$$

(Vasile C., 2006)

Solution. Let

$$A=a_1+a_2+\cdots+a_n-n,\qquad B=\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}-n.$$

By the AM-GM inequality, it follows that $A>0$ and $B>0$. According to the preceding P 5.68, the following inequality holds

$$(a_1+\cdots+a_{n+1}-n-1)\left(\frac{1}{a_1}+\cdots+\frac{1}{a_{n+1}}-n-1\right)+a_1\cdots a_{n+1}+\frac{1}{a_1\cdots a_{n+1}}\geq 2,$$

which is equivalent to

$$(A-1+a_{n+1})\left(B-1+\frac{1}{a_{n+1}}\right)+a_{n+1}+\frac{1}{a_{n+1}}\ge 2,$$

$$\frac{A}{a_{n+1}}+Ba_{n+1}+AB-A-B\ge 0.$$

Choosing

$$a_{n+1}=\sqrt{\frac{A}{B}},$$

we get

$$2\sqrt{AB}+AB-A-B\ge 0,$$

$$AB\ge\left(\sqrt{A}-\sqrt{B}\,\right)^2,$$

$$1\ge\left|\frac{1}{\sqrt{A}}-\frac{1}{\sqrt{B}}\right|.$$

□

P 5.70. *If* $a_1,a_2,\dots,a_n$ *are positive real numbers so that* $a_1a_2\cdots a_n=1$*, then*

$$a_1^{n-1}+a_2^{n-1}+\cdots+a_n^{n-1}+\frac{n^2(n-2)}{a_1+a_2+\cdots+a_n}\ge(n-1)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right).$$

Solution. For $n=2$, the inequality is an identity. Consider further that $n\ge 3$. According to Corollary 5 (case $k=0$), *if* $0<a_1\le a_2\le\cdots\le a_n$ *and*

$$a_1+a_2+\cdots+a_n=constant,\qquad a_1a_2\cdots a_n=1,$$

then the sum $a_1^{n-1}+a_2^{n-1}+\cdots+a_n^{n-1}$ *is minimal and the sum* $\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}$ *is maximal for* $0<a_1\le a_2=\cdots=a_n$. Thus, we only need to prove the homogeneous inequality

$$a_1^{n-1}+a_2^{n-1}+\cdots+a_n^{n-1}+\frac{n^2(n-2)a_1a_2\cdots a_n}{a_1+a_2+\cdots+a_n}\ge(n-1)a_1a_2\cdots a_n\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)$$

for $a_2=\cdots=a_n=1$; that is, to show that $f(x)\ge 0$ for $x\in[0,1]$, where

$$f(x)=x^{n-2}+\frac{n^2(n-2)}{x+n-1}-(n-1)^2,$$

$$\frac{f'(x)}{n-2}=x^{n-3}-\frac{n^2}{(x+n-1)^2}.$$

Since f' is increasing, we have $f'(x)\le f'(1)=0$ for $x\in[0,1]$, f is decreasing on $[0,1]$, hence $f(x)\ge f(1)=0$.

The equality holds for $a_1=a_2=\cdots=a_n=1$. If $n=2$, then the equality holds for $a_1a_2=1$.

□

P 5.71. *If a, b, c are nonnegative real numbers, then*

$$(a+b+c-3)^2 \geq \frac{abc-1}{abc+1}(a^2+b^2+c^2-3).$$

(Vasile C., 2006)

Solution. For $a=0$, the inequality reduces to

$$b^2+c^2+bc+3 \geq 3(b+c),$$

which is equivalent to

$$(b-c)^2+3(b+c-2)^2 \geq 0.$$

For $abc>0$, according to Corollary 5 (case $k=0$ and $m=2$), *if*

$$a+b+c = constant, \qquad abc = constant,$$

then

$$S_3 = a^2+b^2+c^2$$

is minimal and maximal when two of a, b, c are equal. Thus, we only need to prove the desired inequality for $a=b$; that is,

$$(2a+c-3)^2 \geq \frac{a^2c-1}{a^2c+1}(2a^2+c^2-3),$$

which is equivalent to

$$(a-1)^2[ca^2+2c(c-2)a+c^2-3c+3] \geq 0.$$

For $c \geq 2$, the inequality is clearly true. It is also true for $c \leq 2$, because

$$ca^2+2c(c-2)a+c^2-3c+3 = c(a+c-2)^2+(1-c)^2(3-c) \geq 0.$$

The equality holds if two of a, b, c are equal to 1.

□

P 5.72. *If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1+a_2+\cdots+a_n=n$, then*

$$(a_1a_2\cdots a_n)^{\frac{1}{\sqrt{n-1}}}(a_1^2+a_2^2+\cdots+a_n^2) \leq n.$$

(Vasile C., 2006)

Solution. For $n=2$, the inequality is equivalent to

$$(a_1a_2-1)^2 \ge 0.$$

For $n \ge 3$, according to Corollary 5 (case $k=0$, $m=2$), *if* $0 < a_1 \le a_2 \le \cdots \le a_n$ *and*

$$a_1+a_2+\cdots+a_n = n, \qquad a_1a_2\cdots a_n = constant,$$

then the sum

$$S_n = a_1^2+a_2^2+\cdots+a_n^2$$

is maximal for $a_1=a_2=\cdots=a_{n-1}$. Therefore, we only need to prove the homogeneous inequality

$$(a_1a_2\cdots a_n)^{\frac{1}{\sqrt{n-1}}}\cdot\frac{a_1^2+a_2^2+\cdots+a_n^2}{n} \le \left(\frac{a_1+a_2+\cdots+a_n}{n}\right)^{2+\frac{n}{\sqrt{n-1}}}$$

for $a_1=a_2=\cdots=a_{n-1}=1$. The inequality is equivalent to $f(x)\ge 0$ for $x\ge 1$, where

$$f(x)=\left(2+\frac{n}{\sqrt{n-1}}\right)\ln\frac{x+n-1}{n}-\frac{\ln x}{\sqrt{n-1}}-\ln\frac{x^2+n-1}{n}.$$

Let

$$p=\frac{1}{\sqrt{n-1}}.$$

Since

$$\begin{aligned} f'(x) &= \frac{2+np}{x+n-1}-\frac{p}{x}-\frac{2x}{x^2+n-1} \\ &= \frac{(n-1)(x-1)}{x+n-1}\left(\frac{p}{x}-\frac{2}{x^2+n-1}\right) \\ &= \frac{p(n-1)(x-1)(x-\sqrt{n-1})^2}{x(x+n-1)(x^2+n-1)} \ge 0, \end{aligned}$$

$f(x)$ is increasing for $x\ge 1$, hence

$$f(x)\ge f(1)=0.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$.

Remark. For $n=5$, from the homogeneous inequality above, we get the following nice results:

- *If a,b,c,d,e are positive real numbers so that*

$$a^2+b^2+c^2+d^2+e^2=5,$$

then

(a) $$abcde(a^4+b^4+c^4+d^4+e^4)\le 5;$$

(b) $$a+b+c+d+e\ge 5\sqrt[9]{abcde}.$$

□

P 5.73. *If $a_1,a_2,\ldots,a_n$ are positive real numbers such that $a_1+a_2+\cdots+a_n=n-1$, then*

$$\sqrt[n]{\frac{n-1}{a_1a_2\cdots a_n}}\ \ge\ 4\sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n(n-1)}}.$$

(Vasile Cîrtoaje and KaiRain, 2020)

Solution. For $n=2$, we need to show that $a_1+a_2=1$ involves

$$\frac{1}{a_1a_2}\ge 8(a_1^2+a_2)^2,$$

which is equivalent to

$$(4a_1a_2-1)^2\ge 0.$$

For $n\ge 3$, write the inequality in the homogeneous form

$$\left(\frac{a_1+a_2+\cdots+a_n}{n-1}\right)^2\sqrt[n]{\frac{n-1}{a_1a_2\cdots a_n}}\ \ge 4\ \sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n(n-1)}}.$$

According to Corollary 4, for $a_1+a_2+\cdots+a_n=constant$ and $a_1^2+a_2^2+\cdots+a_n^2=constant$, the product $a_1a_2\cdots a_n$ is maximal for $a_1=a_2=\cdots=a_{n-1}\le a_n$. Due to homogeneity, we may set $a_1=a_2=\cdots=a_{n-1}=1$, when the inequality becomes

$$\frac{A(x+n-1)^2}{\sqrt[n]{x}}\ge\sqrt{x^2+n-1},$$

where

$$A=\frac{\sqrt{n}}{4(n-1)^{(3n-2)/(2n)}}\,,\qquad x\ge 1.$$

The inequality is true if $f(x)\ge 0$, where

$$f(x)=\ln A+2\ln(x+n-1)-\frac{1}{n}\ln x-\frac{1}{2}\ln(x^2+n-1).$$

From

$$\begin{aligned}f'(x)&=\frac{2}{x+n-1}-\frac{1}{nx}-\frac{x}{x^2+n-1}\\&=\frac{(n-1)\left[x^3-(n+1)x^2+(2n-1)x-n+1\right]}{nx(x+n-1)(x^2+n-1)}\\&=\frac{(n-1)(x-1)^2(x-n+1)}{nx(x+n-1)(x^2+n-1)},\end{aligned}$$

it follows that f is decreasing on $[1,n-1]$ and increasing on $[n-1,\infty)$, therefore

$$f(x)\ge f(n-1)=0.$$

The equality occurs for $a_1=a_2=\cdots=a_{n-1}=\dfrac{1}{2}$ and $a_n=\dfrac{n-1}{2}$ (or any cyclic permutation).

□

P 5.74. *If* $a_1, a_2, \ldots, a_n$ *are positive real numbers so that* $a_1^3 + a_2^3 + \cdots + a_n^3 = n$, *then*

$$a_1 + a_2 + \cdots + a_n \ge n \sqrt[n+1]{a_1 a_2 \cdots a_n}.$$

(Vasile C., 2007)

Solution. For $n = 2$, we need to show that $a_1^3 + a_2^3 = 2$ involves $(a_1 + a_2)^3 \ge 8a_1a_2$. Let

$$x = a_1 + a_2.$$

From

$$2 = a_1^3 + a_2^3 = x^3 - 3a_1a_2x,$$

we get

$$a_1a_2 = \frac{x^3 - 2}{3x}.$$

Thus,

$$(a_1 + a_2)^3 - 8a_1a_2 = x^3 - \frac{8(x^3 - 2)}{3x} = \frac{(x-2)^2(3x^2 + 4x + 4)}{3x} \ge 0.$$

For $n \ge 3$, according to Corollary 4, *if* $0 < a_1 \le a_2 \le \cdots \le a_n$ *and*

$$a_1 + a_2 + \cdots + a_n = constant, \qquad a_1^3 + a_2^3 + \cdots + a_n^3 = n,$$

then the product

$$P = a_1 a_2 \cdots a_n$$

is maximal for $a_1 = a_2 = \cdots = a_{n-1}$. Therefore, we only need to prove the homogeneous inequality

$$\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^{n+1} \ge a_1 a_2 \cdots a_n \sqrt[3]{\frac{a_1^3 + a_2^3 + \cdots + a_n^3}{n}}$$

for $a_1 = a_2 = \cdots = a_{n-1} = 1$. The inequality is equivalent to $f(x) \ge 0$ for $x \ge 1$, where

$$f(x) = (n+1)\ln\frac{x + n - 1}{n} - \ln x - \frac{1}{3}\ln\frac{x^3 + n - 1}{n}.$$

Since

$$\begin{aligned}
f'(x) &= \frac{n+1}{x+n-1} - \frac{1}{x} - \frac{x^2}{x^3 + n - 1} \\
&= \frac{(n-1)(x-1)(x^3 - x^2 - x + n - 1)}{x(x+n-1)(x^3 + n - 1)} \\
&\ge \frac{(n-1)(x-1)(x^3 - x^2 - x + 1)}{x(x+n-1)(x^3 + n - 1)} \\
&= \frac{(n-1)(x-1)^3(x+1)}{x(x+n-1)(x^3 + n - 1)},
\end{aligned}$$

$f(x)$ is increasing for $x \ge 1$, hence

$$f(x) \ge f(1) = 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 5.75. *Let a, b, c be nonnegative real numbers so that $ab + bc + ca = 3$. If*

$$k \ge 2 - \frac{\ln 4}{\ln 3} \approx 0.738,$$

then

$$a^k + b^k + c^k \ge 3.$$

(Vasile C., 2004)

Solution. Let

$$r = 2 - \frac{\ln 4}{\ln 3}.$$

By the power mean inequality, we have

$$\frac{a^k + b^k + c^k}{3} \ge \left(\frac{a^r + b^r + c^r}{3}\right)^{k/r}.$$

Thus, it suffices to show that

$$a^r + b^r + c^r \ge 3.$$

Since

$$2(ab + bc + ca) = (a + b + c)^2 - (a^2 + b^2 + c^2),$$

according to Corollary 5 (case $k = 2$, $m = r$), *if $a \le b \le c$ and*

$$a + b + c = constant, \qquad a^2 + b^2 + c^2 = constant,$$

then

$$S_3 = a^r + b^r + c^r$$

is minimal for either $a = 0$ or $0 < a \le b = c$.

Case 1: $a = 0$. We need to show that $bc = 3$ implies $b^r + c^r \ge 3$. Indeed, by the AM-GM inequality, we have

$$b^r + c^r \ge 2\sqrt{(bc)^r} = 2 \cdot 3^{r/2} = 3.$$

Case 2: $0 < a \le b = c$. We only need to show that the homogeneous inequality

$$a^r + b^r + c^r \ge 3\left(\frac{ab + bc + ca}{3}\right)^{r/2}$$

holds for $b = c = 1$; that is, to show that $a \in (0,1]$ involves

$$a^r + 2 \ge 3\left(\frac{2a+1}{3}\right)^{r/2},$$

which is equivalent to $f(a) \ge 0$, where

$$f(a) = \ln\frac{a^r+2}{3} - \frac{r}{2}\ln\frac{2a+1}{3}.$$

The derivative

$$f'(a) = \frac{ra^{r-1}}{a^r+2} - \frac{r}{2a+1} = \frac{rg(a)}{a^{1-r}(a^r+2)(2a+1)},$$

where

$$g(a) = a - 2a^{1-r} + 1.$$

From

$$g'(a) = 1 - \frac{2(1-r)}{a^r},$$

it follows that $g'(a) < 0$ for $a \in (0, a_1)$, and $g'(a) > 0$ for $a \in (a_1, 1]$, where

$$a_1 = (2-2r)^{1/r} \approx 0.416.$$

Then, g is strictly decreasing on $[0, a_1]$ and strictly increasing on $[a_1, 1]$. Since $g(0) = 1$ and $g(1) = 0$, there exists $a_2 \in (0,1)$ so that $g(a_2) = 0$, $g(a) > 0$ for $a \in [0, a_2)$, and $g(a) < 0$ for $a \in (a_2, 1]$. Consequently, f is increasing on $[0, a_2]$ and decreasing on $[a_2, 1]$. Since $f(0) = f(1) = 0$, we have $f(a) \ge 0$ for $0 < a \le 1$.

The equality holds for $a = b = c = 1$. If $k = 2 - \dfrac{\ln 4}{\ln 3}$, then the equality holds also for

$$a = 0, \qquad b = c = \sqrt{3}$$

(or any cyclic permutation).

Remark. For $k = 3/4$, we get the following nice results (see P 3.33 in Volume 1):

- *Let a, b, c be positive real numbers.*

  (a) *If* $a^4b^4 + b^4c^4 + c^4a^4 = 3$, *then*

$$a^3 + b^3 + c^3 \ge 3.$$

  (b) *If* $a^3 + b^3 + c^3 = 3$, *then*

$$a^4b^4 + b^4c^4 + c^4a^4 \le 3.$$

□

P 5.76. *Let a,b,c be nonnegative real numbers so that $a+b+c=3$. If*

$$k \ge \frac{\ln 9-\ln 8}{\ln 3-\ln 2} \approx 0.29,$$

then

$$a^k+b^k+c^k \ge ab+bc+ca.$$

(Vasile C., 2005)

Solution. For $k \ge 1$, by Jensen's inequality, we get

$$a^k+b^k+c^k \ge 3\left(\frac{a+b+c}{3}\right)^k = 3 = \frac{1}{3}(a+b+c)^2 \ge ab+bc+ca.$$

Let

$$r = \frac{\ln 9-\ln 8}{\ln 3-\ln 2}.$$

Assume further that

$$r \le k < 1,$$

and write the inequality as

$$2(a^k+b^k+c^k)+a^2+b^2+c^2 \ge 9.$$

By Corollary 5, *if $a \le b \le c$ and*

$$a+b+c=3, \qquad a^2+b^2+c^2 = constant,$$

then the sum

$$S_3 = a^k+b^k+c^k$$

is minimal for either $a=0$ or $0<a\le b=c$. Thus, we only need to prove the desired inequality for these cases.

Case 1: $a=0$. We need to show that $b+c=3$ involves $b^k+c^k \ge bc$. Indeed, by the AM-GM inequality, we have

$$\begin{aligned} b^k+c^k-bc &\ge 2(bc)^{k/2}-bc = (bc)^{k/2}\left[2-(bc)^{1-k/2}\right] \\ &\ge (bc)^{k/2}\left[2-\left(\frac{b+c}{2}\right)^{2-k}\right] = (bc)^{k/2}\left[2-\left(\frac{3}{2}\right)^{2-k}\right] \\ &\ge (bc)^{k/2}\left[2-\left(\frac{3}{2}\right)^{2-r}\right] = 0. \end{aligned}$$

Case 2: $0<a\le b=c$. We only need to show that the homogeneous inequality

$$(a^k+b^k+c^k)\left(\frac{a+b+c}{3}\right)^{2-k} \ge ab+bc+ca$$

holds for $b=c=1$; that is, to show that $a\in(0,1]$ involves

$$(a^k+2)\left(\frac{a+2}{3}\right)^{2-k}\ge 2a+1,$$

which is equivalent to $f(a)\ge 0$, where

$$f(a)=\ln(a^k+2)+(2-k)\ln\frac{a+2}{3}-\ln(2a+1).$$

We have

$$f'(a)=\frac{ka^{k-1}}{a^k+2}+\frac{2-k}{a+2}-\frac{2}{2a+1}=\frac{2g(a)}{a^{1-k}(a^k+2)(2a+1)},$$

where

$$g(a)=a^2+(2k-1)a+k+2(1-k)a^{2-k}-(k+2)a^{1-k},$$

with

$$g'(a)=2a+2k-1+2(1-k)(2-k)a^{1-k}-(k+2)(1-k)a^{-k},$$

$$g''(a)=2+2(1-k)^2(2-k)a^{-k}+k(k+2)(1-k)a^{-k-1}.$$

Since $g''>0$, g' is strictly increasing. From $g'(0_+)=-\infty$ and $g'(1)=3(1-k)+3k^2>0$, it follows that there exists $a_1\in(0,1)$ so that $g'(a_1)=0$, $g'(a)<0$ for $a\in(0,a_1)$ and $g'(a)>0$ for $a\in(a_1,1]$. Therefore, g is strictly decreasing on $[0,a_1]$ and strictly increasing on $[a_1,1]$. Since $g(0)=k>0$ and $g(1)=0$, there exists $a_2\in(0,a_1)$ so that $g(a_2)=0$, $g(a)>0$ for $a\in[0,a_2)$ and $g(a)<0$ for $a\in(a_2,1]$. Consequently, f is increasing on $[0,a_2]$ and decreasing on $[a_2,1]$. Since

$$f(0)=\ln 2+(3-k)\ln\frac{2}{3}\ge \ln 2+(3-r)\ln\frac{2}{3}=0$$

and $f(1)=0$, we get $f(a)\ge 0$ for $0\le a\le 1$.

The equality holds for $a=b=c=1$. If $k=\dfrac{\ln 9-\ln 8}{\ln 3-\ln 2}$, then the equality holds also for

$$a=0,\qquad b=c=\frac{3}{2}$$

(or any cyclic permutation).

□

P 5.77. *If $a_1,a_2,\dots,a_n$ $(n\ge 4)$ are nonnegative numbers so that $a_1+a_2+\cdots+a_n=n$, then*

$$\frac{1}{n+1-a_2a_3\cdots a_n}+\frac{1}{n+1-a_3a_4\cdots a_1}+\cdots+\frac{1}{n+1-a_1a_2\cdots a_{n-1}}\le 1.$$

(Vasile C., 2004)

Solution. Let $a_1 \le a_2 \le \cdots \le a_n$ and

$$e_{n-1} = \left(1 + \frac{1}{n-1}\right)^{n-1}.$$

By the AM-GM inequality, we have

$$a_2 a_3 \cdots a_n \le \left(\frac{a_2 + a_3 + \cdots + a_n}{n-1}\right)^{n-1} \le \left(\frac{a_1 + a_2 + \cdots + a_n}{n-1}\right)^{n-1} = e_{n-1},$$

hence

$$n + 1 - a_2 a_3 \cdots a_n \ge n + 1 - e_{n-1} = (n-2) + (3 - e_{n-1}) > 0.$$

Consider the cases $a_1 = 0$ and $a_1 > 0$.

Case 1: $a_1 = 0$. We need to show that $a_2 + a_3 + \cdots + a_n = n$ involves

$$\frac{1}{n + 1 - a_2 a_3 \cdots a_n} + \frac{n-1}{n+1} \le 1,$$

which is equivalent to

$$a_2 a_3 \cdots a_n \le \frac{n+1}{2}.$$

Since

$$a_2 a_3 \cdots a_n \le \left(\frac{a_2 + a_3 + \cdots + a_n}{n-1}\right)^{n-1} = e_{n-1},$$

it suffices to show that

$$e_{n-1} \le \frac{n+1}{2}.$$

For $n = 4$, we have

$$\frac{n+1}{2} - e_{n-1} = \frac{7}{54} > 0.$$

For $n \ge 5$, we get

$$\frac{n+1}{2} \ge 3 > e_{n-1}.$$

Case 2: $0 < a_1 \le a_2 \le \cdots \le a_n$. Denote

$$a_1 a_2 \cdots a_n = (n+1)r, \qquad r > 0.$$

From $a_2 a_3 \cdots a_n \le e_{n-1}$, we get

$$a_1 \ge a, \qquad a = \frac{(n+1)r}{e_{n-1}} > r.$$

Write the inequality as follows

$$\frac{a_1}{a_1 - r} + \frac{a_2}{a_2 - r} + \cdots + \frac{a_n}{a_n - r} \le n + 1,$$

$$\frac{1}{a_1-r}+\frac{1}{a_2-r}+\cdots+\frac{1}{a_n-r}\le\frac{1}{r},$$

$$f(a_1)+f(a_2)+\cdots+f(a_n)+\frac{1}{r}\ge 0,$$

where

$$f(u)=\frac{-1}{u-r},\quad u\ge a.$$

We will apply Corollary 3 to the function f. We have

$$f'(u)=\frac{1}{(u-r)^2},$$

$$g(x)=f'\left(\frac{1}{x}\right)=\frac{x^2}{(1-rx)^2},\qquad g''(x)=\frac{4rx+2}{(1-rx)^4}>0.$$

According to Corollary 3, *if* $a\le a_1\le a_2\le\cdots\le a_n$ *and*

$$a_1+a_2+\cdots+a_n=n,\qquad a_1a_2\cdots a_n=(n+1)r=constant,$$

then the sum $S_3=f(a_1)+f(a_2)+\cdots+f(a_n)$ *is minimal for* $a\le a_1\le a_2=\cdots=a_n$. Thus, we only need to prove the homogeneous inequality

$$\frac{1}{n+1-\frac{a_2a_3\cdots a_n}{s^{n-1}}}+\frac{1}{n+1-\frac{a_3a_4\cdots a_1}{s^{n-1}}}+\cdots+\frac{1}{n+1-\frac{a_1a_2\cdots a_{n-1}}{s^{n-1}}}\le 1$$

for $0<a_1\le a_2=a_3=\cdots=a_n=1$, where

$$s=\frac{a_1+a_2+\cdots+a_n}{n};$$

that is,

$$\frac{s^{n-1}}{(n+1)s^{n-1}-1}+\frac{(n-1)s^{n-1}}{(n+1)s^{n-1}-a_1}\le 1,\qquad s=\frac{a_1+n-1}{n},$$

which is equivalent to

$$f(s)\ge 0,\qquad s_1<s\le 1,$$

where

$$s_1=\frac{n-1}{n}$$

and

$$f(s)=(n+1)s^{2n-2}-n^2s^n+(n+1)(n-2)s^{n-1}+ns-n+1.$$

We have

$$f'(s)=2(n^2-1)s^{2n-3}-n^3s^{n-1}+(n^2-1)(n-2)s^{n-2}+n,$$

$$f''(s)=(n-1)s^{n-3}g(s),$$

where

$$g(s)=2(2n-3)(n+1)s^{n-1}-n^3s+(n-2)^2(n+1),$$

$$g'(s) = 2(2n-3)(n^2-1)s^{n-2} - n^3.$$

Since

$$\begin{aligned} g'(s) \geq g'(s_1) &= \frac{2n(2n-3)(n+1)}{e_{n-1}} - n^3 \\ &> \frac{2n(2n-3)(n+1)}{3} - n^3 = \frac{n(n^2-2n-6)}{3} > 0, \end{aligned}$$

g is increasing. There are two cases to consider: $g(s_1) \geq 0$ and $g(s_1) < 0$.

Subcase A: $g(s_1) \geq 0$. Then, $g(s) \geq 0$, $f''(s) \geq 0$, f' is increasing. Since $f'(1) = 0$, it follows that $f'(s) \leq 0$ for $s \in [s_1, 1]$, f is decreasing, hence $f(s) \geq f(1) = 0$.

Subcase B: $g(s_1) < 0$. Then, since $g(1) = n^2-2n+4 > 0$, there exists $s_2 \in (s_1, 1)$ so that $g(s_2) = 0$, $g(s) < 0$ for $s \in [s_1, s_2)$ and $g(s) > 0$ for $s \in (s_2, 1]$, f' is decreasing on $[s_1, s_2]$ and increasing on $[s_2, 1]$. We see that $f'(1) = 0$. If $f'(s_1) \leq 0$, then $f'(s) \leq 0$ for $s \in [s_1, 1]$, f is decreasing, hence $f(s) \geq f(1) = 0$. If $f'(s_1) > 0$, then there exists $s_3 \in (s_1, s_2)$ so that $f'(s_3) = 0$, $f'(s) > 0$ for $s \in [s_1, s_3)$ and $g(s) < 0$ for $s \in (s_3, 1]$, hence f is increasing on $[s_1, s_3]$ and decreasing on $[s_3, 1]$. Since $f(1) = 0$, it suffices to show that $f(s_1) \geq 0$. This is true since $s = s_1$ involves $a_1 = 0$, and we have shown that the desired inequality holds for $a_1 = 0$.

The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 5.78. *If a, b, c are nonnegative real numbers so that*

$$a+b+c \geq 2, \qquad ab+bc+ca \geq 1,$$

then

$$\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} \geq 2.$$

(Vasile C., 2005)

Solution. According to Corollary 5 (case $k = 2$ and $m = 1/3$), if $0 \leq a \leq b \leq c$ and

$$a+b+c = constant, \qquad ab+bc+ca = constant,$$

then the sum $S_3 = \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}$ is minimal for either $a = 0$ or $0 < a \leq b = c$.

Case 1: $a = 0$. The hypothesis $ab + bc + ca \geq 1$ implies $bc \geq 1$; consequently,

$$\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} = \sqrt[3]{b} + \sqrt[3]{c} \geq 2\sqrt[6]{bc} \geq 2.$$

Case 2: $0 < a \leq b = c$. If $c \geq 1$, then

$$\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} \geq 2\sqrt[3]{c} \geq 2.$$

If $c < 1$, then

$$\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} \geq a + b + c \geq 2.$$

The equality holds for

$$a = 0, \qquad b = c = 1$$

(or any cyclic permutation).

