Vasile Cîrtoaje

MATHEMATICAL INEQUALITIES

Volume 1

SYMMETRIC POLYNOMIAL INEQUALITIES

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About the author

"The simpler and sharper, the more beautiful."
Vasile Cîrtoaje

Vasile Cîrtoaje is a Professor at the Department of Automatic Control and Computers from Petroleum-Gas University of Ploiesti, Romania, where he teaches university courses such as Control System Theory and Digital Control Systems.

Since 1970, he published many mathematical problems, solutions and articles in the Romanian journals Gazeta Matematica-B, Gazeta Matematica-A and Mathematical Review of Timisoara. In addition, from 2000 to present, Vasile Cîrtoaje has published many interesting problems and articles in *Art of Problem Solving* website, Mathematical Reflections, Crux with Mayhem, Journal of Inequalities and Applications, Journal of Inequalities in Pure and Applied Mathematics, Mathematical Inequalities and Applications, Banach Journal of Mathematical Analysis, Journal of Nonlinear Science and Applications, Journal of Nonlinear Analysis and Application, Australian Journal of Mathematical Analysis and Applications, British Journal of Mathematical and Computer Science,International Journal of Pure and Applied Mathematics, A.M.M.

He collaborated with Titu Andreescu, Gabriel Dospinescu and Mircea Lascu in writing the book *Old and New Inequalities*, with Vo Quoc Ba Can and Tran Quoc Anh in writing the book *Inequalities with Beautiful Solution*, and he wrote his own books *Algebraic Inequalities - Old and New Methods* and *Mathematical Inequalities* (Volume 1 ... 4).

Notice that Vasile Cîrtoaje is the author of some well-known results and strong methods for proving and creating discrete inequalities, such as:

- Half convex function method (HCF method) for Jensen type discrete inequalities;
- Partial convex function method (PCF method) for Jensen type discrete inequalities;
- Jensen type discrete inequalities with ordered variables;
- Equal variable method (EV method) for real or nonnegative variables;
- Arithmetic compensation method (AC method);
- Best lower and upper bounds for Jensen's inequality;
- Necessary and sufficient conditions for symmetric homogeneous polynomial inequalities of degree six in real variables;

- Necessary and sufficient conditions for symmetric homogeneous polynomial inequalities of degree six in nonnegative variables;
- Highest coefficient cancellation method (HCC method) for symmetric homogeneous polynomial inequalities of degree six and eight in real variables;
- Highest coefficient cancellation method for symmetric homogeneous polynomial inequalities of degree six, seven and eight in nonnegative variables;
- Necessary and sufficient conditions for cyclic homogeneous polynomial inequalities of degree four in real variables;
- Necessary and sufficient conditions for cyclic homogeneous polynomial inequalities of degree four in nonnegative variables;
- Strong sufficient conditions for cyclic homogeneous polynomial inequalities of degree four in real or nonnegative variables;
- Inequalities with power-exponential functions.

Foreword

The author, Vasile Cîrtoaje, professor at University of Ploiesti-Romania, has become well-known for his excellent creations in the mathematical inequality field, ever since the time when he was student in high school (in Breaza city, Prahova Valley). As a student (quite some time ago, oh yes!), I was already familiar with the name of Vasile Cîrtoaje. For me, and many others of my age, it is the name of someone who helped me to grow in mathematics, even though I never met him face to face. It is a name synonymous to hard and beautiful problems involving inequalities. When you say Vasile Cîrtoaje (*Vasc* username on the site Art of Problem Solving), you say inequalities. I remember how happy I was when I could manage to solve one of the problems proposed by professor Cîrtoaje in Gazeta Matematica or Revista Matematica Timisoara.

The first three volumes of this book are a great opportunity to see and know many old and new elementary methods for solving mathematical inequalities: Volume 1 - Symmetric polynomial inequalities (in real variables and nonnegative real variables), Volume 2 - Symmetric rational and nonrational inequalities, Volume 3 - Cyclic and noncyclic inequalities. As a rule, the inequalities from each section of these volumes are increasingly ordered by the number of variables: two, three, four, five, six and n-variables.

The last volume (Volume 4) contains new beautiful and efficient original methods for creating and solving inequalities: *half or partially convex function method* - for Jensen's type inequalities, *equal variable method* - for nonnegative or real variables, *arithmetic compensation method* - for symmetric inequalities, *the highest coefficient cancellation method* - for symmetric homogeneous polynomial inequalities of degree six, seven and eight in nonnegative or real variables, *methods* involving either strong or necessary and sufficient conditions - for cyclic homogeneous inequalities of degree four in real or nonnegative variables and so on.

Many problems, the majority I would say, are made up by the author himself. The chapters and volumes are relatively independent, and you can open the book somewhere to solve an inequality or only read its solution. If you carefully make a thorough study of the book, then you will find that your skills in solving inequalities are considerably improved.

The book is a rich and meaningful collection of more than 1000 beautiful inequalities, hints, solutions and methods, some of them being posted in the last ten years by the author and other inventive mathematicians on *Art of Problem Solving* website (Vo Quoc

Ba Can, Pham Kim Hung, Michael Rozenberg, Nguyen Van Quy, Gabriel Dospinescu, Darij Grinberg, Pham Huu Duc, Tran Quoc Anh, Le Huu Dien Khue, Marius Stanean, Cezar Lupu, Nguyen Anh Tuan, Pham Van Thuan, Bin Zhao, Ji Chen etc.)

Most inequalities and methods are old and recent own creations of the author. Among these, I would like to point out the following inequalities:

$$(a^2+b^2+c^2)^2 \ge 3(a^3b+b^3c+c^3a), \quad a,b,c \in \mathbb{R};$$

$$\sum (a-kb)(a-kc)(a-b)(a-c) \ge 0, \quad a,b,c,k \in \mathbb{R};$$

$$\left(\frac{a}{a+b}\right)^2+\left(\frac{b}{b+c}\right)^2+\left(\frac{c}{c+d}\right)^2+\left(\frac{d}{d+a}\right)^2 \ge 1, \quad a,b,c,d \ge 0;$$

$$\sum_{i=1}^{4} \frac{1}{1+a_i+a_i^2+a_i^3} \ge 1, \quad a_1,a_2,a_3,a_4 > 0, \; a_1a_2a_3a_4 = 1;$$

$$\frac{a_1}{a_1+(n-1)a_2}+\frac{a_2}{a_2+(n-1)a_3}+\cdots+\frac{a_n}{a_n+(n-1)a_1} \ge 1, \quad a_1,a_2,\cdots,a_n \ge 0;$$

$$a^{ea}+b^{eb} \ge a^{eb}+b^{ea}, \qquad a,b>0, \quad e \approx 2.7182818.$$

$$a^{3b}+b^{3a} \le 2, \qquad a,b \ge 0, \quad a+b=2.$$

The book represents a rich source of beautiful, serious and profound mathematics, dealing with classical and new approaches and techniques which help the reader to develop his inequality-solving skills, intuition and creativity. As a result, it is suitable for a wide variety of audiences; high school students and teachers, college and university students, mathematics educators and mathematicians will find something of interest here. Each problem has a hint, and many problems have multiple solutions, almost all of which are, not surprisingly, quite ingenious. Almost all inequalities require careful thought and analysis, making the book a rich and rewarding source for anyone interested in Olympiad-type problems and in the development of the inequality field. Many problems and methods can be used in a classroom for advanced high school students as group projects.

What makes this book so attractive? The answer is simple: the great number of inequalities, their quality and freshness, as well as the new approaches and methods for solving mathematical inequalities. Nevertheless, you will find this book to be delightful, inspired, original and enjoyable. Certainly, any interested reader will remark the tenacity, enthusiasm and ability of the author in creating and solving nice and difficult inequalities. And this book is neither more, nor less than a Work of a Master.

The author must be congratulated for publishing such interesting and original inequality book. I highly recommend it.

Marian Tetiva

Contents

Chapter 1

Some Classic and New Inequalities and Methods

1. AM-GM (ARITHMETIC MEAN-GEOMETRIC MEAN) INEQUALITY

If $a_1, a_2, \cdots, a_n$ are nonnegative real numbers, then

$$a_1 + a_2 + \cdots + a_n \geq n\sqrt[n]{a_1 a_2 \cdots a_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

2. WEIGHTED AM-GM INEQUALITY

Let $p_1, p_2, \cdots, p_n$ be positive real numbers satisfying

$$p_1 + p_2 + \cdots + p_n = 1.$$

If $a_1, a_2, \cdots, a_n$ are nonnegative real numbers, then

$$p_1 a_1 + p_2 a_2 + \cdots + p_n a_n \geq a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

3. AM-HM (ARITHMETIC MEAN-HARMONIC MEAN) INEQUALITY

If $a_1, a_2, \cdots, a_n$ are positive real numbers, then

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2,$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

4. POWER MEAN INEQUALITY

The power mean of order k of positive real numbers $a_1, a_2, \cdots, a_n$, that is

$$M_k = \begin{cases} \left(\frac{a_1^k + a_2^k + \cdots + a_n^k}{n}\right)^{\frac{1}{k}}, & k \neq 0 \\ \sqrt[n]{a_1 a_2 \cdots a_n}, & k = 0 \end{cases},$$

is an increasing function with respect to $k \in \mathbb{R}$. For instant, $M_2 \geq M_1 \geq M_0 \geq M_{-1}$ is equivalent to

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}.$$

5. BERNOULLI'S INEQUALITY

For any real number $x \geq -1$, we have

a) $(1+x)^r \geq 1 + rx$ for $r \geq 1$ and $r \leq 0$;

b) $(1+x)^r \leq 1 + rx$ for $0 \leq r \leq 1$.

In addition, if $a_1, a_2, \cdots, a_n$ are real numbers such that either $a_1, a_2, \cdots, a_n \geq 0$ or $-1 \leq a_1, a_2, \cdots, a_n \leq 0$, then

$$(1+a_1)(1+a_2)\cdots(1+a_n) \geq 1 + a_1 + a_2 + \cdots + a_n.$$

6. SCHUR'S INEQUALITY

For any nonnegative real numbers a, b, c and any positive number k, the inequality holds

$$a^k(a-b)(a-c) + b^k(b-c)(b-a) + c^k(c-a)(c-b) \geq 0,$$

with equality for $a = b = c$, and for $a = 0$ and $b = c$ (or any cyclic permutation).
For $k = 1$, we get the third degree Schur's inequality, which can be rewritten as follows

$$a^3 + b^3 + c^3 + 3abc \geq ab(a+b) + bc(b+c) + ca(c+a),$$

$$(a+b+c)^3 + 9abc \geq 4(a+b+c)(ab+bc+ca),$$

$$a^2 + b^2 + c^2 + \frac{9abc}{a+b+c} \geq 2(ab+bc+ca),$$

$$(b-c)^2(b+c-a) + (c-a)^2(c+a-b) + (a-b)^2(a+b-c) \geq 0.$$

For $k = 2$, we get the fourth degree Schur's inequality, which holds for any real numbers a, b, c, and can be rewritten as follows

$$a^4 + b^4 + c^4 + abc(a + b + c) \geq ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2),$$

$$(b - c)^2(b + c - a)^2 + (c - a)^2(c + a - b)^2 + (a - b)^2(a + b - c)^2 \geq 0,$$

$$6abcp \geq (p^2 - q)(4q - p^2),$$

where $p = a + b + c$, $q = ab + bc + ca$.

A generalization of the fourth degree Schur's inequality, which holds for any real numbers a, b, c and any real number m, is the following (*Vasile Cirtoaje*, 2004)

$$\sum (a - mb)(a - mc)(a - b)(a - c) \geq 0,$$

where the equality holds for $a = b = c$, and for $a/m = b = c$ (or any cyclic permutation). This inequality is equivalent to

$$\sum a^4 + m(m + 2)\sum a^2b^2 + (1 - m^2)abc\sum a \geq (m + 1)\sum ab(a^2 + b^2)$$

and

$$\sum (b - c)^2(b + c - a - ma)^2 \geq 0.$$

A more general result is given by the following theorem (*Vasile Cirtoaje*, 2004).

Theorem. *Let*

$$f_4(a, b, c) = \sum a^4 + \alpha\sum a^2b^2 + \beta abc\sum a - \gamma\sum ab(a^2 + b^2),$$

where α, β, γ *are real constants such that*

$$1 + \alpha + \beta = 2\gamma.$$

(a) $f_4(a, b, c) \geq 0$ *for all real numbers* a, b, c *if and only if*

$$1 + \alpha \geq \gamma^2;$$

(b) $f_4(a, b, c) \geq 0$ *for all* $a, b, c \geq 0$ *if and only if*

$$\alpha \geq (\gamma - 1)\max\{2, \gamma + 1\}.$$

7. CAUCHY-SCHWARZ INEQUALITY

For any real numbers $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ we have

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2,$$

with equality if and only if a_i and b_i are proportional for all i.

8. HÖLDER'S INEQUALITY

If x_{ij} $(i = 1, 2, \cdots, m; j = 1, 2, \cdots n)$ are nonnegative real numbers, then

$$\prod_{i=1}^{m}\left(\sum_{j=1}^{n} x_{ij}\right) \ge \left(\sum_{j=1}^{n} \sqrt[m]{\prod_{i=1}^{m} x_{ij}}\right)^{m}.$$

9. CHEBYSHEV'S INEQUALITY

Let $a_1 \ge a_2 \ge \cdots \ge a_n$ be real numbers.

a) If $b_1 \ge b_2 \ge \cdots b_n$, then

$$n\sum_{i=1}^{n} a_i b_i \ge \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right);$$

b) If $b_1 \le b_2 \le \cdots \le b_n$, then

$$n\sum_{i=1}^{n} a_i b_i \le \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right).$$

10. MINKOWSKI'S INEQUALITY

For any real number $k \ge 1$ and any positive real numbers $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$, the following inequalities hold:

$$\sum_{i=1}^{n}\left(a_i^k + b_i^k\right)^{\frac{1}{k}} \ge \left[\left(\sum_{i=1}^{n} a_i\right)^{k} + \left(\sum_{i=1}^{n} b_i\right)^{k}\right]^{\frac{1}{k}};$$

$$\sum_{i=1}^{n}\left(a_i^k + b_i^k + c_i^k\right)^{\frac{1}{k}} \ge \left[\left(\sum_{i=1}^{n} a_i\right)^{k} + \left(\sum_{i=1}^{n} b_i\right)^{k} + \left(\sum_{i=1}^{n} c_i\right)^{k}\right]^{\frac{1}{k}}.$$

11. REARRANGEMENT INEQUALITY

(1) If $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ are two increasing (or decreasing) real sequences, and $(i_1, i_2, \cdots, i_n)$ is an arbitrary permutation of $(1, 2, \cdots, n)$, then

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \ge a_1 b_{i_1} + a_2 b_{i_2} + \cdots + a_n b_{i_n}$$

and

$$n(a_1b_1 + a_2b_2 + \cdots + a_nb_n) \geq (a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n).$$

(2) If $a_1, a_2, \cdots, a_n$ is decreasing and $b_1, b_2, \cdots, b_n$ is increasing, then

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \leq a_1b_{i_1} + a_2b_{i_2} + \cdots + a_nb_{i_n}$$

and

$$n(a_1b_1 + a_2b_2 + \cdots + a_nb_n) \leq (a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n).$$

(3) Let $b_1, b_2, \cdots, b_n$ and $c_1, c_2, \cdots, c_n$ be two real sequences such that

$$b_1 + \cdots + b_k \geq c_1 + \cdots + c_k, \quad k = 1, 2, \cdots, n.$$

If $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, then

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \geq a_1c_1 + a_2c_2 + \cdots + a_nc_n.$$

Notice that all these inequalities follow immediately from the identity

$$\sum_{i=1}^{n} a_i(b_i - c_i) = \sum_{i=1}^{n}(a_i - a_{i+1})\left(\sum_{j=1}^{i} b_j - \sum_{j=1}^{i} c_j\right),$$

where $a_{n+1} = 0$.

12. MACLAURIN'S INEQUALITY and NEWTON'S INEQUALITY

If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers, then

$$S_1 \geq S_2 \geq \cdots \geq S_n \qquad (Maclaurin)$$

and

$$S_k^2 \geq S_{k-1}S_{k+1}, \qquad (Newton)$$

where

$$S_k = \sqrt[k]{\frac{\sum\limits_{1\leq i_1<\cdots<i_k\leq n} a_{i_1}a_{i_2}\cdots a_{i_k}}{\binom{n}{k}}}.$$

13. CONVEX FUNCTIONS

A function f defined on a real interval $\mathbb{I}$ is said to be *convex* if

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$$

for all $x,\, y \in \mathbb{I}$ and any $\alpha,\, \beta \ge 0$ with $\alpha + \beta = 1$. If the inequality is reversed, then f is said to be concave.
If f is differentiable on $\mathbb{I}$, then f is (strictly) convex if and only if the derivative f' is (strictly) increasing. If $f'' \ge 0$ on $\mathbb{I}$, then f is convex on $\mathbb{I}$. Also, if $f'' \ge 0$ on $(a,\, b)$ and f is continuous on $[a,\, b]$, then f is convex on $[a,\, b]$.
A function $f : \mathbb{I} \to \mathbb{R}$ is *half convex* on a real interval $\mathbb{I}$ if there exists a point $s \in \mathbb{I}$ such that f is convex on $\mathbb{I}_{u \le s}$ or $\mathbb{I}_{u \ge s}$.
A function $f : \mathbb{I} \to \mathbb{R}$ is *right partially convex* related to a a point $s \in \mathbb{I}$ if there exists a number $s_0 \in \mathbb{I}$, $s_0 > s$, such that f is convex on $\mathbb{I}_{u \in [s, s_0]}$. Also, a function $f : \mathbb{I} \to \mathbb{R}$ is *left partially convex* related to a point $s \in \mathbb{I}$ if there exists a point $s_0 \in \mathbb{I}$, $s_0 < s$, such that f is convex on $\mathbb{I}_{u \in [s_0, s]}$.

Jensen's inequality. *Let $p_1, p_2, \ldots, p_n$ be positive real numbers. If f is a convex function on a real interval $\mathbb{I}$, then for any $a_1,\, a_2,\, \ldots,\, a_n \in \mathbb{I}$, the inequality holds*

$$\frac{p_1 f(a_1) + p_2 f(a_2) + \cdots + p_n f(a_n)}{p_1 + p_2 + \cdots + p_n} \ge f\left(\frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_1 + p_2 + \cdots + p_n}\right).$$

For $p_1 = p_2 = \cdots = p_n$, Jensen's inequality becomes

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge n f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right).$$

Based on the following three theorems, we can extend this form of Jensen's inequality to half or partially convex functions.

Half Convex Function-Theorem (*Vasile Cirtoaje*, 2004). *Let $f(u)$ be a function defined on a real interval $\mathbb{I}$ and convex on $\mathbb{I}_{u \ge s}$ or $\mathbb{I}_{u \le s}$, where $s \in \mathbb{I}$. The inequality*

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge n f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying $a_1 + a_2 + \ldots + a_n = ns$ if and only if

$$f(x) + (n-1) f(y) \ge n f(s)$$

for all $x, y \in \mathbb{I}$ such that $x + (n-1)y = ns$.

Half Convex Function Theorem for Ordered Variables (*Vasile Cirtoaje*, 2007). *Let $f(u)$ be a function defined on a real interval $\mathbb{I}$ and convex on $\mathbb{I}_{u \ge s}$ / $\mathbb{I}_{u \le s}$, where $s \in \mathbb{I}$. The inequality*

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge n f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ such that $a_1 + a_2 + \cdots + a_n = ns$ and at least $n - m$ of $a_1, a_2, \ldots, a_n$ are smaller/greater than or equal to s if and only if

$$f(x) + m f(y) \ge (1+m) f(s)$$

for all $x, y \in \mathbb{I}$ *such that* $x + my = (1 + m)s$.

Right Partially Convex Function-Theorem (*Vasile Cirtoaje*, 2012). *Let* f *be a function defined on a real interval* $\mathbb{I}$ *and convex on* $[s, s_0]$, *where* $s, s_0 \in \mathbb{I}$, $s < s_0$. *In addition,* f *is decreasing on* $\mathbb{I}_{u \le s_0}$ *and*

$$\min_{u \in \mathbb{I}} f(u) = f(s_0).$$

The inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \cdots, a_n \in \mathbb{I}$ *satisfying* $a_1 + a_2 + \cdots + a_n = ns$ *if and only if*

$$f(x) + (n-1)f(y) \ge nf(s)$$

for all $x, y \in \mathbb{I}$ *such that* $x \le s \le y$ *and* $x + (n-1)y = ns$.

Left Partially Convex Function-Theorem (*Vasile Cirtoaje*, 2012). *Let* f *be a function defined on a real interval* $\mathbb{I}$ *and convex on* $[s_0, s]$, *where* $s_0, s \in \mathbb{I}$, $s_0 < s$. *In addition,* f *is increasing on* $\mathbb{I}_{u \ge s_0}$ *and satisfies*

$$\min_{u \in \mathbb{I}} f(u) = f(s_0).$$

The inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $x_1, x_2, \cdots, x_n \in \mathbb{I}$ *satisfying* $a_1 + a_2 + \cdots + a_n = ns$ *if and only if*

$$f(x) + (n-1)f(y) \ge nf(s)$$

for all $x, y \in \mathbb{I}$ *such that* $x \ge s \ge y$ *and* $x + (n-1)y = ns$.

Right Partially Convex Function Theorem for Ordered Variables (*Vasile Cirtoaje*, 2012). *Let* f *be a function defined on a real interval* $\mathbb{I}$ *and convex on* $[s, s_0]$, *where* $s, s_0 \in \mathbb{I}$, $s < s_0$. *In addition,* f *is decreasing on* $\mathbb{I}_{u \le s_0}$ *and*

$$\min_{u \in \mathbb{I}} f(u) = f(s_0).$$

The inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \cdots, a_n \in \mathbb{I}$ *such that* $a_1 + a_2 + \cdots + a_n = ns$ *and at least* $n - m$ *of* $a_1, a_2, \ldots, a_n$ *are smaller than or equal to* s *if and only if*

$$f(x) + mf(y) \geq (1+m)f(s)$$

for all $x, y \in \mathbb{I}$ *such that* $x \leq s \leq y$ *and* $x + my = (1+m)s$.

Left Partially Convex Function Theorem for Ordered Variables (*Vasile Cirtoaje*, 2012). *Let* f *be a function defined on a real interval* $\mathbb{I}$ *and convex on* $[s_0, s]$, *where* $s_0, s \in \mathbb{I}$, $s_0 < s$. *In addition,* f *is increasing on* $\mathbb{I}_{u \geq s_0}$ *and satisfies*

$$\min_{u \in \mathbb{I}} f(u) = f(s_0).$$

The inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $x_1, x_2, \cdots, x_n \in \mathbb{I}$ *such that* $a_1 + a_2 + \cdots + a_n = ns$ *and at least* $n - m$ *of* $a_1, a_2, \ldots, a_n$ *are greater than or equal to* s *if and only if if and only if*

$$f(x) + mf(y) \geq (1+m)nf(s)$$

for all $x, y \in \mathbb{I}$ *such that* $x \geq s \geq y$ *and* $x + my = (1+m)s$.

In all these theorems, we may replace the hypothesis condition

$$f(x) + mf(y) \geq (1+m)f(s),$$

by the equivalent condition

$$h(x, y) \geq 0 \text{ for all } x, y \in \mathbb{I} \text{ such that } x + my = (1+m),$$

where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \qquad g(u) = \frac{f(u) - f(s)}{u - s}.$$

The following theorem is also useful to prove some symmetric inequalities.

Left Convex-Right Concave Function Theorem (*Vasile Cirtoaje*, 2004). *Let* $a < c$ *be real numbers, let* f *be a continuous function on* $\mathbb{I} = [a, \infty)$, *strictly convex on* $[a, c]$ *and strictly concave on* $[c, \infty)$, *and let*

$$E(a_1, a_2, \ldots, a_n) = f(a_1) + f(a_2) + \cdots + f(a_n).$$

If $a_1, a_2, \ldots, a_n \in \mathbb{I}$ *such that*

$$a_1 + a_2 + \cdots + a_n = S = constant,$$

then

(a) *E is minimum for $a_1 = a_2 = \cdots = a_{n-1} \le a_n$;*

(b) *E is maximum for either $a_1 = a$* or *$a < a_1 \le a_2 = \cdots = a_n$.*

On the other hand, it is known the following result concerning the best upper bound of Jensen's difference.

Best Upper Bound of Jensen's Difference-Theorem (*Vasile Cirtoaje*, 1989). *Let $p_1, p_2, \ldots, p_n$ be fixed positive real numbers, and let f be a convex function on a closed interval $\mathbb{I} = [a, b]$. If $a_1, a_2, \cdots, a_n \in \mathbb{I}$, then Jensen's difference*

$$D = \frac{p_1 f(a_1) + p_2 f(a_2) + \cdots + p_n f(a_n)}{p_1 + p_2 + \cdots + p_n} - f\left(\frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_1 + p_2 + \cdots + p_n}\right)$$

is maximum when some of a_i are equal to a, and the others a_i are equal to b; that is, when all $a_i \in \{a, b\}$.

14. KARAMATA'S MAJORIZATION INEQUALITY

We say that a vector $\overrightarrow{A} = (a_1, a_2, \ldots, a_n)$ with $a_1 \ge a_2 \ge \cdots \ge a_n$ majorizes a vector $\overrightarrow{B} = (b_1, b_2, \ldots, b_n)$ with $b_1 \ge b_2 \ge \cdots \ge b_n$, and write it as

$$\overrightarrow{A} \succ \overrightarrow{B},$$

if

$$\begin{gathered} a_1 \ge b_1, \\ a_1 + a_2 \ge b_1 + b_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_1 + a_2 + \cdots + a_{n-1} \ge b_1 + b_2 + \cdots + b_{n-1}, \\ a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n. \end{gathered}$$

Let f be a convex function on a real interval $\mathbb{I}$. If a decreasingly ordered vector

$$\overrightarrow{A} = (a_1, a_2, \ldots, a_n), \quad a_i \in \mathbb{I},$$

majorizes a decreasingly ordered vector

$$\overrightarrow{B} = (b_1, b_2, \ldots, b_n), \quad b_i \in \mathbb{I},$$

then

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge f(b_1) + f(b_2) + \cdots + f(b_n).$$

15. POPOVICIU'S INEQUALITY

If f is a convex function on a real interval $\mathbb{I}$ and $a_1, a_2, \ldots, a_n \in \mathbb{I}$, then

$$f(a_1) + f(a_2) + \cdots + f(a_n) + n(n-2)f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \ge$$

$$\geq (n-1)[f(b_1)+f(b_2)+\cdots+f(b_n)],$$

where

$$b_i = \frac{1}{n-1}\sum_{j\neq i} a_j, \quad i=1,2,\cdots,n.$$

16. SQUARE PRODUCT INEQUALITY

Let a,b,c be real numbers, and let

$$p=a+b+c, \quad q=ab+bc+ca, \quad r=abc,$$

$$s=\sqrt{p^2-3q}=\sqrt{a^2+b^2+c^2-ab-bc-ca}.$$

From the identity

$$27(a-b)^2(b-c)^2(c-a)^2=4(p^2-3q)^3-(2p^3-9pq+27r)^2,$$

it follows that

$$\frac{-2p^3+9pq-2(p^2-3q)\sqrt{p^2-3q}}{27} \leq r \leq \frac{-2p^3+9pq+2(p^2-3q)\sqrt{p^2-3q}}{27},$$

which is equivalent to

$$\frac{p^3-3ps^2-2s^3}{27} \leq r \leq \frac{p^3-3ps^2+2s^3}{27}.$$

Therefore, for constant p and q, the product r is minimal and maximal when two of a,b,c are equal.

17. SYMMETRIC INEQUALITIES OF DEGREE THREE, FOUR OR FIVE

Let $f_n(a,b,c)$ be a symmetric homogeneous polynomial of degree $n=3$, $n=4$ or $n=5$.

Theorem.