□

P 5.79. *If a, b, c, d are positive real numbers so that $abcd = 1$, then*

$$(a + b + c + d)^4 \geq 36\sqrt{3}\,(a^2 + b^2 + c^2 + d^2).$$

(Vasile C., 2008)

Solution. According to Corollary 5 (case $k = 0$ and $m = 2$), *if $a \leq b \leq c \leq d$ and*

$$a + b + c + d = constant, \qquad abcd = 1,$$

then the sum

$$S_4 = a^2 + b^2 + c^2 + d^2$$

is maximal for $a = b = c \leq d$. Thus, we only need to show that

$$(3a + d)^4 \geq 36\sqrt{3}\,(3a^2 + d^2)$$

for $a^3 d = 1$. Write this inequality as $f(a) \geq 0$, where

$$f(a) = 4\ln\left(3a + \frac{1}{a^3}\right) - \ln\left(3a^2 + \frac{1}{a^6}\right) - \ln 36\sqrt{3}, \qquad 0 < a \leq 1.$$

Since

$$f'(a) = \frac{12(a^4 - 1)}{a(3a^4 + 1)} - \frac{6(a^8 - 1)}{a(3a^8 + 1)} = \frac{6(a^4 - 1)^2(3a^4 - 1)}{a(3a^4 + 1)(3a^8 + 1)},$$

f is decreasing on $[0, 1/\sqrt[4]{3}]$ and increasing on $[1/\sqrt[4]{3}, 1]$; therefore,

$$f(a) \geq f\left(\frac{1}{\sqrt[4]{3}}\right) = 0.$$

The equality holds for

$$a = b = c = \frac{1}{\sqrt[4]{3}}, \qquad d = \sqrt[4]{27}$$

(or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

- *If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1a_2\cdots a_n = 1$, then*

$$(a_1+a_2+\cdots+a_n)^4 \geq \frac{16}{n}\sqrt[n]{(n-1)^{3n-2}}\,(a_1^2+a_2^2+\cdots+a_n^2),$$

with equality for

$$a_1 = a_2 = \cdots = a_{n-1} = \frac{1}{\sqrt[n]{n-1}}, \qquad a_n = \sqrt[n]{(n-1)^{n-1}}$$

(or any cyclic permutation).

□

P 5.80. *If a,b,c are nonnegative real numbers so that $ab+bc+ca=1$, then*

$$\sqrt{33a^2+16}+\sqrt{33b^2+16}+\sqrt{33c^2+16} \leq 9(a+b+c).$$

(Vasile C., 2006)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+297(a+b+c) \geq 0,$$

where

$$f(u) = -\frac{1}{33}\sqrt{33u^2+16}, \qquad u \geq 0.$$

We have

$$g(x) = f'(x) = \frac{-x}{\sqrt{33x^2+16}},$$

$$g''(x) = \frac{33\cdot 48x}{(33x^2+16)^{5/2}}.$$

Since $g''(x) > 0$ for $x > 0$, g is strictly convex on $[0,\infty)$. According to Corollary 1, if $0 \leq a \leq b \leq c$ and

$$a+b+c = constant, \qquad a^2+b^2+c^2 = constant,$$

then the sum

$$S_n = f(a)+f(b)+f(c)$$

is minimal for either $a=0$ or $0 < a \leq b = c$.

Case 1: $a=0$. We need to show that $bc=1$ involves

$$\sqrt{33b^2+16}+\sqrt{33c^2+16} \leq 9(b+c)-4.$$

We see that

$$9(b+c)-4 \geq 18\sqrt{bc}-4 = 14 > 0.$$

By squaring, the inequality becomes

$$\sqrt{528t^2+289} \le 24t^2-36t+25,$$

where

$$t = b+c \ge 2.$$

Since

$$24t^2-36t+25 \ge 6t^2+25,$$

it suffices to show that

$$528t^2+289 \le (6t^2+25)^2,$$

which is equivalent to

$$(t^2-4)(3t^2-7) \ge 0.$$

Case 2: $0 < a \le b = c$. Write the inequality in the homogeneous form

$$\sum \sqrt{33a^2+16(ab+bc+ca)} \le 9(a+b+c).$$

Without loss of generality, assume that $b = c = 1$, when the inequality becomes

$$\sqrt{33a^2+32a+16}+2\sqrt{32a+49} \le 9a+18.$$

By squaring twice, the inequality turns as follows:

$$\sqrt{(33a^2+32a+16)(32a+49)} \le 12a^2+41a+28,$$

$$72a(2a^3-a^2-4a+3) \ge 0,$$

$$72a(a-1)^2(2a+3) \ge 0.$$

The equality holds for $a = b = c = \frac{1}{\sqrt{3}}$, and also for

$$a = 0, \qquad b = c = 1$$

(or any cyclic permutation).

□

P 5.81. *If a,b,c are positive real numbers so that $a+b+c = 3$, then*

$$a^2b^2+b^2c^2+c^2a^2 \le \frac{3}{\sqrt[3]{abc}}.$$

(Vasile C., 2006)

Solution. Write the inequality in the homogeneous form

$$\left(\frac{a+b+c}{3}\right)^{15} \geq abc\left(\frac{a^2b^2+b^2c^2+c^2a^2}{3}\right)^3.$$

Since

$$\begin{aligned} a^2b^2+b^2c^2+c^2a^2 &= (ab+bc+ca)^2-2abc(a+b+c) \\ &= \frac{1}{4}(9-a^2-b^2-c^2)-6abc, \end{aligned}$$

we will apply Corollary 5 (case $k=0$ and $m=2$):

- *If* $0 \leq a \leq b \leq c$ *and*

$$a+b+c=3, \qquad abc=constant,$$

them the sum

$$S_3 = a^2+b^2+c^2$$

is minimal for $0 < a \leq b = c$.

Therefore, we only need to prove the homogeneous inequality for $0 < a \leq 1$ and $b=c=1$. Taking logarithms, we have to show that $f(a) \geq 0$, where

$$f(a) = 15\ln\frac{a+2}{3} - \ln a - 3\ln\frac{2a^2+1}{3}.$$

Since the derivative

$$f'(a) = \frac{15}{a+2} - \frac{1}{a} - \frac{12a}{2a^2+1} = \frac{2(a-1)(2a-1)(4a-1)}{a(a+2)(2a^2+1)}$$

is negative for $a \in \left(0,\frac{1}{4}\right) \cup \left(\frac{1}{2},1\right)$ and positive for $a \in \left(\frac{1}{4},\frac{1}{2}\right)$, f is decreasing on $\left(0,\frac{1}{4}\right] \cup \left[\frac{1}{2},1\right]$ and increasing on $\left[\frac{1}{4},\frac{1}{2}\right]$. Therefore, it suffices to show that $f\left(\frac{1}{4}\right) \geq 0$ and $f(1) \geq 0$. Indeed, we have $f(1)=0$ and

$$f\left(\frac{1}{4}\right) = \ln\frac{3^{12}}{2^{19}} > 0.$$

The equality holds for $a=b=c=1$.

$\square$

P 5.82. *If* $a_1, a_2, \ldots, a_n$ $(n \le 81)$ *are nonnegative real numbers so that*

$$a_1^2 + a_2^2 + \cdots + a_n^2 = a_1^5 + a_2^5 + \cdots + a_n^5,$$

then

$$a_1^6 + a_2^6 + \cdots + a_n^6 \le n.$$

(Vasile C., 2006)

Solution. Setting $a_n = 1$, we obtain the statement for $n-1$ numbers a_i. Consequently, it suffices to prove the inequality for $n = 81$. We need to show that the following homogeneous inequality holds:

$$81(a_1^5 + a_2^5 + \cdots + a_{81}^5)^2 \ge (a_1^6 + a_2^6 + \cdots + a_{81}^6)(a_1^2 + a_2^2 + \cdots + a_{81}^2)^2.$$

According to Corollary 5 (case $k = 3$ and $m = 5/2$), *if* $0 \le a_1 \le a_2 \le \cdots \le a_{81}$ *and*

$$a_1^2 + a_2^2 + \cdots + a_{81}^2 = constant, \qquad a_1^6 + a_2^6 + \cdots + a_{81}^6 = constant,$$

then the sum $a_1^5 + a_2^5 + \cdots + a_{81}^5$ *is minimal for* $a_1 = a_2 = \cdots = a_{80} \le a_{81}$. Therefore, we only need to prove the homogeneous inequality for $a_1 = a_2 = \cdots = a_{80} = 0$ and for $a_1 = a_2 = \cdots = a_{80} = 1$. The first case is trivial. In the second case, denoting a_{81} by x, the homogeneous inequality becomes as follows:

$$81(80 + x^5)^2 \ge (80 + x^6)(80 + x^2)^2,$$

$$x^{10} - 2x^8 - 80x^6 + 162x^5 - x^4 - 160x^2 + 80 \ge 0,$$

$$(x-1)^2(x-2)^2(x^6 + 6x^5 + 21x^4 + 60x^3 + 75x^2 + 60x + 20) \ge 0.$$

Thus, the proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $n = 81$, then the equality holds also for

$$a_1 = a_2 = \cdots = a_{80} = \frac{a_{81}}{2} = \sqrt[3]{\frac{3}{4}}$$

(or any cyclic permutation).

□

P 5.83. *If* a, b, c *are nonnegative real numbers so that* $a + b + c = 3$, *then*

$$1 + \sqrt{1 + a^3 + b^3 + c^3} \ge \sqrt{3(a^2 + b^2 + c^2)}.$$

(Vasile C., 2006)

Solution. Write the inequality as

$$\sqrt{1+a^3+b^3+c^3} \ge \sqrt{3(a^2+b^2+c^2)} - 1.$$

By squaring, we may rewrite the inequality in the homogeneous form

$$a^3+b^3+c^3+2\left(\frac{a+b+c}{3}\right)^2\sqrt{3(a^2+b^2+c^2)} \ge (a+b+c)(a^2+b^2+c^2).$$

According to Corollary 5 (case $k=2$ and $m=3$), *if* $0 \le a \le b \le c$ *and*

$$a+b+c = constant, \qquad a^2+b^2+c^2 = constant,$$

then the sum

$$S_3 = a^3+b^3+c^3$$

is minimal for either $a=0$ *or* $0 < a \le b = c$. Thus, we only need to prove the homogeneous inequality for $a=0$ and for $b=c=1$.

Case 1: $a=0$. We need to show that

$$b^3+c^3+2\left(\frac{b+c}{3}\right)^2\sqrt{3(b^2+c^2)} \ge (b+c)(b^2+c^2).$$

Simplifying by $b+c$, it remains to show that

$$(b+c)\sqrt{b^2+c^2} \ge \frac{3\sqrt{3}}{2}bc.$$

Indeed,

$$(b+c)\sqrt{b^2+c^2} \ge \left(2\sqrt{bc}\right)\sqrt{2bc} \ge \frac{3\sqrt{3}}{2}bc.$$

Case 2: $b=c=1$. We need to prove that

$$(a+2)^2\sqrt{3(a^2+2)} \ge 9(a^2+a+1).$$

By squaring, the inequality becomes

$$a^6+8a^5-a^4-6a^3-17a^2+10a+5 \ge 0,$$

$$(a-1)^2(a^4+10a^3+18a^2+20a+5) \ge 0.$$

The equality holds for $a=b=c=1$.

□

P 5.84. *If a,b,c are nonnegative real numbers so that $a+b+c=3$, then*

$$\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a} \le \sqrt{16+\frac{2}{3}(ab+bc+ca)}.$$

(Lorian Saceanu, 2017)

Solution. Write the inequality in the form

$$f(a)+f(b)+f(c)+\sqrt{16+\frac{2}{3}(ab+bc+ca)} \ge 0,$$

where

$$f(u) = -\sqrt{3-u}, \qquad 0 \le u \le 3.$$

We have

$$g(x) = f'(x) = \frac{1}{2\sqrt{3-x}},$$

$$g''(x) = \frac{3}{8}(3-x)^{-5/2}.$$

Since $g''(x) > 0$ for $x \in [0,3)$, g is strictly convex on $[0,3]$. According to Corollary 1, *if* $0 \le a \le b \le c$ *and*

$$a+b+c=3, \qquad ab+bc+ca = constant,$$

then the sum $S_3 = f(a)+f(b)+f(c)$ is minimal for either $a=0$ or $0<a \le b=c$. Therefore, we only need to prove the homogeneous inequality

$$\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a} \le \sqrt{\frac{16}{3}(a+b+c)+\frac{2(ab+bc+ca)}{a+b+c}}$$

for $a=0$ and $b=c=1$.

Case 1: $a=0$. We need to show that

$$\sqrt{b}+\sqrt{c}+\sqrt{b+c} \le \sqrt{\frac{16}{3}(b+c)+\frac{2bc}{b+c}}.$$

Consider the nontrivial case $b,c>0$, use the substitution

$$x = \sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}, \qquad x \ge 2,$$

and write the inequality as

$$\sqrt{b+c+2\sqrt{bc}}+\sqrt{b+c} \le \sqrt{\frac{16}{3}(b+c)+\frac{2bc}{b+c}},$$

$$\sqrt{x+2}+\sqrt{x} \le \sqrt{\frac{16}{3}x+\frac{2}{x}}.$$

By squaring twice, the inequality becomes as follows:

$$\sqrt{x(x+2)} \le \frac{5}{3}x-1+\frac{1}{x},$$

$$16x^4-48x^3+39x^2-18x+9 \ge 0,$$

$$(x-2)[16x^2(x-1)+7x-4]+1 \ge 0.$$

Case 2: $b=c=1$. We need to prove that

$$2\sqrt{a+1}+\sqrt{2} \le \sqrt{\frac{16}{3}(a+2)+\frac{2(2a+1)}{a+2}}$$

By squaring twice, the inequality becomes as follows:

$$6(a+2)\sqrt{2(a+1)} \le 2a^2+17a+17,$$

$$4a^4-4a^3-3a^2+2a+1 \ge 0,$$

$$(a-1)^2(2a+1)^2 \ge 0.$$

The equality holds for $a=b=c=1$.

□

P 5.85. *If $a,b,c \in [0,4]$ and $ab+bc+ca=4$, then*

$$\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a} \le 3+\sqrt{5}.$$

(Vasile Cîrtoaje, 2019)

First Solution. Denote $s=a+b+c$, consider s fixed and write the inequality as

$$f(a)+f(b)+f(c) \ge -3-\sqrt{5},$$

where

$$f(x)=-\sqrt{s-x}. \quad 0 \le x < s.$$

From

$$g(x)=f'(x)=\frac{1}{2}(s-x)^{-1/2}, \qquad g''(x)=\frac{3}{8}(s-x)^{-5/2}>0,$$

it follows that g is strictly convex. Thus, by Corollary 1 and Note 2, the sum $f(a)+f(b)+f(c)$ is minimal for either $a \le b=c$ or $a=0$.

Case 1: $a \le b=c$. We need to show that $2ac+c^2=4$ yields

$$2\sqrt{a+c}+\sqrt{2c} \le 3+\sqrt{5},$$

that is
$$\sqrt{\frac{2(c^2+1)}{c}}+\sqrt{2c}\le 3+\sqrt{5}.$$
From $2ac+c^2=4$,it follows that
$$\frac{2}{\sqrt{3}}\le c\le 2.$$
Since $\sqrt{2c}\le 2$, it is enough to show that
$$\sqrt{\frac{2(c^2+1)}{c}}\le 1+\sqrt{5},$$
that is
$$c^2-(3+\sqrt{5})c+4\le 0.$$
Indeed,
$$c^2-(3+\sqrt{5})c+4\le c^2-5c+4=(c-1)(c-4)<0.$$
Case 2: $a=0$. We need to show that $bc=4$ yields
$$\sqrt{b}+\sqrt{c}+\sqrt{b+c}\le 3+\sqrt{5}.$$
From $(4-b)(4-c)\ge 0$, we get $b+c\le 5$. Thus,
$$\sqrt{b}+\sqrt{c}+\sqrt{b+c}\le\sqrt{b+c+2\sqrt{bc}}+\sqrt{b+c}$$
$$\le\sqrt{5+2\sqrt{4}}+\sqrt{5}=3+\sqrt{5}.$$
The equality occurs for $a=0$, $b=1$ and $c=4$ (or any permutation).

Second Solution(by *Kiyoras-2001*) Assume that $a\ge b\ge c$, denote
$$S=ab+bc+ca$$
and show that
$$f(a,b,c)\le f\left(a,\frac{S}{a},0\right)\le 3+\sqrt{5},$$
where
$$f(a,b,c)=\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}.$$
The left homogeneous inequality is true because
$$f\left(a,\frac{S}{a},0\right)-f(a,b,c)=$$
$$=\sqrt{a+\frac{S}{a}}-\sqrt{a+b}+\sqrt{\frac{S}{a}}-\sqrt{b+c}+\sqrt{a}-\sqrt{c+a}$$

$$= \frac{\frac{c}{a}(a+b)}{\sqrt{\frac{(a+b)(a+c)}{a}}+\sqrt{a+b}} + \frac{\frac{bc}{a}}{\sqrt{\frac{S}{a}}+\sqrt{b+c}} - \frac{c}{\sqrt{a}+\sqrt{c+a}}$$

$$\geq \frac{c}{a}\left(\frac{\sqrt{a(a+b)}}{\sqrt{a+c}+\sqrt{a}} - \frac{a}{\sqrt{a}+\sqrt{c+a}}\right) \geq 0.$$

Also, the right inequality is true for $S = 4$ and $a, b, c \in [0, 4]$ since $a > 1$ and

$$f\left(a, \frac{4}{a}, 0\right) - 3 - \sqrt{5} =$$

$$= \sqrt{a + \frac{4}{a}} - \sqrt{5} + \frac{2}{\sqrt{a}} + \sqrt{a} - 3$$

$$= \frac{(a-1)\left(1-\frac{4}{a}\right)}{\sqrt{a+\frac{4}{a}}+\sqrt{5}} + (\sqrt{a}-1)\left(1-\frac{2}{\sqrt{a}}\right) \leq 0.$$

□

P 5.86. *If a, b, c are positive real numbers so that $abc = 1$, then*

(a) $$\frac{a+b+c}{3} \geq \sqrt[3]{\frac{2+a^2+b^2+c^2}{5}};$$

(b) $$a^3 + b^3 + c^3 \geq \sqrt{3(a^4+b^4+c^4)}.$$

(Vasile C., 2006)

Solution. (a) According to Corollary 5 (case $k = 0$ and $m = 2$), *if $a \leq b \leq c$ and*

$$a + b + c = constant, \qquad abc = 1,$$

the sum $S_3 = a^2 + b^2 + c^2$ is maximal for $0 < a = b \leq c$. Thus, we only need to show that $a^2c = 1$ involves

$$\frac{2a+c}{3} \geq \sqrt[3]{\frac{2+2a^2+c^2}{5}},$$

which is equivalent to

$$5\left(2a + \frac{1}{a^2}\right)^3 \geq 27\left(2 + 2a^2 + \frac{1}{a^4}\right),$$

$$40a^9 - 54a^8 + 6a^6 + 30a^3 - 27a^2 + 5 \geq 0,$$

$$(a-1)^2(40a^7 + 26a^6 + 12a^5 + 4a^4 - 4a^3 - 12a^2 + 10a + 5) \geq 0.$$

The inequality is true since

$$\begin{aligned}12a^5+4a^4-4a^3-12a^2+10a+5 &> 2a^5+4a^4-4a^3-12a^2+10a\\ &= 2a(a-1)^2(a^2+4a+5)\ge 0.\end{aligned}$$

The equality holds for $a=b=c=1$.

(b) According to Corollary 5 (case $k=0$ and $m=4/3$), *if* $a\le b\le c$ *and*

$$a^3+b^3+c^3=constant, \qquad a^3b^3c^3=1,$$

the sum $S_3=a^4+b^4+c^4$ *is maximal for* $0<a=b\le c.$ Thus, we only need to show that

$$2a^3+c^3\ge\sqrt{3(2a^4+c^4)}$$

for $a^2c=1,\ \ a\le 1.$ The inequality is equivalent to

$$\left(2a^3+\frac{1}{a^6}\right)^2\ge 3\left(2a^4+\frac{1}{a^8}\right).$$

Substituting $a=1/t,\ t\ge 1$, the inequality becomes

$$\left(\frac{2}{t^3}+t^6\right)^2\ge 3\left(\frac{2}{t^4}+t^8\right),$$

which is equivalent to $f(t)\ge 0$, where

$$f(t)=t^{18}-3t^{14}+4t^9-6t^2+4.$$

We have

$$f'(t)=6tg(t), \qquad g(t)=3t^{16}-7t^{12}+6t^7-2,$$

$$g'(t)=6t^6h(t), \qquad h(t)=8t^9-14t^5+7,$$

$$h'(t)=2t^4(36t^2-35).$$

Since $h'(t)>0$ for $t\ge 1$, h is increasing, $h(t)\ge h(1)=1$ for $t\ge 1$, g is increasing, $g(t)\ge g(1)=0$ for $t\ge 1$, f is increasing, hence $f(t)\ge f(1)=0$ for $t\ge 1$.

The equality holds for $a=b=c=1$.

□

P 5.87. *If a,b,c,d are nonnegative real numbers so that $a+b+c+d=4$, then*

$$(a^2+b^2+c^2+d^2-4)(a^2+b^2+c^2+d^2+18)\le 10(a^3+b^3+c^3+d^3-4).$$

(Vasile Cîrtoaje, 2010)

Solution. Apply Corollary 2 for $n = 4$, $k = 2$, $m = 3$:

- *If a, b, c, d are real numbers so that $0 \le a \le b \le c \le d$ and*

$$a + b + c + d = 4, \qquad a^2 + b^2 + c^2 + d^2 = constant,$$

then

$$S_4 = a^3 + b^3 + c^3 + d^3$$

is minimal for either $0 < a \le b = c = d$ or $a = 0$.

Case 1: $0 < a \le b = c = d$. We need to show that $a + 3d = 4$ involves

$$(a^2 + 3d^2 - 4)(a^2 + 3d^2 + 18) \le 10(a^3 + 3d^3 - 4).$$

This inequality is equivalent to

$$(1 - d)^2(1 + d)(4 - 3d) \ge 0,$$

$$(1 - d)^2(1 + d)a \ge 0.$$

Case 2: $a = 0$. Let

$$s = b^2 + c^2 + d^2.$$

We need to show that $b + c + d = 4$ involves

$$(s - 4)(s + 18) \le 10(b^3 + c^3 + d^3 - 4).$$

By the Cauchy-Schwarz inequality, we have

$$s \ge \frac{1}{3}(b + c + d)^2 = \frac{16}{3}$$

and

$$(b + c + d)(b^3 + c^3 + d^3) \ge (b^2 + c^2 + d^2)^2, \qquad b^3 + c^3 + d^3 \ge \frac{s^2}{4}.$$

Thus, it suffices to show that

$$(s - 4)(s + 18) \le 10\left(\frac{s^2}{4} - 4\right),$$

which is equivalent to the obvious inequality

$$(s - 4)(3s - 16) \ge 0.$$

The equality holds for $a = b = c = d = 1$, and also for

$$a = 0, \qquad b = c = d = \frac{4}{3}$$

(or any cyclic permutation).

□

P 5.88. *If a,b,c,d are nonnegative real numbers such that*

$$a+b+c+d=4,$$

then

$$(a^4+b^4+c^4+d^4)^2 \ge (a^2+b^2+c^2+d^2)(a^5+b^5+c^5+d^5).$$

(Vasile C., 2020)

Proof. Consider the inequality

$$(a_1^4+a_2^4+\cdots+a_n^4)^2 \ge (a_1^2+a_2^2+\cdots+a_n^2)(a_1^5+a_2^5+\cdots+a_n^5),$$

where $a_1,a_2,\ldots,a_n$ are nonnegative real numbers such that $a_1+a_2+\cdots+a_n=n$. Write this inequality in the homogeneous form

$$n(a_1^4+a_2^4+\cdots+a_n^4)^2 \ge (a_1+a_2+\cdots+a_n)(a_1^2+a_2^2+\cdots+a_n^2)(a_1^5+a_2^5+\cdots+a_n^5).$$

Replacing $a_1,a_2,\ldots,a_n$ with $x_1^{1/4},x_2^{1/4},\ldots,x_n^{1/4}$, the inequality becomes

$$n(x_1+x_2+\cdots+x_n)^2 \ge$$

$$\ge \left(x_1^{1/4}+x_2^{1/4}+\cdots+x_n^{1/4}\right)\left(x_1^{1/2}+x_2^{1/2}+\cdots+x_n^{1/2}\right)\left(x_1^{5/4}+x_2^{5/4}+\cdots+x_n^{5/4}\right).$$

By Corollary 5 (case $k=5/4$), *if*

$$x_1+x_2+\cdots+x_n=constant, \qquad x_1^{5/4}+x_2^{5/4}+\cdots+x_n^{5/4}=constant,$$

then the sums $x_1^{1/4}+x_2^{1/4}+\cdots+x_n^{1/4}$ and $x_1^{1/2}+x_2^{1/2}+\cdots+x_n^{1/2}$ are maximal for

$$0 \le x_1=x_2=\cdots=x_{n-1}\le x_n.$$

Since the case $a_1=a_2=\cdots=a_{n-1}=0$ is trivial, it suffices to consider the case $a_1=a_2=\cdots=a_{n-1}=1$, when the required inequality becomes $f(a)\ge 0$, where

$$f(a)=(a^4+n-1)^2-(a+n-1)(a^2+n-1)(a^5+n-1), \qquad a\ge 1.$$

We have

$$\frac{f(a)}{n-1}=a^8-a^7-a^6-(n-1)a^5+2na^4-a^3-(n-1)a^2-(n-1)a+n-1$$

$$=a^3A-(n-1)B,$$

where

$$A=a^5-a^4-a^3+2a-1, \qquad B=a^5-2a^4+a^2+a-1.$$

Since

$$A=(a-1)^2(a^3+a^2-1), \qquad B=(a-1)^2(a^3-a-1),$$

we have

$$f(a) = (n-1)(a-1)^2 g(a),$$

where

$$g(a) = a^6 + a^5 - na^3 + (n-1)a + n - 1.$$

The inequality is true if $g(a) \geq 0$. For $n = 4$, we have

$$g(a) = a^6 + a^5 - 4a^3 + 3a + 3 > 2a^5 - 4a^3 + 2a = 2a(a^2-1)^2 \geq 0.$$

The equality occurs for $a = b = c = d = 1$.

Remark 1. Since $g(a) \geq 0$ for $n \leq 16$, the homogeneous inequality is true for all $n \leq 16$.

Remark 2. Since

$$(a_1 + a_2 + \cdots + a_n)(a_1^5 + a_2^5 + \cdots + a_n^5) \leq |(a_1 + a_2 + \cdots + a_n)(a_1^5 + a_2^5 + \cdots + a_n^5)|$$
$$\leq (|a_1| + |a_2| + \cdots + |a_n|)(|a_1|^5 + |a_2|^5 + \cdots + |a_n|^5),$$

the homogeneous inequality is true for $n \leq 16$ and real $a_1, a_2, \ldots, a_n$.

□

P 5.89. *If a, b, c, d are nonnegative real numbers such that*

$$a + b + c + d = 4,$$

then

$$13(a^2 + b^2 + c^2 + d^2)^2 \geq 12(a^4 + b^4 + c^4 + d^4) + 160.$$

(Vasile Cîrtoaje, 2020)

Solution. Write the inequality in the homogeneous form

$$104(a^2 + b^2 + c^2 + d^2)^2 \geq 96(a^4 + b^4 + c^4 + d^4) + 5(a + b + c + d)^4.$$

According to Corollary 5, for $a + b + c + d = constant$ and $a^2 + b^2 + c^2 + d^2 = constant$, the sum

$$S = a^4 + b^4 + c^4 + d^4$$

is maximal when $a \geq b = c = d$. Therefore, it suffices to consider this case. Due to homogeneity, for the nontrivial case $b = c = d \neq 0$, we may consider that $b = c = d = 1$. Thus we only need to prove that

$$104(a^2 + 3)^2 \geq 96(a^4 + 3) + 5(a + 3)^4,$$

which is equivalent to

$$(a-1)^2(a-9)^2 \geq 0.$$

The equality occurs for $a = b = c = d = 1$, and also for $a = 3$ and $b = c = d = \dfrac{1}{3}$ (or any cyclic permutation).

□

P 5.90. *If* $a_1, a_2, \ldots, a_8$ *are nonnegative real numbers, then*

$$19(a_1^2 + a_2^2 + \cdots + a_8^2)^2 \geq 12(a_1 + a_2 + \cdots + a_8)(a_1^3 + a_2^3 + \cdots + a_8^3).$$

(Vasile C., 2007)

Solution. By Corollary 5 (case $n = 8$, $k = 2$, $m = 3$), *if* $0 \leq a_1 \leq a_2 \leq \cdots \leq a_8$ *and*

$$a_1 + a_2 + \cdots + a_8 = constant, \qquad a_1^2 + a_2^2 + \cdots + a_8^2 = constant,$$

then the sum

$$S_8 = a_1^3 + a_2^3 + \cdots + a_8^3$$

is maximal for $a_1 = a_2 = \cdots = a_7 \leq a_8$. Due to homogeneity, we only need to consider the cases $a_1 = a_2 = \cdots = a_7 = 0$ and $a_1 = a_2 = \cdots = a_7 = 1$. For the second case (nontrivial), we need to show that

$$19(7 + a_8^2)^2 \geq 12(7 + a_8)(7 + a_8^3),$$

which is equivalent to

$$a_8^4 - 12a_8^3 + 38a_8^2 - 12a_8 + 49 \geq 0,$$

$$(a_8^2 - 6a_8 + 1)^2 + 48 \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_8 = 0$.

□

P 5.91. *If* a, b, c *are nonnegative real numbers so that*

$$5(a^2 + b^2 + c^2) = 17(ab + bc + ca),$$

then

$$3\sqrt{\frac{3}{5}} \leq \sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} \leq \frac{1+\sqrt{7}}{\sqrt{2}}.$$

(Vasile C., 2006)

Solution. Due to homogeneity, we may assume that $a + b + c = 9$. From the hypothesis $5(a^2 + b^2 + c^2) = 17(ab + bc + ca)$, which is equivalent to

$$27(a^2 + b^2 + c^2) = 17(a + b + c)^2,$$

we get

$$a^2 + b^2 + c^2 = 51.$$

Also, from $2(b^2 + c^2) \geq (b + c)^2$ and

$$b + c = 9 - a, \qquad b^2 + c^2 = 51 - a^2,$$

we get $a \le 7$. Write the desired inequality in the form

$$3\sqrt{\frac{3}{5}} \le f(a)+f(b)+f(c) \le \frac{1+\sqrt{7}}{\sqrt{2}}.$$

where

$$f(u)=\sqrt{\frac{u}{9-u}}, \qquad 0 \le u \le 7.$$

We have

$$g(x)=f'(x)=\frac{9}{2x^{1/2}(9-x)^{3/2}},$$

$$g''(x)=\frac{27(8x^2-36x+81)}{8x^{5/2}(9-x)^{7/2}}.$$

Since $g''(x)>0$ for $x \in (0,7]$, g is strictly convex on $(0,7]$. According to Corollary 1, *if* $0 \le a \le b \le c$ *and*

$$a+b+c=9, \qquad a^2+b^2+c^2=51,$$

then the sum $S_3=f(a)+f(b)+f(c)$ *is maximal for* $a=b \le c$*, and is minimal for either* $a=0$ *or* $0<a \le b=c$.

(a) To prove the right inequality, it suffices to consider the case $a=b \le c$. From

$$a+b+c=9, \qquad a^2+b^2+c^2=51,$$

we get $a=b=1$ and $c=7$, therefore

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}=\frac{1+\sqrt{7}}{\sqrt{2}}.$$

The original right inequality is an equality for $a=b=c/7$ (or any cyclic permutation).

(b) To prove the left inequality, it suffices to consider the cases $a=0$ and $0<a \le b=c$. For $a=0$, from

$$a+b+c=9, \qquad a^2+b^2+c^2=51,$$

we get

$$\frac{b}{c}+\frac{c}{b}=\frac{17}{5},$$

therefore

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}=\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}=\sqrt{\frac{b}{c}+\frac{c}{b}+2}=3\sqrt{\frac{3}{5}}.$$

The case $0<a \le b=c$ is not possible, because from

$$a+b+c=9, \qquad a^2+b^2+c^2=51,$$

we get $a = 7$ and $b = c = 1$, which don't satisfy the condition $a \le b$. The original left inequality is an equality for

$$a = 0, \qquad \frac{b}{c} + \frac{c}{b} = \frac{17}{5}$$

(or any cyclic permutation).

□

P 5.92. *If a, b, c are nonnegative real numbers so that*

$$8(a^2 + b^2 + c^2) = 9(ab + bc + ca),$$

then

$$\frac{19}{12} \le \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \le \frac{141}{88}.$$

(Vasile C., 2006)

Solution. The proof is similar to the one of the preceding P 5.91. Assume that $a + b + c = 15$, which involves $a^2 + b^2 + c^2 = 81$ and $a \in [3, 7]$, then write the inequality in the form

$$\frac{19}{12} \le f(a) + f(b) + f(c) \le \frac{141}{88},$$

where

$$f(u) = \frac{u}{15 - u}, \qquad 3 \le u \le 7.$$

We have

$$g(x) = f'(x) = \frac{1}{5}(15 - x)^2, \qquad g''(x) = \frac{90}{(15 - x)^4}.$$

Since g is strictly convex on $[3, 7]$, according to Corollary 1, *if* $0 \le a \le b \le c$ *and*

$$a + b + c = 15, \qquad a^2 + b^2 + c^2 = 81,$$

then the sum $S_3 = f(a) + f(b) + f(c)$ is maximal for $a = b \le c$, and is minimal for either $a = 0$ or $0 < a \le b = c$.

(a) To prove the right inequality, it suffices to consider the case $a = b \le c$, which involves

$$a = b = 4, \qquad c = 7,$$

and

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{141}{88}.$$

The original right inequality is an equality for $a = b = 4c/7$ (or any cyclic permutation).

(b) To prove the left inequality, it suffices to consider the cases $a = 0$ and $0 < a \le b = c$. The first case is not possible, while the second case involves

$$a = 3, \qquad b = c = 6,$$

and

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{19}{12}.$$

The original left inequality is an equality for $2a = b = c$ (or any cyclic permutation).

□

P 5.93. *If $a, b, c \in (0, 2]$ such that $a + b + c = 3$, then*

$$\sqrt{\frac{2(b+c)}{a} - 1} + \sqrt{\frac{2(c+a)}{b} - 1} + \sqrt{\frac{2(a+b)}{c} - 1} \ge \frac{9}{\sqrt{ab+bc+ca}}.$$

(Vasile C., 2020)

Solution. Write the inequality in the form

$$f(a) + f(b) + f(c) \le \frac{-3\sqrt{3}}{\sqrt{ab+bc+ca}},$$

where

$$f(u) = -\sqrt{\frac{2}{u} - 1}, \qquad 0 < u \le 2.$$

We have $f(0+) = -\infty$ and

$$g(x) = f'(x) = x^{-3/2}(2-x)^{-1/2}, \qquad g'(x) = (2x-3)x^{-5/2}(2-x)^{-3/2},$$

$$g''(x) = (7x^2 - 20x + 15)x^{-7/2}(2-x)^{-5/2} > 0.$$

Since g is strictly convex on $(0, 2)$, according to Corollary 1, Note 1 and Note 2, *if $a \ge b \ge c > 0$ and*

$$a + b + c = 3, \qquad ab + bc + ca = constant,$$

then the sum $S_3 = f(a) + f(b) + f(c)$ is maximal for $a = 2$ or $a \ge b = c$. Thus, it suffices to prove the desired inequality for these cases.

Case 1: $a = 2$. We need to prove the homogeneous inequality

$$\sqrt{\frac{2(b+c)}{a} - 1} + \sqrt{\frac{2(c+a)}{b} - 1} + \sqrt{\frac{2(a+b)}{c} - 1} \ge \frac{3(a+b+c)}{\sqrt{ab+bc+ca}}$$

for

$$a = 2(b+c).$$

The inequality is equivalent to

$$\sqrt{\frac{2b}{c}+1}+\sqrt{\frac{2c}{b}+1}\ge \frac{3\sqrt{3}(b+c)}{\sqrt{2(b+c)^2+bc}}.$$

Let

$$x=\frac{(b+c)^2}{4bc}, \qquad x\ge 1.$$

Since

$$\sqrt{\frac{2c}{b}+1}+\sqrt{\frac{2b}{c}+1}\ge 2\sqrt[4]{\left(\frac{2b}{c}+1\right)\left(\frac{2c}{b}+1\right)}=2\sqrt[4]{8x+1},$$

the inequality becomes

$$\sqrt[4]{8x+1}\ge \frac{3\sqrt{3x}}{\sqrt{8x+1}},$$

$$(8x+1)^3\ge 729x^2.$$

Since

$$8x+1\ge 3(2x+1),$$

it suffices to show that

$$(2x+1)^3\ge 27x^2.$$

This is true because

$$2x+1=x+x+1\ge 3\sqrt[3]{x^2}.$$

Case 2: $a\ge b=c$. We need to show that $a+2c=3$ implies

$$\sqrt{\frac{4c}{a}-1}+2\sqrt{\frac{2(a+c)}{c}-1}\ge \frac{9}{\sqrt{2ac+c^2}},$$

that is

$$\sqrt{\frac{2-a}{a}}+2\sqrt{\frac{1+a}{3-a}}\ge \frac{6}{\sqrt{(1+a)(3-a)}},$$

$$\sqrt{\frac{2-a}{a}}\ge \frac{2(2-a)}{\sqrt{(1+a)(3-a)}}.$$

It is true if

$$\frac{1}{\sqrt{a}}\ge \frac{2\sqrt{2-a}}{\sqrt{(1+a)(3-a)}},$$

which, by squaring, reduces to

$$(a-1)^2\ge 0.$$

The equality occurs for $a=b=c=1$, and also for $a=b=\frac{1}{2}$ and $c=2$ (or any cyclic permutation).

□

P 5.94. *Let a,b,c and x,y,z be nonnegative real numbers such that*

$$x^3+y^3+z^3=a^3+b^3+c^3.$$

Then,

$$\frac{(a+b+c)(x+y+z)}{ab+bc+ca+xy+yz+zx}\geq\sqrt[3]{3}.$$

(Vasile Cîrtoaje, 2019)

Solution. Assume that

$$x+y+z\geq a+b+c$$

and denote

$$t=\frac{x+y+z}{3},\qquad t\geq\frac{a+b+c}{3}.$$

Since

$$\frac{a+b+c}{3}\leq\frac{x+y+z}{3}\leq\sqrt[3]{\frac{x^3+y^3+z^3}{3}}=\sqrt[3]{\frac{a^3+b^3+c^3}{3}},$$

we have

$$t_1\leq t\leq t_2,$$

where

$$t_1=\frac{a+b+c}{3},\qquad t_2=\sqrt[3]{\frac{a^3+b^3+c^3}{3}}.$$

It is enough to prove the inequality

$$\frac{1}{\sqrt[3]{3}}\,(a+b+c)(x+y+z)\geq ab+bc+ca+\frac{1}{3}(x+y+z)^2.$$

For fixed a,b,c, we may write the required inequality as $f(t)\leq 0$, where

$$f(t)=3t^2-\sqrt[3]{9}\,(a+b+c)t+ab+bc+ca$$

is a quadratic convex function. Thus, it is enough to show that $f(t_1)\leq 0$ and $f(t_2)\leq 0$. We have

$$3f(t_1)=3(ab+bc+ca)-\left(\sqrt[3]{9}-1\right)(a+b+c)^2$$

$$\leq 3\left(2-\sqrt[3]{9}\right)(ab+bc+ca)\leq 0.$$

To prove the inequality $f(t_2)\leq 0$, we write it as

$$3t_2^2-\sqrt[3]{9}\,(a+b+c)t_2+ab+bc+ca\leq 0.$$

According to Corollary 5, for $a+b+c=constant$ and $a^n+b^n+c^n=constant$, the sum $a^2+b^2+c^2$ is minimal (hence the sum $ab+bc+ca$ is maximal) for $a\geq$

$b = c$. Thus, due to homogeneity, it is enough to prove the inequality for $a = 1$ and $b = c \le 1$. So, we need to prove that $g(u) \le 0$, where

$$g(u) = u^2 - (2c+1)u + \frac{c^2+2c}{\sqrt[3]{3}},$$

with

$$u = \sqrt[3]{2c^3+1}, \quad c \in [0,1].$$

Consider two cases: $c \in [0,4/5]$ and $c \in [4/5,1]$.

Case 1: $c \in [0,4/5]$. Since $\sqrt[3]{3} > 4/3$, we have

$$g(u) \le u^2 - (2c+1)u + \frac{3(c^2+2c)}{4} = \frac{(2u-3c)(2u-c-2)}{4}.$$

Thus, we need to show that

$$\frac{3c}{2} \le u \le \frac{c+2}{2}.$$

The left inequality is equivalent to

$$c \le \sqrt{\frac{8}{11}}.$$

This is true since

$$c \le \frac{4}{5} < \sqrt{\frac{8}{11}}.$$

The right inequality is equivalent to

$$c(2c+6-5c^2) \ge 0.$$

Case 2: $c \in [4/5,1]$. Since $\sqrt[3]{3} > 7/5$, we have $g(u) < h(u)$, where

$$h(u) = u^2 - (2c+1)u + \frac{5(c^2+2c)}{7}.$$

It suffices to prove that $h(u) \le 0$. From

$$h'(u) = 2u - 2c - 1$$

and

$$(2u)^3 - (2c+1)^3 = 7 + 8c^3 - 12c^2 - 6c \le 7 - 4c^2 - 6c \le 7 - \frac{64}{25} - \frac{24}{5} = \frac{-9}{25} < 0,$$

it follows that $h'(u) < 0$, hence $h(u)$ is a decreasing function. Since

$$u > 1 + \frac{c^3}{3},$$

it follows that

$$h(u) < h\left(1+\frac{c^3}{3}\right) = c\left(\frac{5c}{7}+\frac{c^2}{3}+\frac{c^5}{9}-\frac{4}{7}-\frac{2c^3}{3}\right).$$

Since

$$\frac{5c}{7}+\frac{c^2}{3}+\frac{c^5}{9} \le \frac{5c}{7}+\frac{c}{3}+\frac{c^3}{9} = \frac{22c}{21}+\frac{c^3}{9},$$

it suffices to show that

$$\frac{22c}{21}+\frac{c^3}{9}-\frac{4}{7}-\frac{2c^3}{3} \le 0,$$

that is

$$\frac{22c}{21}-\frac{4}{7}-\frac{5c^3}{9} \le 0.$$

Indeed, we have

$$\frac{4}{7}+\frac{5c^3}{9} = \frac{2}{7}+\frac{2}{7}+\frac{5c^3}{9} \ge 3\sqrt[3]{\frac{20c^3}{49\cdot 9}} > \frac{22c}{21}.$$

Thus, the proof is completed. If $a \ge b \ge c$ and $x \ge y \ge z$, then the equality occurs for $a = b = c = \dfrac{x}{\sqrt[n]{3}}$ and $y = z = 0$, and for $x = y = z = \dfrac{a}{\sqrt[n]{3}}$ and $b = c = 0$.

□

P 5.95. *If a, b, c, d are positive numbers such that*

$$a+b+c+d = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d},$$

then

$$ab+ac+ad+bc+bd+cd+3abcd \ge 9.$$

(Vasile Cîrtoaje, 2019)

Solution. Write the inequality as

$$(a+b+c+d)^2+6abcd \ge 18+a^2+b^2+c^2+d^2$$

and apply Corollary 4 for $k = -1$, and Corollary 5 for $k = -1$ and $m = 2$:

- *If a, b, c, d are positive numbers such that*

$$a+b+c+d = constant\ , \qquad \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} = constant, \qquad a \le b \le c \le d,$$

then the product $abcd$ is minimal and the sum $a^2+b^2+c^2+d^2$ is maximal for $a = b = c \le d$.

Thus, it suffices to consider this case. We need to show that

$$3a+d=\frac{3}{a}+\frac{1}{d}$$

involve

$$a^2+ad+a^3d\geq 3.$$

From the hypothesis, we get

$$d=\frac{3(1-a^2)+\sqrt{9a^4-14a^2+9}}{2a}.$$

So, the required inequality becomes as follows:

$$a^2+(a^2+1)ad\geq 3,$$

$$(a^2+1)\sqrt{9a^4-14a^2+9}\geq 3a^4-2a^2+3,$$

$$(a^2+1)^2(9a^4-14a^2+9)\geq(3a^4-2a^2+3)^2,$$

$$16a^2(a^2-1)^2\geq 0.$$

The equality occurs for $a=b=c=d=1$.

□

P 5.96. *If a_1,a_2,a_3,a_4,a_5 are nonnegative real numbers, then*

$$\frac{(a_1^3+a_2^3+a_3^3+a_4^3+a_5^3)^2}{a_1^4+a_2^4+a_3^4+a_4^4+a_5^4}\geq\frac{1}{2}\sum_{i<j}a_ia_j.$$

(Vasile Cîrtoaje, 2019)

Solution. Write the inequality in the form

$$\frac{4(a_1^3+a_2^3+a_3^3+a_4^3+a_5^3)^2}{a_1^4+a_2^4+a_3^4+a_4^4+a_5^4}+a_1^2+a_2^2+a_3^2+a_4^2+a_5^2\geq(a_1+a_2+a_3+a_4+a_5)^2.$$

According to Corollary 5, for $a_1+a_2+a_3+a_4+a_5=constant$ and $a_1^3+a_2^3+a_3^3+a_4^3+a_5^3=constant$, the sum $a_1^2+a_2^2+a_3^2+a_4^2+a_5^2$ is minimal and the sum $a_1^4+a_2^4+a_3^4+a_4^4+a_5^4$ is maximal for $a_1=a_2=a_3=a_4\leq a_5$.Thus, it is enough to show that

$$\frac{4(4x^3+y^3)^2}{4x^4+y^4}+4x^2+y^2\geq(4x+y)^2,$$

which can be written as

$$4x^6-8x^5y+8x^3y^3-3x^2y^4-2xy^5+y^6\geq 0,$$

$$(x-y)^2(2x^2-y^2)^2\geq 0.$$

The proof is completed. The equality occurs for $a_1=a_2=a_3=a_4=a_5$, and also for $a_1=a_2=a_3=a_4=\dfrac{a_5}{\sqrt{2}}$ (or any cyclic permutation).

□

P 5.97. *If $a_1, a_2, \ldots, a_n \geq 0$ such that*

$$a_1 + a_2 + \cdots + a_n = n,$$

then

$$\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n} \leq \sqrt{2n - 1 + 2\left(1 - \frac{1}{n}\right)\sum_{i<j} a_i a_j}.$$

(Vasile C., 2018)

Proof. Since

$$2\sum_{i<j} a_i a_j = (a_1 + a_2 + \cdots + a_n)^2 - a_1^2 - a_2^2 - \cdots - a_n^2 = n^2 - a_1^2 - a_2^2 - \cdots - a_n^2,$$

we can write the inequality as

$$\left(\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n}\right)^2 \leq n^2 + n - 1 - \left(1 - \frac{1}{n}\right)(a_1^2 + a_2^2 + \cdots + a_n^2).$$

Now, we can apply Corollary 5 for $k = 2$ and $m = 1/2$:

- *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that*

$$a_1 + a_2 + \cdots + a_n = n\ , \quad a_1^2 + a_2^2 + \cdots + a_n^2 = constant, \quad a_1 \leq a_2 \leq \cdots \leq a_n,$$

then the sum

$$\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n}$$

is maximal for $0 \leq a_1 = \cdots = a_{n-1} \leq a_n$.

Thus, it suffices to show that

$$[(n-1)x + y]^2 \leq n^2 + n - 1 - \left(1 - \frac{1}{n}\right)[(n-1)x^4 + y^4].$$

for

$$(n-1)x^2 + y^2 = n, \quad 0 \leq x \leq y$$

Write this inequality in the homogeneous form

$$[(n-1)x + y]^2 \leq \frac{(n^2 + n - 1)\frac{[(n-1)x^2 + y^2]^2}{n} - (n-1)[(n-1)x^4 + y^4]}{(n-1)x^2 + y^2},$$

which is equivalent to

$$(n-1)^2x^4 - 2n(n-1)x^3y + (n^2 + 2n - 2)x^2y^2 - 2nxy^3 + y^4 \geq 0,$$

$$(x-y)^2[(n-1)x - y]^2 \geq 0.$$

The inequality is an equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = \cdots = a_{n-1} = \frac{1}{n-1}$ and $a_n = n - 1$ (or any cyclic permutation).

□

P 5.98. *If $a_1, a_2, \dots, a_n \ge 0$ such that*

$$a_1 + a_2 + \cdots + a_n = \sum_{i<j} a_i a_j > 0,$$

then

$$\frac{(n-1)(n-2)}{2}(a_1 + a_2 + \cdots + a_n) + \sum_{i<j} \sqrt{a_i a_j} \ge n(n-1).$$

(Vasile C., 2020)

Proof. For $n = 2$, we need to show that $a_1 + a_2 = a_1 a_2$ involves $a_1 a_2 \ge 4$. Indeed, this follows from

$$a_1 a_2 = a_1 + a_2 \ge 2\sqrt{a_1 a_2},$$

Since

$$2\sum_{i<j} a_i a_j = (a_1 + a_2 + \cdots + a_n)^2 - a_1^2 - a_2^2 - \cdots - a_n^2$$

and

$$2\sum_{i<j} \sqrt{a_i a_j} = (\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n})^2 - a_1 - a_2 - \cdots - a_n,$$

we can apply Corollary 5 for $k = 2$ and $m = 1/2$:

- *If $a_1, a_2, \dots, a_n$ are nonnegative real numbers so that*

$$a_1 + a_2 + \cdots + a_n = constant\,, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = constant, \quad a_1 \le a_2 \le \cdots \le a_n,$$

then the sum

$$\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n}$$

is minimal for either $0 < a_1 \le a_2 = \cdots = a_n$ or $a_1 = 0$.

Thus, it suffices to consider the case $a_1 = x^2$, $a_2 = \cdots = a_n = y^2$, $0 < x \le y$, and the case $a_1 = 0$. In addition, we will use the induction method.

Case 1: $a_1 = x^2$, $a_2 = \cdots = a_n = y^2$. We need to show that

$$x^2 + (n-1)y^2 = (n-1)x^2y^2 + \frac{(n-1)(n-2)}{2}y^4$$

implies

$$\frac{(n-2)}{2}[x^2 + (n-1)y^2] + xy + \frac{(n-2)}{2}y^2 \ge n,$$

which can be written in the homogeneous form

$$(n-2)x^2 + 2xy + n(n-2)y^2 \ge n\frac{2(n-1)x^2y^2 + (n-1)(n-2)y^4}{x^2 + (n-1)y^2}.$$

For $y = 1$, the inequality becomes

$$(x^2 + n - 1)[(n-2)x^2 + 2x + n(n-2)] \ge 2n(n-1)x^2 + n(n-1)(n-2),$$

$$(n-2)x^4+2x^3-(3n-2)x^2+2(n-1)x \geq 0,$$

$$x(x-1)^2[(n-2)x+2(n-1)] \geq 0.$$

Case 2: $a_1=0$. We need to show that

$$a_2+a_3+\cdots+a_n=\sum_{2\leq i<j} a_i a_j > 0 \tag{1}$$

involves

$$\frac{(n-1)(n-2)}{2}(a_2+a_3+\cdots+a_n)+\sum_{2\leq i<j}\sqrt{a_i a_j} \geq n(n-1). \tag{2}$$

From

$$(a_2+a_3+\cdots+a_n)^2 \leq (n-1)(a_2^2+a_2^3+\cdots+a_n^2)$$

$$=(n-1)(a_2+a_3+\cdots+a_n)^2-2(n-1)\sum_{2\leq i<j} a_i a_j,$$

we get

$$(n-2)(a_2+a_3+\cdots+a_n)^2 \geq 2(n-1)\sum_{2\leq i<j} a_i a_j = 2(n-1)(a_2+a_3+\cdots+a_n),$$

hence

$$a_2+a_3+\cdots+a_n \geq \frac{2(n-1)}{n-2}. \tag{3}$$

On the other hand, by the induction hypothesis, (1) involves

$$\frac{(n-2)(n-3)}{2}(a_2+a_3+\cdots+a_n)+\sum_{2\leq i<j}\sqrt{a_i a_j} \geq (n-1)(n-2).$$

According to this inequality, (2) is true if

$$\frac{(n-1)(n-2)}{2}(a_2+a_3+\cdots+a_n)+(n-1)(n-2)-\frac{(n-2)(n-3)}{2}(a_2+a_3+\cdots+a_n)$$

$$\geq n(n-1),$$

which is equivalent to (3).

The inequality is an equality for $a_1=a_2=\cdots=a_n=\frac{2}{n-1}$, and also for $a_1=0$ and $a_2=a_3=\cdots=a_n=\frac{2}{n-2}$ (or any cyclic permutation).

$\square$

P 5.99. *Let*

$$F(a_1,a_2,\dots,a_n)=n(a_1^2+a_2^2+\cdots+a_n^2)-(a_1+a_2+\cdots+a_n)^2\ ,$$

where $a_1,a_2,\dots,a_n$ are positive real numbers such that $a_1\le a_2\le\cdots\le a_n$ and

$$a_1^2(a_2^2+a_3^2+\cdots+a_n^2)\ge n-1.$$

Then,

$$F(a_1,a_2,\dots,a_n)\ge F\left(\frac{1}{a_1},\frac{1}{a_2},\dots,\frac{1}{a_n}\right).$$

(Vasile C., 2020)

Proof. For $n=2$, we need to show that $a_1a_2\ge 1$ involves

$$(a_1^2a_2^2-1)(a_1-a_2)^2\ge 0,$$

which is clearly true. For $n\ge 3$, write the inequality as

$$n(a_1^2+a_2^2+\cdots+a_n^2)-(a_1+a_2+\cdots+a_n)^2\ge n\left(\frac{1}{a_1^2}+\frac{1}{a_2^2}+\cdots+\frac{1}{a_n^2}\right)-\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)^2.$$

According to Corollary 5 (case $k=-1$), we have:

- *If $a_2,a_3,\dots,a_n$ are positive real numbers so that*

$$a_2+a_3+\cdots+a_n=constant\ ,\quad \frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_n}=constant,\quad a_2\le a_3\le\cdots\le a_n,$$

then the sum $a_2^2+a_3^2+\cdots+a_n^2$ is minimal and the sum $\frac{1}{a_2^2}+\frac{1}{a_3^2}+\cdots+\frac{1}{a_n^2}$ is maximal for $a_2\le a_3=\cdots=a_n$.