(a) *The inequality $f_4(a,b,c) \geq 0$ holds for all real numbers a,b,c if and only if $f_4(a,1,1)\geq 0$ for all real a;*

(b) *The inequality $f_n(a,b,c)\geq 0$ holds for all $a,b,c\geq 0$ if and only if $f_n(a,1,1)\geq 0$ and $f_n(0,b,c)\geq 0$ for all $a,b,c\geq 0$.*

18. SYMMETRIC INEQUALITIES OF DEGREE SIX

Any sixth degree symmetric homogeneous polynomial $f_6(a,b,c)$ can be written in the form

$$f_6(a,b,c)=Ar^2+B(p,q)r+C(p,q),$$

where A is called *the highest coefficient* of f_6, and

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

Theorem (*Vasile Cirtoaje*, 2008). *Let* $A \le 0$.

(a) *The inequality* $f_6(a, b, c) \ge 0$ *holds for all real numbers* a, b, c *if and only if* $f_6(a, 1, 1) \ge 0$ *for all real* a;

(b) *The inequality* $f_6(a, b, c) \ge 0$ *holds for all* $a, b, c \ge 0$ *if and only if* $f_6(a, 1, 1) \ge 0$ *and* $f_6(0, b, c) \ge 0$ *for all* $a, b, c \ge 0$.

For $A > 0$, we can use the *highest coefficient cancellation method* (*Vasile Cirtoaje*, 2008). This method consists in finding some suitable real numbers B, C and D such that the following sharper inequality holds

$$f_6(a, b, c) \ge A\left(r + Bp^3 + Cpq + D\frac{q^2}{p}\right)^2.$$

Because the function g_6 defined by

$$g_6(a, b, c) = f_6(a, b, c) - A\left(r + Bp^3 + Cpq + D\frac{q^2}{p}\right)^2$$

has the highest coefficient $A_1 = 0$, we can prove the inequality $g_6(a, b, c) \ge 0$ using Theorem above.

Notice that sometimes it is useful to break the problem into two parts, $p^2 \le \xi q$ and $p^2 > \xi q$, where ξ is a suitable real number.

19. EQUAL VARIABLE METHOD

The Equal Variable Theorem (EV-Theorem) for nonnegative real variables has the following statement (*Vasile Cirtoaje*, 2005).

EV-Theorem (for nonnegative variables). *Let* $a_1, a_2, \cdots, a_n$ ($n \ge 3$) *be fixed nonnegative real numbers, and let* $x_1 \le x_2 \le \cdots \le x_n$ *be nonnegative real variables such that*

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n,$$

$$x_1^k + x_2^k + \cdots + x_n^k = a_1^k + a_2^k + \cdots + a_n^k,$$

where k *is a real number; for* $k = 0$, *assume that* $x_1x_2\cdots x_n = a_1a_2\cdots a_n > 0$. *Let* $f : (0, \infty) \to \mathbb{R}$ *be a differentiable function such that* $g : (0, \infty) \to \mathbb{R}$ *defined by*

$$g(x) = f'\left(x^{\frac{1}{k-1}}\right)$$

is strictly convex, and let

$$S_n = f(x_1) + f(x_2) + \cdots + f(x_n).$$

(1) *If $k \le 0$, then S_n is maximum for*

$$x_1 = x_2 = \cdots = x_{n-1} \le x_n,$$

and is minimum for

$$0 < x_1 \le x_2 = x_3 = \cdots = x_n;$$

(2) *If $k > 0$ and either f is continuous at $x = 0$ or $f(0_+) = -\infty$, then S_n is maximum for*

$$x_1 = x_2 = \cdots = x_{n-1} \le x_n;$$

(3) *If $k > 0$ and either f is continuous at $x = 0$ or $f(0_+) = \infty$, then S_n is minimum for*

$$x_1 = \cdots = x_{j-1} = 0, \quad x_{j+1} = \cdots = x_n, \quad j \in \{1, 2, \ldots, n\}.$$

For $f(x) = x^m$, we get the following corollary.

EV-COROLLARY (for nonnegative variables). *Let $a_1, a_2, \cdots, a_n$ ($n \ge 3$) be fixed nonnegative real numbers, let $x_1 \le x_2 \le \cdots \le x_n$ be nonnegative real variables such that*

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n,$$

$$x_1^k + x_2^k + \cdots + x_n^k = a_1^k + a_2^k + \cdots + a_n^k,$$

and let

$$S_n = x_1^m + x_2^m + \cdots + x_n^m.$$

Case 1 : $k \le 0$ (*for $k = 0$, assume that $x_1 x_2 \cdots x_n = a_1 a_2 \cdots a_n > 0$*).

(a) *If $m \in (k, 0) \cup (1, \infty)$, then S_n is maximum for*

$$x_1 = x_2 = \cdots = x_{n-1} \le x_n,$$

and is minimum for

$$x_1 \le x_2 = x_3 = \cdots = x_n;$$

(b) *If $m \in (-\infty, k) \cup (0, 1)$, then S_n is minimum for*

$$x_1 = x_2 = \cdots = x_{n-1} \le x_n,$$

and is maximum for

$$x_1 \le x_2 = x_3 = \cdots = x_n.$$

Case 2 : $0 < k < 1$.

(a) *If $m \in (0,k) \cup (1,\infty)$, then S_n is maximum for*

$$x_1 = x_2 = \cdots = x_{n-1} \le x_n,$$

and is minimum for

$$x_1 = \cdots = x_{j-1} = 0, \quad x_{j+1} = \cdots = x_n, \qquad j \in \{1,2,\ldots,n\};$$

(b) *If $m \in (-\infty,0) \cup (k,1)$, then S_n is minimum for*

$$x_1 = x_2 = \cdots = x_{n-1} \le x_n;$$

(c) *If $m \in (k,1)$, then S_n is maximum for*

$$x_1 = \cdots = x_{j-1} = 0, \quad x_{j+1} = \cdots = x_n, \qquad j \in \{1,2,\ldots,n\}.$$

Case $3:\ k > 1$.

(a) *If $m \in (0,1) \cup (k,\infty)$, then S_n is maximum for*

$$x_1 = x_2 = \cdots = x_{n-1} \le x_n,$$

and is minimum for

$$x_1 = \cdots = x_{j-1} = 0, \quad x_{j+1} = \cdots = x_n, \qquad j \in \{1,2,\ldots,n\};$$

(b) *If $m \in (-\infty,0) \cup (1,k)$, then S_n is minimum for*

$$0 \le x_1 = x_2 = \cdots = x_{n-1} \le x_n;$$

(c) *If $m \in (1,k)$, then S_n is maximum for*

$$x_1 = \cdots = x_{j-1} = 0, \quad x_{j+1} = \cdots = x_n, \qquad j \in \{1,2,\ldots,n\}.$$

The Equal Variable Theorem (EV-Theorem) for real variables has the following statement (*Vasile Cirtoaje*, 2012).

EV-Theorem (for real variables). *Let $a_1, a_2, \cdots, a_n$ $(n \ge 3)$ be fixed real numbers, let $x_1 \le x_2 \le \cdots \le x_n$ be real variables such that*

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n,$$

$$x_1^k + x_2^k + \cdots + x_n^k = a_1^k + a_2^k + \cdots + a_n^k,$$

where k is an even positive integer, and let f be a differentiable function on $\mathbb{R}$ such that the associated function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = f'\left(\sqrt[k-1]{x}\right)$$

is strictly convex on $\mathbb{R}$. Then, the sum

$$S_n = f(x_1) + f(x_2) + \cdots + f(x_n)$$

is minimum for $x_2 = x_3 = \cdots = x_n$, and is maximum for $x_1 = x_2 = \cdots = x_{n-1}$.

20. ARITHMETIC COMPENSATION METHOD

The Arithmetic Compensation Theorem (AC-Theorem) has the following statement (*Vasile Cirtoaje*, 2002).

AC-THEOREM. *Let $s > 0$ and let F be a symmetric continuous function on the compact set in $\mathbb{R}^n$*

$$S = \{(x_1, x_2, \cdots, x_n) : x_1 + x_2 + \cdots + x_n = s,\ x_i \ge 0,\ i = 1, 2, \cdots, n\}.$$

If

$$F(x_1, x_2, x_3, \cdots, x_n) \ge$$
$$\ge \min\left\{F\left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}, x_3, \cdots, x_n\right), F(0, x_1+x_2, x_3, \cdots, x_n)\right\}$$

for all $(x_1, x_2, \cdots, x_n) \in S$, then $F(x_1, x_2, x_3, \cdots, x_n)$ is minimal when

$$x_1 = x_2 = \cdots = x_k = \frac{s}{k}, \quad x_{k+1} = \cdots = x_n = 0;$$

that is,

$$F(x_1, x_2, x_3, \cdots, x_n) \ge \min_{1 \le k \le n} F\left(\frac{s}{k}, \cdots, \frac{s}{k}, 0, \cdots, 0\right)$$

for all $(x_1, x_2, \cdots, x_n) \in S$.

Notice that if

$$F(x_1, x_2, x_3, \cdots, x_n) < F\left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}, x_3, \cdots, x_n\right)$$

involves

$$F(x_1, x_2, x_3, \cdots, x_n) \ge F(0, x_1+x_2, x_3, \cdots, x_n),$$

then the hypothesis

$$F(x_1, x_2, x_3, \cdots, x_n) \ge$$
$$\ge \min\left\{F\left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}, x_3, \cdots, x_n\right), F(0, x_1+x_2, x_3, \cdots, x_n)\right\}$$

is satisfied.

21. VASC'S INEQUALITY

If a, b, c are real numbers, then

$$(a^2 + b^2 + c^2)^2 \geq 3(a^3b + b^3c + c^3a),$$

with equality for $a = b = c$, and also for

$$\frac{a}{\sin^2 \frac{4\pi}{7}} = \frac{b}{\sin^2 \frac{2\pi}{7}} = \frac{c}{\sin^2 \frac{\pi}{7}}$$

(or any cyclic permutation) - *Vasile Cirtoaje*, 1991.

The following theorem gives a generalization of Vasc's inequality.

Theorem 0 (*Vasile Cirtoaje*, 2007). *Let*

$$f_4(a, b, c) = \sum a^4 + A\sum a^2b^2 + Babc\sum a + C\sum a^3b + D\sum ab^3,$$

where A, B, C, D are real constants such that $1 + A + B + C + D = 0$. The inequality $f_4(a, b, c) \geq 0$ holds for all real numbers a, b, c if and only if

$$3(1 + A) \geq C^2 + CD + D^2.$$

Notice that

$$\frac{4}{S}f_4(a, b, c) = (U + V + C + D)^2 + 3\left(U - V + \frac{C - D}{3}\right)^2 + \frac{4}{3}(3 + 3A - C^2 - CD - D^2),$$

where

$$S = \sum a^2b^2 - \sum a^2bc, \quad U = \frac{\sum a^3b - \sum a^2bc}{S}, \quad V = \frac{\sum ab^3 - \sum a^2bc}{S}.$$

For $A = B = 0$, $C = -2$ and $D = 1$, we get the following inequality

$$a^4 + b^4 + c^4 + ab^3 + bc^3 + ca^3 \geq 2(a^3b + b^3c + c^3a),$$

with equality for $a = b = c$, and also for

$$\frac{a}{\sin \frac{\pi}{9}} = \frac{b}{\sin \frac{7\pi}{9}} = \frac{c}{\sin \frac{13\pi}{9}}$$

(or any cyclic permutation) - *Vasile Cirtoaje*, 1991.

22. CYCLIC INEQUALITIES OF DEGREE THREE AND FOUR

Consider the third degree cyclic homogeneous polynomial

$$f_3(a, b, c) = \sum a^3 + Babc + C\sum a^2b + D\sum ab^2,$$

where B, C, D are real constants. The following Theorem 1 holds.

Theorem 1 (*Pham Kim Hung*, 2007). *The cyclic inequality $f_3(a,b,c) \ge 0$ holds for all $a,b,c \ge 0$ if and only if*

$$f_3(1,1,1) \ge 0$$

and

$$f_3(a,1,0) \ge 0$$

for all $a \ge 0$.

Consider now the fourth degree cyclic homogeneous polynomial

$$f_4(a,b,c) = \sum a^4 + A\sum a^2b^2 + Babc\sum a + C\sum a^3b + D\sum ab^3,$$

where A, B, C, D are real constants.

The following Theorem 2 states the necessary and sufficient conditions that $f_4(a,b,c) \ge 0$ for all real numbers a, b, c.

Theorem 2 (*Vasile Cirtoaje* and *Yuanzhe Zhou*, 2011). *The inequality $f_4(a,b,c) \ge 0$ holds for all real numbers a,b,c if and only if $g_4(t) \ge 0$ for all $t \ge 0$, where*

$$g_4(t) = 3(2+A-C-D)t^4 - Ft^3 + 3(4-B+C+D)t^2 + 1 + A + B + C + D,$$

$$F = \sqrt{27(C-D)^2 + E^2}, \quad E = 8 - 4A + 2B - C - D.$$

Note that in the special case $f_4(1,1,1) = 0$ (when $1 + A + B + C + D = 0$), Theorem 1 yields Theorem 0 from the preceding section 21.

The following Theorem 3 states some strong sufficient conditions that $f_4(a,b,c) \ge 0$ for all real numbers a, b, c.

Theorem 3 (*Vasile Cirtoaje* and *Yuanzhe Zhou*, 2012). *The inequality $f_4(a,b,c) \ge 0$ holds for all real numbers a,b,c if the following two conditions are satisfied*:

(a) $1 + A + B + C + D \ge 0$;

(b) *there exists a real number $t \in (-\sqrt{3}, \sqrt{3})$ such that $f(t) \ge 0$, where*

$$f(t) = 2Gt^3 - (6 + 2A + B + 3C + 3D)t^2 + 2(1 + C + D)Gt + H,$$

$$G = \sqrt{1 + A + B + C + D}, \quad H = 2 + 2A - B - C - D - C^2 - CD - D^2.$$

The following Theorem 4 states the necessary and sufficient conditions that $f_4(a,b,c) \ge 0$ for all $a,b,c \ge 0$.

Theorem 4 (*Vasile Cirtoaje*, 2013). *Let*

$$E = 8 - 4A + 2B - C - D, \quad F = \sqrt{27(C-D)^2 + E^2},$$

$$g_4(t) = 3(2+A-C-D)t^4 - Ft^3 + 3(4-B+C+D)t^2 + 1 + A + B + C + D,$$

$$g_3(t) = \frac{2E}{F} t^3 + 3t^2 - 1.$$

For $F = 0$, the inequality $f_4(a,b,c) \geq 0$ holds for all $a,b,c \geq 0$ if and only if $g_4(t) \geq 0$ for all $t \in [0,1]$.

For $F \neq 0$, the inequality $f_4(a,b,c) \geq 0$ holds for all $a,b,c \geq 0$ if and only if the following two conditions are satisfied:

(a) $g_4(t) \geq 0$ *for all $t \in [0, t_1]$, where $t_1 \in [1/2, 1]$ such that $g_3(t_1) = 0$;*

(b) $f_4(a,1,0) \geq 0$ *for all $a \geq 0$.*

The following Theorem 5 states some strong sufficient conditions that $f_4(a,b,c) \geq 0$ for all $a,b,c \geq 0$.

Theorem 5 (*Vasile Cirtoaje* and *Yuanzhe Zhou*, 2013). *The inequality $f_4(a,b,c) \geq 0$ holds for all $a,b,c \geq 0$ if*

$$1 + A + B + C + D \geq 0$$

and one of the following two conditions is satisfied:

(a) $3(1+A) \geq C^2 + CD + D^2$;

(b) $3(1+A) < C^2 + CD + D^2$, *and there exists $t \geq 0$ such that*

$$(C+2D)t^2 + 6t + 2C + D \geq 2\sqrt{(t^4 + t^2 + 1)(C^2 + CD + D^2 - 3 - 3A)}.$$

Chapter 2

Symmetric Polynomial Inequalities in Real Variables

2.1 Applications

2.1. Let a, b, c, d be real numbers such that $a^2 + b^2 + c^2 + d^2 = 9$. Prove that

$$a^3 + b^3 + c^3 + d^3 \le 27.$$

2.2. If a, b, c are real numbers such that $a + b + c = 0$, then

$$(2a^2 + bc)(2b^2 + ca)(2c^2 + ab) \le 0.$$

2.3. Let a, b, c be real numbers such that $a + b \ge 0$, $b + c \ge 0$, $c + a \ge 0$. Prove that

$$9(a + b)(b + c)(c + a) \ge 8(a + b + c)(ab + bc + ca).$$

2.4. Let a, b, c be real numbers such that $ab + bc + ca = 3$. Prove that

$$(3a^2 + 1)(3b^2 + 1)(3c^2 + 1) \ge 64.$$

When does equality hold?

2.5. If a and b are real numbers, then

$$3(1 - a + a^2)(1 - b + b^2) \ge 2(1 - ab + a^2b^2).$$

2.6. If a, b, c are real numbers, then

$$3(1-a+a^2)(1-b+b^2)(1-c+c^2) \ge 1+abc+a^2b^2c^2.$$

2.7. If a, b, c are real numbers, then

$$(a^2+b^2+c^2)^3 \ge (a+b+c)(ab+bc+ca)(a^3+b^3+c^3).$$

2.8. If a, b, c are real numbers, then

$$2(a^2+b^2)(b^2+c^2)(c^2+a^2) \ge [ab(a+b)+bc(b+c)+ca(c+a)-2abc]^2.$$

2.9. If a, b, c are real numbers, then

$$(a^2+1)(b^2+1)(c^2+1) \ge 2(ab+bc+ca).$$

2.10. If a, b, c are real numbers, then

$$(a^2+1)(b^2+1)(c^2+1) \ge \frac{5}{16}(a+b+c+1)^2.$$

2.11. If a, b, c are real numbers, then

(a) $$a^6+b^6+c^6-3a^2b^2c^2+2(a^2+bc)(b^2+ca)(c^2+ab) \ge 0;$$

(b) $$a^6+b^6+c^6-3a^2b^2c^2 \ge (a^2-2bc)(b^2-2ca)(c^2-2ab).$$

2.12. If a, b, c are real numbers, then

$$\frac{2}{3}(a^6+b^6+c^6)+a^3b^3+b^3c^3+c^3a^3+abc(a^3+b^3+c^3) \ge 0.$$

2.13. If a, b, c are real numbers, then

$$4(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \ge (a-b)^2(b-c)^2(c-a)^2.$$

2.14. If a, b, c are real numbers, then

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \geq 3(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2).$$

2.15. If a, b, c are real numbers such that $abc > 0$, then

$$4\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right) \geq 9(a+b+c).$$

2.16. If a, b, c are real numbers, then

(a) $$(a^2+2bc)(b^2+2ca)(c^2+2ab) \leq (a^2+b^2+c^2)(ab+bc+ca)^2;$$

(b) $$(2a^2+bc)(2b^2+ca)(2c^2+ab) \leq (a+b+c)^2(a^2b^2+b^2c^2+c^2a^2).$$

2.17. If a, b, c are real numbers such that $ab+bc+ca \geq 0$, then

$$27(a^2+2bc)(b^2+2ca)(c^2+2ab) \leq (a+b+c)^6.$$

2.18. If a, b, c are real numbers such that $a^2+b^2+c^2 = 2$, then

$$(a^2+2bc)(b^2+2ca)(c^2+2ab)+2 \geq 0.$$

2.19. If a, b, c are real numbers such that $a+b+c = 3$, then

$$3(a^4+b^4+c^4)+a^2+b^2+c^2+6 \geq 6(a^3+b^3+c^3).$$

2.20. If a, b, c are real numbers such that $abc = 1$, then

$$3(a^2+b^2+c^2)+2(a+b+c) \geq 5(ab+bc+ca).$$

2.21. If a, b, c are real numbers such that $abc = 1$, then

$$a^2+b^2+c^2+6 \geq \frac{3}{2}\left(a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

2.22. If a, b, c are real numbers, then

$$(1+a^2)(1+b^2)(1+c^2)+8abc \ge \frac{1}{4}(1+a)^2(1+b)^2(1+c)^2.$$

2.23. Let $a,\ b,\ c$ be real numbers such that $a+b+c=0$. Prove that

$$a^{12}+b^{12}+c^{12} \ge \frac{2049}{8}a^4b^4c^4.$$

2.24. If a, b, c are real numbers such that $abc \ge 0$, then

$$a^2+b^2+c^2+2abc+4 \ge 2(a+b+c)+ab+bc+ca.$$

2.25. Let a, b, c be real numbers such that $a+b+c=3$.

(a) If $a, b, c \ge -3$, then

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \ge \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

(b) If $a, b, c \ge -7$, then

$$\frac{1-a}{(1+a)^2}+\frac{1-b}{(1+b)^2}+\frac{1-c}{(1+c)^2} \ge 0.$$

2.26. If a, b, c are real numbers, then

$$a^6+b^6+c^6-3a^2b^2c^2 \ge \frac{1}{2}(a-b)^2(b-c)^2(c-a)^2.$$

2.27. If a, b, c are real numbers, then

$$\left(\frac{a^2+b^2+c^2}{3}\right)^3 \ge a^2b^2c^2+\frac{1}{16}(a-b)^2(b-c)^2(c-a)^2.$$

2.28. If a, b, c are real numbers, then

$$(a^2+b^2+c^2)^3 \ge \frac{108}{5}a^2b^2c^2+2(a-b)^2(b-c)^2(c-a)^2.$$

2.29. If a, b, c are real numbers, then

$$2(a^2+b^2)(b^2+c^2)(c^2+a^2) \geq (a-b)^2(b-c)^2(c-a)^2.$$

2.30. If a, b, c are real numbers, then

$$32(a^2+bc)(b^2+ca)(c^2+ab)+9(a-b)^2(b-c)^2(c-a)^2 \geq 0.$$

2.31. If a, b, c are real numbers, then

$$a^4(b-c)^2+b^4(c-a)^2+c^4(a-b)^2 \geq \frac{1}{2}(a-b)^2(b-c)^2(c-a)^2.$$

2.32. If a, b, c are real numbers, then

$$a^2(b-c)^4+b^2(c-a)^4+c^2(a-b)^4 \geq \frac{1}{2}(a-b)^2(b-c)^2(c-a)^2.$$

2.33. If a, b, c are real numbers, then

$$a^2(b^2-c^2)^2+b^2(c^2-a^2)^2+c^2(a^2-b^2)^2 \geq \frac{3}{8}(a-b)^2(b-c)^2(c-a)^2.$$

2.34. If a, b, c are real numbers such that $ab+bc+ca=3$, then

(a) $$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \geq 3(a+b+c)^2;$$

(b) $$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \geq \frac{3}{2}(a^2+b^2+c^2).$$

2.35. If a, b, c are real numbers, then

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \geq 3(ab+bc+ca)(a^2b^2+b^2c^2+c^2a^2).$$

2.36. If a, b, c are real numbers, not all of the same sign, then

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \geq 3(ab+bc+ca)^3.$$

2.37. If a, b, c are real numbers, then

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \ge \frac{3}{8}(a^2+b^2)(b^2+c^2)(c^2+a^2).$$

2.38. If a, b, c are real numbers, then

$$2(a^2+b^2)(b^2+c^2)(c^2+a^2) \ge (a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2).$$

2.39. If a, b, c are real numbers, then

$$9(1+a^4)(1+b^4)(1+c^4) \ge 8(1+abc+a^2b^2c^2)^2.$$

2.40. If a, b, c are real numbers, then

$$2(1+a^2)(1+b^2)(1+c^2) \ge (1+a)(1+b)(1+c)(1+abc).$$

2.41. If a, b, c are real numbers, then

$$3(a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2) \ge a^3b^3+b^3c^3+c^3a^3.$$

2.42. If a, b, c are nonzero real numbers, then

$$\sum \frac{b^2-bc+c^2}{a^2} + 2\sum \frac{a^2}{bc} \ge \left(\sum a\right)\left(\sum \frac{1}{a}\right).$$

2.43. Let a, b, c be real numbers. Prove that

(a) if $a, b, c \in [0, 1]$, then

$$abc-(b+c-a)(c+a-b)(a+b-c) \le 1;$$

(b) if $a, b, c \in [-1, 1]$, then

$$abc-(b+c-a)(c+a-b)(a+b-c) \le 4.$$

2.44. Let a, b, c be real numbers. Prove that

(a) if $a, b, c \in [0, 1]$, then

$$\sum a^2(a-b)(a-c) \le 1;$$

(b) if $a, b, c \in [-1, 1]$, then

$$\sum a^2(a-b)(a-c) \le 4.$$

2.45. Let a, b, c be real numbers such that

$$ab + bc + ca = abc + 2.$$

Prove that

$$a^2 + b^2 + c^2 - 3 \ge (2+\sqrt{3})(a+b+c-3).$$

2.46. Let a, b, c be real numbers such that

$$(a+b)(b+c)(c+a) = 10.$$

Prove that

$$(a^2+b^2)(b^2+c^2)(c^2+a^2) + 12a^2b^2c^2 \ge 30.$$

2.47. Let a, b, c be real numbers such that

$$(a+b)(b+c)(c+a) = 5.$$

Prove that

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) + 12a^2b^2c^2 \ge 15.$$

2.48. Let a, b, c be real numbers such that $a+b+c = 1$ and $a^3+b^3+c^3 = k$. Prove that

(a) if $k = 25$, then $|a| \le 1$ or $|b| \le 1$ or $|c| \le 1$;

(b) if $k = -11$, then $1 < a \le 2$ or $1 < b \le 2$ or $1 < c \le 2$.

2.49. Let a, b, c be real numbers such that

$$a+b+c=a^3+b^3+c^3=2.$$

Prove that $a, b, c \notin \left[\frac{5}{4}, 2\right]$.

2.50. If a, b, c and k are real numbers, then

$$\sum(a-b)(a-c)(a-kb)(a-kc) \ge 0.$$

2.51. If a, b, c are real numbers, then

$$(b+c-a)^2(c+a-b)^2(a+b-c)^2 \ge (b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2).$$

2.52. If a, b, c are real numbers, then

$$\sum a^2(a-b)(a-c) \ge \frac{(a-b)^2(b-c)^2(c-a)^2}{a^2+b^2+c^2+ab+bc+ca}.$$

2.53. Let $a \le b \le c$ be real numbers such that

$$a+b+c=p, \quad ab+bc+ca=q,$$

where p and q are fixed real numbers satisfying $p^2 \ge 3q$. Prove that the product

$$r=abc$$

is minimal when $b=c$, and is maximal when $a=b$.

2.54. Let a, b, c be real numbers such that $a+b+c=3$. Prove that

$$(ab+bc+ca-3)^2 \ge 27(abc-1).$$

2.55. Let a, b, c be real numbers such that $a+b+c=3$. Prove that

$$(ab+bc+ca)^2+9 \ge 18abc.$$

2.56. If a, b, c are real numbers such that $a^2 + b^2 + c^2 = 9$, then

$$abc + 10 \geq 2(a + b + c).$$

2.57. If a, b, c are real numbers such that

$$a + b + c + abc = 4,$$

then

$$a^2 + b^2 + c^2 + 3 \geq 2(ab + bc + ca).$$

2.58. If a, b, c are real numbers such that

$$ab + bc + ca = 3abc,$$

then

$$4(a^2 + b^2 + c^2) + 9 \geq 7(ab + bc + ca).$$

2.59. If a, b, c are real numbers such that $a + b + c = 3$, then

$$(a^2 + 1)(b^2 + 1)(c^2 + 1) \geq (a + 1)(b + 1)(c + 1).$$

2.60. Let $f_4(a, b, c)$ be a symmetric homogeneous polynomial of degree four. Prove that the inequality $f_4(a, b, c) \geq 0$ holds for all real numbers a, b, c if and only if $f_4(a, 1, 1) \geq 0$ for all real a.