Thus, it suffices to consider the case $a_2\le a_3=\cdots=a_n$. We need to show that if x,y,z are positive real numbers such that $x\le y\le z$ and

$$x^2[y^2+(n-2)z^2]\ge n-1,$$

then

$$n[x^2+y^2+(n-2)z^2]-[x+y+(n-2)z]^2\ge n\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{n-2}{z^2}\right)-\left(\frac{1}{x}+\frac{1}{y}+\frac{n-2}{z}\right)^2,$$

which is equivalent to

$$(x-y)^2+(n-2)(y-z)^2+(n-2)(z-x)^2\ge\frac{(x-y)^2}{x^2y^2}+\frac{(n-2)(y-z)^2}{y^2z^2}+\frac{(n-2)(z-x)^2}{z^2x^2},$$

$$(x-y)^2\left(1-\frac{1}{x^2y^2}\right)+(n-2)(y-z)^2\left(1-\frac{1}{y^2z^2}\right)+(n-2)(z-x)^2\left(1-\frac{1}{z^2x^2}\right)\ge 0.$$

From

$$n-1 \le x^2[y^2+(n-2)z^2] \le (n-1)x^2z^2,$$

it follows that

$$xz \ge 1, \qquad yz \ge 1.$$

Thus, suffices to show that

$$(x-y)^2\left(1-\frac{1}{x^2y^2}\right)+(n-2)(z-x)^2\left(1-\frac{1}{z^2x^2}\right) \ge 0,$$

that is

$$(n-2)\left(1-\frac{x}{z}\right)^2\left(z^2-\frac{1}{x^2}\right) \ge \left(1-\frac{x}{y}\right)^2\left(\frac{1}{x^2}-y^2\right).$$

Since

$$1-\frac{x}{z} \ge 1-\frac{x}{y} \ge 0,$$

it suffices to show that

$$(n-2)\left(z^2-\frac{1}{x^2}\right) \ge \frac{1}{x^2}-y^2,$$

that is equivalent to the hypothesis

$$y^2+(n-2)z^2 \ge \frac{n-1}{x^2}.$$

The equality occurs for $a_1 = a_2 = \cdots = a_n \ge 1$ and for $\dfrac{1}{a_1} = a_2 = \cdots = a_n \ge 1$.

Remark. Since $a_1(a_2+a_3+\cdots+a_n) \ge n-1$ yields $a_1^2(a_2^2+a_3^2+\cdots+a_n^2) \ge n-1$, the inequality is also true for

$$a_1(a_2+a_3+\cdots+a_n) \ge n-1.$$

In addition, it is true in the particular case

$$a_1, a_2, \ldots, a_n \ge 1.$$

□

P 5.100. *Let*

$$F(a_1,a_2,\ldots,a_n) = a_1+a_2+\cdots+a_n - n\sqrt[n]{a_1a_2\cdots a_n},$$

where $a_1,a_2,\ldots,a_n$ are positive real numbers such that $a_1 \le a_2 \le \cdots \le a_n$ and

$$a_1(a_2+a_3+\cdots+a_n) \ge n-1.$$

Then,

$$F(a_1,a_2,\ldots,a_n) \ge F\left(\frac{1}{a_1},\frac{1}{a_2},\ldots,\frac{1}{a_n}\right).$$

(Vasile C., 2020)

Solution. For $n = 2$, we need to show that $a_1 a_2 \ge 1$ involves

$$(a_1a_2 - 1)\left(\sqrt{a_1} - \sqrt{a_2}\right)^2 \ge 0,$$

which is true. For $n \ge 3$, the inequality has the form

$$a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1a_2\cdots a_n} \ge \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - \frac{n}{\sqrt[n]{a_1a_2\cdots a_n}} .$$

According to Corollary 5 (case $k = 0$ and $m = -1$), we have:

- *If $a_2, a_3, \ldots, a_n$ are positive real numbers so that*

$$a_2 + a_3 + \cdots + a_n = constant\ , \qquad a_2a_3\cdots a_n = constant, \qquad a_2 \le a_3 \le \cdots \le a_n\ ,$$

then the sum $\frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n}$ is maximal for $a_2 \le a_3 = \cdots = a_n$.

Thus, we only need to show that

$$x + y + (n-2)z - n\sqrt[n]{xyz^{n-2}} \ge \frac{1}{x} + \frac{1}{y} + \frac{n-2}{z} - \frac{n}{\sqrt[n]{xyz^{n-2}}}$$

for $0 < x \le y \le z$ and $x[y + (n-2)z] \ge n - 1$. Since both sides of the inequality are nonnegative, it suffices to prove the homogeneous inequality

$$\left[x + y + (n-2)z - n\sqrt[n]{xyz^{n-2}}\right] \ge \frac{x[y + (n-2)z]}{n-1}\left[\frac{1}{x} + \frac{1}{y} + \frac{n-2}{z} - \frac{n}{\sqrt[n]{xyz^{n-2}}}\right],$$

that is

$$(n-1)\left[x + y + (n-2)z - n\sqrt[n]{xyz^{n-2}}\right] \ge$$

$$\ge y + (n-2)z + \frac{[y + (n-2)z][(n-2)y + z]}{yz}x - n[y + (n-2)z]\sqrt[n]{\frac{x^{n-1}}{yz^{n-2}}}\ .$$

For fixed y and z, write this inequality as $f(x) \ge 0$, $x \in (0, y]$. We will show that

$$f(x) \ge f(y) \ge 0.$$

To prove that $f(x) \ge f(y)$, we show that $f'(x) \le 0$, which is equivalent to

$$n - 1 - (n-1)\sqrt[n]{\frac{yz^{n-2}}{x^{n-1}}} - \frac{[y + (n-2)z][(n-2)y + z]}{yz} + (n-1)\frac{y + (n-2)z}{\sqrt[n]{xyz^{n-2}}} \le 0\ ,$$

$$(n-2)\left(\frac{y}{z} + \frac{z}{y} + n - 3\right) + (n-1)\sqrt[n]{\frac{yz^{n-2}}{x^{n-1}}} \ge (n-1)\frac{y + (n-2)z}{\sqrt[n]{xyz^{n-2}}}\ .$$

By the AM-GM inequality, we have

$$(n-2)\cdot\left(\frac{y}{z} + \frac{z}{y} + n - 3\right) + (n-1)\sqrt[n]{\frac{yz^{n-2}}{x^{n-1}}} \ge$$

$$\geq (n-1)\sqrt[n-1]{\left(\frac{y}{z}+\frac{z}{y}+n-3\right)^{n-2}\cdot(n-1)\sqrt[n]{\frac{yz^{n-2}}{x^{n-1}}}}\ .$$

Thus, it suffices to show that

$$\sqrt[n-1]{\left(\frac{y}{z}+\frac{z}{y}+n-3\right)^{n-2}\cdot(n-1)\sqrt[n]{\frac{yz^{n-2}}{x^{n-1}}}}\geq\frac{y+(n-2)z}{\sqrt[n]{xyz^{n-2}}}\ ,$$

which is equivalent to

$$(n-1)\left(\frac{y}{z}+\frac{z}{y}+n-3\right)^{n-2}yz^{n-2}\geq[y+(n-2)z]^{n-1}\ .$$

Due to homogeneity, we may set $z=1$, when the inequality becomes

$$(n-1)Ay\geq y+n-2,$$

where

$$A=\left(\frac{y+1/y+n-3}{y+n-2}\right)^{n-2},\quad 0<y\leq 1.$$

By Bernoulli's inequality, we have

$$A=\left(1+\frac{1/y-1}{y+n-2}\right)^{n-2}\geq 1+\frac{(n-2)(1/y-1)}{y+n-2}=\frac{y^2+n-2}{y(y+n-2)}\ ,$$

hence

$$(n-1)Ay-(y+n-2)\geq\frac{(n-1)(y^2+n-2)}{y+n-2}-(y+n-2)$$

$$=\frac{(n-2)(y-1)^2}{y+n-2}\geq 0\ .$$

The inequality $f(y)\geq 0$ has the form

$$2y+(n-2)z-n\sqrt[n]{y^2z^{n-2}}\geq\frac{y[y+(n-2)z]}{n-1}\left[\frac{2}{y}+\frac{n-2}{z}-\frac{n}{\sqrt[n]{y^2z^{n-2}}}\right].$$

Due to homogeneity, we may set $z=1$ (hence $0<y\leq 1$), when the inequality becomes

$$2y+n-2-n\sqrt[n]{y^2}\geq\frac{y(y+n-2)}{n-1}\left(\frac{2}{y}+n-2-\frac{n}{\sqrt[n]{y^2}}\right).$$

Denoting

$$t=\sqrt[n]{y},\qquad 0<t\leq 1,$$

we need to show that $g(t)\geq 0$, where

$$g(t)=(n-1)(2t^n-nt^2+n-2)-(t^n+n-2)[(n-2)t^n-nt^{n-2}+2]$$

$$= -(n-2)t^{2n} + nt^{2n-2} - (n-2)(n-4)t^n + n(n-2)t^{n-2} - n(n-1)t^2 + (n-2)(n-3) .$$

For $n = 3$, we have

$$g(t) = t(1-t)^3(3+3t+t^2) \geq 0,$$

and for $n = 4$, we have

$$g(t) = 2(1-t^2)^3(1+t^2) \geq 0.$$

For $n \geq 5$, we have

$$g'(t) = ntg_1(t),$$

$$g_1(t) = -2(n-2)t^{2n-2} + 2(n-1)t^{2n-4} - (n-2)(n-4)t^{n-2} + (n-2)^2t^{n-4} - 2(n-1),$$

$$g_1'(t) = (n-2)t^{n-5}(1-t^2)[4(n-1)t^n + n - 2] \geq 0 ,$$

hence $g_1(t)$ is increasing, $g_1(t) \leq g_1(1) = 0$, $g'(t) \leq 0$, $g(t)$ is decreasing, $g(t) \geq g(1) = 0$. Thus, the proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n \geq 1$.

Remark 1. Since $a_1^{n-1}a_2a_3\cdots a_n \geq 1$ yields $a_1(a_2+a_3+\cdots+a_n) \geq n-1$, the inequality

$$F(a_1, a_2, \ldots, a_n) \geq F\left(\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}\right)$$

is also valid if $a_1, a_2, \ldots, a_n$ are positive real numbers such that

$$a_1 \leq a_2 \leq \cdots \leq a_n, \qquad a_1^{n-1}a_2a_3\cdots a_n \geq 1.$$

Also, it is valid in the particular case

$$a_1, a_2, \ldots, a_n \geq 1.$$

Remark 2. Since $a_1a_2\cdots a_n \geq 1$, from P 5.100 it follows that

$$a_1 + a_2 + \cdots + a_n \geq \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

for

$$a_1(a_2 + a_3 + \cdots + a_n) \geq n - 1.$$

□

P 5.101. *Let*

$$F(a_1, a_2, \ldots, a_n) = \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} - \frac{a_1 + a_2 + \cdots + a_n}{n} ,$$

where $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 \leq a_2 \leq \cdots \leq a_n$ and

$$a_1^{n-1}(a_2 + a_3 + \cdots + a_n) \geq n - 1.$$

Then,

$$F(a_1, a_2, \dots, a_n) \ge F\left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}\right).$$

(Vasile C., 2020)

Solution. For $n = 2$, we need to show that $a_1 a_2 \ge 1$ involves

$$(a_1 a_2 - 1)(\sqrt{2(a_1^2 + a_2^2)} - a_1 - a_2) \ge 0,$$

which is true. For $n \ge 3$, write the inequality in the form

$$\sqrt{n(a_1^2 + a_2^2 + \cdots + a_n^2)} - (a_1 + a_2 + \cdots + a_n)$$

$$\ge \sqrt{n\left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2}\right)} - \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \ge 0 .$$

According to Corollary 5 (case $k = -1$), we have:

- *If $a_2, a_3, \dots, a_n$ are positive real numbers so that*

$$a_2 + a_3 + \cdots + a_n = constant\ , \quad \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n} = constant, \quad a_2 \le a_3 \le \cdots \le a_n,$$

then the sum $a_2^2 + a_3^2 + \cdots + a_n^2$ is minimal and the sum $\frac{1}{a_2^2} + \frac{1}{a_3^2} + \cdots + \frac{1}{a_n^2}$ is maximal for $a_2 \le a_3 = \cdots = a_n$.

Thus, it suffices to consider the case $a_2 \le a_3 = \cdots = a_n$. We need to show that if x, y, z are positive real numbers such that $x \le y \le z$ and

$$x^{n-1}[y + (n-2)z] \ge n - 1,$$

then $E(x, y, z) \ge 0$, where

$$E(x, y, z) = \sqrt{x^2 + y^2 + (n-2)z^2} - \frac{x + y + (n-2)z}{\sqrt{n}}$$

$$-\sqrt{\frac{1}{x^2} + \frac{1}{y^2} + \frac{n-2}{z^2}} + \frac{1}{\sqrt{n}}\left(\frac{1}{x} + \frac{1}{y} + \frac{n-2}{z}\right).$$

We will show that

$$E(x, y, z) \ge E(x, w, w) \ge 0,$$

where

$$w = \frac{y + (n-2)z}{n-1}, \qquad x \le y \le w \le z.$$

Write the inequality $E(x, y, z) \ge E(x, w, w)$ as follows:

$$\frac{y^2 + (n-2)z^2 - (n-1)w^2}{\sqrt{x^2 + y^2 + (n-2)z^2} + \sqrt{x^2 + (n-1)w^2}} + \frac{1}{\sqrt{n}}\left(\frac{1}{y} + \frac{n-2}{z} - \frac{n-1}{w}\right)$$

$$\geq \frac{\frac{1}{y^2}+\frac{n-2}{z^2}+\frac{n-1}{w^2}}{\sqrt{\frac{1}{x^2}+\frac{1}{y^2}+\frac{n-2}{z^2}}+\sqrt{\frac{1}{x^2}+\frac{n-1}{w^2}}},$$

$$\frac{(n-2)(y-z)^2}{n-1}\cdot\frac{1}{\sqrt{x^2+y^2+(n-2)z^2}+\sqrt{x^2+(n-1)w^2}}+\frac{(n-2)(y-z)^2}{\sqrt{n}yz[y+(n-2)z]}$$

$$\geq \frac{(n-2)(y-z)^2[y^2+2(n-1)yz+(n-2)z^2]}{y^2z^2[y+(n-2)z]^2}\cdot\frac{1}{\sqrt{\frac{1}{x^2}+\frac{1}{y^2}+\frac{n-2}{z^2}}+\sqrt{\frac{1}{x^2}+\frac{n-1}{w^2}}},$$

which is true if

$$\frac{1}{n-1}\cdot\frac{1}{\sqrt{x^2+y^2+(n-2)z^2}+\sqrt{x^2+(n-1)w^2}}+\frac{1}{\sqrt{n}yz[y+(n-2)z]}$$

$$\geq \frac{y^2+2(n-1)yz+(n-2)z^2}{y^2z^2[y+(n-2)z]^2}\cdot\frac{1}{\sqrt{\frac{1}{x^2}+\frac{1}{y^2}+\frac{n-2}{z^2}}+\sqrt{\frac{1}{x^2}+\frac{n-1}{w^2}}}.$$

Since $x \leq y$, it is enough to show that

$$\frac{1}{n-1}\cdot\frac{1}{\sqrt{2y^2+(n-2)z^2}+\sqrt{y^2+(n-1)w^2}}+\frac{1}{\sqrt{n}yz[y+(n-2)z]}$$

$$\geq \frac{y^2+2(n-1)yz+(n-2)z^2}{y^2z^2[y+(n-2)z]^2}\cdot\frac{1}{\sqrt{\frac{2}{y^2}+\frac{n-2}{z^2}}+\sqrt{\frac{1}{y^2}+\frac{n-1}{w^2}}}.$$

In addition, since $w \leq z$, it suffices to show that

$$\frac{1}{n-1}\cdot\frac{1}{\sqrt{2y^2+(n-2)z^2}+\sqrt{y^2+(n-1)z^2}}+\frac{1}{\sqrt{n}yz[y+(n-2)z]}$$

$$\geq \frac{y^2+2(n-1)yz+(n-2)z^2}{y^2z^2[y+(n-2)z]^2}\cdot\frac{1}{\sqrt{\frac{2}{y^2}+\frac{n-2}{z^2}}+\sqrt{\frac{1}{y^2}+\frac{n-1}{z^2}}}.$$

Since

$$y^2+2(n-1)yz+(n-2)z^2=[y^2+(n-2)z^2]+2(n-1)yz,$$

we rewrite the inequality as

$$A+B \geq C+D,$$

where

$$A=\frac{1}{n-1}\cdot\frac{1}{\sqrt{2y^2+(n-2)z^2}+\sqrt{y^2+(n-1)z^2}},$$

$$B=\frac{1}{\sqrt{n}yz[y+(n-2)z]},$$

$$C=\frac{y^2+(n-2)z^2}{y^2z^2[y+(n-2)z]^2}\cdot\frac{1}{\sqrt{\frac{2}{y^2}+\frac{n-2}{z^2}}+\sqrt{\frac{1}{y^2}+\frac{n-1}{z^2}}},$$

$$D = \frac{2(n-1)yz}{y^2z^2[y+(n-2)z]^2} \cdot \frac{1}{\sqrt{\frac{2}{y^2}+\frac{n-2}{z^2}}+\sqrt{\frac{1}{y^2}+\frac{n-1}{z^2}}}\ .$$

We will show that

$$A \ge C, \qquad B \ge D.$$

Since the inequality $B \ge D$ is homogeneous, we may consider $y = 1$ and $z \ge 1$, when it becomes

$$[(n-2)z+1]\left[\sqrt{2z^2+n-2}+\sqrt{z^2+n-1}\right] \ge 2\sqrt{n}(n-1)z\ .$$

Since

$$\sqrt{2z^2+n-2}+\sqrt{z^2+n-1} \ge \frac{2z+n-2}{\sqrt{n}}+\frac{z+n-1}{\sqrt{n}} = \frac{3z+2n-3}{\sqrt{n}}\ ,$$

it is sufficient to show that

$$[(n-2)z+1](3z+2n-3) \ge 2n(n-1),$$

which is equivalent to

$$(z-1)[3(n-2)z+2n^2-4n+3] \ge 0.$$

To show that $A \ge C$, we see that $x^{n-1}[y+(n-2)z] \ge n-1$ yields

$$y^{n-1}[y+(n-2)z] \ge n-1.$$

Thus, it suffices to prove the homogeneous inequality

$$A \ge C_0 C, \qquad C_0 = \left[\frac{y^{n-1}[y+(n-2)z]}{n-1}\right]^{2/n}\ ,$$

that is

$$\frac{1}{\sqrt{2y^2+(n-2)z^2}+\sqrt{y^2+(n-1)z^2}} \ge$$

$$\ge \frac{(n-1)[y^2+(n-2)z^2]}{y^2z^2[y+(n-2)z]} \cdot \frac{C_0}{\sqrt{\frac{2}{y^2}+\frac{n-2}{z^2}}+\sqrt{\frac{1}{y^2}+\frac{n-1}{z^2}}}\ ,$$

Due to homogeneity, we may set $y = 1$, hence $z \ge 1$. The inequality becomes

$$\sqrt{2z^2+n-2}+\sqrt{z^2+n-1} \ge$$

$$\ge \frac{(n-1)[1+(n-2)z^2]C_1}{z[1+(n-2)z]^2}\left[\sqrt{2+(n-2)z^2}+\sqrt{1+(n-1)z^2}\right]\ ,$$

where

$$C_1 = \left[\frac{1+(n-2)z}{n-1}\right]^{2/n}.$$

By Bernoulli's inequality, we have

$$C_1 = \left[1 + \frac{(n-2)(z-1)}{n-1}\right]^{2/n} \le 1 + \frac{2(n-2)(z-1)}{n(n-1)} = \frac{2(n-2)z + n^2 - 3n + 4)}{n(n-1)} .$$

Thus, it suffices to show that

$$\sqrt{2z^2+n-2} + \sqrt{z^2+n-1} \ge$$

$$\ge \frac{[1+(n-2)z^2][2(n-2)z+n^2-3n+4]}{nz[1+(n-2)z]^2}\left[\sqrt{2+(n-2)z^2} + \sqrt{1+(n-1)z^2}\right] .$$

We will show that

$$\sqrt{2z^2+n-2} \ge \frac{[1+(n-2)z^2][2(n-2)z+n^2-3n+4]}{nz[1+(n-2)z]^2}\sqrt{(n-1)z^2+1}$$

and

$$\sqrt{z^2+n-1} \ge \frac{[1+(n-2)z^2][2(n-2)z+n^2-3n+4]}{nz[1+(n-2)z]^2}\sqrt{(n-2)z^2+2} .$$

Since

$$\frac{2z^2+n-2}{(n-1)z^2+1} - \frac{z^2+n-1}{(n-2)z^2+2} = \frac{(n-3)(z^2-1)^2}{[n-1)z^2+1][(n-2)z^2+2]} \ge 0 ,$$

it suffices to prove the second inequality. After squaring and making many calculations, this inequality can be written as $(z-1)P(z) \ge 0$, where $P(z) \ge 0$ for $z \ge 1$.

To complete the proof, we need to show that $E(x,w,w) \ge 0$ for $x^{n-1}w \ge 1$. Write the required inequality as follows:

$$\sqrt{n[x^2+(n-1)w^2]} - [x+(n-1)w] \ge \sqrt{n\left[\frac{1}{x^2} + \frac{n-1}{w^2}\right]} - \left(\frac{1}{x} + \frac{n-1}{w}\right) ,$$

$$\frac{(n-1)(x-w)^2}{\sqrt{x^2+(n-1)w^2} + \frac{x+(n-1)w}{\sqrt{n}}} \ge \frac{1}{xw} \cdot \frac{(n-1)(x-w)^2}{\sqrt{(n-1)x^2+w^2} + \frac{(n-1)x+w}{\sqrt{n}}} .$$

This is true if

$$\sqrt{(n-1)x^2+w^2} + \frac{(n-1)x+w}{\sqrt{n}} \ge \frac{1}{xw} \cdot \left[\sqrt{x^2+(n-1)w^2} + \frac{x+(n-1)w}{\sqrt{n}}\right] .$$

Since $x^{n-1}w \ge 1$, it suffices to prove the homogeneous inequality

$$\sqrt{(n-1)x^2+w^2} + \frac{(n-1)x+w}{\sqrt{n}} \ge \frac{(x^{n-1}w)^{2/n}}{xw} \cdot \left[\sqrt{x^2+(n-1)w^2} + \frac{x+(n-1)w}{\sqrt{n}}\right] .$$

Due to homogeneity, we may set $w = 1$, which yields $x \le 1$. The inequality becomes

$$\sqrt{(n-1)x^2+1}+\frac{(n-1)x+1}{\sqrt{n}} \ge x^{\frac{n-2}{n}}\left[\sqrt{x^2+n-1}+\frac{x+n-1}{\sqrt{n}}\right].$$

We can get this by summing the inequalities

$$\sqrt{(n-1)x^2+1} \ge x^{\frac{n-2}{n}} \cdot \sqrt{x^2+n-1}$$

and

$$\frac{(n-1)x+1}{\sqrt{n}} \ge x^{\frac{n-2}{n}} \cdot \frac{x+n-1}{\sqrt{n}}.$$

Replacing x with x^2 in the second inequality gives the first inequality. Thus,it suffices to prove the second inequality, which can be rewritten as $f(x) \ge 0$, where

$$f(x) = \ln[(n-1)x+1] - \ln(x+n-1) - \frac{n-2}{n}\ln x \, .$$

From

$$f'(x) = \frac{n-1}{(n-1)x+1} - \frac{1}{x+n-1} - \frac{n-2}{nx} = \frac{-(n-1)(n-2)(x-1)^2}{nx[(n-1)x+1]_x+n-1)} \le 0 \, ,$$

it follows that f is decreasing, hence $f(x) \ge f(1) = 0$.
The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n \ge 1$.

Remark. The inequality

$$F(a_1,a_2,\ldots,a_n) \ge F\left(\frac{1}{a_1},\frac{1}{a_2},\ldots,\frac{1}{a_n}\right)$$

is also valid in the particular case

$$a_1,a_2,\ldots,a_n \ge 1.$$

□

P 5.102. *If* $a_1,a_2,\ldots,a_n$ $(n \ge 4)$ *are positive real numbers such that*

$$a_1+a_2+\cdots+a_n = n, \qquad a_n = \max\{a_1,a_2,\ldots,a_n\},$$

then

$$n\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_{n-1}}\right) \ge 4(a_1^2+a_2^2+\cdots+a_n^2)+n(n-5).$$

(Vasile C., 2021)

Solution. Assume that a_n is fixed and $a_1 \le a_2 \le \cdots \le a_n$. According to Corollary 5 (case $k = 2$ and $m = -1$), we have:

- *If $a_1, a_2, \ldots, a_{n-1}$ are positive real numbers so that*

$$a_1+a_2+\cdots+a_{n-1} = constant, \quad a_1^2+a_2^2+\cdots+a_{n-1}^2 = constant, \quad a_1 \le a_2 \le \cdots \le a_{n-1},$$

then the sum $\dfrac{1}{a_1} + \dfrac{1}{a_2} + \cdots + \dfrac{1}{a_{n-1}}$ *is minimal for* $a_1 = a_2 = \cdots = a_{n-2} \le a_{n-1}$.

Therefore, it suffices to consider the case $a_1 = a_2 = \cdots = a_{n-2}$, that is to show that $F(a,b) \ge 0$, where

$$F(a,b) = n\left(\frac{n-2}{a} + \frac{1}{b}\right) - 4(n-2)a^2 - 4b^2 - 4c^2 - n(n-5), \qquad c = n-(n-2)a-b,$$

with a, b positive real numbers such that $a \le b \le c$. From $c \ge b$, we get

$$(n-2)a + 2b \le n.$$

We will show that

$$F(a,b) \ge F(t,t) \ge 0,$$

where

$$t = \frac{(n-2)a+b}{n-1}, \qquad t \le 1.$$

Since

$$F(a,b) - F(t,t) = n\left(\frac{n-2}{a} + \frac{1}{b} - \frac{n-1}{t}\right) - 4\left[(n-2)a^2 + b^2 - (n-1)t^2\right]$$

$$= \frac{n(n-2)(a-b)^2}{(n-1)abt} - \frac{4(n-2)(a-b)^2}{n-1}$$

$$\ge \frac{n(n-2)(a-b)^2}{(n-1)ab} - \frac{4(n-2)(a-b)^2}{n-1}$$

$$= \frac{(n-2)(a-b)^2(n-4ab)}{(n-1)ab},$$

it suffices to show that $4ab \le n$. From

$$n \ge (n-2)a + 2b \ge 2\sqrt{2(n-2)ab},$$

we get

$$4ab - n \le \frac{n^2}{2(n-2)} - n = \frac{n(4-n)}{n-2} \le 0.$$

In addition,

$$F(t,t) = \frac{n(n-1)}{t} - 4(n-1)t^2 - 4[n-(n-1)t]^2 - n(n-5)$$

$$= \frac{n(n-1)(1-t)(1-2t)^2}{t} \geq 0.$$

The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = a_2 = \cdots = a_{n-1} = \frac{1}{2}, \qquad a_n = \frac{n+1}{2}.$$

□

P 5.103. *If a, b, c are nonnegative real numbers so that $ab + bc + ca = 3$, then*

$$\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \leq 1.$$

(Vasile C., 2021)

Solution. Using the substitution

$$m = a + b + c + 1,$$

we have to show that

$$f(a) + f(b) + f(c) \leq 1$$

for

$$a + b + c = m - 1, \qquad a^2 + b^2 + c^2 = (m-1)^2 - 6,$$

$$f(u) = \frac{1}{m-u}, \qquad 0 \leq u < m - 1.$$

From

$$g(x) = f'(x) = \frac{1}{(m-u)^2}\ , \qquad g''(x) = \frac{6}{(m-u)^4}\ ,$$

it follows that $g''(x) > 0$, hence g is strictly convex. For fixed m, by Corollary 1, *if*

$$a + b + c = fixed, \qquad a^2 + b^2 + c^2 = fixed,$$

then the sum

$$S_3 = f(a) + f(b) + f(c)$$

is maximal for $a = b \leq c$. Thus, we only need to prove the inequality for $a = b \leq c$; that is, to show that $a^2 + 2ac = 3$ involves

$$\frac{2}{a+c+1} + \frac{1}{2a+1} \leq 1.$$

Write this inequality as follows

$$\frac{4a}{a^2+2a+3} + \frac{1}{2a+1} \leq 1,$$

$$a(a-1)^2 \geq 0.$$

The equality holds for $a = b = c = 1$.

□

Chapter 6

EV Method for Real Variables

6.1 Theoretical Basis

The Equal Variables Method may be extended to solve some difficult symmetric inequalities in real variables.