2.61. If a, b, c are real numbers, then

$$10(a^4 + b^4 + c^4) + 64(a^2b^2 + b^2c^2 + c^2a^2) \geq 33 \sum ab(a^2 + b^2).$$

2.62. If a, b, c are real numbers such that $a + b + c = 3$, then

$$3(a^4 + b^4 + c^4) + 33 \geq 14(a^2 + b^2 + c^2).$$

2.63. If a, b, c are real numbers such that $a^2 + b^2 + c^2 = 3$, then

$$a^4 + b^4 + c^4 + 3(ab + bc + ca) \le 12.$$

2.64. Let α, β, γ be real numbers such that

$$1 + \alpha + \beta = 2\gamma.$$

The inequality

$$\sum a^4 + \alpha \sum a^2b^2 + \beta abc \sum a \ge \gamma \sum ab(a^2 + b^2)$$

holds for any real numbers a, b, c if and only if

$$1 + \alpha \ge \gamma^2.$$

2.65. If a, b, c are real numbers such that $a^2 + b^2 + c^2 = 2$, then

$$ab(a^2 - ab + b^2 - c^2) + bc(b^2 - bc + c^2 - a^2) + ca(c^2 - ca + a^2 - b^2) \le 1.$$

2.66. If a, b, c are real numbers, then

$$(a + b)^4 + (b + c)^4 + (c + a)^4 \ge \frac{4}{7}(a^4 + b^4 + c^4).$$

2.67. Let a, b, c be real numbers, and let

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

Prove that

$$(3 - p)r + \frac{p^2 + q^2 - pq}{3} \ge q.$$

2.68. If a, b, c are real numbers, then

$$\frac{ab(a + b) + bc(b + c) + ca(c + a)}{(a^2 + 1)(b^2 + 1)(c^2 + 1)} \le \frac{3}{4}.$$

2.69. If a, b, c are real numbers such that $abc > 0$, then

$$\left(a+\frac{1}{a}-1\right)\left(b+\frac{1}{b}-1\right)\left(c+\frac{1}{c}-1\right)+2 \geq \frac{1}{3}(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

2.70. If a, b, c are real numbers, then

$$\left(a^2+\frac{1}{2}\right)\left(b^2+\frac{1}{2}\right)\left(c^2+\frac{1}{2}\right) \geq \left(a+b-\frac{1}{2}\right)\left(b+c-\frac{1}{2}\right)\left(c+a-\frac{1}{2}\right).$$

2.71. If a, b, c are real numbers such that $a+b+c=3$, then

$$\frac{a(a-1)}{8a^2+9}+\frac{b(b-1)}{8b^2+9}+\frac{c(c-1)}{8c^2+9} \geq 0.$$

2.72. If a, b, c are real numbers such that $a+b+c=3$, then

$$\frac{(a-11)(a-1)}{2a^2+1}+\frac{(b-11)(b-1)}{2b^2+1}+\frac{(c-11)(c-1)}{2c^2+1} \geq 0.$$

2.73. If a, b, c are real numbers, then

$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca).$$

2.74. If a, b, c are real numbers such that $ab+bc+ca=3$, then

$$4(a^4+b^4+c^4)+11abc(a+b+c) \geq 45.$$

2.75. Any sixth degree symmetric homogeneous polynomial $f_6(a, b, c)$ can be written in the form

$$f_6(a, b, c) = Ar^2 + B(p, q)r + C(p, q),$$

where A is called *the highest coefficient* of f_6, and

$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc.$$

In the case $A \leq 0$, prove that the inequality $f_6(a, b, c) \geq 0$ holds for all real numbers a, b, c if and only if $f_6(a, 1, 1) \geq 0$ for all real a.

2.76. If a, b, c are real numbers such that $ab + bc + ca = -1$, then

(a) $$5(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) \geq 8;$$

(b) $$(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \geq 1.$$

2.77. If a, b, c are real numbers, then

(a) $$\sum a^2(a-b)(a-c)(a+2b)(a+2c) + (a-b)^2(b-c)^2(c-a)^2 \geq 0;$$

(b) $$\sum a^2(a-b)(a-c)(a-4b)(a-4c) + 7(a-b)^2(b-c)^2(c-a)^2 \geq 0.$$

2.78. If a, b, c are real numbers, then

$$(a^2 + 2bc)(b^2 + 2ca)(c^2 + 2ab) + (a-b)^2(b-c)^2(c-a)^2 \geq 0.$$

2.79. If a, b, c are real numbers, then

$$(2a^2 + 5ab + 2b^2)(2b^2 + 5bc + 2c^2)(2c^2 + 5ca + 2a^2) + (a-b)^2(b-c)^2(c-a)^2 \geq 0.$$

2.80. If a, b, c are real numbers, then

$$\left(a^2 + \frac{2}{3}ab + b^2\right)\left(b^2 + \frac{2}{3}bc + c^2\right)\left(c^2 + \frac{2}{3}ca + a^2\right) \geq \frac{64}{27}(a^2 + bc)(b^2 + ca)(c^2 + ab).$$

2.81. If a, b, c are real numbers, then

$$\sum a^2(a-b)(a-c) \geq \frac{2(a-b)^2(b-c)^2(c-a)^2}{a^2 + b^2 + c^2}.$$

2.82. If a, b, c are real numbers, then

$$\sum (a-b)(a-c)(a-2b)(a-2c) \geq \frac{8(a-b)^2(b-c)^2(c-a)^2}{a^2 + b^2 + c^2}.$$

2.83. If a, b, c are real numbers, no two of which are zero, then

$$\frac{a^2+3bc}{b^2+c^2}+\frac{b^2+3ca}{c^2+a^2}+\frac{c^2+3ab}{a^2+b^2}\geq 0.$$

2.84. If a, b, c are real numbers, no two of which are zero, then

$$\frac{a^2+6bc}{b^2-bc+c^2}+\frac{b^2+6ca}{c^2-ca+a^2}+\frac{c^2+6ab}{a^2-ab+b^2}\geq 0.$$

2.85. If a, b, c are real numbers such that $ab+bc+ca\geq 0$, then

$$\frac{4a^2+23bc}{b^2+c^2}+\frac{4b^2+23ca}{c^2+a^2}+\frac{4c^2+23ab}{a^2+b^2}\geq 0.$$

2.86. If a, b, c are real numbers such that $ab+bc+ca=3$, then

$$20(a^6+b^6+c^6)+43abc(a^3+b^3+c^3)\geq 189.$$

2.87. If a, b, c are real numbers, then

$$4\sum(a^2+bc)(a-b)(a-c)(a-3b)(a-3c)\geq 7(a-b)^2(b-c)^2(c-a)^2.$$

2.88. Let a, b, c be real numbers such that $ab+bc+ca\geq 0$. For any real k, prove that

$$\sum 4bc(a-b)(a-c)(a-kb)(a-kc)+(a-b)^2(b-c)^2(c-a)^2\geq 0.$$

2.89. If a, b, c are real numbers, then

$$\left[(a^2b+b^2c+c^2a)+(ab^2+bc^2+ca^2)\right]^2\geq 4(ab+bc+ca)(a^2b^2+b^2c^2+c^2a^2).$$

2.90. If a, b, c are real numbers such that $a+b+c=3$, then

$$\frac{(a-1)(a-25)}{a^2+23}+\frac{(b-1)(b-25)}{b^2+23}+\frac{(c-1)(c-25)}{c^2+23}\geq 0.$$

2.91. If a, b, c are real numbers such that $abc \neq 0$, then

$$\left(\frac{b+c}{a}\right)^2 + \left(\frac{c+a}{b}\right)^2 + \left(\frac{a+b}{c}\right)^2 > 2.$$

2.92. If a, b, c are real numbers, then

(a) $$(a^2+1)(b^2+1)(c^2+1) \geq \frac{8}{3\sqrt{3}}\,|(a-b)(b-c)(c-a)|;$$

(b) $$(a^2-a+1)(b^2-b+1)(c^2-c+1) \geq |(a-b)(b-c)(c-a)|.$$

2.93. If a, b, c are real numbers such that $a+b+c=3$, then

$$(1-a+a^2)(1-b+b^2)(1-c+c^2) \geq 1.$$

2.94. If a, b, c are real numbers such that $a+b+c=0$, then

$$\frac{a(a-4)}{a^2+2} + \frac{b(b-4)}{b^2+2} + \frac{c(c-4)}{c^2+2} \geq 0.$$

2.95. If a, b, c, d are real numbers, then

$$(1-a+a^2)(1-b+b^2)(1-c+c^2)(1-d+d^2) \geq \left(\frac{1+abcd}{2}\right)^2.$$

2.96. Let a, b, c, d be real numbers such that $abcd > 0$. Prove that

$$\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right)\left(d+\frac{1}{d}\right) \geq (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$$

2.97. Let a, b, c, d be real numbers such that

$$a+b+c+d=4, \qquad a^2+b^2+c^2+d^2=7.$$

Prove that

$$a^3+b^3+c^3+d^3 \leq 16.$$

2.98. Let a,b,c,d be real numbers such that $a+b+c+d=0$. Prove that

$$12(a^4+b^4+c^4+d^4)\le 7(a^2+b^2+c^2+d^2)^2.$$

2.99. Let a,b,c,d be real numbers such that $a+b+c+d=0$. Prove that

$$(a^2+b^2+c^2+d^2)^3\ge 3(a^3+b^3+c^3+d^3)^2.$$

2.100. If a,b,c,d are real numbers such that $abcd=1$. Prove that

$$(1+a^2)(1+b^2)(1+c^2)(1+d^2)\ge (a+b+c+d)^2.$$

2.101. Let a,b,c,d be real numbers such that

$$a^2+b^2+c^2+d^2=4.$$

Prove that

$$(abc)^3+(bcd)^3+(cda)^3+(dab)^3\le 4.$$

2.102. Let a, b, c, d be real numbers such that $a^2+b^2+c^2+d^2=1$. Prove that

$$(1-a)^4+(1-b)^4+(1-c)^4+(1-d)^4\ge a^4+b^4+c^4+d^4.$$

2.103. If $a,b,c,d\ge \frac{-1}{2}$ such that $a+b+c+d=4$, then

$$\frac{1-a}{1-a+a^2}+\frac{1-b}{1-b+b^2}+\frac{1-c}{1-c+c^2}+\frac{1-d}{1-d+d^2}\ge 0.$$

2.104. If $a,b,c,d,e\ge -3$ such that $a+b+c+d+e=5$, then

$$\frac{1-a}{1+a+a^2}+\frac{1-b}{1+b+b^2}+\frac{1-c}{1+c+c^2}+\frac{1-d}{1+d+d^2}+\frac{1-e}{1+e+e^2}\ge 0.$$

2.105. Let a,b,c,d,e be real numbers such that $a+b+c+d+e=0$. Prove that

$$30(a^4+b^4+c^4+d^4+e^4)\ge 7(a^2+b^2+c^2+d^2+e^2)^2.$$

2.106. Let $a_1, a_2, \ldots, a_n \ge -1$ such that $a_1 + a_2 + \cdots + a_n = 0$. Prove that

$$(n-2)(a_1^2 + a_2^2 + \cdots + a_n^2) \ge a_1^3 + a_2^3 + \cdots + a_n^3.$$

2.107. Let $a_1, a_2, \ldots, a_n \ge -1$ such that $a_1 + a_2 + \cdots + a_n = 0$. Prove that

$$(n-2)(a_1^2 + a_2^2 + \cdots + a_n^2) + (n-1(a_1^3 + a_2^3 + \cdots + a_n^3) \ge 0.$$

2.108. Let $a_1, a_2, \ldots, a_n \ge n - 1 - \sqrt{n^2 - n + 1}$ be nonzero real numbers such that $a_1 + a_2 + \cdots + a_n = n$. Prove that

$$\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2} \ge n.$$

2.109. Let $a_1, a_2, \ldots, a_n \le \dfrac{n}{n-2}$ be real numbers such that

$$a_1 + a_2 + \cdots + a_n = n.$$

If k is a positive integer, $k \ge 2$, then

$$a_1^k + a_2^k + \cdots + a_n^k \ge n.$$

2.110. If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are real numbers such that

$$a_1, a_2, \ldots, a_n \ge \frac{-(3n-2)}{n-2}, \quad a_1 + a_2 + \cdots + a_n = n,$$

then

$$\frac{1-a_1}{(1+a_1)^2} + \frac{1-a_2}{(1+a_2)^2} + \cdots + \frac{1-a_n}{(1+a_n)^2} \ge 0.$$

2.111. Let a_1, a_2, $\ldots$, a_n be real numbers. Prove that

(a) $$\frac{(a_1 + a_2 + \cdots + a_n)^2}{(a_1^2+1)(a_2^2+1)\cdots(a_n^2+1)} \le \frac{(n-1)^{n-1}}{n^{n-2}};$$

(b) $$\frac{a_1 + a_2 + \cdots + a_n}{(a_1^2+1)(a_2^2+1)\cdots(a_n^2+1)} \le \frac{(2n-1)^{n-\frac{1}{2}}}{2^n n^{n-1}}.$$

2.2 Solutions

P 2.1. *Let a, b, c, d be real numbers such that $a^2+b^2+c^2+d^2=9$. Prove that*

$$a^3+b^3+c^3+d^3 \le 27.$$

Solution. From $a^2+b^2+c^2+d^2=9$, we get $a^2 \le 9$, $a \le 3$, $a^2(a-3) \le 0$, $a^3 \le 3a^2$. Similarly, $b^3 \le 3b^2$, $c^3 \le 3c^2$ and $d^3 \le 3d^2$. Therefore,

$$a^3+b^3+c^3+d^3 \le 3(a^2+b^2+c^2+d^2)=27.$$

The equality holds for $a=3$ and $b=c=d=0$ (or any cyclic permutation thereof).

□

P 2.2. *If a, b, c are real numbers such that $a+b+c=0$, then*

$$(2a^2+bc)(2b^2+ca)(2c^2+ab) \le 0.$$

First Solution. Among a, b, c there are two with the same sign. Let $bc \ge 0$. We need to show that

$$(2b^2+ca)(2c^2+ab) \le 0.$$

This is equivalent to

$$[2b^2-c(b+c)][2c^2-(b+c)b] \le 0,$$

$$(b-c)^2(2b+c)(b+2c) \ge 0.$$

Since

$$(2b+c)(b+2c)=2(b^2+c^2)+5bc \ge 0,$$

the conclusion follows. The equality holds for $\frac{-a}{2}=b=c$ (or any cyclic permutation).

Second Solution. We have

$$2a^2+bc=(a-b)(a-c)+a(a+b+c)=(a-b)(a-c),$$

$$2b^2+ca=(b-c)(b-a)+b(a+b+c)=(b-c)(b-a),$$

$$2c^2+ab=(c-a)(c-b)+c(a+b+c)=(c-a)(c-b).$$

Therefore,

$$(2a^2+bc)(2b^2+ca)(2c^2+ab)=-(a-b)^2(b-c)^2(c-a)^2 \le 0.$$

□

P 2.3. *Let a, b, c be real numbers such that $a+b \geq 0,\ b+c \geq 0,\ c+a \geq 0$. Prove that*

$$9(a+b)(b+c)(c+a) \geq 8(a+b+c)(ab+bc+ca).$$

(Nguyen Van Huyen, 2014)

Solution. Write the inequality in the form

$$a(b-c)^2 + b(c-a)^2 + c(a-b)^2 \geq 0.$$

For $a, b, c \geq 0$, the inequality is clearly true. Otherwise, without loss of generality, assume that $a \leq b \leq c$. From $a+b \geq 0$, $b+c \geq 0$, $c+a \geq 0$, it follows that $a \leq 0 \leq b \leq c$ and $a+b \geq 0$. Replacing a by $-a$, we need to show that $0 \leq a \leq b \leq c$ involves

$$-a(c-b)^2 + b(c+a)^2 + c(a+b)^2 \geq 0.$$

This is true since

$$-a(c-b)^2 + b(c+a)^2 \geq -b(c-b)^2 + b(c+a)^2 = b(a+b)(a-b+2c) \geq 0.$$

The equality holds for $a = b = c \geq 0$.

□

P 2.4. *Let a, b, c be real numbers such that $ab+bc+ca = 3$. Prove that*

$$(3a^2+1)(3b^2+1)(3c^2+1) \geq 64.$$

When does equality hold?

Solution. Using the substitution

$$a = \frac{x}{\sqrt{3}}, \quad b = \frac{y}{\sqrt{3}}, \quad c = \frac{z}{\sqrt{3}},$$

we need to show that

$$(x^2+1)(y^2+1)(z^2+1) \geq 64$$

for all real x, y, z such that $xy + yz + zx = 9$.

First Solution. Applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned}(x^2+1)(y^2+1)(z^2+1) &= (x^2+1)[(y+z)^2 + (yz-1)^2] \\ &\geq [x(y+z) + (yz-1)]^2 = 64.\end{aligned}$$

The equality holds for $xy + yz + zx = 9$ and $\frac{y+z}{x} = yz - 1$; that is, for

$$y + z = (yz-1)x = \frac{(yz-1)(9-yz)}{y+z},$$

$$(y+z)^2 + (yz-1)(yz-9) = 0,$$

$$(y-z)^2 + (yz-3)^2 = 0,$$

$$y = z = \pm\sqrt{3}.$$

In addition, from $xy + yz + zx = 9$, we get

$$x = y = z = \pm\sqrt{3}.$$

Therefore, the original inequality becomes an equality for

$$a = b = c = \pm 1.$$

Second Solution. We have

$$(x^2+1)(y^2+1)(z^2+1) - 64 = x^2y^2z^2 + \sum x^2y^2 + \sum x^2 - 63$$

$$= x^2y^2z^2 + \left(\sum xy\right)^2 - 2xyz\sum x + \left(\sum x\right)^2 - 2\sum xy - 63$$

$$= x^2y^2z^2 - 2xyz\sum x + \left(\sum x\right)^2 = \left(xyz - \sum x\right)^2 \geq 0.$$

□

P 2.5. *If a and b are real numbers, then*

$$3(1-a+a^2)(1-b+b^2) \geq 2(1-ab+a^2b^2).$$

(Titu Andreescu, 2006)

Solution. Since

$$(1-a+a^2)(1-b+b^2) = (a+b)^2 - (ab+1)(a+b) + (1-ab+a^2b^2),$$

we can write the inequality as

$$3(a+b)^2 - 3(ab+1)(a+b) + a^2b^2 - ab + 1 \geq 0.$$

Clearly, this inequality is true if $a \leq 0$ and $b \leq 0$. Otherwise, write the inequality in the form

$$3(2a+2b-ab-1)^2 + a^2b^2 - 10ab + 1 \geq 0.$$

This inequality is obviously true for $ab \le 0$. Therefore, consider further that $a > 0$ and $b > 0$. If $a^2b^2 - 10ab + 1 \ge 0$, then the inequality is true. Assume now that $a^2b^2 - 10ab + 1 \le 0$. Since

$$2a + 2b - ab - 1 \ge 4\sqrt{ab} - ab - 1$$

and

$$4\sqrt{ab} - ab - 1 = \frac{14ab - a^2b^2 - 1}{4\sqrt{ab} + ab + 1} > \frac{10ab - a^2b^2 - 1}{4\sqrt{ab} + ab + 1} \ge 0,$$

it suffices to show that

$$3(4\sqrt{ab} - ab - 1)^2 + a^2b^2 - 10ab + 1 \ge 0.$$

This inequality is equivalent to

$$(ab - 3\sqrt{ab} + 1)^2 \ge 0,$$

which is obviously true. The equality holds for $a = b = \dfrac{1}{2}(3 \pm \sqrt{5})$.

□

P 2.6. *If a, b, c are real numbers, then*

$$3(1 - a + a^2)(1 - b + b^2)(1 - c + c^2) \ge 1 + abc + a^2b^2c^2.$$

(Vasile Cîrtoaje and Mircea Lascu, 1989)

First Solution. From the identity

$$2(1 - a + a^2)(1 - b + b^2) = 1 + a^2b^2 + (a - b)^2 + (1 - a)^2(1 - b)^2,$$

it follows that

$$2(1 - a + a^2)(1 - b + b^2) \ge 1 + a^2b^2.$$

Thus, it is enough to prove that

$$3(1 + a^2b^2)(1 - c + c^2) \ge 2(1 + abc + a^2b^2c^2).$$

This inequality is equivalent to

$$(3 + a^2b^2)c^2 - (3 + 2ab + 3a^2b^2)c + 1 + 3a^2b^2 \ge 0,$$

which is true, since the quadratic in c has the discriminant

$$D = -3(1 - ab)^4 \le 0.$$

This means that we can write the inequality as

$$[2(3+a^2b^2)c-3-2ab-3a^2b^2]^2+3(1-ab)^4 \ge 0.$$

The equality holds for $a=b=c=1$.

Second Solution. Write the required inequality as

$$3(1-a+a^2)(1-b+b^2)(1-c+c^2)-abc \ge 1+a^2b^2c^2.$$

Substituting a,b,c by $|a|,|b|,|c|$, respectively, the left side of this inequality remains unchanged or decreases, while the right side remains unchanged. Therefore, it suffices to prove the inequality only for $a,b,c \ge 0$. For $a=b=c$, the inequality is true since

$$3(1-a+a^2)^3-(1+a^3+a^6)=(1-a)^4(2-a+2a^2) \ge 0.$$

Multiplying the inequalities

$$\sqrt[3]{3}(1-a+a^2) \ge \sqrt[3]{1+a^3+a^6},$$

$$\sqrt[3]{3}(1-b+b^2) \ge \sqrt[3]{1+b^3+b^6},$$

$$\sqrt[3]{3}(1-c+c^2) \ge \sqrt[3]{1+c^3+c^6},$$

we get

$$3(1-a+a^2)(1-b+b^2)(1-c+c^2) \ge \sqrt[3]{(1+a^3+a^6)(1+b^3+b^6)(1+c^3+c^6)}.$$

Therefore, it suffices to prove that

$$\sqrt[3]{(1+a^3+a^6)(1+b^3+b^6)(1+c^3+c^6)} \ge 1+abc+a^2b^2c^2,$$

which follows immediately from Hölder's inequality.

□

P 2.7. *If a,b,c are real numbers, then*

$$(a^2+b^2+c^2)^3 \ge (a+b+c)(ab+bc+ca)(a^3+b^3+c^3).$$

(Vasile Cîrtoaje, 2007)

Solution. Substituting a,b,c by $|a|,|b|,|c|$, respectively, the left side of the inequality remains unchanged, while the right side either remains unchanged or increases. Therefore, it suffices to prove the inequality only for $a,b,c \ge 0$. Let $p = a+b+c$ and $q = ab+bc+ca$. Since

$$q^2 - 3abcp = \frac{a^2(b-c)^2 + b^2(c-a)^2 + c^2(a-b)^2}{2} \ge 0,$$

we have

$$(a+b+c)(a^3+b^3+c^3) = p(p^3-3pq+3abc) \le p^4 - 3p^2q + q^2.$$

Thus, it suffices to show that

$$(p^2-2q)^3 \ge q(p^4-3p^2q+q^2),$$

which is equivalent to the obvious inequality

$$(p^2-3q)^2(p^2-q) \ge 0.$$

The equality holds for $a=b=c$.

□

P 2.8. *If a,b,c are real numbers, then*

$$2(a^2+b^2)(b^2+c^2)(c^2+a^2) \ge [ab(a+b)+bc(b+c)+ca(c+a)-2abc]^2.$$

(Vo Quoc Ba Can, 2009)

Solution. Since

$$(a^2+b^2)(a^2+c^2) = (a^2+bc)^2 + (ab-ac)^2$$

and

$$2(b^2+c^2) = (b+c)^2 + (b-c)^2,$$

the required inequality follows by applying the Cauchy-Schwarz inequality as follows

$$\begin{aligned} 2(a^2+b^2)(b^2+c^2)(c^2+a^2) &\ge [(a^2+bc)(b+c)+(ab-ac)(b-c)]^2 \\ &= [ab(a+b)+bc(b+c)+ca(c+a)-2abc]^2. \end{aligned}$$

The equality holds when two of a,b,c are equal.

□

P 2.9. *If a, b, c are real numbers, then*

$$(a^2+1)(b^2+1)(c^2+1) \geq 2(ab+bc+ca).$$

First Solution. Substituting a, b, c by $|a|, |b|, |c|$, respectively, the left side of this inequality remains unchanged, while the right side remains unchanged or increases. Therefore, it suffices to prove the inequality only for $a, b, c \geq 0$. Without loss of generality, assume that $a \geq b \geq c \geq 0$. Since

$$2(ab+bc+ca) \leq 3a(b+c) \leq \frac{3(a^2+1)(b+c)}{2},$$

it suffices to prove that

$$2(b^2+1)(c^2+1) \geq 3(b+c),$$

which is equivalent to

$$2(b+c)^2 - 3(b+c) + 2(bc-1)^2 \geq 0.$$

Case 1: $4bc \leq 1$. We have

$$2(b+c)^2 - 3(b+c) + 2(bc-1)^2 = 2\left(b+c-\frac{3}{4}\right)^2 + \frac{(1-4bc)(7-4bc)}{8} \geq 0.$$

Case 2: $4bc \geq 1$. We get the required inequality by summing

$$\frac{9(b+c)^2}{8} - 3(b+c) + 2 \geq 0,$$

and

$$\frac{7(b+c)^2}{8} + 2b^2c^2 - 4bc \geq 0.$$

We have

$$\frac{9(b+c)^2}{8} - 3(b+c) + 2 = \frac{[3(b+c)-4]^2}{8} \geq 0$$

and

$$\frac{7(b+c)^2}{8} + 2b^2c^2 - 4bc \geq \frac{7bc}{2} + 2b^2c^2 - 4bc = \frac{bc(4bc-1)}{2}.$$

Second Solution Write the inequality as

$$(b^2+1)(c^2+1)\left[a - \frac{b+c}{(b^2+1)(c^2+1)}\right]^2 + A \geq 0,$$

where

$$A=(b^2+1)(c^2+1)-2bc-\frac{(b+c)^2}{(b^2+1)(c^2+1)}.$$

We need to show that $A\geq 0$. By virtue of the Cauchy-Schwarz inequality,

$$(b^2+1)(c^2+1)\geq (b+c)^2.$$

Then,

$$A\geq (b^2+1)(c^2+1)-2bc-1=b^2c^2+(b-c)^2\geq 0.$$

□

P 2.10. *If a,b,c are real numbers, then*

$$(a^2+1)(b^2+1)(c^2+1)\geq \frac{5}{16}(a+b+c+1)^2.$$

(Vasile Cîrtoaje, 2006)

First Solution. Replacing a,b,c respectively by $a/2,b/2,c/2$, the inequality becomes

$$(a^2+4)(b^2+4)(c^2+4)\geq 5(a+b+c+2)^2.$$

Since the equality in this inequality holds for $a=b=c=1$, we apply the Cauchy-Schwarz inequality in the form

$$(a+b+c+2)^2\leq (a^2+4)\left[1+\left(\frac{b+c+2}{2}\right)^2\right].$$

Thus, it suffices to prove that

$$(b^2+4)(c^2+4)\geq 5\left[1+\left(\frac{b+c+2}{2}\right)^2\right].$$

This inequality is equivalent to

$$11(b+c)^2-20(b+c)+4b^2c^2-32bc+24\geq 0.$$

Since $4b^2c^2-8bc+4=4(bc-1)^2\geq 0$, it suffices to show that

$$11(b+c)^2-20(b+c)-24bc+20\geq 0.$$

Indeed,

$$\begin{aligned}11(b+c)^2-20(b+c)-24bc+20&\geq 11(b+c)^2-20(b+c)-6(b+c)^2+20\\&=5(b+c-2)^2\geq 0.\end{aligned}$$

Equality in the original inequality occurs for $a=b=c=\dfrac{1}{2}$.

Second Solution. Obviously, among a^2, b^2, c^2 there are two either less than or equal to $\dfrac{1}{4}$, or greater than or equal to $\dfrac{1}{4}$. Let b and c be these numbers; that is,

$$(4b^2-1)(4c^2-1)\ge 0.$$

Then, we have

$$\frac{16}{5}(b^2+1)(c^2+1)=5\left(\frac{4b^2-1}{5}+1\right)\left(\frac{4c^2-1}{5}+1\right)$$

$$\ge 5\left(\frac{4b^2-1}{5}+\frac{4c^2-1}{5}+1\right)=4b^2+4c^2+3.$$

Hence, it suffices to prove that

$$(a^2+1)(4b^2+4c^2+3)\ge (a+b+c+1)^2.$$

Writing this inequality as

$$\left(a^2+\frac{1}{4}+\frac{1}{4}+\frac{1}{2}\right)(1+4b^2+4c^2+2)\ge (a+b+c+1)^2,$$

we recognize the Cauchy-Schwarz inequality.

□

P 2.11. *If a,b,c are real numbers, then*

(a) $$a^6+b^6+c^6-3a^2b^2c^2+2(a^2+bc)(b^2+ca)(c^2+ab)\ge 0;$$

(b) $$a^6+b^6+c^6-3a^2b^2c^2\ge (a^2-2bc)(b^2-2ca)(c^2-2ab).$$

Solution. (a) Since

$$(a^2+bc)(b^2+ca)(c^2+ab)=2a^2b^2c^2+\sum a^3b^3+abc\sum a^3,$$

we can write the desired inequality as follows

$$\sum a^6+2\sum a^3b^3+2abc\sum a^3+a^2b^2c^2\ge 0,$$

$$(\sum a^3)^2+2abc\sum a^3+a^2b^2c^2\ge 0,$$

$$(\sum a^3 + abc)^2 \ge 0.$$

The equality holds for $a^3 + b^3 + c^3 + abc = 0$.

(b) Since

$$(a^2 - 2bc)(b^2 - 2ca)(c^2 - 2ab) = -7a^2b^2c^2 - 2\sum a^3b^3 + 4abc\sum a^3,$$

we can write the desired inequality as follows

$$\sum a^6 + 2\sum a^3b^3 - 4abc\sum a^3 + 4a^2b^2c^2 \ge 0,$$

$$(\sum a^3)^2 - 4abc\sum a^3 + 4a^2b^2c^2 \ge 0,$$

$$(\sum a^3 - 2abc)^2 \ge 0.$$

The equality holds for $a^3 + b^3 + c^3 - 2abc = 0$.

□

P 2.12. *If a, b, c are real numbers, then*

$$\frac{2}{3}(a^6 + b^6 + c^6) + a^3b^3 + b^3c^3 + c^3a^3 + abc(a^3 + b^3 + c^3) \ge 0.$$

Solution. Write the inequality as follows

$$\frac{4}{3}(a^6 + b^6 + c^6) + 2(a^3b^3 + b^3c^3 + c^3a^3) + 2abc(a^3 + b^3 + c^3) \ge 0,$$

$$\frac{1}{3}(a^6 + b^6 + c^6) + (a^3 + b^3 + c^3)^2 + 2abc(a^3 + b^3 + c^3) \ge 0.$$

By virtue of the AM-GM inequality, we have

$$\frac{1}{3}(a^6 + b^6 + c^6) \ge a^2b^2c^2.$$

Therefore, it suffices to show that

$$a^2b^2c^2 + (a^3 + b^3 + c^3)^2 + 2abc(a^3 + b^3 + c^3) \ge 0,$$

which is equivalent to

$$(abc + a^3 + b^3 + c^3)^2 \ge 0.$$

The equality holds for $-a = b = c$ (or any cyclic permutation).

□

P 2.13. *If a, b, c are real numbers, then*

$$4(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \ge (a-b)^2(b-c)^2(c-a)^2.$$

(Vasile Cîrtoaje, 2009)

Solution. Using the identity

$$4xy = (x+y)^2 - (x-y)^2,$$

we have

$$4(a^2+ab+b^2)(a^2+ac+c^2) = [(2a^2+ab+ac+2bc)+(b-c)^2]^2 - (a+b+c)^2(b-c)^2$$
$$= (2a^2+ab+ac+2bc)^2 + 3a^2(b-c)^2.$$

From this result and

$$4(b^2+bc+c^2) = (b-c)^2 + 3(b+c)^2,$$

we get

$$16\prod(a^2+ab+b^2) = \left[(2a^2+ab+ac+2bc)^2 + 3a^2(b-c)^2\right]\left[(b-c)^2 + 3(b+c)^2\right].$$

So, the Cauchy-Schwarz inequality gives

$$16\prod(a^2+ab+b^2) \ge \left[(2a^2+ab+ac+2bc)(b-c) + 3a(b-c)(b+c)\right]^2$$
$$= 4(b-c)^2(a-b)^2(a-c)^2.$$

The equality holds for $ab(a+b)+bc(b+c)+ca(c+a)=0$.

Remark. The inequality is a consequence of the identity

$$4\prod(a^2+ab+b^2) = 3[ab(a+b)+bc(b+c)+ca(c+a)]^2 + (a-b)^2(b-c)^2(c-a)^2.$$

□

P 2.14. *If a, b, c are real numbers, then*

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \ge 3(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2).$$

(Gabriel Dospinescu, 2009)

Solution *(by Vo Quoc Ba Can).* As we have shown in the proof of the preceding P 2.13,

$$16\prod(a^2+ab+b^2)=\left[(2a^2+ab+ac+2bc)^2+3a^2(b-c)^2\right]\left[3(b+c)^2+(b-c)^2\right].$$

Thus, by the Cauchy-Schwarz inequality, we get

$$16\prod(a^2+ab+b^2)\ge 3[(b+c)(2a^2+ab+ac+2bc)+a(b-c)^2]^2$$
$$=12[(a^2b+b^2c+c^2a)+(ab^2+bc^2+ca^2)]^2.$$

To prove the desired inequality, it suffices to show that

$$[(a^2b+b^2c+c^2a)+(ab^2+bc^2+ca^2)]^2\ge 4(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2),$$

which follows immediately from the AM-GM inequality. The equality holds when two of a,b,c are equal.

Remark. The inequality is a consequence of the identity

$$\prod(a^2+ab+b^2)=3(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2)+(a-b)^2(b-c)^2(c-a)^2.$$

□

P 2.15. *If a,b,c are real numbers such that $abc>0$, then*

$$4\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right)\ge 9(a+b+c).$$

Solution. Write the inequality as

$$4(a^2+1)(b^2+1)(c^2+1)\ge 9abc(a+b+c).$$

First Solution. It is easy to check that the equality occurs for $a=b=c=\sqrt{2}$. Therefore, using the substitution $a=x\sqrt{2}$, $b=y\sqrt{2}$, $c=z\sqrt{2}$, the inequality can be written as

$$(2x^2+1)(2y^2+1)2z^2+1)\ge 9xyz(x+y+z).$$

Since

$$(xy+yz+zx)^2-3xyz(x+y+z)=\frac{1}{2}\sum x^2(y-z)^2\ge 0,$$

it suffices to prove the stronger inequality

$$(2x^2+1)(2y^2+1)(2z^2+1)\ge 3(xy+yz+zx)^2.$$

Let

$$A=(y^2-1)(z^2-1),\quad B=(z^2-1)(x^2-1),\quad C=(x^2-1)(y^2-1).$$

From

$$ABC = (x^2-1)^2(y^2-1)^2(z^2-1)^2 \ge 0,$$

it follows that at least one of A, B, C is nonnegative. Due to symmetry, assume that

$$A = (y^2-1)(z^2-1) \ge 0.$$

Applying the Cauchy-Schwarz inequality, we have

$$(xy+yz+zx)^2 \le (x^2+1+x^2)(y^2+y^2z^2+z^2).$$

Therefore, it suffices to show that

$$(2y^2+1)(2z^2+1) \ge 3(y^2+y^2z^2+z^2),$$

which reduces to the obvious inequality

$$(y^2-1)(z^2-1) \ge 0.$$

Second Solution. Since

$$4(b^2+1)(c^2+1) - 3[(b+c)^2+b^2c^2] = (b-c)^2+(bc-2)^2 \ge 0,$$

it suffices to show that

$$(a^2+1)[(b+c)^2+b^2c^2] \ge 3abc(a+b+c).$$

Applying the Cauchy-Schwarz inequality, we get

$$(a^2+1)[(b+c)^2+b^2c^2] \ge [a(b+c)+bc]^2 \ge 3abc(a+b+c).$$

□

P 2.16. *If a, b, c are real numbers, then*

(a) $$(a^2+2bc)(b^2+2ca)(c^2+2ab) \le (a^2+b^2+c^2)(ab+bc+ca)^2;$$

(b) $$(2a^2+bc)(2b^2+ca)(2c^2+ab) \le (a+b+c)^2(a^2b^2+b^2c^2+c^2a^2).$$

(Vasile Cîrtoaje, 2005)

Solution. (a) Let $q = ab + bc + ca$. Since

$$a^2 + 2bc = q + (a - b)(a - c),$$

$$b^2 + 2ca = q + (b - c)(b - a),$$

$$c^2 + 2ab = q + (c - a)(c - b),$$

we can write the required inequality as follows

$$q^2(a^2 + b^2 + c^2) \geq [q + (a - b)(a - c)][q + (b - c)(b - a)][q + (c - a)(c - b],$$

$$q^2(a^2 + b^2 + c^2) \geq q^3 + q^2 \sum (a - b)(a - c) - (a - b)^2(b - c)^2(c - a)^2.$$

Since

$$\sum (a - b)(a - c) = a^2 + b^2 + c^2 - q,$$

the inequality reduces to the obvious form

$$(a - b)^2(b - c)^2(c - a)^2 \geq 0.$$

The equality holds for $a = b$, or $b = c$, or $c = a$.

(b) Since For $a = 0$, this inequality reduces to $b^2c^2(b - c)^2 \geq 0$. Otherwise, for $abc \neq 0$, the inequality follows from the inequality in (a) by substituting a, b, c with $1/a$, $1/b$, $1/c$, respectively. The equality occurs for $a = b$, or $b = c$, or $c = a$.

□

P 2.17. *If a, b, c are real numbers such that $ab + bc + ca \geq 0$, then*

$$27(a^2 + 2bc)(b^2 + 2ca)(c^2 + 2ab) \leq (a + b + c)^6.$$

Solution. In virtue of the AM-GM inequality, we have

$$\begin{aligned}(a + b + c)^6 &= [a^2 + b^2 + c^2 + (ab + bc + ca) + (ab + bc + ca)]^3 \\ &\geq 27(a^2 + b^2 + c^2)(ab + bc + ca)^2.\end{aligned}$$

Thus, the required inequality follows immediately from the inequality (a) in P 2.16. The equality holds for $a = b = c$.

□

P 2.18. *If a,b,c are real numbers such that $a^2+b^2+c^2=2$, then*

$$(a^2+2bc)(b^2+2ca)(c^2+2ab)+2\geq 0.$$

(Vasile Cîrtoaje, 2011)

Solution *(by Vo Quoc Ba Can)*. If a,b,c have the same sign, then the inequality is trivial. Otherwise, since the inequality is symmetric and does not change by substituting $-a,-b,-c$ for a,b,c, respectively, it suffices to consider the case where $a\leq 0$ and $b,c\geq 0$. Substituting $-a$ for a, we need to prove that

$$(a^2+2bc)(b^2-2ac)(c^2-2ab)+2\geq 0$$

for all $a,b,c\geq 0$ satisfying $a^2+b^2+c^2=2$. If b^2-2ac and c^2-2ab have the same sign, then the inequality is also trivial. Due to symmetry in b and c, we may assume that

$$b^2-2ac\geq 0\geq c^2-2ab.$$

On the other hand, it is easy to check that the desired inequality becomes an equality for $a=b=1$ and $c=0$, when $a^2+2bc=b^2-2ac=ab-c^2/2$. Then, we rewrite the desired inequality in the form

$$(a^2+2bc)(b^2-2ac)\left(ab-\frac{c^2}{2}\right)\leq 1.$$

Using the AM-GM inequality, we have

$$27(a^2+2bc)(b^2-2ac)\left(ab-\frac{c^2}{2}\right)\leq\left[(a^2+2bc)+(b^2-2ac)+\left(ab-\frac{c^2}{2}\right)\right]^3.$$

Thus, it suffices to prove that

$$(a^2+2bc)+(b^2-2ac)+\left(ab-\frac{c^2}{2}\right)\leq 3.$$

This inequality can be written in the homogeneous form

$$2(a^2+2bc)+2(b^2-2ac)+(2ab-c^2)\leq 3(a^2+b^2+c^2),$$

which is equivalent to $(a-b+2c)^2\geq 0$. The equality occurs for for $a=0$ and $b+c=0$ (or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization.

- *Let a,b,c be real numbers such that $a^2+b^2+c^2=2$. If $0<k\leq 2$, then*

$$(a^2+kbc)(b^2+kca)(c^2+kab)+k\geq 0.$$

□

P 2.19. *If a,b,c are real numbers such that $a+b+c=3$, then*

$$3(a^4+b^4+c^4)+a^2+b^2+c^2+6\ge 6(a^3+b^3+c^3).$$

(Vasile Cîrtoaje, 2006)

Solution. Write the inequality as $F(a,b,c)\ge 0$, where

$$F(a,b,c)=3(a^4+b^4+c^4)+(a^2+b^2+c^2)-6(a^3+b^3+c^3)+6.$$

Due to symmetry, we may assume that $a\le b\le c$. To prove the required inequality, we use the mixing variable method. More precisely, we show that

$$F(a,b,c)\ge F(a,x,x)\ge 0,$$

where $x=(b+c)/2$, $x\ge 1$. We have

$$F(a,b,c)-F(a,x,x)=3(b^4+b^4-2x^4)+(b^2+c^2-2x^2)-6(b^3+c^3-2x^3)$$

$$=3[(b^2+c^2)^2-4x^4]+6(x^4-b^2c^2)+(b^2+c^2-2x^2)-6(b^3+c^3-2x^3)$$

$$=(b^2+c^2-2x^2)[3(b^2+c^2+2x^2)+1]+6(x^2-bc)(x^2+bc)-12x(b^2+c^2-bc-x^2).$$

Since

$$b^2+c^2-2x^2=\frac{1}{2}(b-c)^2,$$

$$x^2-bc=\frac{1}{4}(b-c)^2,$$

$$b^2+c^2-bc-x^2=\frac{3}{4}(b-c)^2,$$

we have

$$F(a,b,c)-F(a,x,x)=\frac{1}{2}(b-c)^2[3(b^2+c^2+2x^2)+1+3(x^2+bc)-18x]$$

$$=\frac{1}{2}(b-c)^2[3(x^2-bc)+18x(x-1)+1]\ge 0.$$

Also,

$$F(a,x,x)=F(3-2x,x,x)=6(x-1)^2(3x-4)^2\ge 0.$$

This completes the proof. The equality holds for $a=b=c=1$, and for $a=1/3$ and $b=c=4/3$ (or any cyclic permutation).

□

P 2.20. *If a,b,c are real numbers such that $abc=1$, then*

$$3(a^2+b^2+c^2)+2(a+b+c)\ge 5(ab+bc+ca).$$

Solution. From $abc = 1$, it follows that either $a, b, c > 0$, or one of a, b, c is positive and the others are negative. In the last case, due to symmetry, we may assume that $a > 0$ and $b, c < 0$.

Case 1: $a, b, c > 0$. Let $p = a + b + c$ and $q = ab + bc + ca$. By the AM-GM inequality

$$a + b + c \geq 3\sqrt[3]{abc},$$

we get $p \geq 3$, while by Schur's inequality

$$p^3 + 9abc \geq 4pq,$$

we get

$$q \leq \frac{p^3 + 9}{4p}.$$

Write the required inequality as

$$3(p^2 - 2q) + 2p \geq 5q,$$

$$3p^2 + 2p \geq 11q.$$

This is true since

$$3p^2 + 2p - 11q \geq 3p^2 + 2p - \frac{11(p^3 + 9)}{4p} = \frac{(p-3)(p^2 + 11p + 33)}{4p} \geq 0.$$

Case 2: $a > 0$ and $b, c < 0$. Substituting $-b$ for b and $-c$ for c, we need to prove that

$$3(a^2 + b^2 + c^2) + 2a + 5a(b + c) \geq 2(b + c) + 5bc$$

for $a, b, c > 0$ satisfying $abc = 1$. It suffices to show that

$$3(b^2 + c^2) - 5bc \geq (2 - 5a)(b + c).$$

Since

$$\frac{3(b^2 + c^2) - 5bc}{b + c} \geq \frac{b + c}{4} \geq \frac{\sqrt{bc}}{2} = \frac{1}{2\sqrt{a}},$$

we need to prove that

$$\frac{1}{2\sqrt{a}} \geq 2 - 5a.$$

Indeed, by the AM-GM inequality,

$$5a + \frac{1}{2\sqrt{a}} = 5a + \frac{1}{4\sqrt{a}} + \frac{1}{4\sqrt{a}} \geq 3\sqrt[3]{5a \cdot \frac{1}{4\sqrt{a}} \cdot \frac{1}{4\sqrt{a}}} > 2.$$

This completes the proof. The equality holds for $a = b = c = 1$.

□

P 2.21. *If a,b,c are real numbers such that $abc=1$, then*

$$a^2+b^2+c^2+6\geq\frac{3}{2}\left(a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

Solution. Substituting a,b,c by $|a|,|b|,|c|$, respectively, the left side of the inequality remains unchanged, while the right side either remains unchanged or increases. Therefore, it suffices to prove the inequality only for $a,b,c>0$.

First Solution. Write the inequality in the form

$$3(6x^2-3x+4)\geq 7(ab+bc+ca),$$

where $x=\dfrac{a+b+c}{3}$. By virtue of the AM-GM inequality, we get $x\geq 1$. The third degree Schur's inequality states that

$$(a+b+c)^3+9abc\geq 4(a+b+c)(ab+bc+ca),$$

which is equivalent to

$$ab+bc+ca\leq\frac{3(3x^3+1)}{4x}.$$

Therefore, it suffices to show that

$$3(6x^2-3x+4)\geq\frac{21(3x^3+1)}{4x}.$$

This inequality reduces to $(x-1)(3x^2-9x+7)\geq 0$, which is true because

$$3x^2-9x+7=3(x-\frac{3}{2})^2+\frac{1}{4}>0.$$

The equality holds for $a=b=c=1$.

Second Solution. Due to symmetry, assume that $a=\min\{a,b,c\}$, and then use the mixing variable method. Let

$$F(a,b,c)=a^2+b^2+c^2+6-\frac{3}{2}\left(a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

We will show that

$$F(a,b,c)\geq F(a,\sqrt{bc},\sqrt{bc})\geq 0,$$

where $x = \sqrt{bc}$ $(x \ge 1)$. We have

$$\begin{aligned} F(a,b,c) - F(a,\sqrt{bc},\sqrt{bc}) &= (b-c)^2 - \frac{3}{2}\left(b + c - 2\sqrt{bc} + \frac{1}{b} + \frac{1}{c} - \frac{2}{\sqrt{bc}}\right) \\ &= \frac{1}{2}(\sqrt{b} - \sqrt{c})^2\left[2(\sqrt{b} + \sqrt{c})^2 - 3 - \frac{3}{bc}\right] \\ &\ge \frac{1}{2}(\sqrt{b} - \sqrt{c})^2\left(8\sqrt{bc} - 3 - \frac{3}{bc}\right) \\ &\ge \frac{1}{2}(\sqrt{b} - \sqrt{c})^2(8 - 3 - 3) \ge 0 \end{aligned}$$

and

$$\begin{aligned} F(a,\sqrt{bc},\sqrt{bc}) &= F(\frac{1}{x^2}, x, x) \\ &= \frac{x^6 - 6x^5 + 12x^4 - 6x^3 - 3x^2 + 2}{2x^4} \\ &= \frac{(x-1)^2(x^4 - 4x^3 + 3x^2 + 4x + 2}{2x^4} \\ &= \frac{(x-1)^2[(x^2 - 2x - 1)^2 + x^2 + 1]}{2x^4} \ge 0. \end{aligned}$$

□

P 2.22. *If a, b, c are real numbers, then*

$$(1 + a^2)(1 + b^2)(1 + c^2) + 8abc \ge \frac{1}{4}(1 + a)^2(1 + b)^2(1 + c)^2.$$

Solution. It is easy to check that

$$(1 + a^2)(1 + b^2)(1 + c^2) + 8abc = (1 + abc)^2 + (a + bc)^2 + (b + ca)^2 + (c + ab)^2.$$

Thus, using the Cauchy-Schwarz inequality, we have

$$(1 + a^2)(1 + b^2)(1 + c^2) + 8abc \ge \frac{[(1 + abc) + (a + bc) + (b + ca) + (c + ab)]^2}{4}$$

$$= \frac{1}{4}(1 + a)^2(1 + b)^2(1 + c)^2.$$

The equality holds for $b = c = 1$ (or any cyclic permutation), and also for $a = b = c = -1$.

□

P 2.23. *Let a, b, c be real numbers such that $a+b+c=0$. Prove that*

$$a^{12}+b^{12}+c^{12} \ge \frac{2049}{8}a^4b^4c^4.$$

Solution. Consider only the nontrivial case $abc \ne 0$, and rewrite the inequality as follows

$$a^{12}+b^{12}+(a+b)^{12} \ge \frac{2049}{8}a^4b^4(a+b)^4,$$

$$(a^6+b^6)^2-2a^6b^6+(a^2+b^2+2ab)^6 \ge \frac{2049}{8}a^4b^4(a^2+b^2+2ab)^2.$$

Let us denote

$$d=\frac{a^2+b^2}{ab}, \quad |d| \ge 2.$$

Since

$$a^6+b^6=(a^2+b^2)^3-3a^2b^2(a^2+b^2),$$

the inequality can be restated as

$$(d^3-3d)^2-2+(d+2)^6 \ge \frac{2049}{8}(d+2)^2,$$

which is equivalent to

$$(d-2)(2d+5)^2(4d^3+12d^2+87d+154) \ge 0.$$

Since this inequality is obvious for $d \ge 2$, we only need to show that

$$4d^3+12d^2+87d+154 \le 0$$

for $d \le -2$. Indeed,

$$\begin{aligned}4d^3+12d^2+87d+154 &< 4d^3+12d^2+85d+154\\ &=(d+2)[(2d+1)^2+76] \le 0.\end{aligned}$$

The equality holds for $a=b=-c/2$ (or any cyclic permutation).

□

P 2.24. *If a,b,c are real numbers such that $abc \ge 0$, then*

$$a^2+b^2+c^2+2abc+4 \ge 2(a+b+c)+ab+bc+ca.$$

(Vasile Cîrtoaje, 2012)

Solution. Let us denote

$$x = a(b-1)(c-1), \quad y = b(c-1)(a-1), \quad z = c(a-1)(b-1).$$

Since

$$xyz = abc(a-1)^2(b-1)^2(c-1)^2 \geq 0,$$

at least one of x, y, z is nonnegative; let $a(1-b)(1-c) \geq 0$. Thus, we have

$$abc \geq a(b+c-1),$$

and it suffices to show that

$$a^2 + b^2 + c^2 + 2a(b+c-1) + 4 \geq 2(a+b+c) + ab + bc + ca,$$

which is equivalent to

$$a^2 - (4-b-c)a + b^2 + c^2 - bc - 2(b+c) + 4 \geq 0,$$

$$\left(a - 2 + \frac{b+c}{2}\right)^2 + \frac{3}{4}(b-c)^2 \geq 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 0$ and $b = c = 2$ (or any cyclic permutation).

□

P 2.25. *Let a, b, c be real numbers such that $a + b + c = 3$.*

(a) If $a, b, c \geq -3$, then

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

(b) If $a, b, c \geq -7$, then

$$\frac{1-a}{(1+a)^2} + \frac{1-b}{(1+b)^2} + \frac{1-c}{(1+c)^2} \geq 0.$$

(Vasile Cîrtoaje, 2012)

Solution. Assume that $a = \min\{a, b, c\}$ and denote

$$t = \frac{b+c}{2}, \quad a + 2t = 3.$$

(a) From $a, b, c \geq -3$ and $a + b + c = 3$, it follows that

$$-3 \leq a \leq \frac{a+b+c}{3} = 1.$$

We will show that

$$E(a,b,c) \ge E(a,t,t) \ge 0,$$

where

$$E(a,b,c) = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c}.$$

We have

$$E(a,b,c) - E(a,t,t) = \frac{1}{b^2} + \frac{1}{c^2} - \frac{2}{t^2} - \left(\frac{1}{b} + \frac{1}{c} - \frac{2}{t}\right)$$

$$= \frac{(b-c)^2(b^2+c^2+4bc)}{b^2c^2(b+c)^2} - \frac{(b-c)^2}{bc(b+c)}$$

$$= \frac{(b-c)^2[(b+c)^2 - bc(b+c-2)]}{b^2c^2(b+c)^2} \ge 0,$$

since

$$(b+c)^2 - bc(b+c-2) = (b+c)^2 - bc(1-a)$$

$$\ge (b+c)^2 - \frac{(b+c)^2(1-a)}{4} = \frac{(b+c)^2(3+a)}{4} \ge 0.$$

Also,

$$E(a,t,t) = \frac{1-a}{a^2} + \frac{2(1-t)}{t^2} = \frac{3(a-1)^2(a+3)}{a^2(3-a)^2} \ge 0.$$

The equality holds for $a = b = c = 1$, and also for $a = -3$ and $b = c = 3$ (or any cyclic permutation).

(b) From

$$t \ge \frac{a+b+c}{3} = 1, \quad t = \frac{3-a}{2} \le 5,$$

it follows that

$$t \in [1,5].$$

We will show that

$$E(a,b,c) \ge E(a,t,t) \ge 0,$$

where

$$E(a,b,c) = \frac{1-a}{(1+a)^2} + \frac{1-b}{(1+b)^2} + \frac{1-c}{(1+c)^2}.$$

Write the left inequality as follows:

$$\left[\frac{1-b}{(1+b)^2} - \frac{1-t}{(1+t)^2}\right] + \left[\frac{1-c}{(1+c)^2} - \frac{1-t}{(1+t)^2}\right] \ge 0,$$

$$(b-c)\left[\frac{(b-1)t-b-3}{(1+b)^2} - \frac{(c-1)t-c-3}{(1+c)^2}\right] \ge 0,$$

$$(b-c)^2[(3+b+c-bc)t+3(b+c)+bc]\ge 0,$$
$$(b-c)^2[2t^2+9t+5-bc(t-1)]\ge 0.$$

The last inequality is true since

$$2t^2+9t+5-bc(t-1)\ge 2t^2+9t+5-t^2(t-1)=(5-t)(1+t)^2\ge 0.$$

Also, we have

$$E(a,t,t)=\frac{1-a}{(1+a)^2}+\frac{2(1-t)}{(1+t)^2}=\frac{t-1}{2(2-t)^2}+\frac{2(1-t)}{(1+t)^2}$$
$$=\frac{3(1-t)^2(5-t)}{2(2-t)^2(1+t)^2}\ge 0.$$

The proof is completed. The equality occurs for $a=b=c=1$, and also for $a=-7$ and $b=c=5$ (or any cyclic permutation).

□

P 2.26. *If a,b,c are real numbers, then*

$$a^6+b^6+c^6-3a^2b^2c^2\ge\frac{1}{2}(a-b)^2(b-c)^2(c-a)^2.$$

(Sungyoon Kim, 2006)

Solution. Applying the Cauchy-Schwarz inequality, we have

$$a^6+b^6+c^6-3a^2b^2c^2=\frac{1}{2}(a^2+b^2+c^2)[(b^2-c^2)^2+(c^2-a^2)^2+(a^2-b^2)^2]$$
$$\ge\frac{1}{2}[a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)]^2$$
$$=\frac{1}{2}(a-b)^2(b-c)^2(c-a)^2.$$

Thus, the proof is completed. The equality holds for $a=b=c$, and for $a=0$ and $b+c=0$ (or any cyclic permutation).