EV-Theorem (*Vasile Cirtoaje*, 2010). *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be fixed real numbers, and let*

$$x_1 \le x_2 \le \cdots \le x_n$$

so that

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n, \qquad x_1^k + x_2^k + \cdots + x_n^k = a_1^k + a_2^k + \cdots + a_n^k,$$

where k is an even positive integer. If f is a differentiable function on $\mathbb{R}$ so that the joined function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = f'\left(\sqrt[k-1]{x}\right)$$

is strictly convex on $\mathbb{R}$, then the sum

$$S_n = f(x_1) + f(x_2) + \cdots + f(x_n)$$

is minimum for $x_2 = x_3 = \cdots = x_n$, and is maximum for $x_1 = x_2 = \cdots = x_{n-1}$.

To prove this theorem, we will use EV-Lemma and EV-Proposition below.

EV-Lemma. *Let a, b, c be fixed real numbers, not all equal, and let x, y, z be real numbers satisfying*

$$x \le y \le z, \quad x + y + z = a + b + c, \quad x^k + y^k + z^k = a^k + b^k + c^k,$$

where k is an even positive integer. Then, there exist two real numbers m and M so that $m < M$ and

(1) $y \in [m, M]$;

(2) *$y = m$ if and only if $x = y$;*
(3) *$y = M$ if and only if $y = z$.*

Proof. We show first, by contradiction method, that $x < z$. Indeed, if $x = z$, then

$$x = z \;\Rightarrow\; x = y = z \;\Rightarrow\; x^k + y^k + z^k = 3\left(\frac{x+y+z}{3}\right)^k$$
$$\Rightarrow\; a^k + b^k + c^k = 3\left(\frac{a+b+c}{3}\right)^k \;\Rightarrow\; a = b = c,$$

which is false. Notice that the last implication follows from Jensen's inequality

$$a^k + b^k + c^k \ge 3\left(\frac{a+b+c}{3}\right)^k,$$

with equality if and only if $a = b = c$.

According to the relations

$$x + z = a + b + c - y, \quad x^k + z^k = a^k + b^k + c^k - y^k,$$

we may consider x and z as functions of y. From

$$x' + z' = -1, \quad x^{k-1}x' + z^{k-1}z' = -y^{k-1},$$

we get

$$x' = \frac{y^{k-1} - z^{k-1}}{z^{k-1} - x^{k-1}}, \quad z' = \frac{y^{k-1} - x^{k-1}}{x^{k-1} - z^{k-1}}. \tag{*}$$

The two-sided inequality

$$x(y) \le y \le z(y)$$

is equivalent to the inequalities $f_1(y) \le 0$ and $f_2(y) \ge 0$, where

$$f_1(y) = x(y) - y, \quad f_2(y) = z(y) - y.$$

Using (*), we get

$$f_1'(y) = \frac{y^{k-1} - z^{k-1}}{z^{k-1} - x^{k-1}} - 1$$

and

$$f_2'(y) = \frac{y^{k-1} - x^{k-1}}{x^{k-1} - z^{k-1}} - 1.$$

Since $f_1'(y) \le -1$ and $f_2'(y) \le -1$, f_1 and f_2 are strictly decreasing. Thus, the inequality $f_1(y) \le 0$ involves $y \ge m$, where m is the root of the equation $x(y) = y$, while the inequality $f_2(y) \ge 0$ involves $y \le M$, where M is the root of the equation $z(y) = y$. Moreover, $y = m$ if and only if $x = y$, and $y = M$ if and only if $y = z$.

EV-Proposition. *Let a, b, c be fixed real numbers, and let x, y, z be real numbers satisfying*

$$x \le y \le z, \quad x + y + z = a + b + c, \quad x^k + y^k + z^k = a^k + b^k + c^k,$$

where k is an even positive integer. If f is a differentiable function on $\mathbb{R}$ so that the joined function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = f'\left(\sqrt[k-1]{x}\right)$$

is strictly convex on $\mathbb{R}$, then the sum

$$S = f(x) + f(y) + f(z)$$

is minimum if and only if $y = z$, and is maximum if and only if $x = y$.

Proof. If $a = b = c$, then

$$a = b = c \quad \Rightarrow \quad a^k + b^k + c^k = 3\left(\frac{a+b+c}{3}\right)^k$$
$$\Rightarrow \quad x^k + y^k + z^k = 3\left(\frac{x+y+z}{3}\right)^k \quad \Rightarrow \quad x = y = z.$$

Consider further that a, b, c are not all equal. As it is shown in the proof of EV-Lemma, we have $x < z$. According to the relations

$$x + z = a + b + c - y, \quad x^k + z^k = a^k + b^k + c^k - y^k,$$

we may consider x and z as functions of y. Thus, we have

$$S = f(x(y)) + f(y) + f(z(y)) := F(y).$$

According to EV-Lemma, it suffices to show that F is maximum for $y = m$ and is minimum for $y = M$. Using (*), we have

$$\begin{aligned} F'(y) &= x'f'(x) + f'(y) + z'f'(z) \\ &= \frac{y^{k-1} - z^{k-1}}{z^{k-1} - x^{k-1}} g(x^{k-1}) + g(y^{k-1}) + \frac{y^{k-1} - x^{k-1}}{x^{k-1} - z^{k-1}} g(z^{k-1}), \end{aligned}$$

which, for $x < y < z$, is equivalent to

$$\begin{aligned} \frac{F'(y)}{(y^{k-1} - x^{k-1})(y^{k-1} - z^{k-1})} &= \frac{g(x^{k-1})}{(x^{k-1} - y^{k-1})(x^{k-1} - z^{k-1})} \\ &+ \frac{g(y^{k-1})}{(y^{k-1} - z^{k-1})(y^{k-1} - x^{k-1})} + \frac{g(z^{k-1})}{(z^{k-1} - x^{k-1})(z^{k-1} - y^{k-1})}. \end{aligned}$$

Since g is strictly convex, the right hand side is positive. Moreover, since

$$(y^{k-1} - x^{k-1})(y^{k-1} - z^{k-1}) < 0,$$

we have $F'(y) < 0$ for $y \in (m, M)$, hence F is strictly decreasing on $[m, M]$. Therefore, F is maximum for $y = m$ and is minimum for $y = M$.

Proof of EV-Theorem. For $n = 3$, EV-Theorem follows immediately from EV-Proposition. Consider next that $n \geq 4$. Since $X = (x_1, x_2, \ldots, x_n)$ is defined in EV-Theorem as a compact set in $\mathbb{R}^n$, S_n attains its minimum and maximum values. Using this property and EV-Proposition, we can prove EV-Theorem via contradiction. Thus, for the sake of contradiction, assume that S_n attains its maximum at $(b_1, b_2, \ldots, b_n)$, where $b_1 \leq b_2 \leq \cdots \leq b_n$ and $b_1 < b_{n-1}$. Let x_1, x_{n-1} and x_n be real numbers so that

$$x_1 \leq x_{n-1} \leq x_n, \quad x_1 + x_{n-1} + x_n = b_1 + b_{n-1} + b_n, \quad x_1^k + x_{n-1}^k + x_n^k = b_1^k + b_{n-1}^k + b_n^k.$$

According to EV-Proposition, the sum $f(x_1)+f(x_{n-1})+f(x_n)$ is maximum for $x_1 = x_{n-1}$, when

$$f(x_1) + f(x_{n-1}) + f(x_n) > f(b_1) + f(b_{n-1}) + f(b_n).$$

This result contradicts the assumption that S_n attains its maximum value at $(b_1, b_2, \ldots, b_n)$ with $b_1 < b_{n-1}$. Similarly, we can prove that S_n is minimum for $x_2 = x_3 = \cdots = x_n$.

Taking $k = 2$ in EV-Theorem, we obtain the following corollary.

Corollary 1. *Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be fixed real numbers, and let $x_1, x_2, \ldots, x_n$ be real variables so that*

$$x_1 \leq x_2 \leq \cdots \leq x_n,$$
$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n,$$
$$x_1^2 + x_2^2 + \cdots + x_n^2 = a_1^2 + a_2^2 + \cdots + a_n^2.$$

If f is a differentiable function on $\mathbb{R}$ so that the derivative f' is strictly convex on $\mathbb{R}$, then the sum

$$S_n = f(x_1) + f(x_2) + \cdots + f(x_n)$$

is minimum for $x_2 = x_3 = \cdots = x_n$, and is maximum for $x_1 = x_2 = \cdots = x_{n-1}$.

Corollary 2. *Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be fixed real numbers, and let $x_1, x_2, \ldots, x_n$ be real variables so that*

$$x_1 \leq x_2 \leq \cdots \leq x_n,$$
$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n,$$
$$x_1^k + x_2^k + \cdots + x_n^k = a_1^k + a_2^k + \cdots + a_n^k,$$

where k is an even positive integer. For any positive odd number m, $m > k$, the power sum

$$S_n = x_1^m + x_2^m + \cdots + x_n^m$$

is minimum for $x_2 = x_3 = \cdots = x_n$, and is maximum for $x_1 = x_2 = \cdots = x_{n-1}$.

Proof. We apply the EV-Theorem the function $f(u) = u^m$. The joined function

$$g(x) = f'\left(\sqrt[k-1]{x}\right) = m\sqrt[k-1]{x^{m-1}}$$

is strictly convex on $\mathbb{R}$ because its derivative

$$g'(x) = \frac{m(m-1)}{k-1} \sqrt[k-1]{x^{m-k}}$$

is strictly increasing on $\mathbb{R}$.

Theorem 1. *Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be fixed real numbers, and let $x_1, x_2, \ldots, x_n$ be real variables so that*

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n,$$

$$x_1^2 + x_2^2 + \cdots + x_n^2 = a_1^2 + a_2^2 + \cdots + a_n^2.$$

The power sum

$$S_n = x_1^4 + x_2^4 + \cdots + x_n^4$$

is minimum and maximum when the set $(x_1, x_2, \ldots, x_n)$ has at most two distinct values.

To prove this theorem, we will use Proposition 1 below.

Proposition 1. *Let a, b, c be fixed real numbers, and let x, y, z be real numbers so that*

$$x + y + z = a + b + c, \quad x^2 + y^2 + z^2 = a^2 + b^2 + c^2.$$

The power sum

$$S = x^4 + y^4 + z^4$$

is minimum and maximum when two of x, y, z are equal

Proof. The proof is based on EV-Lemma. Without loss of generality, assume that $x \leq y \leq z$. For the nontrivial case when a, b, c are not all equal (which involves $x < z$), consider the function of y

$$F(y) = x^4(y) + y^4 + z^4(y).$$

According to (*), we have

$$\begin{aligned} F'(y) &= 4x^3x' + 4y^3 + 4z^3z' = 4x^3\frac{y-z}{z-x} + 4y^3 + 4z^3\frac{y-x}{x-z} \\ &= 4(x+y+z)(y-x)(y-z) = 4(a+b+c)(y-x)(y-z). \end{aligned}$$

There are three cases to consider.

Case 1: $a + b + c < 0$. Since $F'(y) > 0$ for $x < y < z$, F is strictly increasing on $[m, M]$.

Case 2: $a + b + c > 0$. Since $F'(y) < 0$ for $x < y < z$, F is strictly decreasing on $[m, M]$.

Case 3: $a + b + c = 0$. Since $F'(y) = 0$, F is constant on $[m, M]$.

In all cases, F is monotonic on $m, M]$. Therefore, F is minimum and maximum for $y = m$ or $y = M$; that is, when $x = y$ or $y = z$ (see EV-Lemma). Notice that for $a+b+c \neq 0$, F is strictly monotonic on $[m, M]$, hence F is minimum and maximum if and only if $y = m$ or $y = M$; that is, if and only if $x = y$ or $y = z$.

Proof of Theorem 1. For $n = 3$, Theorem 1 follows from Proposition 1. In order to prove Theorem 1 for any $n \geq 4$, we will use the contradiction method. For the sake of contradiction, assume that $(b_1, b_2, \ldots, b_n)$ is an extreme point having at least three distinct components; let us say $b_1 < b_2 < b_3$. Let x_1, x_2 and x_3 be real numbers so that

$$x_1 \leq x_2 \leq x_3, \quad x_1 + x_2 + x_3 = b_1 + b_2 + b_3 \quad x_1^2 + x_2^2 + x_3^2 = b_1^2 + b_2^2 + b_3^2.$$

We need to consider two cases.

Case 1: $b_1 + b_2 + b_3 \neq 0$. According to Proposition 1, the sum $x_1^4 + x_2^4 + x_3^4$ is extreme only when two of x_1, x_2, x_3 are equal, which contradicts the assumption that the sum $x_1^4 + x_2^4 + \cdots + x_n^4$ attains its extreme value at $(b_1, b_2, \ldots, b_n)$ with $b_1 < b_2 < b_3$.

Case 2: $b_1 + b_2 + b_3 = 0$. There exist three real numbers x_1, x_2, x_3 so that $x_1 = x_2$ and

$$x_1 + x_2 + x_3 = b_1 + b_2 + b_3 = 0, \quad x_1^2 + x_2^2 + x_3^2 = b_1^2 + b_2^2 + b_3^2.$$

Letting $x_1 = x_2 := x$ and $x_3 := y$, we have $2x + y = 0$, $x \neq y$. According to Proposition 1, the sum $x_1^4 + x_2^4 + x_3^4$ is constant (equal to $b_1^4 + b_2^4 + b_3^4$). Thus, $(x, x, y, b_4, \ldots, b_n)$ is also an extreme point. According to our hypothesis, this extreme point has at least three distinct components. Therefore, among the numbers $b_4, \ldots, b_n$ there is one, let us say b_4, so that x, y and b_4 are distinct. Since

$$x + y + b_4 = -x + b_4 \neq 0,$$

we have a case similar to Case 1, which leads to a contradiction.

Theorem 2. *Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be fixed real numbers, and let $x_1, x_2, \ldots, x_n$ be real variables so that*

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n,$$

$$x_1^2 + x_2^2 + \cdots + x_n^2 = a_1^2 + a_2^2 + \cdots + a_n^2.$$

For $m \in \{6, 8\}$, the power sum

$$S_n = x_1^m + x_2^m + \cdots + x_n^m$$

is maximum when the set $(x_1, x_2, \ldots, x_n)$ has at most two distinct values.

Theorem 2 can be proved using Proposition 2 below, in a similar way as the EV-Theorem.

Proposition 2. *Let a,b,c be fixed real numbers, let x,y,z be real numbers so that*

$$x+y+z=a+b+c, \quad x^2+y^2+z^2=a^2+b^2+c^2.$$

For $m\in\{6,8\}$, the power sum

$$S_m=x^m+y^m+z^m$$

is maximum if and only if two of x,y,z are equal.

Proof. Consider the nontrivial case where a,b,c are not all equal. Let

$$p=a+b+c, \qquad q=ab+bc+ca, \qquad r=xyz.$$

Since $x+y+z=p$ and $xy+yz+zx=q$, from

$$(x-y)^2(y-z)^2(z-x)^2\ge 0,$$

which is equivalent to

$$27r^2+2(2p^3-9pq)r-p^2q^2+4q^3\le 0,$$

we get $r\in[r_1,r_2]$, where

$$r_1=\frac{9pq-2p^3-2(p^2-3q)\sqrt{p^2-3q}}{27}, \quad r_2=\frac{9pq-2p^3+2(p^2-3q)\sqrt{p^2-3q}}{27}.$$

From

$$-27(r-r_1)(r-r_2)=(x-y)^2(y-z)^2(z-x)^2\ge 0,$$

it follows that the product $r=xyz$ attains its minimum value r_1 and its maximum value r_2 only when two of x,y,z are equal. For fixed p and q, we have

$$S_6=3r^2+f_6(p,q)r+h_6(p,q):=g_6(r),$$

$$S_8=4(3p^2-2q)r^2+f_8(p,q)r+h_8(p,q):=g_8(r).$$

Since

$$3p^2-2q=\frac{7}{3}p^2+\frac{2}{3}(p^2-3q)>0,$$

the functions g_6 and g_8 are strictly convex, hence are maximum only for $r=r_1$ or $r=r_2$; that is, only when two of x,y,z are equal.

Open problem. *Theorem 2 is valid for any integer number $m\ge 3$.*

Note. The EV-Theorem for real variables and Corollary 1 are also valid under the conditions in Note 2 and Note 3 from the preceding chapter 5, where $m,M\in\mathbb{R}$.

6.2 Applications

6.1. If a,b,c,d are real numbers so that $a+b+c+d=4$, then

$$\left(a^2+b^2+c^2+d^2+\frac{8}{3}\right)^2 \ge 4\left(a^3+b^3+c^3+d^3+\frac{64}{9}\right).$$

6.2. If a,b,c,d are real numbers so that $a+b+c+d=4$, then

$$(a^2+b^2+c^2+d^2-4)\left(a^2+b^2+c^2+d^2+\frac{76}{3}\right) \ge 8(a^3+b^3+c^3+d^3-4).$$

6.3. If a,b,c are real numbers so that $a+b+c=3$, then

$$(a^2+b^2+c^2-3)(a^2+b^2+c^2+93) \ge 24(a^3+b^3+c^3-3).$$

6.4. If a,b,c,d are real numbers so that $a+b+c+d=4$, then

$$(a^2+b^2+c^2+d^2-4)(a^2+b^2+c^2+d^2+116) \ge 24(a^3+b^3+c^3+d^3-4).$$

6.5. Let a,b,c,d be real numbers so that $a+b+c+d=4$, and let

$$E=a^2+b^2+c^2+d^2-4, \qquad F=a^3+b^3+c^3+d^3-4.$$

Prove that

$$E\left(\sqrt{\frac{E}{3}}+3\right) \ge F.$$

6.6. Let $a_1,a_2,\ldots,a_n$ be real numbers so that

$$a_1+a_2+\cdots+a_n=0, \qquad a_1^2+a_2^2+\cdots+a_n^2=n(n-1).$$

If m is an odd number ($m \ge 3$), then

$$n-1-(n-1)^m \le a_1^m+a_2^m+\cdots+a_n^m \le (n-1)^m-n+1.$$

6.7. Let $a_1, a_2, \ldots, a_n$ be real numbers so that

$$a_1 + a_2 + \cdots + a_n = 1, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n^2 + n - 1.$$

If m is an odd number ($m \geq 3$), then

$$(n-1)\left(1+\frac{2}{n}\right)^m - \left(n-\frac{2}{n}\right)^m \leq a_1^m + a_2^m + \cdots + a_n^m \leq n^m - n + 1.$$

6.8. Let $a_1, a_2, \ldots, a_n$ be real numbers so that

$$a_1 + a_2 + \cdots + a_n = 1, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n^2 - 3n + 3.$$

If m is an odd number ($m \geq 3$), then

$$n - 1 - (n-2)^m \leq a_1^m + a_2^m + \cdots + a_n^m \leq \left(n - 2 + \frac{2}{n}\right)^m - (n-1)\left(1 - \frac{2}{n}\right)^m.$$

6.9. Let $a_1, a_2, \ldots, a_n$ be real numbers so that

$$a_1 + a_2 + \cdots + a_n = a_1^2 + a_2^2 + \cdots + a_n^2 = n - 1.$$

If m is an odd number ($m \geq 3$), then

$$n - 1 \leq a_1^m + a_2^m + \cdots + a_n^m \leq (n-1)\left(1 - \frac{2}{n}\right)^m + \left(2 - \frac{2}{n}\right)^m.$$

6.10. Let $a_1, a_2, \ldots, a_n$ be real numbers so that

$$a_1 + a_2 + \cdots + a_n = n + 1, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n + 3.$$

If m is an odd number ($m \geq 3$), then

$$\left(\frac{2}{n}\right)^m + (n-1)\left(1 + \frac{2}{n}\right)^m \leq a_1^m + a_2^m + \cdots + a_n^m \leq 2^m + n - 1.$$

6.11. If $a_1, a_2, \ldots, a_n$ are real numbers so that

$$a_1 + a_2 + \cdots + a_n = a_1^4 + a_2^4 + \cdots + a_n^4 = n - 1,$$

then

$$a_1^5 + a_2^5 + \cdots + a_n^5 \geq n - 1.$$

6.12. If a, b, c are real numbers so that $a^2 + b^2 + c^2 = 3$, then

$$a^3 + b^3 + c^3 + 3 \geq 2(a + b + c).$$

6.13. If $a_1, a_2, \ldots, a_n$ are real numbers so that

$$a_1 + a_2 + \cdots + a_n = 0, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n(n-1),$$

then

$$a_1^4 + a_2^4 + \cdots + a_n^4 \leq n(n-1)(n^2 - 3n + 3).$$

6.14. If $a_1, a_2, \ldots, a_n$ are real numbers so that

$$a_1 + a_2 + \cdots + a_n = n + 1, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = 4n^2 + n - 1,$$

then

$$a_1^4 + a_2^4 + \cdots + a_n^4 \leq 16n^4 + n - 1.$$

6.15. If n is an odd number and $a_1, a_2, \ldots, a_n$ are real numbers so that

$$a_1 + a_2 + \cdots + a_n = 0, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n(n^2 - 1),$$

then

$$a_1^4 + a_2^4 + \cdots + a_n^4 \geq n(n^2 - 1)(n^2 + 3).$$

6.16. If $a_1, a_2, \ldots, a_n$ are real numbers so that

$$a_1 + a_2 + \cdots + a_n = n^2 - n - 1, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n^3 + 2n^2 - n - 1,$$

then

$$a_1^4 + a_2^4 + \cdots + a_n^4 \geq n^4 + (n-1)(n+1)^4.$$

6.17. If $a_1, a_2, \ldots, a_n$ are real numbers so that

$$a_1 + a_2 + \cdots + a_n = n^2 - 2n - 1, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n^3 + 2n + 1,$$

then

$$a_1^4 + a_2^4 + \cdots + a_n^4 \geq (n+1)^4 + (n-1)n^4.$$

6.18. If $a_1, a_2, \ldots, a_n$ are real numbers so that

$$a_1 + a_2 + \cdots + a_n = n^2 - 3n - 2, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n^3 + 2n^2 - 3n - 2,$$

then

$$a_1^4 + a_2^4 + \cdots + a_n^4 \ge 2n^4 + (n-2)(n+1)^4.$$

6.19. If a, b, c, d are real numbers so that $a + b + c + d = 4$, then

$$(a^2 + b^2 + c^2 + d^2 - 4)(a^2 + b^2 + c^2 + d^2 + 36) \le 12(a^4 + b^4 + c^4 + d^4 - 4).$$

6.20. If $a_1, a_2, \ldots, a_n$ are real numbers so that

$$a_1 + a_2 + \cdots + a_n = 0, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n(n-1),$$

then

$$a_1^6 + a_2^6 + \cdots + a_n^6 \le (n-1)^6 + n - 1.$$

6.21. If $a_1, a_2, \ldots, a_n$ are real numbers so that

$$a_1 + a_2 + \cdots + a_n = 1, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n^2 + n - 1,$$

then

$$a_1^6 + a_2^6 + \cdots + a_n^6 \le n^6 + n - 1.$$

6.22. If $a_1, a_2, \ldots, a_n$ are real numbers so that

$$a_1 + a_2 + \cdots + a_n = 0, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n(n-1),$$

then

$$a_1^8 + a_2^8 + \cdots + a_n^8 \le (n-1)^8 + n - 1.$$

6.23. If $a_1, a_2, \ldots, a_n$ are real numbers so that

$$a_1 + a_2 + \cdots + a_n = 1, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n^2 + n - 1,$$

then

$$a_1^8 + a_2^8 + \cdots + a_n^8 \le n^8 + n - 1.$$

6.24. Let $a_1, a_2, \ldots, a_n$ $(n \geq 2)$ be real numbers (not all equal), and let

$$A = \frac{a_1 + a_2 + \cdots + a_n}{n}, \quad B = \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}, \quad C = \frac{a_1^3 + a_2^3 + \cdots + a_n^3}{n}.$$

Then,

$$\frac{1}{4}\left(1 - \sqrt{1 + \frac{2n^2}{n-1}}\right) \leq \frac{B^2 - AC}{B^2 - A^4} \leq \frac{1}{4}\left(1 + \sqrt{1 + \frac{2n^2}{n-1}}\right).$$

6.25. If a, b, c, d are real numbers so that

$$a + b + c + d = 2,$$

then

$$a^4 + b^4 + c^4 + d^4 \leq 40 + \frac{3}{4}(a^2 + b^2 + c^2 + d^2)^2.$$

6.26. If a, b, c, d, e are real numbers, then

$$a^4 + b^4 + c^4 + d^4 + e^4 \leq \frac{31 + 18\sqrt{3}}{8}(a + b + c + d + e)^4 + \frac{3}{4}(a^2 + b^2 + c^2 + d^2 + e^2)^2.$$

6.27. Let $a, b, c, d, e \neq \frac{-5}{4}$ be real numbers so that $a + b + c + d + e = 5$. Then,

$$\frac{a(a-1)}{(4a+5)^2} + \frac{b(b-1)}{(4b+5)^2} + \frac{c(c-1)}{(4c+5)^2} + \frac{d(d-1)}{(4d+5)^2} + \frac{e(e-1)}{(4e+5)^2} \geq 0.$$

6.28. If a, b, c are real numbers so that

$$a + b + c = 9, \qquad ab + bc + ca = 15,$$

then

$$\frac{19}{175} \leq \frac{1}{b^2 + bc + c^2} + \frac{1}{c^2 + ca + a^2} + \frac{1}{a^2 + ab + b^2} \leq \frac{7}{19}.$$

6.29. If a, b, c are real numbers so that

$$8(a^2 + b^2 + c^2) = 9(ab + bc + ca),$$

then

$$\frac{419}{175} \leq \frac{a^2}{b^2 + bc + c^2} + \frac{b^2}{c^2 + ca + a^2} + \frac{c^2}{a^2 + ab + b^2} \leq \frac{311}{19}.$$

6.30. Let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 + a_2 + \cdots + a_n = n$. If $n \leq 10$, then

$$2(a_1^2 + a_2^2 + \cdots + a_n^2)^2 - n(a_1^3 + a_2^3 + \cdots + a_n^3) \geq n^2.$$

6.3 Solutions

P 6.1. *If a,b,c,d are real numbers so that $a+b+c+d=4$, then*

$$\left(a^2+b^2+c^2+d^2+\frac{8}{3}\right)^2 \ge 4\left(a^3+b^3+c^3+d^3+\frac{64}{9}\right).$$

(Vasile Cîrtoaje, 2010)

Solution. Apply Corollary 2 for $n=4$, $k=2$, $m=3$:

- *If a,b,c,d are real numbers so that $a \le b \le c \le d$ and*

$$a+b+c+d=4, \qquad a^2+b^2+c^2+d^2 = constant,$$

then

$$S_4 = a^3+b^3+c^3+d^3$$

is maximum for $a=b=c \le d$.

Thus, we only need to show that $3a+d=4$ involves

$$\left(3a^2+d^2+\frac{8}{3}\right)^2 \ge 4\left(3a^3+d^3+\frac{64}{9}\right).$$

This inequality is equivalent to

$$(a-1)^2(3a-2)^2 \ge 0.$$

The equality holds for $a=b=c=d=1$, and also for

$$a=b=c=\frac{2}{3}, \qquad d=2$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- *If $a_1,a_2,\ldots,a_n$ are real numbers so that*

$$a_1+a_2+\cdots+a_n=n,$$

then

$$\left(a_1^2+a_2^2+\cdots+a_n^2+\frac{n^3}{8n-8}\right)^2 \ge n\left(a_1^3+a_2^3+\cdots+a_n^3\right)+\frac{n^4(n^2+16n-16)}{64(n-1)^2},$$

with equality for $a_1=a_2=\cdots=a_n=1$, and also for

$$a_1=a_2=\cdots=a_{n-1}=\frac{n}{2n-2}, \qquad a_n=\frac{n}{2}$$

(or any cyclic permutation).