□

P 2.27. *If a,b,c are real numbers, then*

$$\left(\frac{a^2+b^2+c^2}{3}\right)^3\ge a^2b^2c^2+\frac{1}{16}(a-b)^2(b-c)^2(c-a)^2.$$

(Vasile Cîrtoaje, 2011)

Solution *(by Vo Quoc Ba Can)*. Without loss of generality, assume that b and c have the same sign. Denote $x=\sqrt{(b^2+c^2)/2}$. Since

$$\begin{aligned}\left(\frac{a^2+b^2+c^2}{3}\right)^3-a^2b^2c^2&=\left(\frac{a^2+2x^2}{3}\right)^3-a^2x^4+a^2(x^4-b^2c^2)\\&=\frac{(a^2-x^2)^2(a^2+8x^2)}{27}+a^2(x^4-b^2c^2)\\&=\frac{(2a^2-b^2-c^2)^2(a^2+4b^2+4c^2)}{108}+\frac{a^2(b^2-c^2)^2}{4},\end{aligned}$$

the desired inequality can be rewritten as

$$(2a^2-b^2-c^2)^2(a^2+4b^2+4c^2)\ge\frac{27}{4}(b-c)^2[(a-b)^2(a-c)^2-4a^2(b+c)^2].$$

According to the inequalities $x^2-y^2\le 2x(x+y)$ and $2xy\le\frac{1}{2}(x+y)^2$, we have

$$\begin{aligned}(a-b)^2(a-c)^2-4a^2(b+c)^2&\le 2(a-b)(a-c)[(a-b)(a-c)+2a(b+c)]\\&=2(a^2-b^2)(a^2-c^2)\le\frac{1}{2}(2a^2-b^2-c^2)^2.\end{aligned}$$

Therefore, it suffices to show that

$$8(a^2+4b^2+4c^2)\ge 27(b-c)^2,$$

which is equivalent to the obvious inequality $8a^2+5b^2+5c^2+54bc\ge 0$. The equality holds for $a=b=c$, and for $-a=b=c$ (or any cyclic permutation).

□

P 2.28. *If a,b,c are real numbers, then*

$$(a^2+b^2+c^2)^3\ge\frac{108}{5}a^2b^2c^2+2(a-b)^2(b-c)^2(c-a)^2.$$

(Vo Quoc Ba Can and Vasile Cîrtoaje, 2011)

Solution. Write the inequality as $f(a,b,c)\ge 0$, where

$$f(a,b,c)=(a^2+b^2+c^2)^3-\frac{108}{5}a^2b^2c^2-2(a-b)^2(b-c)^2(c-a)^2.$$

Without loss of generality, assume that b and c have the same sign. Since $f(-a,-b,-c)=f(a,b,c)$, we may consider $b\ge 0$, $c\ge 0$. In addition, for $a>0$, we have

$$f(a,b,c)-f(-a,b,c)=8a(b+c)(a^2+bc)(b-c)^2\ge 0.$$

Therefore, it suffices to prove the desired inequality for $a \le 0$, $b \ge 0$, $c \ge 0$. For $b = c = 0$, the inequality is trivial. Otherwise, due to homogeneity, we may assume that $b + c = 1$. Denoting $x = bc$, we can write the desired inequality as follows:

$$(a^2+1-2x)^3 \ge \frac{108}{5}a^2x^2 + 2(1-2x)(a^2-a+x)2,$$

$$\frac{2}{5}(4a-5)^2x^2 + 2(a+1)(a^3-9a^2+5a-3)x + (a+1)^2(a^4-2a^3+4a^2-2a+1) \ge 0.$$

This inequality holds if

$$\frac{2}{5}(4a-5)^2(a^4-2a^3+4a^2-2a+1) \ge (a^3-9a^2+5a-3)^2.$$

Since

$$10(a^4-2a^3+4a^2-2a+1) = (a+1)^2 + (3a^2-4a+3)^2 \ge (3a^2-4a+3)^2,$$

it suffices to prove that

$$(4a-5)^2(3a^2-4a+3)^2 \ge 25(a^3-9a^2+5a-3)^2.$$

This is true for $a \le 0$ if

$$(5-4a)(3a^2-4a+3) \ge 5(-a^3+9a^2-5a+3),$$

which reduces to

$$a(a+1)^2 \le 0.$$

Thus, the proof is completed. The equality holds for $a = 0$ and $b + c = 0$ (or any cyclic permutation).

□

P 2.29. *If a, b, c are real numbers, then*

$$2(a^2+b^2)(b^2+c^2)(c^2+a^2) \ge (a-b)^2(b-c)^2(c-a)^2.$$

(Vasile Cîrtoaje, 2011)

First Solution. Since

$$2(a^2+b^2) = (a-b)^2 + (a+b)^2$$

and

$$(b^2+c^2)(c^2+a^2) = (ab+c^2)^2 + (bc-ac)^2,$$

by virtue of the Cauchy-Schwarz inequality, we have

$$\begin{aligned}2(a^2+b^2)(b^2+c^2)(c^2+a^2) &\ge [(a-b)(ab+c^2)+(a+b)(bc-ac)]^2\\ &=(a^2b+b^2c+c^2a-ab^2-bc^2-ca^2)^2\\ &=(a-b)^2(b-c)^2(c-a)^2.\end{aligned}$$

This completes the proof. The equality holds for $(a-b)(bc-ac)=(a+b)(ab+c^2)$, which is equivalent to

$$(a+b+c)(ab+bc+ca)=5abc.$$

Second Solution. Making the substitution

$$x=\sum ab^2=ab^2+bc^2+ca^2,\quad y=\sum a^2b=a^2b+b^2c+c^2a,$$

we have

$$\sum a^2b^4=(\sum ab^2)^2-2abc\sum a^2b=x^2-2abcy,$$

$$\sum a^4b^2=(\sum a^2b)^2-2abc\sum ab^2=y^2-2abcx,$$

and hence

$$\begin{aligned}\prod(a^2+b^2)&=\sum a^2b^4+\sum a^4b^2+2a^2b^2c^2\\ &=x^2+y^2-2abc(x+y)+2a^2b^2c^2.\end{aligned}$$

Then, the desired inequality is equivalent to

$$2[x^2+y^2-2abc(x+y)+2a^2b^2c^2]\ge (x-y)^2,$$

$$(x+y)^2-4abc(x+y)+4a^2b^2c^2\ge 0,$$

$$(x+y-2abc)^2\ge 0.$$

The last inequality is obviously true.

□

P 2.30. *If a,b,c are real numbers, then*

$$32(a^2+bc)(b^2+ca)(c^2+ab)+9(a-b)^2(b-c)^2(c-a)^2\ge 0.$$

(Vasile Cîrtoaje, 2011)

Solution *(by Vo Quoc Ba Can)*. For $a, b, c \geq 0$, the inequality is trivial. Otherwise, since the inequality is symmetric and does not change by substituting $-a, -b, -c$ for a, b, c, we may assume that $a \leq 0$ and $b, c \geq 0$. Substituting $-a$ for a, we need to prove that

$$32(a^2 + bc)(b^2 - ac)(c^2 - ab) + 9(a + b)^2(a + c)^2(b - c)^2 \geq 0$$

for all $a, b, c \geq 0$. By the AM-GM inequality, we have

$$(a + b)^2(a + c)^2 = [a(b + c) + (a^2 + bc)]^2 \geq 4a(b + c)(a^2 + bc).$$

Thus, it suffices to prove that

$$8(b^2 - ac)(c^2 - ab) + 9a(b + c)(b - c)^2 \geq 0.$$

Since

$$(b^2 - ac)(c^2 - ab) = bc(bc + a^2) - a(b^3 + c^3)$$
$$\geq 2abc\sqrt{bc} - a(b^3 + c^3) = -a(b\sqrt{b} - c\sqrt{c})^2,$$

it is enough to show that

$$9(b + c)(b - c)^2 - 8(b\sqrt{b} - c\sqrt{c})^2 \geq 0.$$

Setting $\sqrt{b} = x$ and $\sqrt{c} = y$, the inequality can be rewritten as

$$(x - y)^2[9(x^2 + y^2)(x + y)^2 - 8(x^2 + xy + y^2)^2] \geq 0.$$

This follows from the Cauchy-Schwarz inequality as follows

$$9(x^2 + y^2)(x + y)^2 = 9[(x - y)^2 + 2xy][(x - y)^2 + 4xy]$$
$$\geq 9[(x - y)^2 + 2\sqrt{2}xy]^2 \geq 9\left[\frac{2\sqrt{2}}{3}(x - y)^2 + 2\sqrt{2}xy\right]^2$$
$$= 8(x^2 + xy + y^2)^2 \geq 0.$$

The equality occurs when two of a, b, c are zero, and when $-a = b = c$ (or any cyclic permutation).

□

P 2.31. *If a, b, c are real numbers, then*

$$a^4(b - c)^2 + b^4(c - a)^2 + c^4(a - b)^2 \geq \frac{1}{2}(a - b)^2(b - c)^2(c - a)^2.$$

(Vasile Cîrtoaje, 2011)

Solution. Since

$$b^4(c-a)^2+c^4(a-b)^2 \ge \frac{1}{2}[b^2(c-a)+c^2(a-b)]^2$$
$$= \frac{1}{2}(b-c)^2(bc-ca-ab)^2,$$

it suffices to prove that

$$2a^4+(ab-bc+ca)^2 \ge (a-b)^2(a-c)^2,$$

which is equivalent to

$$a^2(a^2-2bc+2ca+2ab) \ge 0.$$

Therefore, the desired inequality is true if

$$a^2-2bc+2ca+2ab \ge 0.$$

Indeed, from

$$\sum(a^2-2bc+2ca+2ab)=\sum(a+b+c)^2 \ge 0,$$

due to symmetry, we may assume that $a^2-2bc+2ca+2ab \ge 0$. Thus, the proof is completed. The equality occurs when $a=b=c$, when two of a,b,c are equal to zero, and when $a=0$ and $b+c=0$ (or any cyclic permutation).

□

P 2.32. *If a,b,c are real numbers, then*

$$a^2(b-c)^4+b^2(c-a)^4+c^2(a-b)^4 \ge \frac{1}{2}(a-b)^2(b-c)^2(c-a)^2.$$

(Vasile Cîrtoaje, 2011)

Solution. Let us denote

$$x=\sum ab^2=ab^2+bc^2+ca^2, \quad y=\sum a^2b=a^2b+b^2c+c^2a.$$

Since

$$\sum a^2b^4=(\sum ab^2)^2-2abc\sum a^2b=x^2-2abcy,$$

$$\sum a^4b^2=(\sum a^2b)^2-2abc\sum ab^2=y^2-2abcx,$$

$$\sum a^2b^2(a^2+b^2)=x^2+y^2-2abc(x+y),$$

we have

$$\begin{aligned}\sum a^2(b-c)^4 &= \sum a^2(b^4-4b^3c+6b^2c^2-4bc^3+c^4)\\ &= \sum a^2b^2(a^2+b^2)-4abc(\sum ab^2+\sum a^2b)+18a^2b^2c^2\\ &= x^2+y^2-6abc(x+y)+18a^2b^2c^2.\end{aligned}$$

Therefore, we can write the desired inequality as

$$x^2+y^2-6abc(x+y)+18a^2b^2c^2 \ge \frac{1}{2}(x-y)^2,$$

which is equivalent to the obvious inequality

$$(x+y-6abc)^2 \ge 0.$$

The equality holds for $a(b-c)^2+b(c-a)^2+c(a-b)^2=0$.

□

P 2.33. *If a,b,c are real numbers, then*

$$a^2(b^2-c^2)^2+b^2(c^2-a^2)^2+c^2(a^2-b^2)^2 \ge \frac{3}{8}(a-b)^2(b-c)^2(c-a)^2.$$

(Vasile Cîrtoaje, 2011)

Solution. We see that the inequality remains unchanged and the product

$$(a+b)(b+c)(c+a)$$

changes its sign by replacing a,b,c with $-a,-b,-c$, respectively. Thus, without loss of generality, we may assume that $(a+b)(b+c)(c+a)\ge 0$. According to this condition, at least one of a,b,c is nonnegative. So, we may consider $a\ge 0$, and hence

$$a(a+b)(b+c)(c+a)\ge 0.$$

In virtue of the Cauchy-Schwarz inequality, we get

$$b^2(c^2-a^2)^2+c^2(a^2-b^2)^2 \ge \frac{1}{2}\left[b(c^2-a^2)+c(a^2-b^2)\right]^2 = \frac{1}{2}(b-c)^2(a^2+bc)^2.$$

Thus, it suffices to show that

$$2a^2(b+c)^2+(a^2+bc)^2 \ge \frac{3}{4}(a-b)^2(a-c)^2,$$

which is equivalent to

$$(a+b)(a+c)[a^2+5a(b+c)+bc] \ge 0,$$

$$(a+b)(a+c)[(a+b)(a+c)+4a(b+c)] \ge 0,$$

$$(a+b)^2(a+c)^2+4a(a+b)(b+c)(c+a) \ge 0.$$

Since the last inequality is clearly true, the proof is completed. The equality holds for $a=b=c$, for $-a=b=c$ (or any cyclic permutation), and for $b=c=0$ (or any cyclic permutation).

□

P 2.34. *If a,b,c are real numbers such that $ab+bc+ca=3$, then*

(a) $$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \ge 3(a+b+c)^2;$$

(b) $$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \ge \frac{3}{2}(a^2+b^2+c^2).$$

(Vasile Cîrtoaje, 1995)

Solution. Let

$$p=a+b+c, \quad q=ab+bc+ca, \quad r=abc.$$

We have

$$\prod(b^2+bc+c^2) = \prod[(b+c)^2-bc]$$
$$= \prod(b+c)^2 - \sum bc(a+b)^2(a+c)^2 + abc\sum a(b+c)^2 - a^2b^2c^2.$$

Since

$$\prod(b+c)^2 = (pq-r)^2 = r^2-2pqr+p^2q^2,$$

$$\sum bc(a+b)^2(a+c)^2 = \sum bc(a^2+q)^2 = r\sum a^3+2pqr+q^2$$
$$= r(3r+p^3-3pq)+2pqr+q^2 = 3r^2+(p^3-pq)r+q^3,$$

$$abc\sum a(b+c)^2 = r(3r+pq) = 3r^2+pqr,$$

we get

$$\prod(b^2+bc+c^2) = (p^2-q)q^2-p^3r.$$

(a) Write the inequality as follows

$$3\prod(b^2+bc+c^2) \ge (a+b+c)^2(ab+bc+ca)^2,$$

$$(2p^2-3q)q^2-3p^3r\geq 0,$$

$$q^2(p^2-3q)+p^2(q^2-3pr)\geq 0,$$

$$q^2\sum(b-c)^2+p^2\sum a^2(b-c)^2\geq 0.$$

Clearly, the last inequality holds for all real a,b,c. The equality holds when $a=b=c=\pm 1$.

(b) Write the inequality in the homogeneous forms

$$2\prod(b^2+bc+c^2)\geq (a^2+b^2+c^2)(ab+bc+ca)^2,$$

$$2(p^2-q)q^2-2p^3r-(p^2-2q)q^2\geq 0,$$

$$p^2(q^2-2pr)\geq 0,$$

$$(a+b+c)^2(a^2b^2+b^2c^2+c^2a^2)\geq 0.$$

The equality holds when $a+b+c=0$ and $ab+bc+ca=3$.

□

P 2.35. *If a,b,c are real numbers, then*

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)\geq 3(ab+bc+ca)(a^2b^2+b^2c^2+c^2a^2).$$

(Vasile Cîrtoaje, 2011)

Solution. As we have shown in the proof of the preceding P 2.34,

$$\prod(b^2+bc+c^2)=(p^2-q)q^2-p^3r,$$

where

$$p=a+b+c,\quad q=ab+bc+ca,\quad r=abc.$$

Thus, we can write the desired inequality as

$$(p^2-q)q^2-p^3r\geq 3q(q^2-2pr),$$

$$q^2(p^2-4q)+p(6q-p^2)r\geq 0.$$

Consider further two cases: $6q-p^2\geq 0$ and $6q-p^2\leq 0$.

Case 1: $6q-p^2\geq 0$. By Schur's inequality of degree four, we have

$$6pr\geq (p^2-q)(4q-p^2).$$

Therefore, it suffices to show that

$$6q^2(p^2-4q)+(6q-p^2)(p^2-q)(4q-p^2)\ge 0,$$

which is equivalent to the obvious inequality

$$(p^2-4q)^2(p^2-3q)\ge 0.$$

Case 2: $6q-p^2\le 0$. Since $3pr\le q^2$, it suffices to show that

$$3q^2(p^2-4q)+(6q-p^2)q^2\ge 0,$$

which is equivalent to the obvious inequality

$$q^2(p^2-3q)\ge 0.$$

The inequality is an equality when $a=b=c$, and when $a=0$ and $b=c$ (or any cyclic permutation).

□

P 2.36. *If a,b,c are real numbers, not all of the same sign, then*

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)\ge 3(ab+bc+ca)^3.$$

(Vasile Cîrtoaje, 2011)

Solution. Since the inequality is symmetric and does not change by substituting $-a,-b,-c$ for a,b,c, we may assume that $a\le 0$ and $b,c\ge 0$. Substituting $-a$ for a, we need to prove that

$$(a^2-ab+b^2)(b^2+bc+c^2)(c^2-ca+a^2)\ge 3(bc-ab-ac)^3$$

for $a,b,c\ge 0$. Since the left hand side of this inequality is nonnegative, consider further the nontrivial case

$$bc-ab-ac>0.$$

Since

$$b^2+bc+c^2-3(bc-ab-ac)=(b-c)^2+3a(b+c)\ge 0,$$

it suffices to show that

$$(a^2-ab+b^2)(a^2-ac+c^2)\ge (bc-ab-ac)^2.$$

First Solution. From $bc-ab-ac>0$, it follows that $a=\min\{a,b,c\}$. Since $a^2-ab+b^2\ge (b-a)^2$ and $a^2-ac+c^2\ge (c-a)^2$, it suffices to show that

$$(b-a)^2(c-a)^2\ge (bc-ab-ac)^2.$$

This is true if $(b-a)(c-a) \ge bc-ab-ac$; indeed,

$$(b-a)(c-a)-(bc-ab-ac)=a^2 \ge 0.$$

The original inequality is an equality when two of a,b,c are zero, and when $a=0$ and $b=c$ (or any cyclic permutation).

Second Solution. Since

$$4(a^2-ab+b^2)=(a+b)^2+3(a-b)^2,$$

$$4(a^2-ac+c^2)=(a+c)^2+3(a-c)^2,$$

by the Cauchy-Schwarz inequality, we get

$$16(a^2-ab+b^2)(b^2+bc+c^2) \ge [(a+b)(a+c)+3(a-b)(a-c)]^2.$$

Thus, we only need to show that

$$(a+b)(a+c)+3(a-b)(a-c) \ge 4(bc-ab-ac),$$

which is equivalent to the obvious inequality $a(2a+b+c) \ge 0$.

□

P 2.37. *If a,b,c are real numbers, then*

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \ge \frac{3}{8}(a^2+b^2)(b^2+c^2)(c^2+a^2).$$

(Vasile Cîrtoaje, 2011)

Solution. If a,b,c have the same sign, then the inequality follows from

$$a^2+ab+b^2 \ge a^2+b^2, \quad b^2+bc+c^2 \ge b^2+c^2, \quad c^2+ca+a^2 \ge c^2+a^2.$$

Consider now that a,b,c have not the same sign. Since the inequality is symmetric and does not change by substituting $-a,-b,-c$ for a,b,c, we may assume that $a \le 0$ and $b,c \ge 0$. Substituting $-a$ for a, we need to prove that

$$(a^2-ab+b^2)(a^2-ac+c^2)(b^2+bc+c^2) \ge \frac{3}{8}(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

for $a,b,c \ge 0$. Write this inequality in the form

$$[(a^2+b^2)+(a-b)^2][(a^2+c^2)+(a-c)^2][2(b^2+c^2)+2bc] \ge$$

$$\ge 3(a^2+b^2)(b^2+c^2)(c^2+a^2).$$

It suffices to show that

$$2(b^2+c^2)[(a-b)^2(a^2+c^2)+(a-c)^2(a^2+b^2)]+2bc(a^2+b^2)(a^2+c^2)\geq$$

$$\geq (a^2+b^2)(b^2+c^2)(c^2+a^2),$$

which is equivalent to

$$2(b^2+c^2)[(a-b)^2(a^2+c^2)+(a-c)^2(a^2+b^2)]\geq (b-c)^2(a^2+b^2)(a^2+c^2),$$

or

$$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2)}{a^2+c^2}\geq \frac{(b-c)^2}{2(b^2+c^2)}.$$

Consider further two cases.

Case 1: $2a^2\leq b^2+c^2$. By the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2)}{a^2+c^2}\geq \frac{[(b-a)+(a-c)]^2}{(a^2+b^2)+(a^2+c^2)}=\frac{(b-c)^2}{2a^2+b^2+c^2}.$$

Thus, it suffices to show that

$$\frac{1}{2a^2+b^2+c^2}\geq \frac{1}{2(b^2+c^2)},$$

which reduces to $b^2+c^2\geq 2a^2$.

Case 2: $2a^2\geq b^2+c^2$. By the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{a^2+c^2}\geq \frac{[c(b-a)+b(a-c)]^2}{c^2(a^2+b^2)+b^2(a^2+c^2)}=\frac{a^2(b-c)^2}{a^2(b^2+c^2)+2b^2c^2}.$$

Therefore, it suffices to prove that

$$\frac{a^2}{a^2(b^2+c^2)+2b^2c^2}\geq \frac{1}{2(b^2+c^2)}.$$

This reduces to $a^2(b^2+c^2)\geq 2b^2c^2$, which is true because

$$2a^2(b^2+c^2)-4b^2c^2\geq (b^2+c^2)^2-4b^2c^2=(b^2-c^2)^2\geq 0.$$

Thus, the proof is completed. The equality holds when two of a,b,c are zero, and when $-a=b=c$ (or any cyclic permutation).

□

P 2.38. *If a, b, c are real numbers, then*

$$2(a^2+b^2)(b^2+c^2)(c^2+a^2) \ge (a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2).$$

(Vasile Cîrtoaje, 2014)

Solution. Since the inequality is symmetric and does not change by substituting $-a, -b, -c$ for a, b, c, we may assume that $a \le 0$ and $b, c \ge 0$. Substituting $-a$ for a, we need to prove that

$$2(a^2+b^2)(b^2+c^2)(c^2+a^2) \ge (b^2-bc+c^2)(c^2+ca+a^2)(a^2+ab+b^2)$$

for $a, b, c \ge 0$. Using the notation

$$A = b^2+c^2, \quad B = c^2+a^2, \quad C = a^2+b^2,$$

we can write the inequality as follows:

$$2ABC \ge (A-bc)(B+ca)(C+ab),$$

$$ABC + a^2b^2c^2 \ge ab(AB-c^2C) + ac(AC-b^2B) - bc(BC-a^2A),$$

$$ABC + a^2b^2c^2 \ge ab(c^4+a^2b^2) + ac(b^4+a^2c^2) - bc(a^4+b^2c^2).$$

It suffices to show that

$$ABC + a^2b^2c^2 \ge ab(c^4+a^2b^2) + ac(b^4+a^2c^2) + bc(a^4+b^2c^2).$$

Moreover, since $2ab \le a^2+b^2$, $2ac \le a^2+c^2$, $2bc \le b^2+c^2$, it is enough to prove that

$$2ABC + 2a^2b^2c^2 \ge (a^2+b^2)(c^4+a^2b^2) + (a^2+c^2)(b^4+a^2c^2) + (b^2+c^2)(a^4+b^2c^2).$$

Indeed, this inequality reduces to the obvious inequality

$$6a^2b^2c^2 \ge 0.$$

The equality holds when two of a, b, c are zero.

□

P 2.39. *If a, b, c are real numbers, then*

$$9(1+a^4)(1+b^4)(1+c^4) \ge 8(1+abc+a^2b^2c^2)^2.$$

(Vasile Cîrtoaje, 2004)

Solution. Substituting a, b, c by $|a|, |b|, |c|$, respectively, the left side of the inequality remains unchanged, while the right side either remains unchanged or increases. Therefore, it suffices to prove the inequality only for $a, b, c \geq 0$. If $a = b = c$, then the inequality reduces to

$$9(1+a^4)^3 \geq 8(1+a^3+a^6)^2,$$

$$9(a^2+\frac{1}{a^2})^3 \geq 8(a^3+\frac{1}{a^3}+1)^2.$$

Setting $a+\dfrac{1}{a} = x$, this inequality can be written as follows

$$9(x^2-2)^3 \geq 8(x^3-3x+1)^2,$$

$$x^6-6x^4-16x^3+36x^2+48x-80 \geq 0,$$

$$(x-2)^2[x(x^3-8)+4(x^3-5)+6x^2] \geq 0.$$

Since $x \geq 2$, the last inequality is clearly true. Multiplying now the inequalities

$$9(1+a^4)^3 \geq 8(1+a^3+a^6)^2,$$

$$9(1+b^4)^3 \geq 8(1+b^3+b^6)^2,$$

$$9(1+c^4)^3 \geq 8(1+c^3+c^6)^2,$$

we get

$$[9(1+a^4)(1+b^4)(1+c^4)]^3 \geq 8^3(1+a^3+a^6)^2(1+b^3+b^6)^2(1+c^3+c^6)^2.$$

According to Hölder's inequality

$$(1+a^3+a^6)(1+b^3+b^6)(1+c^3+c^6) \geq (1+abc+a^2b^2c^2)^3,$$

the conclusion follows. The equality holds for $a = b = c = 1$.

□

P 2.40. *If a, b, c are real numbers, then*

$$2(1+a^2)(1+b^2)(1+c^2) \geq (1+a)(1+b)(1+c)(1+abc).$$

(Vasile Cîrtoaje, 2001)

Solution. Substituting a, b, c by $|a|, |b|, |c|$, respectively, the left side of the inequality remains unchanged, while the right side either remains unchanged or increases. Therefore, it suffices to prove the inequality only for $a, b, c \geq 0$.

First Solution. For $a = b = c$, the inequality reduces to

$$2(1 + a^2)^3 \geq (1 + a)^3(1 + a^3).$$

This is true, since

$$2(1 + a^2)^3 - (1 + a)^3(1 + a^3) = (1 - a)^4(1 + a + a^2) \geq 0.$$

Multiplying the inequalities

$$2(1 + a^2)^3 \geq (1 + a)^3(1 + a^3),$$

$$2(1 + b^2)^3 \geq (1 + b)^3(1 + b^3),$$

$$2(1 + c^2)^3 \geq (1 + c)^3(1 + c^3),$$

we get

$$8(1 + a^2)^3(1 + b^2)^3(1 + c^2)^3 \geq (1 + a)^3(1 + b)^3(1 + c)^3(1 + a^3)(1 + b^3)(1 + c^3).$$

Using this result, we still have to show that

$$(1 + a^3)(1 + b^3)(1 + c^3) \geq (1 + abc)^3,$$

which is just Hölder's inequality. We can also prove this inequality using the AM-GM inequality. To do this, we write the inequality as

$$(a^3b^3 + b^3c^3 + c^3a^3 - 3a^2b^2c^2) + (a^3 + b^3 + c^3 - 3abc) \geq 0.$$

The equality holds for $a = b = c = 1$.

Second Solution. We use the substitution

$$a = \frac{1 - x}{1 + x}, \quad b = \frac{1 - y}{1 + y}, \quad c = \frac{1 - z}{1 + z},$$

where $x, y, z \in (-1, \ 1]$. Since

$$\frac{1 + a^2}{1 + a} = \frac{1 + x^2}{1 + x}, \quad \frac{1 + b^2}{1 + b} = \frac{1 + y^2}{1 + y}, \quad \frac{1 + c^2}{1 + c} = \frac{1 + z^2}{1 + z}$$

and

$$1 + abc = \frac{2(1 + xy + yz + zx)}{(1 + x)(1 + y)(1 + z)},$$

the required inequality becomes

$$(1+x^2)(1+y^2)(1+z^2) \geq 1+xy+yz+zx,$$

$$x^2y^2+y^2z^2+z^2x^2+x^2+y^2+z^2 \geq xy+yz+zx,$$

$$x^2y^2+y^2z^2+z^2x^2+\frac{1}{2}(x-y)^2+\frac{1}{2}(y-z)^2+\frac{1}{2}(z-x)^2 \geq 0.$$

□

P 2.41. *If a,b,c are real numbers, then*

$$3(a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2) \geq a^3b^3+b^3c^3+c^3a^3.$$

(Titu Andreescu, 2006)

Solution. Substituting a,b,c by $|a|,|b|,|c|$, respectively, the left side of the inequality remains unchanged or decreases, while the right side remains unchanged or increases. Therefore, it suffices to prove the inequality for $a,b,c \geq 0$. If $a=0$, then the inequality reduces to $b^2c^2(b-c)^2 \geq 0$. Consider further then $a,b,c>0$. We first show that

$$3(a^2-ab+b^2)^3 \geq a^6+a^3b^3+b^6.$$

Indeed, setting $x=\dfrac{a}{b}+\dfrac{b}{a}$, $x \geq 2$, we can write this inequality as

$$3(x-1)^3 \geq x^3-3x+1,$$

$$(x-2)^2(2x-1) \geq 0.$$

Using this inequality, together with the similar ones, we have

$$27(a^2-ab+b^2)^3(b^2-bc+c^2)^3(c^2-ca+a^2)^3 \geq$$
$$\geq (a^6+a^3b^3+b^6)(b^6+b^3c^3+c^6)(c^6+c^3a^3+a^6).$$

Therefore, it suffices to show that

$$(a^6+a^3b^3+b^6)(b^6+b^3c^3+c^6)(c^6+c^3a^3+a^6) \geq (a^3b^3+b^3c^3+c^3a^3)^3.$$

Writing this inequality in the form

$$(a^3b^3+b^6+a^6)(b^6+b^3c^3+c^6)(a^6+c^6+c^3a^3) \geq (a^3b^3+b^3c^3+c^3a^3)^3,$$

we see that it is just Hölder's inequality. The equality holds when $a=b=c$, when $a=0$ and $b=c$ (or any cyclic permutation), and when two of a,b,c are 0.