□

P 6.2. *If a, b, c, d are real numbers so that $a+b+c+d=4$, then*

$$(a^2+b^2+c^2+d^2-4)\left(a^2+b^2+c^2+d^2+\frac{76}{3}\right) \ge 8(a^3+b^3+c^3+d^3-4).$$

(Vasile Cîrtoaje, 2010)

Solution. As shown in the preceding P 6.1, we only need to show that

$$3a+d=4$$

involves

$$(3a^2+d^2-4)\left(3a^2+d^2+\frac{76}{3}\right) \ge 8(3a^3+d^3-4).$$

This inequality is equivalent to

$$(a-1)^2(3a-1)^2 \ge 0.$$

The equality holds for $a=b=c=d=1$, and also for

$$a=b=c=\frac{1}{3}, \qquad d=3$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- *If $a_1, a_2, \ldots, a_n$ are real numbers so that*

$$a_1+a_2+\cdots+a_n=n,$$

then

$$\left(a_1^2+\cdots+a_n^2-n\right)\left[a_1^2+\cdots+a_n^2+\frac{n(n^2+n-1)}{n-1}\right] \ge 2n\left(a_1^3+\cdots+a_n^3-n\right),$$

with equality for $a_1=a_2=\cdots=a_n=1$, and also for

$$a_1=a_2=\cdots=a_{n-1}=\frac{1}{n-1}, \qquad a_n=n-1$$

(or any cyclic permutation).

□

P 6.3. *If a, b, c are real numbers so that $a+b+c=3$, then*

$$(a^2+b^2+c^2-3)(a^2+b^2+c^2+93) \ge 24(a^3+b^3+c^3-3).$$

(Vasile Cîrtoaje, 2010)

Solution. As shown in the proof of P 6.1, we only need to show that

$$2a + c = 3$$

involves

$$(2a^2 + c^2 - 3)(2a^2 + c^2 + 93) \geq 24(2a^3 + c^3 - 3).$$

This inequality is equivalent to

$$(a^2 - 1)^2 \geq 0.$$

The equality holds for $a = b = c = 1$, and also for

$$a = b = -1, \qquad c = 5$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- *Let a, b, c be real numbers so that $a + b + c = 3$. For any real k, the following inequality holds*

$$(a^2 + b^2 + c^2 - 3)(a^2 + b^2 + c^2 + 6k^2 + 36k - 3) \geq 12k(a^3 + b^3 + c^3 - 3),$$

with equality for $a = b = c = 1$, and also for

$$a = b = 1 - k, \qquad c = 1 + 2k$$

(or any cyclic permutation).

□

P 6.4. *If a, b, c, d are real numbers so that $a + b + c + d = 4$, then*

$$(a^2 + b^2 + c^2 + d^2 - 4)(a^2 + b^2 + c^2 + d^2 + 116) \geq 24(a^3 + b^3 + c^3 + d^3 - 4).$$

(Vasile Cîrtoaje, 2010)

Solution. As shown in the proof of P 6.1, we only need to show that

$$3a + d = 4$$

involves

$$(3a^2 + d^2 - 4)(3a^2 + d^2 + 116) \geq 24(3a^3 + d^3 - 4).$$

This inequality is equivalent to

$$(a^2 - 1)^2 \geq 0.$$

The equality holds for $a=b=c=d=1$, and also for

$$a=b=c=-1, \qquad d=7$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- *Let $a_1,a_2,\ldots,a_n$ be real numbers so that*

$$a_1+a_2+\cdots+a_n=n.$$

If k is a real number, then

$$\frac{k(a_1^3+\cdots+a_n^3-n)}{a_1^2+\cdots+a_n^2-n} \le \frac{a_1^2+\cdots+a_n^2+n(n-1)(n-2)^2k^2+6n(n-1)k-n}{2n(n-1)},$$

with equality for

$$a_1=\cdots=a_{n-1}=1-(n-2)k, \qquad a_n=1+(n-1)(n-2)k$$

(or any cyclic permutation).

For $k=\dfrac{-6}{n-2}$, we get the following nice inequality

$$\left(a_1^2+a_2^2+\cdots+a_n^2-n\right)^2+\frac{12n(n-1)}{n-2}\left(a_1^3+a_2^3+\cdots+a_n^3-n\right)\ge 0,$$

with equality for $a_1=a_2=\cdots=a_n=1$, and also for

$$a_1=\cdots=a_{n-1}=7, \qquad a_n=7-6n$$

(or any cyclic permutation).

□

P 6.5. *Let a,b,c,d be real numbers so that $a+b+c+d=4$, and let*

$$E=a^2+b^2+c^2+d^2-4, \qquad F=a^3+b^3+c^3+d^3-4.$$

Prove that

$$E\left(\sqrt{\frac{E}{3}}+3\right)\ge F.$$

(Vasile Cîrtoaje, 2016)

Solution. As shown in the proof of P 6.1, we only need to prove the desired inequality for $3a+d=4$ and

$$E = 3a^2 + d^2 - 4, \qquad F = 3a^3 + d^3 - 4.$$

Since

$$E = 12(1-a)^2, \qquad F = 12(5-2a)(1-a)^2,$$

we get

$$E\left(\sqrt{\frac{E}{3}}+3\right) - F = 12(1-a)^2(2|1-a|+3) - 12(5-2a)(1-a)^2$$
$$= 24(1-a)^2[|1-a| - (1-a)] \geq 0.$$

The equality holds for

$$a = b = c = \frac{4-d}{3} \leq 1$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- *Let $a_1, a_2, \ldots, a_n$ be real numbers so that $a_1 + a_2 + \cdots + a_n = n$, and let*

$$E = a_1^2 + a_2^2 + \cdots + a_n^2 - n, \qquad F = a_1^3 + a_2^3 + \cdots + a_n^3 - n.$$

Then,

$$E\left[(n-2)\sqrt{\frac{E}{n(n-1)}}+3\right] \geq F,$$

with equality for

$$a_1 = \cdots = a_{n-1} = \frac{n-a_n}{n-1} \leq 1$$

(or any cyclic permutation).

□

P 6.6. *Let $a_1, a_2, \ldots, a_n$ be real numbers so that*

$$a_1 + a_2 + \cdots + a_n = 0, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n(n-1).$$

If m is an odd number ($m \geq 3$), then

$$n - 1 - (n-1)^m \leq a_1^m + a_2^m + \cdots + a_n^m \leq (n-1)^m - n + 1.$$

(Vasile Cîrtoaje, 2010)

Solution. Without loss of generality, assume that

$$a_1 \le a_2 \le \cdots \le a_n.$$

(a) Consider the right inequality. For $n = 2$, we need to show that

$$a_1 + a_2 = 0, \qquad a_1^2 + a_2^2 = 2$$

implies

$$a_1^m + a_2^m \le 0.$$

We have

$$a_1 = -1, \qquad a_2 = 1,$$

therefore $a_1^m + a_2^m = 0$. Assume now that $n \ge 3$. According to Corollary 2, the sum

$$S_n = a_1^m + a_2^m + \cdots + a_n^m$$

is maximum for $a_1 = a_2 = \cdots = a_{n-1}$. Thus, we only need to show that

$$(n-1)a + b = 0, \qquad (n-1)a^2 + b^2 = n(n-1), \qquad a \le b$$

involve

$$(n-1)a^m + b^m \le (n-1)^m - n + 1.$$

From the equations above, we get

$$a = -1, \qquad b = n - 1;$$

therefore,

$$(n-1)a^m + b^m = (n-1)(-1)^m + (n-1)^m = (n-1)^m - n + 1.$$

The equality holds for

$$a_1 = \cdots = a_{n-1} = -1, \qquad a_n = n - 1$$

(or any cyclic permutation).

(b) The left inequality follows from the right inequality by replacing $a_1, a_2, \ldots, a_n$ with $-a_1, -a_2, \ldots, -a_n$, respectively. The equality holds for

$$a_1 = -n + 1, \qquad a_2 = a_3 = \cdots = a_n = 1$$

(or any cyclic permutation).

□

P 6.7. *Let $a_1, a_2, \ldots, a_n$ be real numbers so that*

$$a_1 + a_2 + \cdots + a_n = 1, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n^2 + n - 1.$$

If m is an odd number ($m \ge 3$), then

$$(n-1)\left(1+\frac{2}{n}\right)^m - \left(n-\frac{2}{n}\right)^m \le a_1^m + a_2^m + \cdots + a_n^m \le n^m - n + 1.$$

(Vasile Cîrtoaje, 2010)

Solution. Without loss of generality, assume that

$$a_1 \le a_2 \le \cdots \le a_n.$$

For $n = 2$, we need to show that

$$a_1 + a_2 = 1, \qquad a_1^2 + a_2^2 = 5,$$

implies

$$2^m - 1 \le a_1^m + a_2^m \le 2^m - 1.$$

We have

$$a_1 = -1, \qquad a_2 = 2,$$

for which $a_1^m + a_2^m = 2^m - 1$. Assume now that $n \ge 3$.

(a) Consider the right inequality. According to Corollary 2, the sum

$$S_n = a_1^m + a_2^m + \cdots + a_n^m$$

is maximum for $a_1 = a_2 = \cdots = a_{n-1}$. Thus, we only need to show that

$$(n-1)a + b = 1, \qquad (n-1)a^2 + b^2 = n^2 + n - 1, \qquad a \le b$$

involve

$$(n-1)a^m + b^m \le n^m - n + 1.$$

From the equations above, we get

$$a = -1, \qquad b = n;$$

therefore,

$$(n-1)a^m + b^m = (n-1)(-1)^m + n^m = n^m - n + 1.$$

The equality holds for

$$a_1 = a_2 = \cdots = a_{n-1} = -1, \qquad a_n = n$$

(or any cyclic permutation).

(b) Consider the left inequality. According to Corollary 2, the sum

$$S_n = a_1^m + a_2^m + \cdots + a_n^m$$

is minimum for $a_2 = a_3 = \cdots = a_n$. Thus, we only need to show that

$$a + (n-1)b = 1, \qquad a^2 + (n-1)b^2 = n^2 + n - 1, \qquad a \le b$$

involve

$$a^m + (n-1)b^m \ge (n-1)\left(1+\frac{2}{n}\right)^m - \left(n-\frac{2}{n}\right)^m.$$

From the equations above, we get

$$a = -n + \frac{2}{n}, \qquad b = 1 + \frac{2}{n};$$

therefore,

$$\begin{aligned} a^m + (n-1)b^m &= \left(-n+\frac{2}{n}\right)^m + (n-1)\left(1+\frac{2}{n}\right)^m \\ &= (n-1)\left(1+\frac{2}{n}\right)^m - \left(n-\frac{2}{n}\right)^m. \end{aligned}$$

The equality holds for

$$a_1 = -n + \frac{2}{n}, \qquad a_2 = a_3 = \cdots = a_n = 1 + \frac{2}{n}$$

(or any cyclic permutation).

□

P 6.8. *Let $a_1, a_2, \ldots, a_n$ be real numbers so that*

$$a_1 + a_2 + \cdots + a_n = 1, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n^2 - 3n + 3.$$

If m is an odd number ($m \ge 3$), then

$$n - 1 - (n-2)^m \le a_1^m + a_2^m + \cdots + a_n^m \le \left(n-2+\frac{2}{n}\right)^m - (n-1)\left(1-\frac{2}{n}\right)^m.$$

(Vasile Cîrtoaje, 2010)

***Solution*.** Without loss of generality, assume that

$$a_1 \le a_2 \le \cdots \le a_n.$$

For $n = 2$, we need to show that

$$a_1 + a_2 = 1, \qquad a_1^2 + a_2^2 = 1,$$

implies

$$1 \le a_1^m + a_2^m \le 1.$$

We have

$$a_1 = 0, \qquad a_2 = 1,$$

when $a_1^m + a_2^m = 1$. Assume now that $n \ge 3$.

(a) Consider the left inequality. According to Corollary 2, the sum

$$S_n = a_1^m + a_2^m + \cdots + a_n^m$$

is minimum for $a_2 = a_3 = \cdots = a_n$. Thus, we only need to show that

$$a + (n-1)b = 1, \qquad a^2 + (n-1)b^2 = n^2 - 3n + 3, \qquad a \le b$$

involve

$$a^m + (n-1)b^m \le n - 1 - (n-2)^m.$$

From the equations above, we get

$$a = 2 - n, \qquad b = 1;$$

therefore,

$$a^m + (n-1)b^m = (2-n)^m + n - 1 = n - 1 - (n-2)^m.$$

The equality holds for

$$a_1 = 2 - n, \qquad a_2 = a_3 = \cdots = a_n = 1$$

(or any cyclic permutation).

(b) Consider the right inequality. According to Corollary 2, the sum

$$S_n = a_1^m + a_2^m + \cdots + a_n^m$$

is maximum for $a_1 = a_2 = \cdots = a_{n-1}$. Thus, we only need to show that

$$(n-1)a + b = 1, \qquad (n-1)a^2 + b^2 = n^2 - 3n + 3, \qquad a \le b$$

involve

$$(n-1)a^m + b^m \le \left(n - 2 + \frac{2}{n}\right)^m - (n-1)\left(1 - \frac{2}{n}\right)^m.$$

From the equations above, we get

$$a = -1 + \frac{2}{n}, \qquad b = n - 2 + \frac{2}{n};$$

therefore,

$$(n-1)a^m + b^m = (n-1)\left(-1+\frac{2}{n}\right)^m + \left(n-2+\frac{2}{n}\right)^m$$
$$= \left(n-2+\frac{2}{n}\right)^m - (n-1)\left(1-\frac{2}{n}\right)^m.$$

The equality holds for

$$a_1 = \cdots = a_{n-1} = -1+\frac{2}{n}, \qquad a_n = n-2+\frac{2}{n}$$

(or any cyclic permutation).

□

P 6.9. *Let $a_1, a_2, \ldots, a_n$ be real numbers so that*

$$a_1 + a_2 + \cdots + a_n = a_1^2 + a_2^2 + \cdots + a_n^2 = n-1.$$

If m is an odd number ($m \ge 3$), then

$$n-1 \le a_1^m + a_2^m + \cdots + a_n^m \le (n-1)\left(1-\frac{2}{n}\right)^m + \left(2-\frac{2}{n}\right)^m.$$

(Vasile Cîrtoaje, 2010)

Solution. Without loss of generality, assume that

$$a_1 \le a_2 \le \cdots \le a_n.$$

For $n = 2$, we need to show that

$$a_1 + a_2 = 1, \qquad a_1^2 + a_2^2 = 1,$$

implies

$$1 \le a_1^m + a_2^m \le 1.$$

The above equations involve

$$a_1 = 0, \qquad a_2 = 1,$$

hence $a_1^m + a_2^m = 1$. Assume now that $n \ge 3$.

(a) Consider the left inequality. According to Corollary 2, the sum

$$S_n = a_1^m + a_2^m + \cdots + a_n^m$$

is minimum for $a_2 = a_3 = \cdots = a_n$. Thus, we only need to show that

$$a + (n-1)b = n-1, \qquad a^2 + (n-1)b^2 = n-1, \qquad a \le b$$

involve
$$a^m + (n-1)b^m \geq n-1.$$
From the equations above, we get
$$a = 0, \qquad b = 1;$$
therefore,
$$a^m + (n-1)b^m = n-1.$$
The equality holds for
$$a_1 = 0, \qquad a_2 = \cdots = a_n = 1$$
(or any cyclic permutation).

(b) Consider the right inequality. According to Corollary 2, the sum
$$S_n = a_1^m + a_2^m + \cdots + a_n^m$$
is maximum for $a_1 = a_2 = \cdots = a_{n-1}$. Thus, we only need to show that
$$(n-1)a + b = n-1, \qquad (n-1)a^2 + b^2 = n-1, \qquad a \leq b$$
involve
$$(n-1)a^m + b^m \leq (n-1)\left(1 - \frac{2}{n}\right)^m + \left(2 - \frac{2}{n}\right)^m.$$
From the equations above, we get
$$a = 1 - \frac{2}{n}, \qquad b = 2 - \frac{2}{n},$$
when
$$(n-1)a^m + b^m = (n-1)\left(1 - \frac{2}{n}\right)^m + \left(2 - \frac{2}{n}\right)^m.$$
The equality holds for
$$a_1 = a_2 = \cdots = a_{n-1} = 1 - \frac{2}{n}, \qquad a_n = 2 - \frac{2}{n}$$
(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- *Let $a_1, a_2, \ldots, a_n$ be real numbers so that*
$$a_1 + a_2 + \cdots + a_n = k, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n^2 + (2k-1)n + k(k-2),$$
where k is a real number, $k \geq -n$. If m is an odd number ($m \geq 3$), then
$$\left(\frac{2k}{n} + 1 - n - k\right)^m + (n-1)\left(\frac{2k}{n} + 1\right)^m \leq a_1^m + a_2^m + \cdots + a_n^m \leq (n+k-1)^m - n + 1.$$

The left inequality is an equality for

$$a_1 = \frac{2k}{n} + 1 - n - k, \qquad a_2 = \cdots = a_n = \frac{2k}{n} + 1$$

(or any cyclic permutation). The right inequality is an equality for

$$a_1 = \cdots = a_{n-1} = -1, \qquad a_n = n + k - 1$$

(or any cyclic permutation).

For $k = 0$ and $k = 1$, we get the inequalities in P 6.6 and P 6.7, respectively. For $k = -1$ and $k = -n+1$, by replacing k with $-k$ and $a_1, a_2, \ldots, a_n$ with $-a_1, -a_2, \ldots, -a_n$, we get the inequalities in P 6.8 and P 6.9, respectively.

□

P 6.10. *Let $a_1, a_2, \ldots, a_n$ be real numbers so that*

$$a_1 + a_2 + \cdots + a_n = n + 1, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n + 3.$$

If m is an odd number ($m \geq 3$), then

$$\left(\frac{2}{n}\right)^m + (n-1)\left(1 + \frac{2}{n}\right)^m \leq a_1^m + a_2^m + \cdots + a_n^m \leq 2^m + n - 1.$$

(Vasile Cîrtoaje, 2010)

Solution. Without loss of generality, assume that

$$a_1 \leq a_2 \leq \cdots \leq a_n.$$

For $n = 2$, we need to show that

$$a_1 + a_2 = 3, \qquad a_1^2 + a_2^2 = 5,$$

implies

$$2^m + 1 \leq a_1^m + a_2^m \leq 2^m + 1.$$

We get

$$a_1 = 1, \qquad a_2 = 2,$$

when $a_1^m + a_2^m = 2^m + 1$. Assume now that $n \geq 3$.

(a) Consider the left inequality. According to Corollary 2, the sum

$$S_n = a_1^m + a_2^m + \cdots + a_n^m$$

is minimum for $a_2 = a_3 = \cdots = a_n$. Thus, we only need to show that

$$a + (n-1)b = n + 1, \qquad a^2 + (n-1)b^2 = n + 3, \qquad a \leq b$$

involve

$$a^m + (n-1)b^m \geq \left(\frac{2}{n}\right)^m + (n-1)\left(1+\frac{2}{n}\right)^m.$$

From the equations

$$a + (n-1)b = n+1, \qquad a^2 + (n-1)b^2 = n+3,$$

we get

$$a = \frac{2}{n}, \qquad b = 1 + \frac{2}{n};$$

therefore,

$$a^m + (n-1)b^m = \left(\frac{2}{n}\right)^m + (n-1)\left(1+\frac{2}{n}\right)^m.$$

The equality holds for

$$a_1 = \frac{2}{n}, \qquad a_2 = \cdots = a_n = 1 + \frac{2}{n}$$

(or any cyclic permutation).

(b) Consider the right inequality. According to Corollary 2, the sum

$$S_n = a_1^m + a_2^m + \cdots + a_n^m$$

is maximum for $a_1 = a_2 = \cdots = a_{n-1}$. Thus, we only need to show that

$$(n-1)a + b = n+1, \qquad (n-1)a^2 + b^2 = n+3, \qquad a \leq b$$

involve

$$(n-1)a^m + b^m \leq 2^m + n - 1.$$

From the equations

$$(n-1)a + b = n+1, \qquad (n-1)a^2 + b^2 = n+3,$$

we get

$$a = 1, \qquad b = 2;$$

therefore,

$$(n-1)a^m + b^m = n - 1 + 2^m.$$

The equality holds for

$$a_1 = \cdots = a_{n-1} = 1, \qquad a_n = 2$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- *Let $a_1, a_2, \ldots, a_n$ be real numbers so that*

$$a_1 + a_2 + \cdots + a_n = k, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n^2 - (2k+1)n + k(k+2),$$

where k is a positive number, $k > n$. If m is an odd number ($m \ge 3$), then

$$\left(\frac{2k}{n} - 1 + n - k\right)^m + (n-1)\left(\frac{2k}{n} - 1\right)^m \le a_1^m + a_2^m + \cdots + a_n^m \le (k-n+1)^m + n - 1.$$

The left inequality is an equality for

$$a_1 = \frac{2k}{n} - 1 + n - k, \qquad a_2 = \cdots = a_n = \frac{2k}{n} - 1$$

(or any cyclic permutation). The right inequality is an equality for

$$a_1 = \cdots = a_{n-1} = 1, \qquad a_n = k - n + 1$$

(or any cyclic permutation).

For $k = n + 1$, we get the inequalities in P 6.10.

□

P 6.11. *If $a_1, a_2, \ldots, a_n$ are real numbers so that*

$$a_1 + a_2 + \cdots + a_n = a_1^4 + a_2^4 + \cdots + a_n^4 = n - 1,$$

then

$$a_1^5 + a_2^5 + \cdots + a_n^5 \ge n - 1.$$

(Vasile Cîrtoaje, 2010)

Solution. For $n = 2$, we need to show that

$$a_1 + a_2 = 1, \qquad a_1^4 + a_2^4 = 1,$$

implies

$$a_1^5 + a_2^5 \ge 1.$$

We have

$$a_1 = 0, \qquad a_2 = 1,$$

or

$$a_1 = 1, \qquad a_2 = 0.$$

For each of these cases, the inequality is an equality. Assume now that $n \ge 3$ and

$$a_1 \le a_2 \le \cdots \le a_n.$$

According to Corollary 2, the sum

$$S_n = a_1^5 + a_2^5 + \cdots + a_n^5$$

is minimum for $a_2 = a_3 = \cdots = a_n$. Thus, we only need to show that

$$a + (n-1)b = a^4 + (n-1)b^4 = n-1, \qquad a \le b$$

involve

$$a^5 + (n-1)b^5 \ge n-1.$$

The equations

$$a + (n-1)b = n-1, \qquad a^4 + (n-1)b^4 = n-1,$$

are equivalent to

$$(1-b)[(n-1)^3(1-b)^3 - 1 - b - b^2 - b^3] = 0, \qquad a = (n-1)(1-b);$$

that is,

$$b = 1, \qquad a = 0,$$

and

$$a^3 = 1 + b + b^2 + b^3, \qquad a = (n-1)(1-b).$$

For the second case, the condition $a \le b$ involves

$$b^3 \ge 1 + b + b^2 + b^3,$$

which is not possible. Therefore, it suffices to show that

$$a^5 + (n-1)b^5 \ge n-1$$

for $a = 0$ and $b = 1$, that is clearly true. Thus, the proof is completed. The equality holds for

$$a_1 = 0, \qquad a_2 = \cdots = a_n = 1$$

(or any cyclic permutation).

□

P 6.12. *If a, b, c are real numbers so that*

$$a^2 + b^2 + c^2 = 3,$$

then

$$a^3 + b^3 + c^3 + 3 \ge 2(a + b + c).$$

(Vasile Cîrtoaje, 2010)

Solution. Assume that
$$a \le b \le c.$$
According to Corollary 2, for $a \le b \le c$ and
$$a+b+c = constant, \qquad a^2+b^2+c^2=3,$$
the sum
$$S_3 = a^3+b^3+c^3$$
is minimum for $a \le b = c$. Thus, we only need to show that
$$a^2+2b^2=3, \qquad a \le b,$$
involves
$$a^3+2b^3+3 \ge 2(a+2b).$$
We will show this by two methods. From $a^2+2b^2=3$ and $a \le b$, it follows that
$$-\sqrt{3} \le a \le 1, \quad -\sqrt{\frac{3}{2}} < b \le \sqrt{\frac{3}{2}}.$$

Method 1. Write the desired inequality as
$$a^3+b(3-a^2)+3 \ge 2(a+2b),$$
$$a^3-2a+3 \ge b(a^2+1).$$
For $a \ge 0$, we have
$$a^3-2a+3 \ge -2a+3 > 0,$$
and for $a \le 0$, we have
$$a^3-2a+3 = a(a^2-3)+a+3 = -2ab^2+a+3 \ge a+3 > 0.$$
Thus, it suffices to show that
$$(a^3-2a+3)^2 \ge b^2(a^2+1)^2,$$
which is equivalent to
$$2(a^3-2a+3)^2 \ge (3-a^2)(a^2+1)^2,$$
$$(a-1)^2 f(a) \ge 0,$$
where
$$f(a) = a^4+2a^3+2a+5.$$
We need to prove that $f(a) \ge 0$. For $a \ge -1$, we have
$$f(a) = (a+2)(a^3+2)+1 > 0.$$

For $a \le -1$, we have

$$f(a) = (a+1)^2(a+2)^2 + g(a), \qquad g(a) = -4a^3 - 13a^2 - 10a + 1.$$

It suffices to show that $g(a) \ge 0$. Since

$$g(a) = -(a+1)\left(2a + \frac{7}{2}\right)^2 + 5h(a), \qquad h(a) = a^2 + \frac{13}{4}a + \frac{53}{20}$$

and

$$h(a) = \left(a + \frac{13}{8}\right)^2 + \frac{3}{320} > 0,$$

the conclusion follows. The equality holds for $a = b = c = 1$.

Method 2. Write the desired inequality as follows:

$$2(a^3 - 2a + 1) + 4(b^3 - 2b + 1) \ge 0,$$

$$2(a^3 - 2a + 1) + 4(b^3 - 2b + 1) \ge a^2 + 2b^2 - 3,$$

$$(2a^3 - a^2 - 4a + 3) + 2(b^3 - b^2 - 4b + 3) \ge 0,$$

$$(a-1)^2(2a+3) + 2(b-1)^2(2b+3) \ge 0.$$

Since $2b + 3 > 0$, the inequality is true for $a \ge -3/2$. Consider further that

$$-\sqrt{3} \le a \le \frac{-3}{2},$$

and rewrite the desired inequality as follows:

$$2(a^3 - 2a + 1) + 4(b^3 - 2b + 1) + 4(a^2 + 2b^2 - 3) \ge 0,$$

$$(2a^3 + 4a^2 - 4a - 2) + 2(2b^3 + 4b^2 - 4b - 2) \ge 0,$$

$$\left(2a^3 + 4a^2 - 4a - \frac{33}{4}\right) + \left(4b^3 + 8b^2 - 8b + \frac{9}{4}\right) \ge 0,$$

$$(2a+3)\left(a^2 + \frac{1}{2}a - \frac{11}{4}\right) + f(b) \ge 0,$$

where

$$f(b) = 4b^3 + 8b^2 - 8b + \frac{9}{4}.$$

Since $2a + 3 \le 0$ and

$$a^2 + \frac{1}{2}a - \frac{11}{4} \le 3 + \frac{1}{2}a - \frac{11}{4} = \frac{1}{4}(2a+1) < 0,$$

it suffices to show that $f(b) \ge 0$. For $b \ge 0$, we have

$$f(b) > 8b^2 - 8b + 2 = 2(2b-1)^2 \ge 0,$$

and for $b \le 0$, we have

$$f(b) > 4b^3 + 8b^2 = 4b^2(b+2) \ge 0.$$

□

P 6.13. *If $a_1, a_2, \dots, a_n$ are real numbers so that*

$$a_1 + a_2 + \cdots + a_n = 0, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n(n-1),$$

then

$$a_1^4 + a_2^4 + \cdots + a_n^4 \le n(n-1)(n^2 - 3n + 3).$$

(Vasile Cîrtoaje, 2010)

Solution. For $n = 2$, we need to show that

$$a_1 + a_2 = 0, \qquad a_1^2 + a_2^2 = 2,$$

implies

$$a_1^4 + a_2^4 \le 2.$$

We have

$$a_1 = -1, \qquad a_2 = 1,$$

or

$$a_1 = 1, \qquad a_2 = -1.$$

For each of these cases, the desired inequality is an equality. Assume now that $n \ge 3$. According to Theorem 1, the sum

$$S_n = a_1^4 + a_2^4 + \cdots + a_n^4$$

is maximum for

$$a_1 = \cdots = a_j, \qquad a_{j+1} = \cdots = a_n,$$

where $j \in \{1, 2, \dots, n-1\}$. Thus, we only need to show that

$$ja_1 + (n-j)a_n = 0, \qquad ja_1^2 + (n-j)a_n^2 = n(n-1)$$

involve

$$ja_1^4 + (n-j)a_n^4 \le n(n-1)(n^2 - 3n + 3).$$

From the equations above, we get

$$a_1^2 = \frac{(n-j)(n-1)}{j}, \qquad a_n^2 = \frac{j(n-1)}{n-j};$$

therefore,

$$ja_1^4 + (n-j)a_n^4 = \frac{(n-j)^3 + j^3}{j(n-j)}(n-1)^2 = \left[\frac{n^2}{j(n-j)} - 3\right] n(n-1)^2.$$

Since

$$j(n-j) - (n-1) = (j-1)(n-j-1) \ge 0,$$

we get

$$ja_1^4 + (n-j)a_n^4 \le \left[\frac{n^2}{n-1} - 3\right] n(n-1)^2 = n(n-1)(n^2 - 3n + 3).$$

The equality holds for

$$a_1 = -n+1, \qquad a_2 = \cdots = a_n = 1$$

and for

$$a_1 = n-1, \qquad a_2 = \cdots = a_n = -1$$

(or any cyclic permutation).