□

P 2.42. *If a, b, c are nonzero real numbers, then*

$$\sum \frac{b^2 - bc + c^2}{a^2} + 2\sum \frac{a^2}{bc} \geq \left(\sum a\right)\left(\sum \frac{1}{a}\right).$$

(Vasile Cîrtoaje, 2010)

Solution. We have

$$\sum \frac{b^2 - bc + c^2}{a^2} + 2\sum \frac{a^2}{bc} = \sum \left(\frac{b^2 - bc + c^2}{a^2} + \frac{b^2}{ca} + \frac{c^2}{ab} \right)$$
$$= \sum \frac{(b^2 - bc + c^2)(ab + bc + ca)}{a^2 bc}$$
$$= \frac{ab + bc + ca}{a^2 b^2 c^2} \sum bc(b^2 - bc + c^2).$$

Then, we can write the inequality as

$$(ab + bc + ca)[\sum bc(b^2 - bc + c^2) - abc \sum a] \geq 0.$$

Since

$$\sum bc(b^2 - bc + c^2) - abc \sum a = (\sum bc)(\sum a^2) - \sum b^2c^2 - 2abc \sum a$$
$$= (\sum bc)(\sum a^2) - (\sum bc)^2$$
$$= (\sum bc)(\sum a^2 - \sum bc),$$

the inequality is equivalent to

$$(ab + bc + ca)^2(a^2 + b^2 + c^2 - ab - bc - ca) \geq 0,$$

which is true. The equality holds for $a = b = c$, and for $ab + bc + ca = 0$.

□

P 2.43. *Let a, b, c be real numbers. Prove that*

(a) if $a, b, c \in [0, 1]$, then

$$abc - (b + c - a)(c + a - b)(a + b - c) \leq 1;$$

(b) if $a, b, c \in [-1, 1]$, then

$$abc - (b + c - a)(c + a - b)(a + b - c) \leq 4.$$

(Vasile Cîrtoaje, 2011)

Solution. We will show that if $a,b,c \in [m,M]$, where $M \ge 0$, then

$$abc-(b+c-a)(c+a-b)(a+b-c) \le M(M-m)^2.$$

Without loss of generality, assume that

$$M \ge a \ge b \ge c \ge m.$$

We have two cases to consider.

Case 1: $a \le 0$. The required inequality is true, since

$$abc-(b+c-a)(c+a-b)(a+b-c) \le 0 \le M(M-m)^2.$$

Indeed, substituting $-a,-b,-c$ for a,b,c, respectively, the left inequality can be restated as

$$abc \ge (b+c-a)(c+a-b)(a+b-c),$$

where $a,b,c \ge 0$. This is just the well-known Schur's inequality of degree three.

Case 2: $a > 0$. Since $(M-m)^2 \ge (a-b)^2$ and $M \ge a$, we have

$$M(M-m)^2 \ge a(a-c)^2.$$

Therefore, it suffices to show that

$$abc-(b+c-a)(c+a-b)(a+b-c) \le a(a-c)^2,$$

which is equivalent to

$$(b-c)[a^2+(b-2c)a-b^2+c^2] \ge 0.$$

This is true, since

$$\begin{aligned} a^2+(b-2c)a-b^2+c^2 &= (a-b)(a+2b-2c)+(b-c)^2 \\ &\ge 2(a-b)(b-c)+(b-c)^2 \ge 0. \end{aligned}$$

Thus, the proof is completed. The equality holds when two of a,b,c are equal to m, and the other is equal to M.

□

P 2.44. *Let a,b,c be real numbers. Prove that*

(a) if $a,b,c \in [0,1]$, then

$$\sum a^2(a-b)(a-c) \le 1;$$

(b) if $a,b,c \in [-1,1]$, then

$$\sum a^2(a-b)(a-c) \le 4.$$

(Vasile Cîrtoaje, 2011)

Solution. We will show that if $a,b,c \in [m,M]$, then

$$\sum a^2(a-b)(a-c) \le (M-m)^2 \cdot \max\{m^2, M^2\}.$$

Without loss of generality, assume that

$$M \ge a \ge b \ge c \ge m.$$

Since $b^2(b-c)(b-a) \le 0$, $(a-c)^2 \le (M-m)^2$, and $\max\{a^2,c^2\} \le \max\{m^2,M^2\}$, it suffices to show that

$$a^2(a-b)(a-c) + c^2(c-a)(c-b) \le (a-c)^2 \cdot \max\{a^2,c^2\}.$$

This is equivalent to

$$(a-c)^2(a^2+c^2+ac-ab-bc-\max\{a^2,c^2\}) \le 0.$$

Case 1: $a^2 \ge c^2$. From $a^2-c^2=(a-c)(a+c) \ge 0$, it follows that $a+c \ge 0$. Then,

$$a^2+c^2+ac-ab-bc-\max\{a^2,c^2\} = (a+c)(c-b) \le 0.$$

Case 2: $a^2 \le c^2$. From $a^2-c^2=(a-c)(a+c) \le 0$, it follows that $a+c \le 0$. Then,

$$a^2+c^2+ac-ab-bc-\max\{a^2,c^2\} = (a+c)(a-b) \le 0.$$

Thus, the proof is completed. For $M^2 \ge m^2$, the equality holds when two of a,b,c are equal to m, and the other is equal to M. For $M^2 \le m^2$, the equality holds when two of a,b,c are equal to M, and the other is equal to m.

□

P 2.45. *Let a,b,c be real numbers such that*

$$ab+bc+ca = abc+2.$$

Prove that

$$a^2+b^2+c^2-3 \ge (2+\sqrt{3})(a+b+c-3).$$

(Vasile Cîrtoaje, 2011)

Solution. Substituting $a+1$, $b+1$, $c+1$ for a, b, c, respectively, we need to prove that

$$a+b+c = abc$$

implies

$$a^2+b^2+c^2 \ge \sqrt{3}(a+b+c).$$

This inequality is true if

$$(a^2+b^2+c^2)^2 \ge 3(a+b+c)^2,$$

which is equivalent to the homogeneous inequality

$$(a^2+b^2+c^2)^2 \ge 3abc(a+b+c).$$

Since

$$(ab+bc+ca)^2 - 3abc(a+b+c) = \frac{1}{2}\sum a^2(b-c)^2 \ge 0,$$

it suffices to prove that

$$(a^2+b^2+c^2)^2 \ge (ab+bc+ca)^2,$$

which is equivalent to

$$(a^2+b^2+c^2-ab-bc-ca)(a^2+b^2+c^2+ab+bc+ca) \ge 0.$$

This inequality is true, since

$$2(a^2+b^2+c^2-ab-bc-ca) = (a-b)^2+(b-c)^2+(c-a)^2 \ge 0,$$

$$2(a^2+b^2+c^2+ab+bc+ca) = (a+b)^2+(b+c)^2+(c+a)^2 \ge 0.$$

The equality holds for $a=b=c=1$, and for $a=b=c=1+\sqrt{3}$.

□

P 2.46. *Let a,b,c be real numbers such that*

$$(a+b)(b+c)(c+a) = 10.$$

Prove that

$$(a^2+b^2)(b^2+c^2)(c^2+a^2) + 12a^2b^2c^2 \ge 30.$$

(Vasile Cîrtoaje, 2011)

***Solution**.* Since

$$2(b^2+c^2) = (b+c)^2+(b-c)^2$$

and

$$(a^2+b^2)(a^2+c^2) = (a^2+bc)^2 + a^2(b-c)^2,$$

by virtue of the Cauchy-Schwarz inequality, we have

$$\begin{aligned} 2(a^2+b^2)(b^2+c^2)(c^2+a^2) &\ge [(b+c)(a^2+bc)+a(b-c)^2]^2 \\ &= [(a+b)(b+c)(c+a)-4abc]^2 \\ &= 4(5-2abc)^2. \end{aligned}$$

Thus, it suffices to show that

$$(5-2abc)^2+6a^2b^2c^2 \ge 15.$$

This inequality is equivalent to $(abc-1)^2=0.$ The homogeneous inequality

$$10(a^2+b^2)(b^2+c^2)(c^2+a^2)+120a^2b^2c^2 \ge 3(a+b)^2(b+c)^2(c+a)^2$$

becomes an equality for $\frac{a}{k}=b=c$ (or any cyclic permutation), where $k+\frac{1}{k}=3.$

□

P 2.47. *Let a,b,c be real numbers such that*

$$(a+b)(b+c)(c+a)=5.$$

Prove that

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)+12a^2b^2c^2 \ge 15.$$

(Vasile Cîrtoaje, 2011)

Solution. Since

$$b^2+bc+c^2=\frac{3}{4}(b+c)^2+\frac{1}{4}(b-c)^2$$

and

$$(a^2+ab+b^2)(a^2+ac+c^2)=\left(a^2+\frac{ab+ac}{2}+bc\right)^2+\frac{3}{4}a^2(b-c)^2,$$

by the Cauchy-Schwarz inequality, we have

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \ge$$

$$\ge \left[\frac{\sqrt{3}}{2}(b+c)\left(a^2+\frac{ab+ac}{2}+bc\right)+\frac{\sqrt{3}}{4}a(b-c)^2\right]^2$$

$$=\frac{3}{4}[(a+b)(b+c)(c+a)-2abc]^2=\frac{3}{4}(5-2abc)^2.$$

Thus, it suffices to show that

$$\frac{3}{4}(5-2abc)^2+12a^2b^2c^2 \ge 15,$$

which is equivalent to $(2abc-1)^2=0.$ The homogeneous inequality

$$5(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)+60a^2b^2c^2 \ge 3(a+b)^2(b+c)^2(c+a)^2$$

becomes an equality for $\frac{a}{k}=b=c$ (or any cyclic permutation), where $k+\frac{1}{k}=3.$

□

P 2.48. *Let a, b, c be real numbers such that $a+b+c=1$ and $a^3+b^3+c^3=k$. Prove that*

(a) if $k=25$, then $|a|\le 1$ or $|b|\le 1$ or $|c|\le 1$;

(b) if $k=-11$, then $1<a\le 2$ or $1<b\le 2$ or $1<c\le 2$.

(Vasile Cîrtoaje, 2011)

Solution. Without loss of generality, assume that $a\le b\le c$. If $b=1$, then $a+c=0$, and hence

$$k=a^3+b^3+c^3=1+a^3+c^3=1,$$

which is false in (a) and (b), too. From $(b-a)(b-c)\le 0$, we get

$$b^2-(a+c)b+ac\le 0,$$

which is equivalent to

$$2b^2-b+ac\le 0.$$

(a) It suffices to show that $|b|\le 1$. We have

$$25-b^3=a^3+c^3=(a+c)^3-3ac(a+c)=(1-b)^3-3ac(1-b),$$

which yields

$$ac=\frac{b^2-b-8}{1-b}.$$

Thus, the inequality $2b^2-b+ac\le 0$ is equivalent to

$$\frac{(1+b)(4-3b+b^2)}{1-b}\ge 0,$$

which involves $-1\le b<1$, and hence $|b|\le 1$. The equality $|b|=1$ holds for $a=b=-1$ and $c=3$.

(b) It suffices to show that $1<b\le 2$. We have

$$-11-b^3=a^3+c^3=(a+c)^3-3ac(a+c)=(1-b)^3-3ac(1-b),$$

which yields

$$ac=\frac{b^2-b+4}{1-b}.$$

Thus, the inequality $2b^2-b+ac\le 0$ is equivalent to

$$\frac{(b-2)(b^2+1)}{1-b}\ge 0,$$

which involves $1<b\le 2$. The equality $b=2$ holds for $a=-3$ and $b=c=2$.

□

P 2.49. *Let a, b, c be real numbers such that*

$$a + b + c = a^3 + b^3 + c^3 = 2.$$

Prove that $a, b, c \notin \left[\frac{5}{4}, 2\right]$.

(Vasile Cîrtoaje, 2011)

Solution. From

$$2 = a^3 + b^3 + c^3 = a^3 + (b + c)^2 - 3bc(b + c) = a^3 + (2 - a)^2 - 3bc(2 - a),$$

we get

$$bc = \frac{2(1 - a)^2}{2 - a}.$$

Thus,we can write the false inequality $4bc > (b + c)^2$ as

$$\frac{8(1 - a)^2}{2 - a} > (2 - a)^2,$$

$$\frac{a(a^2 + 2a - 4)}{2 - a} > 0,$$

$$a \in (-1 - \sqrt{5}, 0) \cup (-1 + \sqrt{5}, 2).$$

In addition, the case $a = 2$ is also not possible, because it involves $b + c = 0$ and $b^3 + c^3 = -6$, which is false. Therefore,

$$a, b, c \notin (-1 - \sqrt{5}, 0) \cup (-1 + \sqrt{5}, 2].$$

Since

$$\left[\frac{5}{4}, 2\right] \subset (-1 - \sqrt{5}, 0) \cup (-1 + \sqrt{5}, 2],$$

the conclusion follows.

□

P 2.50. *If a, b, c and k are real numbers, then*

$$\sum (a - b)(a - c)(a - kb)(a - kc) \geq 0.$$

(Vasile Cîrtoaje, 2005)

Solution. For $a=b=c$, the equality holds. Otherwise, using the substitution $m=k+2$, $u=(1-k)a$, $b=a+x$, $c=a+y$, the inequality can be written as

$$Au^2+Bu+c\geq 0,$$

where

$$A=x^2-xy+y^2,$$

$$B=(x+y)(2A-mxy),$$

$$C=(x+y)^2(A-mxy)+m^2x^2y^2.$$

The quadratic Au^2+Bu+C has the discriminant

$$D=B^2-4AC=-3m^2x^2y^2(x-y)^2.$$

Since $A>0$ and $D\leq 0$, the conclusion follows. The equality holds for $a=b=c$, and for $a/k=b=c$ or any cyclic permutation.

Remark 1. The inequality is equivalent to

$$\sum a^4+k(k+2)\sum a^2b^2+(1-k^2)abc\sum a\geq (k+1)\sum ab(a^2+b^2).$$

For $k=0$, we get Schur's inequality of degree four

$$a^4+b^4+c^4+abc(a+b+c)\geq \sum ab(a^2+b^2).$$

For $k=1$, we get the inequality

$$a^4+b^4+c^4+3(a^2b^2+b^2c^2+c^2a^2)\geq 2\sum ab(a^2+b^2),$$

with equality for $a=b=c$.
For $k=2$, we get the inequality

$$a^4+b^4+c^4+8(a^2b^2+b^2c^2+c^2a^2)\geq 3(ab+bc+ca)(a^2+b^2+c^2),$$

which can be rewritten as

$$9(a^4+b^4+c^4)+126(a^2b^2+b^2c^2+c^2a^2)\geq 5(a+b+c)^4,$$

with equality for $a=b=c$, and for $a/2=b=c$ (or any cyclic permutation).

Remark 2. The inequality in P 2.50 is equivalent to

$$\sum(a-b)^2(a+b-c-kc)^2\geq 0.$$

□

P 2.51. *If a, b, c are real numbers, then*

$$(b+c-a)^2(c+a-b)^2(a+b-c)^2 \geq (b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2).$$

(Poland, 1992)

Solution. Consider the nontrivial case where $a \geq b \geq c$ and $b^2+c^2-a^2 \geq 0$. We get the desired inequality by multiplying the inequalities

$$(c+a-b)^2(a+b-c)^2 \geq (c^2+a^2-b^2)(a^2+b^2-c^2),$$

$$(b+c-a)^2(a+b-c)^2 \geq (b^2+c^2-a^2)(a^2+b^2-c^2),$$

$$(b+c-a)^2(c+a-b)^2 \geq (b^2+c^2-a^2)(c^2+a^2-b^2).$$

We have

$$(c+a-b)^2(a+b-c)^2-(c^2+a^2-b^2)(a^2+b^2-c^2)=$$

$$=[a^2-(b-c)^2]^2-a^4+(b^2-c^2)^2=2(b-c)^2(b^2+c^2-a^2) \geq 0.$$

The equality holds for $a=b=c$, for $a=0$ and $b=c$ (or any cyclic permutation), and for $a=0$ and $b+c=0$ (or any cyclic permutation).

□

P 2.52. *If a, b, c are real numbers, then*

$$\sum a^2(a-b)(a-c) \geq \frac{(a-b)^2(b-c)^2(c-a)^2}{a^2+b^2+c^2+ab+bc+ca}.$$

Solution *(by Michael Rozenberg)*. Since

$$\sum a^2(a-b)(a-c)=\frac{1}{2}\sum(b-c)^2(b+c-a)^2,$$

we can write the inequality in the form

$$\left[\sum(b+c)^2\right]\left[\sum(b-c)^2(b+c-a)^2\right] \geq 4(a-b)^2(b-c)^2(c-a)^2.$$

Using now the Cauchy-Schwarz inequality, is suffices to show that

$$\left[\sum(b+c)(b-c)(b+c-a)\right]^2 \geq 4(a-b)^2(b-c)^2(c-a)^2,$$

which is an identity. The equality holds for $a=b=c$, for $a=0$ and $b=c$ (or any cyclic permutation), and for $a=0$ and $b+c=0$ (or any cyclic permutation).

□

P 2.53. *Let $a \le b \le c$ be real numbers such that*

$$a+b+c=p, \quad ab+bc+ca=q,$$

where p and q are fixed real numbers satisfying $p^2 \ge 3q$. Prove that the product

$$r = abc$$

is minimal when $b=c$, and is maximal when $a=b$.

First Solution. We show first that $a \in [a_1,\ a_2]$, where

$$a_1 = \frac{p-2\sqrt{p^2-3q}}{3}, \quad a_2 = \frac{p-\sqrt{p^2-3q}}{3}.$$

From

$$\begin{aligned}(b-c)^2 &= (b+c)^2-4bc = (b+c)^2+4a(b+c)-4q \\ &= (p-a)^2+4a(p-a)-4q = -3a^2+2pa+p^2-4q \ge 0,\end{aligned}$$

we get $a \ge a_1$, with equality for $b=c$. Similarly, from

$$(a-b)(a-c) = a^2-2a(b+c)+q = a^2-2a(p-a)+q = 3a^2-2pa+q \ge 0,$$

we get $a \le a_2$, with equality for $a=b$. On the other hand, from

$$abc = a[q-a(b+c)] = aq-a^2(p-a) = a^3-pa^2+qa,$$

we get

$$r(a) = a^3-pa^2+qa.$$

Since $r'(a) = 3a^2-2pa+q = (a-b)(a-c) \ge 0$, $r(a)$ is strictly increasing on $[a_1,a_2]$, and hence $r(a)$ is minimal for $a=a_1$, when $b=c$, and is maximal for $a=a_2$, when $a=b$.

Second Solution. From

$$(a-b)^2(b-c)^2(c-a)^2 = -27r^2+2(9pq-2p^3)r+p^2q^2-4q^3 \ge 0,$$

we get $r_1 \le r \le r_2$, where

$$r_1 = \frac{9pq-2p^3-2(p^2-3q)\sqrt{p^2-3q}}{27},$$

$$r_2 = \frac{9pq-2p^3+2(p^2-3q)\sqrt{p^2-3q}}{27}.$$

Obviously, r attains its minimal and maximal values when two of a, b, c are equal; that is, when either $a = b$ or $b = c$. For $a = b$, from $a + b + c = p$ and $ab + bc + ca = q$, we get

$$a = b = \frac{p - \sqrt{p^2 - 3q}}{3}, \quad c = \frac{p + 2\sqrt{p^2 - 3q}}{3},$$

$$r = \frac{(p - \sqrt{p^2 - 3q})^2(p + 2\sqrt{p^2 - 3q})}{27} = r_2.$$

Similar, for $b = c$, we get

$$b = c = \frac{p + \sqrt{p^2 - 3q}}{3}, \quad a = \frac{p - 2\sqrt{p^2 - 3q}}{3},$$

$$r = \frac{(p + \sqrt{p^2 - 3q})^2(p - 2\sqrt{p^2 - 3q})}{27} = r_1.$$

Remark. Using this result, we can prove the following generalization:

- *Let $a_1, a_2, \ldots, a_n$ be real numbers such that*

$$a_1 + a_2 + \cdots + a_n = p, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = p_1,$$

where p and p_1 are fixed real numbers satisfying $p^2 \le np_1$. Then, the product

$$r = a_1 a_2 \cdots a_n$$

is minimal and maximal when $n-1$ numbers of $a_1, a_2, \ldots, a_n$ are equal.

Assume, by the sake of contradiction, that the product r is minimal/maximal when three of $a_1, a_2, \ldots, a_n$ are distinct, let $a_1 < a_2 < a_3$. According to P .53, the product r can be increased/decreased (which is a contradiction) by choosing some suitable non-distinct numbers b_1, b_2, b_3 such that

$$b_1 + b_2 + b_3 = a_1 + a_2 + a_3, \quad b_1^2 + b_2^2 + b_3^2 = a_1^2 + a_2^2 + a_3^2.$$

□

P 2.54. *Let a, b, c be real numbers such that $a + b + c = 3$. Prove that*

$$(ab + bc + ca - 3)^2 \ge 27(abc - 1).$$

(Vasile Cîrtoaje, 2011)

First Solution. Let $q = ab + bc + ca$. We need to show that $(q-3)^2 + 27 \geq 27abc$. According to P 2.53, for fixed q, the product abc is maximal when two of a, b, c are equal. Therefore, it suffices to prove the desired inequality for $b = c$; that is, to show that $(2ab + b^2 - 3)^2 \geq 27(ab^2 - 3)$ for $a + 2b = 3$. This inequality is equivalent to

$$(b-1)^2(b+2)^2 \geq 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 7$ and $b = c = -2$ (or any cyclic permutation).

Second Solution. Assume that $a = \max\{a, b, c\}$, $a \geq 1$. Since

$$3 - ab - bc - ca \geq 3 - a(b+c) - \frac{1}{4}(b+c)^2 = 3 - a(3-a) - \frac{1}{4}(3-a)^2 = \frac{3}{4}(a-1)^2$$

and

$$abc - 1 \leq \frac{1}{4}a(b+c)^2 - 1 = \frac{1}{4}a(3-a)^2 - 1 = \frac{1}{4}(a-1)^2(a-4),$$

it suffices to prove that

$$\frac{9}{16}(a-1)^4 \geq \frac{27}{4}(a-1)^2(a-4),$$

which is equivalent to

$$(a-1)^2(a-7)^2 \geq 0.$$

□

P 2.55. *Let a, b, c be real numbers such that $a + b + c = 3$. Prove that*

$$(ab + bc + ca)^2 + 9 \geq 18abc.$$

(Vasile Cîrtoaje, 2011)

First Solution. Let $q = ab+bc+ca$. We need to show that $q^2+9 \geq 18abc$. According to P 2.53, for fixed q, the product abc is maximal when two of a, b, c are equal. Therefore, it suffices to prove the desired inequality for $b = c$; that is, to show that $(2ab+b^2)^2+9 \geq 18ab^2$ for $a + 2b = 3$. This inequality is equivalent to

$$(b-1)^2(b+1)^2 \geq 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 5$ and $b = c = -1$ (or any cyclic permutation).

Second Solution. Using the substitution $a = x + 1$, $b = y + 1$, $c = z + 1$, we need to show that

$$(xy + yz + zx)^2 \geq 12(xy + yz + zx) + 18xyz,$$

where x,y,z are real numbers such that $x+y+z=0$. Substituting x by $-y-z$, the inequality can be restated as

$$(y^2+yz+z^2)^2+12(y^2+yz+z^2)\geq -18yz(y+z).$$

Since

$$y^2+yz+z^2\geq \frac{3}{4}(y+z)^2\geq 3yz,$$

it suffices to show that

$$9y^2z^2+9(y+z)^2\geq -18yz(y+z),$$

which is equivalent to

$$9(yz+y+z)^2\geq 0.$$

□

P 2.56. *If a,b,c are real numbers such that $a^2+b^2+c^2=9$, then*

$$abc+10\geq 2(a+b+c).$$

(Vietnam TST, 2002)

Solution. Let $p=a+b+c$, $q=ab+bc+ca$ and $r=abc$. We need to show that $r\geq 2p-10$ for $p^2-2q=9$. According to P 2.53, for fixed p and q, r is minimal when two of a,b,c are equal. Therefore, it suffices to prove the desired inequality for $b=c$; that is, to show that $ab^2+10\geq 2(a+2b)$ for $a^2+2b^2=9$. Since

$$2(ab^2+10)-4(a+2b)=a(9-a^2)+20-4(a+2b)=20+5a-a^3-8b,$$

we need to show that

$$20+5a-a^3\geq 4\sqrt{2(9-a^2)}$$

for $-3\leq a\leq 3$. For $-3\leq a\leq 0$, we have $20+5a-a^3\geq 5(4+a)>0$, and for $0\leq a\leq 3$, we have $20+5a-a^3=4(5-a)+a(9-a^2)>0$. Therefore it suffices to prove that

$$(20+5a-a^3)^2\geq 32(9-a^2).$$

This is true, since it is equivalent to $(a+1)^2f(a)\geq 0$, where

$$\begin{aligned} f(a)&=a^4-2a^3-7a^2-24a+112\\ &=4+12(3-a)+(3-a)^2(a^2+4a+8)>0. \end{aligned}$$

The equality holds for $a=-1$ and $b=c=2$ (or any cyclic permutation).

□

P 2.57. *If a, b, c are real numbers such that*

$$a+b+c+abc=4,$$

then

$$a^2+b^2+c^2+3 \geq 2(ab+bc+ca).$$

(Vasile Cîrtoaje, 2011)

Solution. Without loss of generality, assume that $a \geq b \geq c$. The case $a \leq 0$ is not possible, since it involves $a+b+c+abc \leq 0 < 4$. If $a \geq 0 \geq b \geq c$, then

$$a^2+b^2+c^2+3-2(ab+bc+ca) \geq a^2+(b-c)^2+3 > 0.$$

Also, if $a \geq b \geq 0 \geq c$, then

$$a^2+b^2+c^2+3-2(ab+bc+ca) \geq (a-b)^2+c^2+3 > 0.$$

Consider further that $a \geq b \geq c \geq 0$. Let $p = a+b+c$, $q = ab+bc+ca$ and $r = abc$. We need to show that $p^2+3 \geq 4q$ for $p+r=4$.

First Solution. By Schur's inequality of degree three, we have $p^3+9r \geq 4pq$. Therefore, we get

$$p(p^2+3-4q) \geq p^3+3p-(p^3+9r) = 12(p-3).$$

To complete the proof, we need to show that $p \geq 3$. By virtue of the AM-GM inequality, we get

$$p^3 \geq 27r,$$

$$p^3 \geq 27(4-p),$$

$$(p-3)(p^2+3p+36) \geq 0,$$

$$p \geq 3.$$

The equality holds for $a = b = c = 1$.

Second Solution. For the sake of contradiction, assume that $p^2+3 < 4q$. Then, it suffices to show that $p+r > 4$. According to P 2.53, for fixed p and q, r is minimal when two of a, b, c are equal. Therefore, it suffices to consider that $b = c$; that is, to prove that $a+2b \geq 0$ and $a^2+3 < 4ab$ imply $a+2b+ab^2 > 4$. From $a^2+3 < 4ab$, it follows that a and b have the same sign. In addition, from $a+2b \geq 0$, it follows that $a > 0$ and $b > 0$. Therefore, it suffices to prove that $a+2b+ab^2 > 4$ if $a > 0$ and $b > \frac{a^2+3}{4a}$. Indeed,

$$\begin{aligned} a+2b+ab^2-4 &> a+\frac{a^2+3}{2a}+\frac{(a^2+3)^2}{16a}-4 \\ &= \frac{(a-1)^2(a^2+2a+33)}{16a} \geq 0. \end{aligned}$$

□

P 2.58. *If a,b,c are real numbers such that*

$$ab+bc+ca=3abc,$$

then

$$4(a^2+b^2+c^2)+9 \ge 7(ab+bc+ca).$$

(Vasile Cîrtoaje, 2011)

Solution. If one of a,b,c is 0, then the inequality is trivial. Otherwise, write the inequality in the homogeneous form

$$4(a^2+b^2+c^2)+\frac{81a^2b^2c^2}{(ab+bc+ca)^2} \ge 7(ab+bc+ca),$$

or

$$81a^2b^2c^2 \ge (ab+bc+ca)^2A,$$

where

$$A=7(ab+bc+ca)-4(a^2+b^2+c^2).$$

First Solution (by *Vo Quoc Ba Can*). Consider the nontrivial case $A>0$. Substituting a,b,c by $|a|,|b|,|c|$, respectively, the left side of the inequality remains unchanged, while the right side remains unchanged or increases. Therefore, it suffices to prove the inequality only for $a,b,c>0$ and $A>0$. Assume that $a \ge b \ge c > 0$. There are two cases to consider.

Case 1: $4b^2 \le 3ab+3bc+ca$. Since

$$4(ab+bc+ca)^2A \le \left[\frac{(ab+bc+ca)^2}{b}+bA\right]^2,$$

it suffices to show that

$$18abc \ge \frac{(ab+bc+ca)^2}{b}+bA,$$

which is equivalent to the obvious inequality

$$(a-b)(b-c)(3ab+3bc+ca-4b^2) \ge 0.$$

Case 2: $4b^2 > 3ab+3bc+ca$. Since

$$4(ab+bc+ca)^2A \le \left[\frac{(ab+bc+ca)^2}{a}+aA\right]^2,$$

it suffices to show that

$$18abc \ge \frac{(ab+bc+ca)^2}{a}+aA,$$

which is equivalent to

$$(a-b)(a-c)(4a^2-3ab-bc-3ca) \ge 0.$$

This is true, since

$$4a^2-3ab-bc-3ca = (4b^2-3ab-3bc-ca)+2(a-b)(2a+2b-c) > 0.$$

The equality holds for $a=b=c=1$, and for $a=1/2$ and $b=c=2$ (or any cyclic permutation).

Second Solution. Let $p=a+b+c$, $q=ab+bc+ca$, $r=abc$. Since $A=15q-4p^2$,we need to show that

$$81r^2 \ge q^2(15q-4p^2)$$

for all real numbers a,b,c. For fixed p and q, r^2 is minimal when $r=0$, or when r is either minimal or maximal. For $a=0$, the inequality is true since

$$2(15q-4p^2) = -b^2-c^2-7(b-c)^2 \le 0.$$

According to P 2.53, r is minimal and maximal when two of a,b,c are equal. Therefore, due to symmetry and homogeneity, it suffices to prove the inequality for $b=c=1$. In this case, the inequality can be written as

$$(a-1)^2(4a-1)^2 \ge 0.$$

□

P 2.59. *If a,b,c are real numbers such that $a+b+c=3$, then*

$$(a^2+1)(b^2+1)(c^2+1) \ge (a+1)(b+1)(c+1).$$

(Tran Quoc Anh, 2010)

Solution. Write the inequality as

$$a^2b^2c^2+a^2b^2+b^2c^2+c^2a^2+a^2+b^2+c^2 \ge abc+ab+bc+ca+3.$$

Since $a^2b^2c^2 \ge 2abc-1$, it suffices to prove that

$$abc+a^2b^2+b^2c^2+c^2a^2+a^2+b^2+c^2 \ge ab+bc+ca+4,$$

which is equivalent to

$$5(1-abc) \ge q(3-q),$$

where $q = ab + bc + ca$. For fixed q, according to P 2.53, the product abc is maximal when two of a, b, c are equal. Therefore, due to symmetry, it suffices to prove the inequality for $b = c$. We have $a = 3 - 2b$ and

$$\begin{aligned} 5(1-abc)-q(3-q) &= 5(1-ab^2)-(2ab+b^2)(3-2ab-b^2) \\ &= (b-1)^2(9b^2-8b+5) \geq 0. \end{aligned}$$

The equality holds for $a = b = c = 1$.

□

P 2.60. *Let $f_4(a, b, c)$ be a symmetric homogeneous polynomial of degree four. Prove that the inequality $f_4(a, b, c) \geq 0$ holds for all real numbers a, b, c if and only if $f_4(a, 1, 1) \geq 0$ for all real a.*

Solution. Let $p = a+b+c$, $q = ab+bc+ca$ and $r = abc$. Any symmetric homogeneous polynomial $f_4(a, b, c)$ can be written as

$$f_4(a, b, c) = Apr + Bp^4 + Cp^2q + Dq^2,$$

where A, B, C, D are real constants. For fixed p and q, the linear function $g(r) = Apr + Bp^4 + Cp^2q + Dq^2$ is minimal when r is either minimal or maximal. By P 2.53, r is minimal and maximal when two of a, b, c are equal. Since $f_4(a, b, c)$ is symmetric, homogeneous and satisfies $f_4(-a, -b, -c) = f_4(a, b, c)$, it follows that the inequality $f_4(a, b, c) \geq 0$ holds for all real numbers a, b, c if and only if $f_4(a, 1, 1) \geq 0$ and $f_4(a, 0, 0) \geq 0$ for all real a. Notice that the condition "$f_4(a, 0, 0) \geq 0$ for all real a" is not necessary because it follows from the condition "$f_4(a, 1, 1) \geq 0$ for all real a" as follows:

$$f_4(a, 0, 0) = \lim_{t \to 0} f_4(a, t, t) = \lim_{t \to 0} t^4 f_4(a/t, 1, 1) \geq 0.$$

Remark. Similarly, we can prove the following more general statement, where $f_4(a, b, c)$ is only a symmetric polynomial (homogeneous or non-homogeneous).

- *Let $f_4(a, b, c)$ be a symmetric polynomial function of degree $n = 4$. The inequality $f_4(a, b, c) \geq 0$ holds for all real numbers a, b, c if and only if $f_4(a, b, b) \geq 0$ for all real numbers a and b.*

Notice that a function $f(a, b, c)$ is symmetric if it is unchanged by any permutation of its variables. A function $f(a, b, c)$ is a polynomial function if it is a polynomial in one variable when the other two variables are fixed. In addition, $f(a, b, c)$ is a polynomial of degree n if $f(a, a, a)$ is a polynomial of degree n.

□

P 2.61. *If a, b, c are real numbers, then*

$$10(a^4+b^4+c^4)+64(a^2b^2+b^2c^2+c^2a^2) \ge 33\sum ab(a^2+b^2).$$

(Vasile Cîrtoaje, 2008)

Solution. According to P 2.60, it suffices to prove the required inequality for $b=c=1$, when it becomes

$$5a^4-33a^3+64a^2-33a+9 \ge 0,$$

$$(a-3)^2(5a^2-3a+1) \ge 0.$$

This is true since

$$5a^2-3a+1 = 5(a-\frac{3}{10})^2+\frac{11}{20} > 0.$$

The equality holds for $a/3=b=c$ (or any cyclic permutation).

□

P 2.62. *If a, b, c are real numbers such that $a+b+c=3$, then*

$$3(a^4+b^4+c^4)+33 \ge 14(a^2+b^2+c^2).$$

(Vasile Cîrtoaje, 2009)

First Solution. Write the inequality as $F(a,b,c) \ge 0$, where

$$F(a,b,c) = 3(a^4+b^4+c^4)+33-14(a^2+b^2+c^2).$$

Due to symmetry, we may assume that $a \le b \le c$. Let us denote $x=(b+c)/2$, $x \ge 1$. To prove the desired inequality, we use the mixing variable method. We will show that

$$F(a,b,c) \ge F(a,x,x) \ge 0.$$

We have

$$F(a,b,c)-F(a,x,x) = 3(b^4+b^4-2x^4)-14(b^2+c^2-2x^2)$$

$$= 3[(b^2+c^2)^2-4x^4]+6(x^4-b^2c^2)-14(b^2+c^2-2x^2)$$

$$= (b^2+c^2-2x^2)[3(b^2+c^2+2x^2)-14]+6(x^2-bc)(x^2+bc).$$

Since

$$b^2+c^2-2x^2 = 2(x^2-bc) = \frac{1}{2}(b-c)^2,$$

we get

$$F(a,b,c)-F(a,x,x)=\frac{1}{2}(b-c)^2[3(b^2+c^2+2x^2)-14+3(x^2+bc)]$$
$$=\frac{1}{2}(b-c)^2[3(x^2-bc)+18x^2-14]\geq 0.$$

Also,

$$F(a,x,x)=F(3-2x,x,x)=6(x-1)^2(3x-5)^2\geq 0.$$

This completes the proof. The equality holds for $a=b=c=1$, and for $a=-1/3$ and $b=c=5/3$ (or any cyclic permutation).

Second Solution. Write the inequality in the homogeneous form

$$81(a^4+b^4+c^4)+11(a+b+c)^4\geq 42(a^2+b^2+c^2)(a+b+c)^2.$$

According to P 2.60, it suffices to prove this inequality for $b=c=1$, when it becomes

$$25a^4-40a^3+6a^2+8a+1\geq 0,$$

$$(a-1)^2(5a+1)^2\geq 0.$$

□

P 2.63. *If a,b,c are real numbers such that $a^2+b^2+c^2=3$, then*

$$a^4+b^4+c^4+3(ab+bc+ca)\leq 12.$$

Solution. Write the inequality in the homogeneous form

$$3(a^4+b^4+c^4)+3(ab+bc+ca)(a^2+b^2+c^2)\leq 4(a^2+b^2+c^2)^2.$$

According to P 2.60, it suffices to prove this inequality for $b=c=1$, when it becomes

$$a^4-6a^3+13a^2-12a+4\geq 0,$$

$$(a-1)^2(a-2)^2\geq 0.$$

The equality holds for $a=b=c=\pm 1$, for $a=\sqrt{2}$ and $b=c=\sqrt{2}/2$ (or any cyclic permutation), and $a=-\sqrt{2}$ and $b=c=-\sqrt{2}/2$ (or any cyclic permutation).

□

P 2.64. *Let α, β, γ be real numbers such that*

$$1+\alpha+\beta=2\gamma.$$

The inequality

$$\sum a^4+\alpha\sum a^2b^2+\beta abc\sum a\geq\gamma\sum ab(a^2+b^2)$$

holds for any real numbers a, b, c if and only if

$$1+\alpha\geq\gamma^2.$$

(Vasile Cîrtoaje, 2009)

Solution. Let

$$f_4(a,b,c)=\sum a^4+\alpha\sum a^2b^2+\beta abc\sum a-\gamma\sum ab(a^2+b^2.$$

According to P 2.60, the inequality $f_4(a,b,c)\geq 0$ holds for any real numbers a, b, c if and only if $f_4(a,1,1)\geq 0$ for any real a. From

$$f_4(a,1,1)=(a-1)^2[(a-\gamma+1)^2+1+\alpha-\gamma^2],$$

the conclusion follows. The equality holds for $a=b=c$.

Remark. For $\gamma=k+1$ and $\alpha=k(k+2)$, we get

$$\sum a^4+k(k+2)\sum a^2b^2+(1-k^2)abc\sum a\geq(k+1)\sum ab(a^2+b^2),\quad k\in\mathbb{R},$$

which is equivalent to the elegant inequality from P 2.50, namely

$$\sum(a-b)(a-c)(a-kb)(a-kc)\geq 0,$$

where the equality holds for $a=b=c$, and also for $a/k=b=c$ (or any cyclic permutation). In addition, for $k=0$, we get Schur's inequality of degree four

$$\sum a^2(a-b)(a-c),$$

with equality for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 2.65. *If a, b, c are real numbers such that $a^2+b^2+c^2=2$, then*

$$ab(a^2-ab+b^2-c^2)+bc(b^2-bc+c^2-a^2)+ca(c^2-ca+a^2-b^2)\leq 1.$$

Solution. Write the inequality in the homogeneous form

$$(a^2+b^2+c^2)^2 \ge 4\sum ab(a^2-ab+b^2-c^2).$$

According to P 2.60, it suffices to prove this inequality for $b=c=1$, when it can be written as

$$a^2(a-4)^2 \ge 0.$$

The equality holds for

$$a^2+b^2+c^2 = 2(ab+bc+ca).$$

□

P 2.66. *If a,b,c are real numbers, then*

$$(a+b)^4+(b+c)^4+(c+a)^4 \ge \frac{4}{7}(a^4+b^4+c^4).$$

(Vietnam TST, 1996)

Solution. Denote the left side of the inequality by $f_4(a,b,c)$. According to P 2.60, it suffices to prove that $f_4(a,1,1) \ge 0$ for all real a. Indeed,

$$f_4(a,1,1) = \frac{2}{7}(5a^4+28a^3+42a^2+28a+59) > 0$$

since, for the nontrivial case $a<0$, we have

$$5a^4+28a^3+42a^2+28a+59 = (5a^2-2a)(a+3)^2+9(a+\frac{23}{9})^2+\frac{2}{9} > 0.$$

The equality holds for $a=b=c=0$.

□

P 2.67. *Let a,b,c be real numbers, and let*

$$p=a+b+c, \quad q=ab+bc+ca, \quad r=abc.$$

Prove that

$$(3-p)r+\frac{p^2+q^2-pq}{3} \ge q.$$

(Vasile Cîrtoaje, 2011)

First Solution. Write the inequality as

$$(p^2-3q)+(q^2-3pr)\geq pq-9r,$$

$$\frac{1}{2}\sum(b-c)^2+\frac{1}{2}\sum a^2(b-c)^2\geq\sum a(b-c)^2.$$

According to the AM-GM inequality, it suffices to prove that

$$\sqrt{\left[\sum(b-c)^2\right]\left[\sum a^2(b-c)^2\right]}\geq\sum a(b-c)^2.$$

Clearly, this inequality follows immediately from the Cauchy-Schwarz inequality. The equality holds for $a=b=c$, and for $b=c=1$ (or any cyclic permutation).

Second Solution Write the inequality as $f_4(a,b,c)\geq 0$, where

$$f_4(a,b,c)=3(3-p)r+p^2+q^2-pq-3q.$$

is a symmetric polynomial of degree four in a,b,c. According to Remark from the proof of P 2.60, it suffices to prove that $f_4(a,b,b)\geq 0$ for all real numbers a and b. Indeed, we have

$$f_4(a,b,b)=(a-b)^2(b-1)^2\geq 0.$$

□

P 2.68. *If a,b,c are real numbers, then*

$$\frac{ab(a+b)+bc(b+c)+ca(c+a)}{(a^2+1)(b^2+1)(c^2+1)}\leq\frac{3}{4}.$$

(Vasile Cîrtoaje, 2011)

First Solution. We try to get a stronger homogeneous inequality of third order. According to the AM-GM inequality, we have

$$(a^2+1)(b^2+1)(c^2+1)=(a^2b^2c^2+1)+(a^2b^2+b^2c^2+c^2a^2)+(a^2+b^2+c^2)$$

$$\geq 2abc+2\sqrt{(a^2b^2+b^2c^2+c^2a^2)(a^2+b^2+c^2)}.$$

Therefore, it suffices to prove that

$$3abc+3\sqrt{(a^2b^2+b^2c^2+c^2a^2)(a^2+b^2+c^2)}\geq 2\sum ab(a+b).$$

Indeed, using the identity

$$9(a^2+b^2+c^2)=\sum(2a+2b-c)^2$$

and the Cauchy-Schwarz inequality, we get

$$3\sqrt{(a^2b^2+b^2c^2+c^2a^2)(a^2+b^2+c^2)}=$$

$$=\sqrt{\left[\sum a^2b^2\right]\left[\sum(2a+2b-c)^2\right]}\ge\sum ab(2a+2b-c)$$

$$=2\sum ab(a+b)-3abc.$$

The equality holds for $a=b=c=1$.

Second Solution Since the equality holds for $a=b=c=1$, we write the inequality as

$$3(abc-1)^2+f_4(a,b,c)\ge 0,$$

where

$$f_4(a,b,c)=\sum a^2b^2+\sum a^2+2abc-\frac{4}{3}\sum ab(a+b)$$

is a symmetric polynomial of degree four. Clearly, it suffices to prove that $f_4(a,b,c)\ge 0$. According to Remark from P 2.60, it suffices to prove that $f_4(a,b,b)\ge 0$ for all real numbers a and b. Indeed, we have

$$3f_4(a,b,b)=(6b^2-8b+3)a^2-2b^2a+b^2(3b^2-8b+6)$$

$$=(6b^2-8b+3)\left(a-\frac{b^2}{6b^2-8b+3}\right)^2+\frac{18b^2(b-1)^4}{6b^2-8b+3}\ge 0.$$

Remark. The inequality is equivalent to

$$3(abc-1)^2+\sum(a-1)^2(b-c)^2+(ab+bc+ca-a-b-c)^2\ge 0.$$

□

P 2.69. *If a,b,c are real numbers such that $abc>0$, then*

$$\left(a+\frac{1}{a}-1\right)\left(b+\frac{1}{b}-1\right)\left(c+\frac{1}{c}-1\right)+2\ge\frac{1}{3}(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

(Vasile Cîrtoaje, 2011)

Solution. Let $p=a+b+c$, $q=ab+bc+ca$ and $r=abc$. Multiplying by abc, we can rewrite the inequality as

$$r^2+(4-p-q)r+p^2+q^2-\frac{4pq}{3}-p-q+1\ge 0.$$

Since the equality holds for $a=b=c=1$, that is, for $p=q=3$ and $r=1$, we write the inequality as

$$\left(r-1+\frac{p-q}{2}\right)^2+f(p,q,r)\ge 0,$$

where

$$\begin{aligned}12f(p,q,r)&=24(3-p)r+9(p^2+q^2)-10pq-24q\\&\ge 24(3-p)r+8(p^2+q^2)-8pq-24q\end{aligned}$$

Thus, it suffices to prove that $f_4(a,b,c)\ge 0$, where

$$f_4(a,b,c)=3(3-p)r+p^2+q^2-pq-3q\ge 0.$$

According to Remark from P 2.60, it suffices to prove that $f_4(a,b,b)\ge 0$ for all real numbers a and b. Indeed, we have

$$f_4(a,b,b)=(b-1)^2(a-b)^2\ge 0.$$

The equality holds for $a=b=c=1$.

Remark. The inequalities in P 2.68 and P 2.69 are particular cases of the following more general statement (*Vasile Cîrtoaje*, 2011).

- *Let a,b,c be real numbers such that $abc>0$. If $-2\le k\le 1$, then*

$$\left(a+\frac{1}{a}+k\right)\left(b+\frac{1}{b}+k\right)\left(c+\frac{1}{c}+k\right)+(1-k)(2+k)^2\ge$$

$$\ge\frac{1}{3}(2+k)^2(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

□

P 2.70. *If a,b,c are real numbers, then*

$$\left(a^2+\frac{1}{2}\right)\left(b^2+\frac{1}{2}\right)\left(c^2+\frac{1}{2}\right)\ge\left(a+b-\frac{1}{2}\right)\left(b+c-\frac{1}{2}\right)\left(c+a-\frac{1}{2}\right).$$

(Vasile Cîrtoaje, 2011)

Solution. Since the equality holds for $a=b=c=1$, we write the inequality as

$$\left(abc+\frac{1}{2}-\frac{a+b+c}{2}\right)^2+f_4(a,b,c)\ge 0,$$

where

$$f_4(a,b,c) = \prod\left(a^2+\frac{1}{2}\right) - \prod\left(a+b-\frac{1}{2}\right) - \left(abc+\frac{1}{2}-\frac{a+b+c}{2}\right)^2$$

is a symmetric polynomial of degree four. Clearly, it suffices to prove that $f_4(a,b,c) \geq 0$. According to Remark from P 2.60, it suffices to prove that $f_4(a,b,b) \geq 0$ for all real numbers a and b. Indeed, we have

$$2f_4(a,b,b) = [(2b-1)a - b(2-b)]^2 \geq 0.$$

Remark. The inequality is equivalent to

$$(2abc+1-a-b-c)^2 + 2(ab+bc+ca-a-b-c)^2 \geq 0.$$

□

P 2.71. *If a,b,c are real numbers such that $a+b+c=3$, then*

$$\frac{a(a-1)}{8a^2+9} + \frac{b(b-1)}{8b^2+9} + \frac{c(c-1)}{8c^2+9} \geq 0.$$

(Vasile Cîrtoaje, 2013)

Solution *(by Michael Rozenberg).* We see that the equality holds for $a=b=c$, and for $a=3/2$ and $b=c=3/4$ (or any cyclic permutation). Let k be a positive constant, $k>3$. Write the inequality as

$$\sum \frac{(k^2-8)a(a-1)}{8a^2+9} \geq 0,$$

$$\sum \left[\frac{(k^2-8)a(a-1)}{8a^2+9} + 1\right] \geq 3.$$

Choosing

$$k = 3+\sqrt{17},$$

the inequality can be written as

$$\sum \frac{(ka-3)^2}{8a^2+9} \geq 3.$$

Let m be a real constant. According to the Cauchy-Schwarz inequality, we have

$$\sum \frac{(ka-3)^2}{8a^2+9} \geq \frac{[\sum (ka-3)(ma+3)]^2}{\sum (ma+3)^2(8a^2+9)},$$

with equality for

$$\frac{ka-3}{(8a^2+9)(ma+3)} = \frac{kb-3}{(8b^2+9)(mb+3)} = \frac{kc-3}{(8c^2+9)(mc+3)}.$$

Choosing $m = k$, these conditions are satisfied for $a = 3/2$ and $b = c = 3/4$. Therefore, it suffices to show that

$$[k^2(a^2+b^2+c^2)-27]^2 \ge 3\sum(ka+3)^2(8a^2+9).$$

Write this inequality in the homogeneous form $f_4(a,b,c) \ge 0$, where

$$f_4(a,b,c) = [k^2(a^2+b^2+c^2)-3(a+b+c)^2]^2$$
$$-3\sum(ka+a+b+c)^2[8a^2+(a+b+c)^2].$$

According to P 2.60, it suffices to prove that $f_4(a,1,1) \ge 0$ for all real a. Indeed, this inequality is equivalent to $(a-1)^2(a-2)^2 \ge 0$.

□

P 2.72. *If a,b,c are real numbers such that $a+b+c=3$, then*

$$\frac{(a-11)(a-1)}{2a^2+1} + \frac{(b-11)(b-1)}{2b^2+1} + \frac{(c-11)(c-1)}{2c^2+1} \ge 0.$$

(Vasile Cîrtoaje, 2013)

Solution. Write the inequality as

$$\sum\left[\frac{(a-11)(a-1)}{2a^2+1}+1\right] \ge 3,$$

$$\sum\frac{(a-2)^2}{2a^2+1} \ge 1.$$

According to the Cauchy-Schwarz inequality, we have

$$\sum\frac{(a-2)^2}{2a^2+1} \ge \frac{[\sum(a-2)^2]^2}{\sum(a-2)^2(2a^2+1)},$$

Therefore, it suffices to show that

$$(a^2+b^2+c^2)^2 \ge 2\sum a^4 - 8\sum a^3 + 9\sum a^2 - 4\sum a + 12.$$

Write this inequality in the homogeneous form $f_4(a,b,c) \ge 0$, where

$$f_4(a,b,c) = 3(a^2+b^2+c^2)^2 - 6\sum a^4 + 8\left(\sum a^3\right)\left(\sum a\right) - 3\left(\sum a^2\right)\left(\sum a\right)^2$$
$$= 2\left[\sum a^4 + \sum ab(a^2+b^2) - 3abc\sum a\right].$$

According to P 2.60, it suffices to prove that $f_4(a,1,1) \ge 0$ for all real a. Indeed,

$$f_4(a,1,1) = 2(a-1)^2(a+2)^2 \ge 0.$$

The equality holds for $a = b = c = 1$.

□

P 2.73. *If a,b,c are real numbers, then*

$$(a^2+2)(b^2+2)(c^2+2) \ge 9(ab+bc+ca).$$

(Vasile Cîrtoaje, 1994)

Solution. We will prove the sharper inequality $f_4(a,b,c) \ge 0$, where

$$f_4(a,b,c) = (a^2+2)(b^2+2)(c^2+2) - 9(ab+bc+ca) - \left(abc - \frac{a+b+c}{3}\right)^2.$$

Since $f_4(a,b,c)$ is a symmetric polynomial of degree four, according to Remark from P 2.60, it suffices to prove that $f_4(a,b,b) \ge 0$ for all real numbers a and b. For fixed b, this inequality is equivalent to $f(a) \ge 0$, where

$$f(a) = 7(6b^2+5)a^2 + 2b(6b^2-83)a + 18b^4 - 13b^2 + 72.$$

Clearly, it is true for all real a if and only if

$$7(6b^2+5)(18b^4-13b^2+72) \ge b^2(6b^2-83)^2.$$

Indeed, we have

$$7(6b^2+5)(18b^4-13b^2+72) - b^2(6b^2-83)^2 = 360(b^2-1)^2(2b^2+7) \ge 0.$$

The equality holds for $a = b = c = 1$.

□

P 2.74. *If a,b,c are real numbers such that $ab+bc+ca = 3$, then*

$$4(a^4+b^4+c^4) + 11abc(a+b+c) \ge 45.$$

(Vasile Cîrtoaje, 2014)

Solution. Write the inequality in the homogeneous form

$$4(a^4+b^4+c^4)+11abc(a+b+c)\ge 5(ab+bc+ca)^2.$$

It suffices to prove that there exists a positive number k such that $f_4(a,b,c)\ge 0$, where

$$\begin{aligned} f_4(a,b,c) =& 4(a^4+b^4+c^4)+11abc(a+b+c)-5(ab+bc+ca)^2 \\ & -k(ab+bc+ca)(a^2+b^2+c^2-ab-bc-ca). \end{aligned}$$

According to P 2.60, the inequality $f_4(a,b,c)\ge 0$ holds for all real a,b,c if and only if $f_4(a,1,1)\ge 0$ for all real a. We have

$$\begin{aligned} f_4(a,1,1) &= (a-1)^2(2a+1)(2a+3)-k(2a+1)(a-1)^2 \\ &= (a-1)^2(2a+1)(2a+3-k). \end{aligned}$$

Setting $k=2$, we get

$$f_4(a,1,1)=(a-1)^2(2a+1)^2\ge 0.$$

The equality holds for $a=b=c=\pm 1$.

□

P 2.75. *Any sixth degree symmetric homogeneous polynomial $f_6(a,b,c)$ can be written in the form*

$$f_6(a,b,c)=Ar^2+B(p,q)r+C(p,q),$$

where A is called the highest coefficient of f_6, and

$$p=a+b+c,\qquad q=ab+bc+ca,\qquad r=abc.$$

In the case $A\le 0$, prove that the inequality $f_6(a,b,c)\ge 0$ holds for all real numbers a,b,c if and only if $f_6(a,1,1)\ge 0$ for all real a.