□

P 6.14. *If $a_1, a_2, \ldots, a_n$ are real numbers so that*

$$a_1 + a_2 + \cdots + a_n = n+1, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = 4n^2 + n - 1,$$

then

$$a_1^4 + a_2^4 + \cdots + a_n^4 \le 16n^4 + n - 1.$$

(Vasile Cîrtoaje, 2010)

Solution. Replacing n by $2n+1$ in the preceding P 6.13, we get the following statement:

- *If $a_1, a_2, \ldots, a_{2n+1}$ are real numbers so that*

$$a_1 + a_2 + \cdots + a_{2n+1} = 0, \qquad a_1^2 + a_2^2 + \cdots + a_{2n+1}^2 = 2n(2n+1),$$

then

$$a_1^4 + a_2^4 + \cdots + a_{2n+1}^4 \le 2n(2n+1)(4n^2 - 2n + 1),$$

with equality for

$$a_1 = -2n, \qquad a_2 = \cdots = a_{2n+1} = 1$$

and for

$$a_1 = 2n, \qquad a_2 = \cdots = a_{2n+1} = -1$$

(or any cyclic permutation).

Putting

$$a_{n+1} = \cdots = a_{2n+1} = -1,$$

it follows that

$$a_1 + a_2 + \cdots + a_n - n - 1 = 0, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 + n + 1 = 2n(2n+1)$$

involve
$$a_1^4 + a_2^4 + \cdots + a_n^4 + n + 1 \le 2n(2n+1)(4n^2 - 2n + 1).$$
This is equivalent to the desired statement. The equality holds for
$$a_1 = 2n, \qquad a_2 = \cdots = a_n = -1$$
(or any cyclic permutation).

□

P 6.15. *If n is an odd number and $a_1, a_2, \ldots, a_n$ are real numbers so that*
$$a_1 + a_2 + \cdots + a_n = 0, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n(n^2 - 1),$$
then
$$a_1^4 + a_2^4 + \cdots + a_n^4 \ge n(n^2 - 1)(n^2 + 3).$$

(Vasile Cîrtoaje, 2010)

Solution. According to Theorem 1, the sum
$$S_n = a_1^4 + a_2^4 + \cdots + a_n^4$$
is minimum for
$$a_1 = \cdots = a_j, \qquad a_{j+1} = \cdots = a_n,$$
where $j \in \{1, 2, \ldots, n-1\}$. Thus, we only need to show that
$$ja_1 + (n-j)a_n = 0, \qquad ja_1^2 + (n-j)a_n^2 = n(n^2 - 1)$$
involve
$$ja_1^4 + (n-j)a_n^4 \le n(n^2 - 1)(n^2 + 3).$$
From the equations above, we get
$$a_1^2 = \frac{(n-j)(n^2-1)}{j}, \qquad a_n^2 = \frac{j(n^2-1)}{n-j};$$
therefore,
$$ja_1^4 + (n-j)a_n^4 = \frac{(n-j)^3 + j^3}{j(n-j)}(n^2-1)^2 = \left[\frac{n^2}{j(n-j)} - 3\right] n(n^2-1)^2.$$
Since
$$\frac{n^2-1}{4} - j(n-j) = \frac{(n-2j)^2 - 1}{4} \ge 0,$$
we get
$$ja_1^4 + (n-j)a_n^4 \ge \left(\frac{4n^2}{n^2-1} - 3\right) n(n^2-1)^2 = n(n^2-1)(n^2+3).$$

The equality holds when $\frac{n-1}{2}$ of $a_1, a_2, \ldots, a_n$ are equal to $-n-1$ and the other $\frac{n+1}{2}$ are equal to $n-1$, and also when $\frac{n-1}{2}$ of $a_1, a_2, \ldots, a_n$ are equal to $n+1$ and the other $\frac{n+1}{2}$ are equal to $-n+1$.

□

P 6.16. *If $a_1, a_2, \ldots, a_n$ are real numbers so that*

$$a_1 + a_2 + \cdots + a_n = n^2 - n - 1, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n^3 + 2n^2 - n - 1,$$

then

$$a_1^4 + a_2^4 + \cdots + a_n^4 \ge n^4 + (n-1)(n+1)^4.$$

(Vasile Cîrtoaje, 2010)

Solution. Replacing $a_1, a_2, \ldots, a_n$ by $2a_1, 2a_2, \ldots, 2a_n$ and then n by $2n+1$, the preceding P 6.15 becomes as follows:

- *If $a_1, a_2, \ldots, a_{2n+1}$ are real numbers so that*

$$a_1 + a_2 + \cdots + a_{2n+1} = 0, \qquad a_1^2 + a_2^2 + \cdots + a_{2n+1}^2 = n(n+1)(2n+1),$$

then

$$a_1^4 + a_2^4 + \cdots + a_{2n+1}^4 \ge n(n+1)(2n+1)(n^2+n+1),$$

with equality when n of $a_1, a_2, \ldots, a_{2n+1}$ are equal to $-n-1$ and the other $n+1$ are equal to n, and also when n of $a_1, a_2, \ldots, a_{2n+1}$ are equal to $n+1$ and the other $n+1$ are equal to $-n$.

Putting

$$a_{n+1} = \cdots = a_{2n} = -n, \qquad a_{2n+1} = n+1,$$

it follows that

$$a_1 + a_2 + \cdots + a_n + n(-n) + (n+1) = 0$$

and

$$a_1^2 + a_2^2 + \cdots + a_n^2 + n(-n)^2 + (n+1)^2 = n(n+1)(2n+1)$$

involve

$$a_1^4 + a_2^4 + \cdots + a_n^4 + n(-n)^4 + (n+1)^4 \le n(n+1)(2n+1)(n^2+n+1).$$

This is equivalent to the desired statement. The equality holds for

$$a_1 = \cdots = a_{n-1} = n+1, \qquad a_n = -n$$

(or any cyclic permutation).

□

P 6.17. *If $a_1, a_2, \dots, a_n$ are real numbers so that*

$$a_1 + a_2 + \cdots + a_n = n^2 - 2n - 1, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n^3 + 2n + 1,$$

then

$$a_1^4 + a_2^4 + \cdots + a_n^4 \ge (n+1)^4 + (n-1)n^4.$$

(Vasile Cîrtoaje, 2010)

Solution. As shown in the proof of the preceding P 6.16, the following statement holds:

- *If $a_1, a_2, \dots, a_{2n+1}$ are real numbers so that*

$$a_1 + a_2 + \cdots + a_{2n+1} = 0, \qquad a_1^2 + a_2^2 + \cdots + a_{2n+1}^2 = n(n+1)(2n+1),$$

then

$$a_1^4 + a_2^4 + \cdots + a_{2n+1}^4 \ge n(n+1)(2n+1)(n^2 + n + 1),$$

with equality when n of $a_1, a_2, \dots, a_{2n+1}$ are equal to $-n-1$ and the other $n+1$ are equal to n, and also when n of $a_1, a_2, \dots, a_{2n+1}$ are equal to $n+1$ and the other $n+1$ are equal to $-n$.

Putting

$$a_{n+1} = \cdots = a_{2n-1} = -n - 1, \qquad a_{2n} = a_{2n+1} = n,$$

it follows that

$$a_1 + a_2 + \cdots + a_n + (n-1)(-n-1) + 2n = 0$$

and

$$a_1^2 + a_2^2 + \cdots + a_n^2 + (n-1)(-n-1)^2 + 2n^2 = n(n+1)(2n+1)$$

involve

$$a_1^4 + a_2^4 + \cdots + a_n^4 + (n-1)(-n-1)^4 + 2n^4 \le n(n+1)(2n+1)(n^2 + n + 1),$$

which is equivalent to the desired statement. The equality holds for

$$a_1 = -n - 1, \qquad a_2 = \cdots = a_n = n$$

(or any cyclic permutation).

□

P 6.18. *If $a_1, a_2, \dots, a_n$ are real numbers so that*

$$a_1 + a_2 + \cdots + a_n = n^2 - 3n - 2, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n^3 + 2n^2 - 3n - 2,$$

then

$$a_1^4 + a_2^4 + \cdots + a_n^4 \ge 2n^4 + (n-2)(n+1)^4.$$

(Vasile Cîrtoaje, 2010)

Solution. As shown in the proof of P 6.16, the following statement holds:

- *If $a_1, a_2, \ldots, a_{2n+1}$ are real numbers so that*

$$a_1 + a_2 + \cdots + a_{2n+1} = 0, \qquad a_1^2 + a_2^2 + \cdots + a_{2n+1}^2 = n(n+1)(2n+1),$$

then

$$a_1^4 + a_2^4 + \cdots + a_{2n+1}^4 \geq n(n+1)(2n+1)(n^2+n+1),$$

with equality when n of $a_1, a_2, \ldots, a_{2n+1}$ are equal to $-n-1$ and the other $n+1$ are equal to n, and also when n of $a_1, a_2, \ldots, a_{2n+1}$ are equal to $n+1$ and the other $n+1$ are equal to $-n$.

Putting

$$a_{n+1} = \cdots = a_{2n-1} = -n, \qquad a_{2n} = a_{2n+1} = n+1,$$

it follows that

$$a_1 + a_2 + \cdots + a_n + (n-1)(-n) + 2(n+1) = 0$$

and

$$a_1^2 + a_2^2 + \cdots + a_n^2 + (n-1)(-n)^2 + 2(n+1)^2 = n(n+1)(2n+1)$$

involve

$$a_1^4 + a_2^4 + \cdots + a_n^4 + (n-1)(-n)^4 + 2(n+1)^4 \leq n(n+1)(2n+1)(n^2+n+1),$$

which is equivalent to the desired statement. The equality holds for

$$a_1 = a_2 = -n, \qquad a_3 = \cdots = a_n = n+1$$

(or any permutation).

□

P 6.19. *If a, b, c, d are real numbers so that $a+b+c+d=4$, then*

$$(a^2+b^2+c^2+d^2-4)(a^2+b^2+c^2+d^2+36) \leq 12(a^4+b^4+c^4+d^4-4).$$

(Vasile Cîrtoaje, 2010)

Solution. By Theorem 1, for $a+b+c+d=4$ and $a^2+b^2+c^2+d^2 = constant$, the sum $a^4+b^4+c^4+d^4$ is maximum when the set (a,b,c,d) has at most two distinct values. Therefore, it suffices to consider the following two cases.

Case 1: $a=b$ and $c=d$. We need to show that $a+c=2$ involves

$$(a^2+c^2-2)(a^2+c^2+18) \leq 6(a^4+c^4-2).$$

Since

$$a^2+c^2-2=(a+c)^2-2ac-2=2(1-ac), \qquad a^2+c^2+18=2(11-ac),$$

$$a^4+c^4-2=(a^2+c^2)^2-2a^2c^2-2=2(1-ac)(7-ac),$$

the inequality becomes

$$(1-ac)(11-ac)\le 3(1-ac)(7-ac),$$

$$(1-ac)(5-ac)\ge 0.$$

It is true because

$$ac\le \frac{1}{4}(a+c)^2=1.$$

Case 2: $b=c=d$. We need to show that $a+3b=4$ involves

$$(a^2+3b^2-4)(a^2+3b^2+36)\le 12(a^4+3b^4-4).$$

Since

$$a^2+3b^2-4=12(b-1)^2, \qquad a^2+3b^2+36=4(3b^2-6b+13),$$

$$a^4+3b^4-4=(4-3b)^4+3b^4-4=12(b-1)^2(7b^2-22b+21),$$

the inequality becomes

$$(b-1)^2[(3b^2-6b+13)\le 3(b-1)^2(7b^2-22b+21),$$

$$(b-1)^2(3b-5)^2\ge 0.$$

The equality holds for $a=b=c=d=1$, and also for

$$a=-1, \qquad b=c=d=\frac{5}{3}$$

(or any cyclic permutation).

□

P 6.20. *If $a_1,a_2,\ldots,a_n$ are real numbers so that*

$$a_1+a_2+\cdots+a_n=0, \qquad a_1^2+a_2^2+\cdots+a_n^2=n(n-1),$$

then

$$a_1^6+a_2^6+\cdots+a_n^6\le (n-1)^6+n-1.$$

(Vasile Cîrtoaje, 2010)

Solution. For $n=2$, we need to show that

$$a_1+a_2=0, \qquad a_1^2+a_2^2=2,$$

implies

$$a_1^6+a_2^6\le 2.$$

We have

$$a_1=-1, \qquad a_2=1,$$

or

$$a_1=1, \qquad a_2=-1.$$

For each of these cases, the desired inequality is an equality. According to Theorem 2, the sum

$$S_n=a_1^6+a_2^6+\cdots+a_n^6$$

is maximum for

$$a_1=\cdots=a_j, \qquad a_{j+1}=\cdots=a_n,$$

where $j\in\{1,2,\ldots,n-1\}$. Thus, we only need to show that

$$ja_1+(n-j)a_n=0, \qquad ja_1^2+(n-j)a_n^2=n(n-1)$$

involve

$$ja_1^6+(n-j)a_n^6\le(n-1)^6+n-1.$$

From the equations above, we get

$$a_1^2=\frac{(n-j)(n-1)}{j}, \qquad a_n^2=\frac{j(n-1)}{n-j}.$$

Thus, the desired inequality becomes

$$\frac{(n-j)^5+j^5}{j^2(n-j)^2}\le\frac{(n-1)^5+1}{(n-1)^2},$$

$$\frac{(n-j)^4-(n-j)^3j+(n-j)^2j^2-(n-j)j^3+j^4}{j^2(n-j)^2}\le$$

$$\le\frac{(n-1)^4-(n-1)^3+(n-1)^2-(n-1)+1}{(n-1)^2},$$

$$\frac{(n-j)^2}{j^2}-\frac{n-j}{j}-\frac{j}{n-j}+\frac{j^2}{(n-j)^2}\le(n-1)^2-(n-1)-\frac{1}{n-1}+\frac{1}{(n-1)^2},$$

which can be written as

$$f(a)\ge f(b),$$

where

$$f(x)=x^2-x-\frac{1}{x}+\frac{1}{x^2},$$

$$a = n - 1, \quad b = \frac{n}{j} - 1.$$

Since $a \geq b$ and

$$ab - 1 = (n-1)\left(\frac{n}{j} - 1\right) - 1 = n\left(\frac{n-1}{j} - 1\right) \geq 0,$$

we have

$$\begin{aligned} f(a) - f(b) &= (a-b)\left(a + b - 1 + \frac{1}{ab} - \frac{a+b}{a^2b^2}\right) \\ &= (a-b)\left(1 - \frac{1}{ab}\right)\left[(a+b)\left(1 + \frac{1}{ab}\right) - 1\right] \geq 0. \end{aligned}$$

The equality holds for

$$a_1 = -n + 1, \quad a_2 = \cdots = a_n = 1,$$

and for

$$a_1 = n - 1, \quad a_2 = \cdots = a_n = -1$$

(or any cyclic permutation).

□

P 6.21. *If $a_1, a_2, \ldots, a_n$ are real numbers so that*

$$a_1 + a_2 + \cdots + a_n = 1, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n^2 + n - 1,$$

then

$$a_1^6 + a_2^6 + \cdots + a_n^6 \leq n^6 + n - 1.$$

(Vasile Cîrtoaje, 2010)

Solution. The inequality follows from the preceding P 6.20 by replacing n with $n+1$, and then making $a_{n+1} = -1$. The equality holds for

$$a_1 = n, \quad a_2 = \cdots = a_n = -1$$

(or any cyclic permutation).

□

P 6.22. *If $a_1, a_2, \ldots, a_n$ are real numbers so that*

$$a_1 + a_2 + \cdots + a_n = 0, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n(n-1),$$

then

$$a_1^8 + a_2^8 + \cdots + a_n^8 \leq (n-1)^8 + n - 1.$$

(Vasile Cîrtoaje, 2010)

Solution. For $n=2$, we need to show that

$$a_1+a_2=0, \qquad a_1^2+a_2^2=2,$$

implies

$$a_1^8+a_2^8 \le 2.$$

We have

$$a_1=-1, \qquad a_2=1,$$

or

$$a_1=1, \qquad a_2=-1.$$

For each of these cases, the desired inequality is an equality. According to Theorem 2, the sum

$$S_n=a_1^8+a_2^8+\cdots+a_n^8$$

is maximum for

$$a_1=\cdots=a_j, \qquad a_{j+1}=\cdots=a_n,$$

where $j\in\{1,2,\ldots,n-1\}$. Thus, we only need to show that

$$ja_1+(n-j)a_n=0, \qquad ja_1^2+(n-j)a_n^2=n(n-1)$$

involve

$$ja_1^8+(n-j)a_n^8 \le (n-1)^8+n-1.$$

From the equations above, we get

$$a_1^2=\frac{(n-j)(n-1)}{j}, \qquad a_n^2=\frac{j(n-1)}{n-j}.$$

Thus, the desired inequality becomes

$$\frac{(n-j)^7+j^7}{j^3(n-j)^3} \le \frac{(n-1)^7+1}{(n-1)^4},$$

$$\frac{(n-j)^3}{j^3}-\frac{(n-j)^2}{j^2}+\frac{n-j}{j}+\frac{j}{n-j}-\frac{j^2}{(n-j)^2}+\frac{j^3}{(n-j)^3} \le$$
$$\le (n-1)^3-(n-1)^2+(n-1)+\frac{1}{n-1}-\frac{1}{(n-1)^2}+\frac{1}{(n-1)^3},$$

$$f(a)\ge f(b),$$

where

$$a=n-1, \quad b=\frac{n}{j}-1,$$

$$f(x)=x^3-x^2+x+\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}, \quad x>0.$$

Since

$$f(x)=(t-1)(t^2-2), \qquad t=x+\frac{1}{x}\geq 2,$$

it suffices to show that

$$a+\frac{1}{a}\geq b+\frac{1}{b}.$$

We have $a\geq b$,

$$ab-1=(n-1)\left(\frac{n}{j}-1\right)-1=n\left(\frac{n-1}{j}-1\right)\geq 0,$$

therefore

$$a+\frac{1}{a}-b-\frac{1}{b}=(a-b)\left(1-\frac{1}{ab}\right)\geq 0.$$

The equality holds for

$$a_1=-n+1, \qquad a_2=\cdots=a_n=1$$

and for

$$a_1=n-1, \qquad a_2=\cdots=a_n=-1$$

(or any cyclic permutation).

□

P 6.23. *If $a_1,a_2,\ldots,a_n$ are real numbers so that*

$$a_1+a_2+\cdots+a_n=1, \qquad a_1^2+a_2^2+\cdots+a_n^2=n^2+n-1,$$

then

$$a_1^8+a_2^8+\cdots+a_n^8\leq n^8+n-1.$$

(Vasile Cîrtoaje, 2010)

Solution. The inequality follows from the preceding P 6.22 by replacing n with $n+1$, and making $a_{n+1}=-1$. The equality holds for

$$a_1=n, \qquad a_2=\cdots=a_n=-1$$

(or any cyclic permutation).

□

P 6.24. *Let $a_1, a_2, \ldots, a_n$ $(n \ge 2)$ be real numbers (not all equal), and let*

$$A = \frac{a_1 + a_2 + \cdots + a_n}{n}, \quad B = \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}, \quad C = \frac{a_1^3 + a_2^3 + \cdots + a_n^3}{n}.$$

Then,

$$\frac{1}{4}\left(1 - \sqrt{1 + \frac{2n^2}{n-1}}\right) \le \frac{B^2 - AC}{B^2 - A^4} \le \frac{1}{4}\left(1 + \sqrt{1 + \frac{2n^2}{n-1}}\right).$$

(Vasile Cîrtoaje, 2010)

Solution. It is well-known that $B > A^2$, hence $B^2 > A^4$.

(a) For $n = 2$, the right inequality reduces to $(a_1^2 - a_2^2)^2 \ge 0$. Consider further that $n \ge 3$. Since the right inequality remains unchanged by replacing $a_1, a_2, \ldots, a_n$ with $-a_1, -a_2, \ldots, -a_n$, we may suppose that $A \ge 0$. Assuming that

$$A = constant, \qquad B = constant,$$

we only need to consider the case when C is minimum. Thus, according to Corollary 2, it suffices to prove the required inequality for $a_1 < a_2 = a_3 = \cdots = a_n$. Setting

$$a_1 := a, \qquad a_2 = a_3 = \cdots = a_n := b, \quad a < b,$$

the inequality becomes

$$\frac{\left[\dfrac{a^2 + (n-1)b^2}{n}\right]^2 - \dfrac{a + (n-1)b}{n} \cdot \dfrac{a^3 + (n-1)b^3}{n}}{\left[\dfrac{a^2 + (n-1)b^2}{n}\right]^2 - \left[\dfrac{a + (n-1)b}{n}\right]^4} \le \frac{1}{4}\left(1 + \sqrt{1 + \frac{2n^2}{n-1}}\right),$$

After dividing the numerator and denominator of the left fraction by $(a-b)^2$, the inequality reduces to

$$\frac{-4n^2ab}{(n+1)a^2 + 2(n-1)ab + (2n^2 - 3n + 1)b} \le 1 + \sqrt{1 + \frac{2n^2}{n-1}},$$

$$\frac{-2ab}{(n+1)a^2 + 2(n-1)ab + (2n^2 - 3n + 1)b} \le \frac{1}{\sqrt{(n^2-1)(2n-1)} - n + 1},$$

$$\left(a + \sqrt{\frac{2n^2 - 3n + 1}{n+1}}\, b\right)^2 \ge 0.$$

The equality holds for

$$-\sqrt{\frac{n+1}{(n-1)(2n-1)}}\, a_1 = a_2 = \cdots = a_n$$

(or any cyclic permutation).

(b) For $n=2$, the left inequality reduces to $(a_1-a_2)^4 \geq 0$. For $n \geq 3$, the proof is similar to the one of the right inequality. The equality holds for

$$\sqrt{\frac{n+1}{(n-1)(2n-1)}}\, a_1 = a_2 = \cdots = a_n$$

(or any cyclic permutation).

□

P 6.25. *If a,b,c,d are real numbers so that*

$$a+b+c+d=2,$$

then

$$a^4+b^4+c^4+d^4 \leq 40+\frac{3}{4}(a^2+b^2+c^2+d^2)^2.$$

(Vasile Cîrtoaje, 2010)

Solution. Write the inequality in the homogeneous form

$$10(a+b+c+d)^4+3(a^2+b^2+c^2+d^2)^2 \geq 4(a^4+b^4+c^4+d^4).$$

By Theorem 1, for $a+b+c+d = constant$ and $a^2+b^2+c^2+d^2 = constant$, the sum $a^4+b^4+c^4+d^4$ is maximum when the set (a,b,c,d) has at most two distinct values. Therefore, it suffices to consider the following two cases.
Case 1: $a=b$ and $c=d$. The inequality reduces to

$$41(a^2+c^2)^2+160ac(a^2+c^2)+164a^2c^2 \geq 0,$$

which can be written in the obvious form

$$(a^2+c^2)^2+40(a^2+c^2+2ac)^2+4a^2c^2 \geq 0.$$

Case 2: $b=c=d$. The inequality reduces to the obvious form

$$(a+5b)^2(3a^2+10ab+11b^2) \geq 0.$$

Since the homogeneous inequality becomes an equality for

$$\frac{-a}{5}=b=c=d$$

(or any cyclic permutation), the original inequality is an equality for

$$a=5, \qquad b=c=d=-1$$

(or any cyclic permutation).

□

P 6.26. *If a,b,c,d,e are real numbers, then*

$$a^4+b^4+c^4+d^4+e^4 \le \frac{31+18\sqrt{3}}{8}(a+b+c+d+e)^4+\frac{3}{4}(a^2+b^2+c^2+d^2+e^2)^2.$$

(Vasile Cîrtoaje, 2010)

Solution. We proceed as in the proof of the preceding P 6.25. Taking into account Theorem 1, it suffices to consider the cases $b=c=d=e$, and $a=b$ and $c=d=e$.

Case 1: $b=c=d=e$. Due to homogeneity, we may consider $b=c=d=e=0$ and $b=c=d=e=1$. The first case is trivial. In the second case, the inequality becomes

$$a^4+4 \le \frac{31+18\sqrt{3}}{8}(a+4)^4+\frac{3}{4}(a^2+4)^2,$$

$$\left(a+2+2\sqrt{3}\right)^2\left[f(a)+2\sqrt{3}\,g(a)\right] \ge 0,$$

where

$$f(a)=29a^2+164a+272, \qquad g(a)=9a^2+50a+76.$$

It suffices to show that $f(a)\ge 0$ and $g(a)\ge 0$. Indeed, we have

$$f(a)>25a^2+164a+269=\left(5a+\frac{82}{5}\right)^2+\frac{1}{25}>0,$$

$$g(a)>9a^2+50a+70=\left(3a+\frac{25}{3}\right)^2+\frac{5}{9}>0.$$

Case 2: $a=b$ and $c=d=e$. It suffices to show that

$$a^4+b^4+c^4+d^4+e^4 \le \frac{3}{4}(a^2+b^2+c^2+d^2+e^2)^2,$$

which reduces to

$$2a^4+3c^4 \le \frac{3}{4}(2a^2+3c^2)^2,$$

$$3(2a^2+3c^2)^2 \ge 4(2a^4+3c^4),$$

$$4a^4+36a^2c^2+15c^4 \ge 0.$$

The equality holds for

$$\frac{-a}{2(1+\sqrt{3})}=b=c=d=e$$

(or any cyclic permutation).

□

P 6.27. *Let $a,b,c,d,e \ne \frac{-5}{4}$ be real numbers so that $a+b+c+d+e=5$. Then,*

$$\frac{a(a-1)}{(4a+5)^2}+\frac{b(b-1)}{(4b+5)^2}+\frac{c(c-1)}{(4c+5)^2}+\frac{d(d-1)}{(4d+5)^2}+\frac{e(e-1)}{(4e+5)^2}\ge 0.$$

(Vasile Cîrtoaje, 2010)

Solution. Write the inequality as

$$\sum\left[\frac{180a(a-1)}{(4a+5)^2}+1\right]\ge 5,$$

$$\sum\frac{(14a-5)^2}{(4a+5)^2}\ge 5.$$

By the Cauchy-Schwarz inequality, we have

$$\sum\frac{(14a-5)^2}{(4a+5)^2}\ge\frac{\left[\sum(4a+5)(14a-5)\right]^2}{\sum(4a+5)^4}.$$

Therefore, it suffices to show that

$$\left(56\sum a^2+125\right)^2\ge 5\sum(4a+5)^4.$$

Using the substitution

$$a_1=\frac{4a+5}{9},a_2=\frac{4b+5}{9},\ldots,a_5=\frac{4e+5}{9},$$

we need to prove that $a_1+a_2+a_3+a_4+a_5=5$ involves

$$\left(7\sum_{i=1}^5 a_i^2-25\right)^2\ge 20\sum_{i=1}^5 a_i^4.$$

Rewrite this inequality in the homogeneous form

$$\left[7\sum_{i=1}^5 a_i^2-\left(\sum_{i=1}^5 a_i\right)^2\right]^2\ge 20\sum_{i=1}^5 a_i^4.$$

By Theorem 1, for $a_1+a_2+a_3+a_4+a_5=5$ and $a_1^2+a_2^2+a_3^2+a_4^2+a_5^2=constant$, the sum $a_1^4+a_2^4+a_3^4+a_4^4+a_5^4$ is maximum when the set (a_1,a_2,a_3,a_4,a_5) has at most two distinct values. Therefore, we need to consider the following two cases.

Case 1: $a_1=x$ and $a_2=a_3=a_4=a_5=y$. The homogeneous inequality reduces to

$$(3x^2+6y^2-4xy)^2\ge 5(x^4+4y^4),$$

which is equivalent to the obvious inequality

$$(x-y)^2(x-2y)^2 \geq 0.$$

Case 2: $a_1 = a_2 = x$ and $a_3 = a_4 = a_5 = y$. The homogeneous inequality becomes

$$(5x^2+6y^2-6xy)^2 \geq 5(2x^4+3y^4),$$

which is equivalent to the obvious inequality

$$(x-y)^2[5(x-y)^2+2y^2] \geq 0.$$

The equality holds for $a=b=c=d=e=1$, and also for

$$a=\frac{5}{2}, \qquad b=c=d=e=\frac{5}{8}$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization.

- *Let $x_1, x_2, \ldots, x_n \neq -k$ be real numbers so that $x_1+x_2+\cdots+x_n=n$, where*

$$k \geq \frac{n}{2\sqrt{n-1}}.$$

Then,

$$\frac{x_1(x_1-1)}{(x_1+k)^2}+\frac{x_2(x_2-1)}{(x_2+k)^2}+\cdots+\frac{x_n(x_n-1)}{(x_n+k)^2} \geq 0,$$

with equality for $x_1=x_2=\cdots=x_n=1$. If $k=\dfrac{n}{2\sqrt{n-1}}$, then the equality holds also for

$$x_1=\frac{n}{2}, \qquad x_2=\cdots=x_n=\frac{n}{2(n-1)}$$

(*or any cyclic permutation*).

□

P 6.28. *If a, b, c are real numbers so that*

$$a+b+c=9, \qquad ab+bc+ca=15,$$

then

$$\frac{19}{175} \leq \frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2}+\frac{1}{a^2+ab+b^2} \leq \frac{7}{19}.$$

(Vasile C., 2011)

Solution. From
$$(b+c)^2 \ge 4bc$$
and
$$b+c=9-a, \qquad bc=15-a(b+c)=15-a(9-a)=a^2-9a+15,$$
we get $a \le 7$. Since
$$b^2+bc+c^2=(a+b+c)(b+c)-(ab+bc+ca)=9(9-a)-15=3(22-3a),$$
we may write the inequality in the form
$$\frac{57}{175} \le f(a)+f(b)+f(c) \le \frac{21}{19}.$$
where
$$f(u)=\frac{1}{22-3u}, \qquad u \le 7.$$
We have
$$g(x)=f'(x)=\frac{3}{(22-3x)^2},$$
$$g''(x)=\frac{162}{(22-3x)^4}.$$
Since $g''(x)>0$ for $x \le 7$, g is strictly convex on $(-\infty,7]$. According to Corollary 1, *if $a \le b \le c$ and*
$$a+b+c=9, \qquad a^2+b^2+c^2=51,$$
then the sum $S_3=f(a)+f(b)+f(c)$ is maximum for $a=b \le c$, and is minimum for $a \le b=c$.

(a) To prove the right inequality, it suffices to consider the case $a=b \le c$. From
$$a+b+c=9, \qquad ab+bc+ca=15,$$
we get $a=b=1$ and $c=7$, therefore
$$\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2}+\frac{1}{a^2+ab+b^2}=\frac{7}{19}.$$
The original right inequality is an equality for $a=b=1$ and $c=7$ (or any cyclic permutation).

(b) To prove the left inequality, it suffices to consider the case $a \le b=c$, which involves $a=-1$ and $b=c=5$, hence
$$\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2}+\frac{1}{a^2+ab+b^2}=\frac{19}{175}.$$
The original left inequality is an equality for $a=-1$ and $b=c=5$ (or any cyclic permutation).

□

P 6.29. *If a, b, c are real numbers so that*

$$8(a^2+b^2+c^2)=9(ab+bc+ca),$$

then

$$\frac{419}{175} \le \frac{a^2}{b^2+bc+c^2}+\frac{b^2}{c^2+ca+a^2}+\frac{c^2}{a^2+ab+b^2} \le \frac{311}{19}.$$

(Vasile C., 2011)

Solution. Due to homogeneity, we may assume that

$$a+b+c=9, \qquad a^2+b^2+c^2=51.$$

Next, the proof is similar to the one of the preceding P 6.28. Write the inequality in the form

$$\frac{1257}{175} \le f(a)+f(b)+f(c) \le \frac{933}{19}.$$

where

$$f(u)=\frac{u^2}{22-3u}, \qquad u \le 7.$$

We have

$$g(x)=f'(x)=\frac{-3x^2+44x}{(22-3x)^2}, \qquad g''(x)=\frac{8712}{(22-3x)^4}.$$

Since g is strictly convex on $(-\infty, 7]$, according to Corollary 1, the sum $S_3 = f(a)+f(b)+f(c)$ is maximum for $a=b\le c$, and is minimum for $a \le b = c$.

(a) To prove the right inequality, it suffices to consider the case $a=b\le c$, which involves

$$a=b=1, \qquad c=7,$$

and

$$\frac{a^2}{b^2+bc+c^2}+\frac{b^2}{c^2+ca+a^2}+\frac{c^2}{a^2+ab+b^2}=\frac{311}{19}.$$

The original right inequality is an equality for $a=b=c/7$ (or any cyclic permutation).

(b) To prove the left inequality, it suffices to consider the case $a \le b = c$, which involves $a=-1$ and $b=c=5$, hence

$$\frac{a^2}{b^2+bc+c^2}+\frac{b^2}{c^2+ca+a^2}+\frac{c^2}{a^2+ab+b^2}=\frac{419}{175}.$$

The original left inequality is an equality for $-5a=b=c$ (or any cyclic permutation).

□

P 6.30. *Let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 + a_2 + \cdots + a_n = n$. If $n \le 10$, then*

$$2(a_1^2 + a_2^2 + \cdots + a_n^2)^2 - n(a_1^3 + a_2^3 + \cdots + a_n^3) \ge n^2.$$

(Vasile Cîrtoaje, 2020)

Solution. Write the inequality in the homogeneous form

$$2n^2(a_1^2+a_2^2+\cdots+a_n^2)^2-n^2(a_1+a_2+\cdots+a_n)(a_1^3+a_2^3+\cdots+a_n^3) \ge (a_1+a_2+\cdots+a_n)^4.$$

According to Corollary 2, for $a_1+a_2+\cdots+a_n = constant > 0$ and $a_1^2+a_2^2+\cdots+a_n^2 = constant$, the sum

$$S = a_1^3 + a_2^3 + \cdots + a_n^3$$

is maximal when $n-1$ of $a_1, a_2, \ldots, a_n$ are equal. Therefore, it suffices to consider the case $a_2 = a_3 = \cdots = a_n$. Due to homogeneity, for the nontrivial case $a_2 = a_3 = \cdots = a_n \ne 0$, we may consider that $a_2 = a_3 = \cdots = a_n = 1$. Thus we only need to prove that

$$2n^2(a_1^2 + n - 1)^2 - n^2(a_1 + n - 1)(a_1^3 + n - 1) \ge (a_1 + n - 1)^4,$$

which is equivalent to

$$(a_1 - 1)^2(Aa_1^2 - Ba_1 + C) \ge 0,$$

where

$$A = n(n+1), \qquad B = n(n^2 - 2n + 2), \qquad C = n(n-1)(2n-1).$$

The inequality is true because

$$4AC - B^2 = n^4(-n^2 + 12n - 12) \ge 0.$$

The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$.

□

Appendix A

Glosar

1. AM-GM (ARITHMETIC MEAN-GEOMETRIC MEAN) INEQUALITY

If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers, then

$$a_1 + a_2 + \cdots + a_n \geq n\sqrt[n]{a_1 a_2 \cdots a_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

2. WEIGHTED AM-GM INEQUALITY

Let $p_1, p_2, \ldots, p_n$ be positive real numbers satisfying

$$p_1 + p_2 + \cdots + p_n = 1.$$

If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers, then

$$p_1 a_1 + p_2 a_2 + \cdots + p_n a_n \geq a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

3. AM-HM (ARITHMETIC MEAN-HARMONIC MEAN) INEQUALITY

If $a_1, a_2, \ldots, a_n$ are positive real numbers, then

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2,$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

4. POWER MEAN INEQUALITY

The power mean of order k of positive real numbers $a_1, a_2, \ldots, a_n$,

$$M_k = \begin{cases} \left(\frac{a_1^k + a_2^k + \cdots + a_n^k}{n}\right)^{\frac{1}{k}}, & k \neq 0 \\ \sqrt[n]{a_1 a_2 \cdots a_n}, & k = 0 \end{cases},$$

is an increasing function with respect to $k \in \mathbb{R}$. For instant, $M_2 \geq M_1 \geq M_0 \geq M_{-1}$ is equivalent to

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{n}{\dfrac{1}{a_1} + \dfrac{1}{a_2} + \cdots + \dfrac{1}{a_n}}.$$

5. BERNOULLI'S INEQUALITY

For any real number $x \geq -1$, we have

a) $(1+x)^r \geq 1 + rx$ for $r \geq 1$ and $r \leq 0$;

b) $(1+x)^r \leq 1 + rx$ for $0 \leq r \leq 1$.

If $a_1, a_2, \ldots, a_n$ are real numbers such that either $a_1, a_2, \ldots, a_n \geq 0$ or

$$-1 \leq a_1, a_2, \ldots, a_n \leq 0,$$

then

$$(1+a_1)(1+a_2)\cdots(1+a_n) \geq 1 + a_1 + a_2 + \cdots + a_n.$$

6. SCHUR'S INEQUALITY

For any nonnegative real numbers a, b, c and any positive number k, the inequality holds

$$a^k(a-b)(a-c) + b^k(b-c)(b-a) + c^k(c-a)(c-b) \geq 0,$$

with equality for $a = b = c$, and for $a = 0$ and $b = c$ (or any cyclic permutation). For $k = 1$, we get the third degree Schur's inequality, which can be rewritten as follows

$$a^3 + b^3 + c^3 + 3abc \geq ab(a+b) + bc(b+c) + ca(c+a),$$

$$(a+b+c)^3 + 9abc \geq 4(a+b+c)(ab+bc+ca),$$

$$a^2 + b^2 + c^2 + \frac{9abc}{a+b+c} \geq 2(ab+bc+ca),$$

$$(b-c)^2(b+c-a)+(c-a)^2(c+a-b)+(a-b)^2(a+b-c) \ge 0.$$

For $k=2$, we get the fourth degree Schur's inequality, which holds for any real numbers a, b, c, and can be rewritten as follows

$$a^4+b^4+c^4+abc(a+b+c) \ge ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2),$$

$$a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2 \ge (ab+bc+ca)(a^2+b^2+c^2-ab-bc-ca),$$

$$(b-c)^2(b+c-a)^2+(c-a)^2(c+a-b)^2+(a-b)^2(a+b-c)^2 \ge 0,$$

$$6abcp \ge (p^2-q)(4q-p^2), \qquad p=a+b+c, \;\; q=ab+bc+ca.$$

A generalization of the fourth degree Schur's inequality, which holds for any real numbers a, b, c and any real number m, is the following (*Vasile Cirtoaje*, 2004)

$$\sum(a-mb)(a-mc)(a-b)(a-c) \ge 0,$$

with equality for $a=b=c$, and also for $a/m=b=c$ (or any cyclic permutation). This inequality is equivalent to

$$\sum a^4+m(m+2)\sum a^2b^2+(1-m^2)abc\sum a \ge (m+1)\sum ab(a^2+b^2),$$

$$\sum(b-c)^2(b+c-a-ma)^2 \ge 0.$$

7. CAUCHY-SCHWARZ INEQUALITY

If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers, then

$$(a_1^2+a_2^2+\cdots+a_n^2)(b_1^2+b_2^2+\cdots+b_n^2) \ge (a_1b_1+a_2b_2+\cdots+a_nb_n)^2,$$

with equality for

$$\frac{a_1}{b_1}=\frac{a_2}{b_2}=\cdots=\frac{a_n}{b_n}.$$

Notice that the equality conditions are also valid for $a_i=b_i=0$, where $1 \le i \le n$.

8. HÖLDER'S INEQUALITY

If x_{ij} $(i=1,2,\cdots,m; j=1,2,\cdots n)$ are nonnegative real numbers, then

$$\prod_{i=1}^{m}\left(\sum_{j=1}^{n} x_{ij}\right) \ge \left(\sum_{j=1}^{n}\sqrt[m]{\prod_{i=1}^{m} x_{ij}}\right)^m.$$

9. CHEBYSHEV'S INEQUALITY

Let $a_1 \geq a_2 \geq \cdots \geq a_n$ be real numbers.

a) If $b_1 \geq b_2 \geq \cdots b_n$, then

$$n \sum_{i=1}^{n} a_i b_i \geq \left(\sum_{i=1}^{n} a_i \right) \left(\sum_{i=1}^{n} b_i \right);$$

b) If $b_1 \leq b_2 \leq \cdots \leq b_n$, then

$$n \sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i \right) \left(\sum_{i=1}^{n} b_i \right).$$

10. REARRANGEMENT INEQUALITY

(1) If $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are two increasing (or decreasing) real sequences, and $(i_1, i_2, \cdots, i_n)$ is an arbitrary permutation of $(1, 2, \cdots, n)$, then

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \geq a_1 b_{i_1} + a_2 b_{i_2} + \cdots + a_n b_{i_n}$$

and

$$n(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n) \geq (a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n).$$

(2) If $(a_1, a_2, \ldots, a_n)$ is decreasing and $(b_1, b_2, \ldots, b_n)$ is increasing, then

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \leq a_1 b_{i_1} + a_2 b_{i_2} + \cdots + a_n b_{i_n}$$

and

$$n(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n) \leq (a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n).$$

(3) Let $b_1, b_2, \ldots, b_n)$ and $(c_1, c_2, \ldots, c_n)$ be two real sequences such that

$$b_1 + \cdots + b_i \geq c_1 + \cdots + c_i, \quad i = 1, 2, \cdots, n.$$

If $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, then

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \geq a_1 c_1 + a_2 c_2 + \cdots + a_n c_n.$$

Notice that all these inequalities follow immediately from the identity

$$\sum_{i=1}^{n} a_i (b_i - c_i) = \sum_{i=1}^{n} (a_i - a_{i+1}) \left(\sum_{j=1}^{i} b_j - \sum_{j=1}^{i} c_j \right), \qquad a_{n+1} = 0.$$

11. SQUARE PRODUCT INEQUALITY

Let a, b, c be real numbers, and let

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc,$$

$$s = \sqrt{p^2 - 3q} = \sqrt{a^2 + b^2 + c^2 - ab - bc - ca}.$$

From the identity

$$(a-b)^2(b-c)^2(c-a)^2 = -27r^2 + 2(9pq - 2p^3)r + p^2q^2 - 4q^3,$$

it follows that

$$\frac{-2p^3 + 9pq - 2(p^2-3q)\sqrt{p^2-3q}}{27} \le r \le \frac{-2p^3 + 9pq + 2(p^2-3q)\sqrt{p^2-3q}}{27},$$

which is equivalent to

$$\frac{p^3 - 3ps^2 - 2s^3}{27} \le r \le \frac{p^3 - 3ps^2 + 2s^3}{27}.$$

Therefore, for constant p and q, the product r is minimum and maximum when two of a, b, c are equal.

12. KARAMATA'S MAJORIZATION INEQUALITY

Let f be a convex function on a real interval $\mathbb{I}$. If a decreasingly ordered sequence

$$A = (a_1, a_2, \ldots, a_n), \quad a_i \in \mathbb{I},$$

majorizes a decreasingly ordered sequence

$$B = (b_1, b_2, \ldots, b_n), \quad b_i \in \mathbb{I},$$

then

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge f(b_1) + f(b_2) + \cdots + f(b_n).$$

We say that a sequence $A = (a_1, a_2, \ldots, a_n)$ with $a_1 \ge a_2 \ge \cdots \ge a_n$ majorizes a sequence $B = (b_1,\ b_2,\ \ldots,\ b_n)$ with $b_1 \ge b_2 \ge \cdots \ge b_n$, and write it as

$$A \succ B,$$

if

$$\begin{gathered} a_1 \ge b_1, \\ a_1 + a_2 \ge b_1 + b_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_1 + a_2 + \cdots + a_{n-1} \ge b_1 + b_2 + \cdots + b_{n-1}, \\ a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n. \end{gathered}$$

13. CONVEX FUNCTIONS

A function f defined on a real interval $\mathbb{I}$ is said to be *convex* if

$$f(\alpha x+\beta y)\le \alpha f(x)+\beta f(y)$$

for all $x,\ y\in\mathbb{I}$ and any $\alpha,\ \beta\ge 0$ with $\alpha+\beta=1.$ If the inequality is reversed, then f is said to be concave.
If f is differentiable on $\mathbb{I}$, then f is (strictly) convex if and only if the derivative f' is (strictly) increasing. If $f''\ge 0$ on $\mathbb{I}$, then f is convex on $\mathbb{I}$. Also, if $f''\ge 0$ on $(a,\ b)$ and f is continuous on $[a,\ b]$, then f is convex on $[a,\ b]$.

Jensen's inequality. *Let $p_1,p_2,\dots,p_n$ be positive real numbers. If f is a convex function on a real interval $\mathbb{I}$, then for any $a_1,\ a_2,\ \dots,\ a_n\in\mathbb{I}$, the inequality holds*

$$\frac{p_1f(a_1)+p_2f(a_2)+\cdots+p_nf(a_n)}{p_1+p_2+\cdots+p_n}\ge f\left(\frac{p_1a_1+p_2a_2+\cdots+p_na_n}{p_1+p_2+\cdots+p_n}\right).$$

For $p_1=p_2=\cdots=p_n$, Jensen's inequality becomes

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right).$$

Right Half Convex Function Theorem (*Vasile Cîrtoaje*, 2004). *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\ge s}$, where $s\in int(\mathbb{I})$. The inequality*

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$$

holds for all $a_1,a_2,\dots,a_n\in\mathbb{I}$ satisfying

$$a_1+a_2+\cdots+a_n=ns$$

if and only if

$$f(x)+(n-1)f(y)\ge nf(s)$$

for all $x,y\in\mathbb{I}$ such that $x\le s\le y$ and $x+(n-1)y=ns$.

Left Half Convex Function Theorem (*Vasile Cîrtoaje*, 2004). *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\le s}$, where $s\in int(\mathbb{I})$. The inequality*

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$$

holds for all $a_1,a_2,\dots,a_n\in\mathbb{I}$ satisfying

$$a_1+a_2+\cdots+a_n=ns$$

if and only if

$$f(x)+(n-1)f(y)\ge nf(s)$$

for all $x, y \in \mathbb{I}$ such that $x \ge s \ge y$ and $x+(n-1)y = ns$.

Left Convex-Right Concave Function Theorem (*Vasile Cîrtoaje*, 2004). *Let $a \le c$ be real numbers, let f be a continuous function defined on $\mathbb{I} = [a, \infty)$, strictly convex on $[a, c]$ and strictly concave on $[c, \infty)$, and let*

$$E(a_1, a_2, \ldots, a_n) = f(a_1) + f(a_2) + \cdots + f(a_n).$$

If $a_1, a_2, \ldots, a_n \in \mathbb{I}$ such that

$$a_1 + a_2 + \cdots + a_n = S = constant,$$

then

(a) *E is minimum for $a_1 = a_2 = \cdots = a_{n-1} \le a_n$;*
(b) *E is maximum for either $a_1 = a$* or *$a < a_1 \le a_2 = \cdots = a_n$.*

Right Half Convex Function Theorem for Ordered Variables (*Vasile Cîrtoaje*, 2008). *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\ge s}$, where $s \in int(\mathbb{I})$. The inequality*

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying

$$a_1 + a_2 + \cdots + a_n = ns$$

and

$$a_1 \le a_2 \le \cdots \le a_m \le s, \quad m \in \{1, 2, \ldots, n-1\},$$

if and only if

$$f(x) + (n-m)f(y) \ge (1+n-m)f(s)$$

for all $x, y \in \mathbb{I}$ such that

$$x \le s \le y, \quad x + (n-m)y = (1+n-m)s.$$

Left Half Convex Function Theorem for Ordered Variables (*Vasile Cîrtoaje*, 2008). *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\le s}$, where $s \in int(\mathbb{I})$. The inequality*

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying

$$a_1 + a_2 + \cdots + a_n = ns$$

and

$$a_1 \ge a_2 \ge \cdots \ge a_m \ge s, \quad m \in \{1, 2, \ldots, n-1\},$$

if and only if

$$f(x)+(n-m)f(y) \geq (1+n-m)f(s)$$

for all $x, y \in \mathbb{I}$ *such tht*

$$x \geq s \geq y, \quad x+(n-m)y=(1+n-m)s.$$

Right Partially Convex Function Theorem (*Vasile Cîrtoaje*, 2012). *Let* f *be a real function defined on an interval* $\mathbb{I}$ *and convex on* $[s, s_0]$, *where* $s, s_0 \in \mathbb{I}$, $s < s_0$. *In addition,* f *is decreasing on* $\mathbb{I}_{\leq s_0}$ *and* $f(u) \geq f(s_0)$ *for* $u \in \mathbb{I}$. *The inequality*

$$f(a_1)+f(a_2)+\cdots+f(a_n) \geq nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ *satisfying*

$$a_1+a_2+\cdots+a_n=ns$$

if and only if

$$f(x)+(n-1)f(y) \geq nf(s)$$

for all $x, y \in \mathbb{I}$ *such that* $x \leq s \leq y$ *and* $x+(n-1)y=ns$.

Left Partially Convex Function Theorem (*Vasile Cîrtoaje*, 2012). *Let* f *be a real function defined on an interval* $\mathbb{I}$ *and convex on* $[s_0, s]$, *where* $s_0, s \in \mathbb{I}$, $s_0 < s$. *In addition,* f *is increasing on* $\mathbb{I}_{\geq s_0}$ *and* $f(u) \geq f(s_0)$ *for* $u \in \mathbb{I}$. *The inequality*

$$f(a_1)+f(a_2)+\cdots+f(a_n) \geq nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ *satisfying*

$$a_1+a_2+\cdots+a_n=ns$$

if and only if

$$f(x)+(n-1)f(y) \geq nf(s)$$

for all $x, y \in \mathbb{I}$ *such that* $x \geq s \geq y$ *and* $x+(n-1)y=ns$.

Right Partially Convex Function Theorem for Ordered Variables (*Vasile Cirtoaje*, 2014). *Let* f *be a real function defined on an interval* $\mathbb{I}$ *and convex on* $[s, s_0]$, *where* $s, s_0 \in \mathbb{I}$, $s < s_0$. *In addition,* f *is decreasing on* $\mathbb{I}_{\leq s_0}$ *and* $f(u) \geq f(s_0)$ *for* $u \in \mathbb{I}$. *The inequality*

$$f(a_1)+f(a_2)+\cdots+f(a_n) \geq nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ *satisfying*

$$a_1+a_2+\cdots+a_n=ns$$

and

$$a_1 \le a_2 \le \cdots \le a_m \le s, \quad m \in \{1, 2, \ldots, n-1\},$$

if and only if

$$f(x) + (n-m)f(y) \ge (1+n-m)f(s)$$

for all $x, y \in \mathbb{I}$ such that $x \le s \le y$ and $x + (n-m)y = (1+n-m)s$.

Left Partially Convex Function Theorem for Ordered Variables (*Vasile Cirtoaje*, 2014). *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $[s_0, s]$, where $s_0, s \in \mathbb{I}$, $s_0 < s$. In addition, f is increasing on $\mathbb{I}_{\ge s_0}$ and $f(u) \ge f(s_0)$ for $u \in \mathbb{I}$. The inequality*

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying

$$a_1 + a_2 + \cdots + a_n = ns$$

and

$$a_1 \ge a_2 \ge \cdots \ge a_m \ge s, \quad m \in \{1, 2, \ldots, n-1\},$$

if and only if

$$f(x) + (n-m)f(y) \ge (1+n-m)f(s)$$

for all $x, y \in \mathbb{I}$ such that $x \ge s \ge y$ and $x + (n-m)y = (1+n-m)s$.

Equal Variables Theorem for Nonnegative Variables (*Vasile Cirtoaje*, 2005). *Let $a_1, a_2, \ldots, a_n$ ($n \ge 3$) be fixed nonnegative real numbers, and let*

$$0 \le x_1 \le x_2 \le \cdots \le x_n$$

such that

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n, \quad x_1^k + x_2^k + \cdots + x_n^k = a_1^k + a_2^k + \cdots + a_n^k,$$

where k is a real number ($k \ne 1$); for $k = 0$, assume that

$$x_1 x_2 \cdots x_n = a_1 a_2 \cdots a_n.$$

Let f be a real-valued function, continuous on $[0, \infty)$ and differentiable on $(0, \infty)$, such that the associated function

$$g(x) = f'\left(x^{\frac{1}{k-1}}\right)$$

is strictly convex on $(0, \infty)$. Then, the sum

$$S_n = f(x_1) + f(x_2) + \cdots + f(x_n)$$

is maximum for

$$x_1 = x_2 = \cdots = x_{n-1} \le x_n,$$

and is minimum for

$$0 < x_1 \le x_2 = x_3 = \cdots = x_n$$

or

$$0 = x_1 = \cdots = x_j \le x_{j+1} \le x_{j+2} = \cdots = x_n, \quad j \in \{1, 2, \ldots, n-1\}.$$

Equal Variables Theorem for Real Variables (*Vasile Cirtoaje*, 2010). *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be fixed real numbers, and let*

$$0 \le x_1 \le x_2 \le \cdots \le x_n$$

such that

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n, \quad x_1^k + x_2^k + \cdots + x_n^k = a_1^k + a_2^k + \cdots + a_n^k,$$

where k is an even positive integer. If f is a differentiable function on $\mathbb{R}$ such that the associated function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = f'\left(\sqrt[k-1]{x}\right)$$

is strictly convex on $\mathbb{R}$, then the sum

$$S_n = f(x_1) + f(x_2) + \cdots + f(x_n)$$

is minimum for $x_2 = x_3 = \cdots = x_n$, and is maximum for $x_1 = x_2 = \cdots = x_{n-1}$.

Best Upper Bound of Jensen's Difference Theorem (*Vasile Cirtoaje*, 1990). *Let $p_1, p_2, \ldots, p_n$ $(n \ge 3)$ be fixed positive real numbers, and let f be a convex function on $\mathbb{I} = [a, b]$. If $a_1, a_2, \ldots, a_n \in \mathbb{I}$, then Jensen's difference*

$$\frac{p_1 f(a_1) + p_2 f(a_2) + \cdots + p_n f(a_n)}{p_1 + p_2 + \cdots + p_n} - f\left(\frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_1 + p_2 + \cdots + p_n}\right)$$

is maximum when all $a_i \in \{a, b\}$.

Appendix B

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