(Vasile Cîrtoaje, 2006)

Solution. For $A\le 0$ and fixed p and q,

$$g(r)=Ar^2+B(p,q)r+C(p,q)$$

is a concave quadratic function of r. Therefore, $g(r)$ is minimal when r is minimal or maximal. By P 2.53, r is minimal and maximal when two of a,b,c are equal. Since $f_6(a,b,c)$ is symmetric, homogeneous and satisfies $f_6(-a,-b,-c)=f_6(a,b,c)$, it follows that the inequality $f_6(a,b,c)\ge 0$ holds for all real numbers a,b,c if and only if $f_6(a,1,1)\ge 0$ and $f_6(a,0,0)\ge 0$ for all real a. Notice that the condition "$f_6(a,0,0)\ge 0$

for all real a" is not necessary because it follows from the condition "$f_6(a,1,1) \geq 0$ for all real a" as follows:

$$f_6(a,0,0) = \lim_{t\to 0} f_6(a,t,t) = \lim_{t\to 0} t^6 f_6(a/t,1,1) \geq 0.$$

Remark 1. In order to write the polynomial $f_6(a,b,c)$ given by (A) as a function of p, q and r, we can use the following relations:

$$\begin{aligned}
\sum a^3 &= 3r + p^3 - 3pq,\\
\sum ab(a+b) &= -3r + pq,\\
\sum a^3b^3 &= 3r^2 - 3pqr + q^3,\\
\sum a^2b^2(a^2+b^2) &= -3r^2 - 2(p^3 - 2pq)r + p^2q^2 - 2q^3,\\
\sum ab(a^4+b^4) &= -3r^2 - 2(p^3 - 7pq)r + p^4q - 4p^2q^2 + 2q^3,\\
\sum a^6 &= 3r^2 + 6(p^3 - 2pq)r + p^6 - 6p^4q + 9p^2q^2 - 2q^3.
\end{aligned}$$

According to these relations, the highest coefficient A of the polynomuial $f_6(a,b,c)$ has the expression

$$(B) \qquad A = 3(A_1 - A_2 - A_3 + A_4 + A_5 - A_6 + A_7).$$

Remark 2. The polynomial

$$P_1(a,b,c) = \sum (A_1a^2 + A_2bc)(B_1a^2 + B_2bc)(C_1a^2 + C_2bc)$$

has the highest coefficient

$$A = 3(A_1 + A_2)(B_1 + B_2)(C_1 + C_2) = P_1(1,1,1).$$

Indeed, since

$$\begin{aligned}
P_1(a,b,c) =& A_1B_1C_1 \sum a^6 + A_2B_2C_2 \sum b^3c^3 + \left(\sum A_1B_1C_2\right) abc \sum a^3\\
&+ 3\left(\sum A_1B_2C_2\right) a^2b^2c^2,
\end{aligned}$$

we have

$$\begin{aligned}
A &= 3A_1B_1C_1 + 3A_2B_2C_2 + 3\sum A_1B_1C_2 + 3\sum A_1B_2C_2\\
&= 3(A_1 + A_2)(B_1 + B_2)(C_1 + C_2).
\end{aligned}$$

Similarly, we can show that the polynomial

$$P_2(a,b,c) = \sum (A_1a^2 + A_2bc)(B_1b^2 + B_2ca)(C_1c^2 + C_2ab)$$

has the highest coefficient

$$A = 3(A_1 + A_2)(B_1 + B_2)(C_1 + C_2) = P_2(1,1,1),$$

and the polynomial

$$P_3(a,b,c) = (A_1a^2 + A_2bc)(A_1b^2 + A_2ca)(A_1c^2 + A_2ab)$$

has the highest coefficient

$$A = (A_1 + A_2)^3 = P_3(1,1,1).$$

With regard to

$$P_4(a,b,c) = (a-b)^2(b-c)^2(c-a)^2,$$

from

$$P_4(a,b,c) = (p^2 - 2q - c^2 - 2ab)(p^2 - 2q - a^2 - 2bc)(p^2 - 2q - b^2 - 2ca),$$

it follows that P_4 has the same highest coefficient as $(-c^2-2ab)(-a^2-2bc)(-b^2-2ca)$; that is,

$$A = (-1-2)^3 = -27.$$

Remark 3. We can extend the statement in P 2.75 as follows:

• *Let $f_6(a,b,c)$ be a sixth degree symmetric homogeneous polynomial having the highest coefficient $A \le 0$, and let k_1, k_2 be two fixed real numbers. The inequality $f_6(a,b,c) \ge 0$ holds for all real numbers a,b,c satisfying*

$$k_1(a+b+c)^2 + k_2(ab+bc+ca) \ge 0,$$

if and only if $f_6(a,1,1) \ge 0$ for all real a satisfying $k_1(a+2)^2 + k_2(2a+1) \ge 0$.

Notice that the condition "$f_6(a,0,0) \ge 0$ for all real a satisfying $k_1a^2 \ge 0$" is not necessary because it follows from the condition "$f_6(a,1,1) \ge 0$ for all real a satisfying $k_1(a+2)^2 + k_2(2a+1) \ge 0$". Indeed, for the non-trivial case $k_1 \ge 0$, when the condition "$f_6(a,0,0) \ge 0$ for all real a satisfying $k_1a^2 \ge 0$" becomes "$f_6(a,0,0) \ge 0$ for all real a", we have

$$f_6(a,0,0) = \lim_{t\to 0} f_6(a,t,t) = \lim_{t\to 0} t^6 f_6(a/t,1,1) \ge 0.$$

Remark 4. The statement in P 2.75 and its extension in Remark 3 are also valid in the more general case when $f_6(a,b,c)$ is a symmetric homogeneous function of the form

$$f_6(a,b,c) = Ar^2 + B(p,q)r + C(p,q),$$

where $B(p,q)$ and $C(p,q)$ are rational functions.

□

P 2.76. *If a, b, c are real numbers such that $ab + bc + ca = -1$, then*

(a) $$5(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) \geq 8;$$

(b) $$(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \geq 1.$$

(Vasile Cîrtoaje, 2011)

Solution. Let $p = a + b + c$ and $q = ab + bc + ca$.

(a) Write the inequality in the homogeneous form $f_6(a, b, c) \geq 0$, where

$$f_6(a, b, c) = 5(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) + 8(ab + bc + ca)^3.$$

From

$$\prod(b^2 + c^2) = \prod(p^2 - 2q - a^2),$$

it follows that $f_6(a, b, c)$ has the highest coefficient $A = -5$. Then, by P 2.75, it suffices to prove that $f_6(a, 1, 1) \geq 0$ for all real a. Indeed, we have

$$f_6(a, 1, 1) = 2(a + 3)^2(5a^2 + 2a + 1) \geq 0.$$

The homogeneous inequality $f_6(a, b, c) \geq 0$ is an equality for $-a/3 = b = c$ (or any cyclic permutation), and for $b = c = 0$ (or any cyclic permutation). The original inequality becomes an equality for $a = -3/\sqrt{5}$ and $b = c = 1/\sqrt{5}$ (or any cyclic permutation), and for $a = 3/\sqrt{5}$ and $b = c = -1/\sqrt{5}$ (or any cyclic permutation)

(b) Write the inequality in the homogeneous form $f_6(a, b, c) \geq 0$, where

$$f_6(a, b, c) = \prod(b^2 + bc + c^2) + (ab + bc + ca)^3.$$

First Solution. From

$$\prod(b^2 + bc + c^2) = \prod(p^2 - 2q + bc - a^2),$$

it follows that $f_6(a, b, c)$ has the same highest coefficient as $P_3(a, b, c)$, where

$$P_3(a, b, c) = \prod(bc - a^2);$$

that is, according to Remark 2 from P 2.75,

$$A = P_3(1, 1, 1) = (1 - 1)^3 = 0.$$

Then, by P 2.75, it suffices to prove that $f_6(a, 1, 1) \geq 0$ for all real a. Indeed, we have

$$f_6(a, 1, 1) = (a + 2)^2(3a^2 + 2a + 1) \geq 0.$$

The homogeneous inequality $f_6(a,b,c) \ge 0$ is an equality when $a+b+c=0$, and when $b=c=0$ (or any cyclic permutation). The original inequality becomes an equality for $ab+bc+ca=-1$ and $a+b+c=0$.

Second Solution. As we have shown in the proof of P 2.35,

$$\prod(b^2+bc+c^2)=(p^2-q)q^2-p^3r,$$

where

$$p=a+b+c, \quad q=ab+bc+ca, \quad r=abc.$$

Therefore,

$$f_6(a,b,c)=p^2(q^2-pr)=\frac{1}{2}p^2\sum a^2(b+c)^2 \ge 0.$$

□

P 2.77. *If a,b,c are real numbers, then*

(a) $$\sum a^2(a-b)(a-c)(a+2b)(a+2c)+(a-b)^2(b-c)^2(c-a)^2 \ge 0;$$

(b) $$\sum a^2(a-b)(a-c)(a-4b)(a-4c)+7(a-b)^2(b-c)^2(c-a)^2 \ge 0.$$

(Vasile Cîrtoaje, 2008)

Solution. Let $p=a+b+c$ and $q=ab+bc+ca$. Consider the more general inequality $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c)=f(a,b,c)+m(a-b)^2(b-c)^2(c-a)^2 \ge 0,$$

$$f(a,b,c)=\sum a^2(a-b)(a-c)(a-kb)(a-kc).$$

Since

$$f(a,b,c)=\sum a^2(a^2+2bc-q)[a^2+(k+k^2)bc-kq],$$

$f(a,b,c)$ has the same highest coefficient as $P_1(a,b,c)$, where

$$P_1(a,b,c)=\sum a^2(a^2+2bc)[a^2+(k+k^2)bc].$$

According to Remark 2 from P 2.75, $f_6(a,b,c)$ has the highest coefficient

$$A=P_1(1,1,1)-27m=9(k^2+k+1-3m).$$

(a) For $k=-2$ and $m=1$, we get $A=0$. Then, by P 2.75, it suffices to prove the original inequality for $b=c=1$; that is,

$$a^2(a-1)^2(a+2)^2 \ge 0.$$

The equality holds for $a = b = c$, for $a + b + c = 0$, and for $a = 0$ and $b = c$ (or any cyclic permutation).

(b) For $k = 4$ and $m = 7$, we get $A = 0$. Then, by P 2.75, it suffices to prove the original inequality for $b = c = 1$; that is,

$$a^2(a-1)^2(a-4)^2 \geq 0.$$

The equality holds for $a = b = c$, and for $a^2 + b^2 + c^2 = 2(ab + bc + ca)$.

Remark. The inequalities in P 2.77 are equivalent to

$$\left(\sum a\right)^2 \left[\sum a^4 + abc\sum a - \sum ab(a^2+b^2)\right] \geq 0$$

and

$$\left(\sum a^2 - \sum ab\right)\left(\sum a^2 - 2\sum ab\right)^2 \geq 0,$$

respectively.

□

P 2.78. *If a, b, c are real numbers, then*

$$(a^2+2bc)(b^2+2ca)(c^2+2ab)+(a-b)^2(b-c)^2(c-a)^2 \geq 0.$$

(Vasile Cîrtoaje, 2011)

First Solution. Denote the left side of the inequality by $f_6(a,b,c)$. According to Remark 2 from P 2.75, $f_6(a,b,c)$ has the highest coefficient

$$A = (1+2)^3 - 27 = 0.$$

Then, by P 2.75, it suffices to prove that $f_6(a,1,1) \geq 0$ for all real a. Indeed,

$$f_6(a,1,1) = (a^2+2)(2a+1)^2 \geq 0.$$

The equality holds for $ab + bc + ca = 0$.

Second Solution (by *Vo Quoc Ba Can*). Without loss of generality, assume that b and c have the same sign. Since

$$\begin{aligned}(a-b)^2(a-c)^2 &= \frac{1}{4}[(a^2+2bc)+(a^2-2ab-2ac)]^2 \\ &\geq (a^2+2bc)(a^2-2ab-2ac)\end{aligned}$$

and $a^2 + 2bc \geq 0$, it suffices to prove that

$$(b^2+2ca)(c^2+2ab)+(b-c)^2(a^2-2ab-2ac) \geq 0.$$

This inequality is equivalent to

$$(b+c)^2a^2+2bc(b+c)a+b^2c^2 \ge 0,$$

or

$$[(b+c)a+bc]^2 \ge 0,$$

which is clearly true. The equality holds for $ab+bc+ca=0$.

Remark 1. The inequality is equivalent to

$$(a^2+b^2+c^2)(ab+bc+ca)^2 \ge 0.$$

Remark 2. The inequality in P 2.78 is a particular case of the following more general statement.

- *If a,b,c are real numbers and*

$$\alpha_k = \begin{cases} \dfrac{9k^2(k^2-k+1)}{4(k+1)^3}, & 1 \le k \le 2 \\ \dfrac{k^2}{4}, & k \ge 2 \end{cases},$$

then

$$(a^2+kbc)(b^2+kca)(c^2+kab)+\alpha_k(a-b)^2(b-c)^2(c-a)^2 \ge 0,$$

with equality for $-ka=b=c$, and for $b=c=0$ (or any cyclic permutation).

□

P 2.79. *If a,b,c are real numbers, then*

$$(2a^2+5ab+2b^2)(2b^2+5bc+2c^2)(2c^2+5ca+2a^2)+(a-b)^2(b-c)^2(c-a)^2 \ge 0.$$

(Vasile Cîrtoaje, 2011)

Solution. Let $p=a+b+c$ and $q=ab+bc+ca$. Write the inequality as $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c)=f(a,b,c)+(a-b)^2(b-c)^2(c-a)^2,$$

$$f(a,b,c)=\prod(2b^2+5bc+2c^2).$$

Since

$$f(a,b,c)=\prod(2p^2-4q+5bc-2a^2),$$

$f(a,b,c)$ has the same highest coefficient as $P_3(a,b,c)$, where

$$P_3(a,b,c)=\prod(5bc-2a^2).$$

Therefore, according to Remark 2 from P 2.75, $f_6(a,b,c)$ has the highest coefficient

$$A = P_3(1,1,1) - 27 = 0.$$

Then, by P 2.75, it suffices to prove that $f_6(a,1,1) \ge 0$ for all real a. Indeed,

$$f_6(a,1,1) = 9(2a^2 + 5a + 2)^2 \ge 0.$$

The equality holds for $a + b + c = 0$, and also for $ab + bc + ca = 0$.

Remark 1. The inequality in P 2.79 is equivalent to

$$(a+b+c)^2(ab+bc+ca)^2 \ge 0.$$

Remark 2. The following more general statement holds.

- *Let a,b,c be real numbers. If $k > -2$, then*

$$4\prod(b^2 + kbc + c^2) \ge (2-k)(a-b)^2(b-c)^2(c-a)^2.$$

Notice that this inequality is equivalent to

$$(k+2)[(a+b+c)(ab+bc+ca) - (5-2k)abc]^2 \ge 0.$$

□

P 2.80. *If a,b,c are real numbers, then*

$$\left(a^2 + \frac{2}{3}ab + b^2\right)\left(b^2 + \frac{2}{3}bc + c^2\right)\left(c^2 + \frac{2}{3}ca + a^2\right) \ge \frac{64}{27}(a^2+bc)(b^2+ca)(c^2+ab).$$

Solution. Let $p = a+b+c$, $q = ab+bc+ca$ and

$$f(a,b,c) = \left(a^2 + \frac{2}{3}ab + b^2\right)\left(b^2 + \frac{2}{3}bc + c^2\right)\left(c^2 + \frac{2}{3}ca + a^2\right).$$

We need to prove that $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c) = f(a,b,c) - \frac{64}{27}(a^2+bc)(b^2+ca)(c^2+ab).$$

Since

$$f(a,b,c) = \left(p^2 - 2q + \frac{2}{3}ab - c^2\right)\left(p^2 - 2q + \frac{2}{3}bc - a^2\right)\left(p^2 - 2q + \frac{2}{3}ca - b^2\right),$$

$f_6(a,b,c)$ has the same highest coefficient as

$$\left(\frac{2}{3}ab-c^2\right)\left(\frac{2}{3}bc-a^2\right)\left(\frac{2}{3}ca-b^2\right)-\frac{64}{27}(a^2+bc)(b^2+ca)(c^2+ab);$$

that is, according to Remark 2 from P 2.75,

$$A=\left(\frac{2}{3}-1\right)^3-\frac{64}{27}(1+1)^3<0.$$

Then, it suffices to prove that $f_6(a,1,1)\ge 0$ for all real a (see P 2.75). Indeed,

$$f_6(a,1,1)=\frac{8}{3}\left(a^2+\frac{2}{3}a+1\right)^2-\frac{64}{27}(a^2+1)(a+1)^2=\frac{8}{27}(a-1)^4\ge 0.$$

The equality holds for $a=b=c$.

□

P 2.81. *If a,b,c are real numbers, then*

$$\sum a^2(a-b)(a-c)\ge \frac{2(a-b)^2(b-c)^2(c-a)^2}{a^2+b^2+c^2}.$$

Solution. Let $p=a+b+c$, $q=ab+bc+ca$, $r=abc$ and

$$f_6(a,b,c)=(a^2+b^2+c^2)\sum a^2(a-b)(a-c)-2(a-b)^2(b-c)^2(c-a)^2.$$

Clearly, $f_6(a,b,c)$ has the highest coefficient

$$A=-2(-27)=54.$$

Since $A>0$, we will use the *highest coefficient cancellation method*. It is easy to check that

$$f(1,1,1)=0,\quad f(0,1,1)=0.$$

Therefore, we define the symmetric homogeneous polynomial of degree three

$$P(a,b,c)=r+Bp^3+Cpq$$

such that $P(1,1,1)=0$ and $P(0,1,1)=0$; that is,

$$P(a,b,c)=r+\frac{1}{9}p^3-\frac{4}{9}pq.$$

We will prove the sharper inequality $g_6(a,b,c)\ge 0$, where

$$g_6(a,b,c)=f_6(a,b,c)-54P^2(a,b,c).$$

Since $g_6(a,b,c)$ has the highest coefficient $A_1 = 0$, it suffices to show that $g_6(a,1,1) \ge 0$ for all real a (see P 2.75). Indeed, we have

$$f_6(a,1,1) = a^2(a^2+2)(a-1)^2, \quad P(a,1,1) = \frac{1}{81}a(a-1)^2,$$

hence

$$g_6(a,1,1) = f_6(a,1,1) - 54P(a,1,1) = \frac{1}{3}a^2(a-1)^2(a+2)^2 \ge 0.$$

The equality holds for $a = b = c$, for $a = 0$ and $b = c$ (or any cyclic permutation), and also for $a = 0$ and $b + c = 0$ (or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization (*Vasile Cîrtoaje, 2014*).

- *Let x, y, z be real numbers. If $k \in [-1, 2)$, then*

$$\sum x^2(x-y)(x-z) \ge \frac{(2-k)(x-y)^2(y-z)^2(z-x)^2}{x^2+y^2+z^2+k(xy+yz+zx)},$$

with equality for $x = y = z$, and for $x = 0$ and $y^2 = z^2$ (or any cyclic permutation).

□

P 2.82. *If a, b, c are real numbers, then*

$$\sum (a-b)(a-c)(a-2b)(a-2c) \ge \frac{8(a-b)^2(b-c)^2(c-a)^2}{a^2+b^2+c^2}.$$

Solution. Let

$$f_6(a,b,c) = (a^2+b^2+c^2)\sum (a-b)(a-c)(a-2b)(a-2c) - 8(a-b)^2(b-c)^2(c-a)^2.$$

Clearly, $f_6(a,b,c)$ has the highest coefficient

$$A = (-8)(-27) = 216.$$

Since $A > 0$, we will use the *highest coefficient cancellation method*. Since

$$f(1,1,1) = 0, \quad f(2,1,1) = 0,$$

we define the symmetric homogeneous polynomial of degree three

$$P(a,b,c) = abc + B(a+b+c)^3 + C(a+b+c)(ab+bc+ca)$$

such that $P(1,1,1) = 0$ and $P(2,1,1) = 0$. We get $B = 1/18$ and $C = -5/18$, hence

$$P(a,b,c) = abc + \frac{1}{18}(a+b+c)^3 - \frac{5}{18}(a+b+c)(ab+bc+ca).$$

Consider now the sharper inequality $g_6(a,b,c) \geq 0$, where

$$g_6(a,b,c) = f_6(a,b,c) - 216P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient $A_1 = 0$. By P 2.75, it suffices to prove that $g_6(a,1,1) \geq 0$ for all real a. We have

$$f_6(a,1,1) = (a^2+2)(a-1)^2(a-2)^2, \quad P(a,1,1) = \frac{1}{18}(a-1)^2(a-2),$$

hence

$$g_6(a,1,1) = f_6(a,1,1) - 216P^2(a,1,1) = \frac{1}{3}(a-1)^2(a^2-4)^2 \geq 0.$$

The equality holds for $a = b = c$, for $a = 0$ and $b + c = 0$ (or any cyclic permutation), and also for $a/2 = b = c$ (or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization (*Vasile Cîrtoaje, 2014*).

- *Let x, y, z be real numbers. If $k \in \mathbb{R}$, then*

$$\sum (x-y)(x-z)(x-ky)(x-kz) \geq \frac{(k+2)^2(x-y)^2(y-z)^2(z-x)^2}{2(x^2+y^2+z^2)}, \tag{2.1}$$

with equality for $x = y = z$, for $x/k = y = z$ (or any cyclic permutation) if $k \neq 0$, and for $x = 0$ and $y + z = 0$ (or any cyclic permutation).

□

P 2.83. *If a, b, c are real numbers, no two of which are zero, then*

$$\frac{a^2+3bc}{b^2+c^2} + \frac{b^2+3ca}{c^2+a^2} + \frac{c^2+3ab}{a^2+b^2} \geq 0.$$

(Vasile Cîrtoaje, 2014)

Solution. Write the inequality as $f_6(a,b,c) \geq 0$, where

$$f_6(a,b,c) = \sum (a^2+3bc)(a^2+b^2)(a^2+c^2).$$

Let

$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc.$$

From
$$f_6(a,b,c) = \sum(a^2+3bc)(p^2-2q-c^2)(p^2-2q-b^2),$$
it follows that $f_6(a,b,c)$ has the same highest coefficient A as $f(a,b,c)$, where
$$f(a,b,c) = \sum(a^2+3bc)b^2c^2 = 3r^2+3\sum b^3c^3 = 12r^2-9pqr+3q^3;$$
that is, $A=12$. Since $A>0$, we will use the *highest coefficient cancellation method*. It is easy to check that
$$f_6(-1,1,1) = 0.$$
So, we define the homogeneous polynomial
$$P(a,b,c) = r+Bp^3+(B-1)pq,$$
which satisfies the property $P(-1,1,1)=0$. We will show that there is at least a real value of B such that the following sharper inequality holds
$$f_6(a,b,c) \ge 12P^2(a,b,c).$$
Let us denote
$$g_6(a,b,c) = f_6(a,b,c)-12P^2(a,b,c).$$
Clearly, $g_6(a,b,c)$ has the highest coefficient $A_1=0$. By P 2.75, it suffices to prove that $g_6(a,1,1) \ge 0$ for all real a. We have
$$f_6(a,1,1) = (a+1)^2(a^2+1)(a^2-2a+7)$$
and
$$P(a,1,1) = (a+1)[B(a+2)(a+5)-2(a+1)],$$
hence
$$g_6(a,1,1) = f_6(a,1,1)-12P^2(a,1,1) = (a+1)^2 g(a),$$
where
$$g(a) = (a^2+1)(a^2-2a+7)-12[B(a+2)(a+5)-2(a+1)]^2.$$
Choosing $B=1/4$, we get
$$4g(a) = a^2(a-1)^2+4(4a^2+a+4) > 0,$$
hence $g_6(a,1,1) \ge 0$ for all real a. The proof is completed. The equality holds for $-a=b=c$ (or any cyclic permutation).

□

P 2.84. *If a,b,c are real numbers, no two of which are zero, then*

$$\frac{a^2+6bc}{b^2-bc+c^2}+\frac{b^2+6ca}{c^2-ca+a^2}+\frac{c^2+6ab}{a^2-ab+b^2}\ge 0.$$

(Vasile Cîrtoaje, 2014)

Solution. Write the inequality as $f_6(a,b,c)\ge 0$, where

$$f_6(a,b,c)=\sum(a^2+6bc)(a^2-ab+b^2)(a^2-ac+c^2).$$

Let

$$p=a+b+c,\quad q=ab+bc+ca,\quad r=abc.$$

From

$$f_6(a,b,c)=\sum(a^2+6bc)(p^2-2q-c^2-ab)(p^2-2q-b^2-ac),$$

it follows that $f_6(a,b,c)$ has the same highest coefficient A as $f(a,b,c)$, where

$$f(a,b,c)=\sum(a^2+6bc)(b^2+ca)(c^2+ab);$$

that is, according to Remark 2 from P 2.75,

$$A=f(1,1,1)=84.$$

Since $A>0$, we use the *highest coefficient cancellation method*. We will show that there are two real numbers B and C such that the following sharper inequality holds

$$f_6(a,b,c)\ge 84P^2(a,b,c),$$

where

$$P(a,b,c)=r+Bp^3+Cpq.$$

Let us denote

$$g_6(a,b,c)=f_6(a,b,c)-84P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient equal to zero. Then, it suffices to prove that $g_6(a,1,1)\ge 0$ for all real a.

We have

$$g_6(a,1,1)=f_6(a,1,1)-84P^2(a,1,1),$$

where

$$f_6(a,1,1)=(a^2-a+1)(a^2+a+1)(a^2-2a+8)$$

and

$$P(a,1,1)=a+B(a+2)^3+C(a+2)(2a+1).$$

Let us denote $g(a)=g_6(a,1,1)$. Since $g(-2)=0$, we can have $g(a)\ge 0$ in the vicinity of $a=-2$ only if $g'(-2)=0$, which involves $C=-61/168$. On the other hand, from $g(1) = 0$, we get $B = 155/1512$. Using these values of B and C, the inequality $g_6(a,1,1) \geq 0$ is equivalent to

$$27216(a^2 - a + 1)(a^2 + a + 1)(a^2 - 2a + 8) \geq$$

$$\geq \left[155(a+2)^3 - 549(a+2)(2a+1) + 1512a\right]^2;$$

that is,

$$(a+2)^2(a-1)^2(3191a^2 - 8734a + 49391) \geq 0,$$

which is true for all real a.

The proof is completed. The equality holds for $a = b+c = 0$ (or any cyclic permutation).

□

P 2.85. *If a, b, c are real numbers such that $ab + bc + ca \geq 0$, then*

$$\frac{4a^2 + 23bc}{b^2 + c^2} + \frac{4b^2 + 23ca}{c^2 + a^2} + \frac{4c^2 + 23ab}{a^2 + b^2} \geq 0.$$

(Vasile Cîrtoaje, 2014)

Solution. Write the inequality as $f_6(a,b,c) \geq 0$, where

$$f_6(a,b,c) = \sum (4a^2 + 23bc)(a^2 + b^2)(a^2 + c^2).$$

Let

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

From

$$f_6(a,b,c) = \sum (4a^2 + 23bc)(p^2 - 2q - c^2)(p^2 - 2q - b^2),$$

it follows that $f_6(a,b,c)$ has the same highest coefficient A as $f(a,b,c)$, where

$$f(a,b,c) = \sum (4a^2 + 23bc)b^2c^2 = 12r^2 + 23\sum b^3c^3 = 81r^2 - 69pqr + 23q^3;$$

that is, $A = 81$. Since $A > 0$, we will use the *highest coefficient cancellation method*. It is easy to check that

$$f(-1,2,2) = 0.$$

Therefore, define the homogeneous polynomial

$$P(a,b,c) = r + \frac{4}{27}p^3 + Cpq,$$

which satisfies the property $P(-1,2,2) = 0$. We will show that there is at least a real C such that the following sharper inequality holds for $ab + bc + ca \geq 0$:

$$f_6(a,b,c) \geq 81P^2(a,b,c).$$

Let us denote

$$g_6(a,b,c) = f_6(a,b,c) - 81P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient $A_1 = 0$. Then, by Remark 3 from P 2.75, it suffices to prove that $g_6(a,1,1) \ge 0$ for all real a such that $2a+1 \ge 0$.
We have

$$f_6(a,1,1) = (2a+1)(a^2+1)(2a^3 - a^2 + 14a + 39),$$

$$P(a,1,1) = \frac{1}{27}(2a+1)[2a^2 + (27C+11)a + 54C + 32],$$

$$g_6(a,1,1) = f_6(a,1,1) - 81P^2(a,1,1).$$

From the condition $g_6(1,1,1) = 0$, we get $C = -1/3$. For this value of C, we find

$$P(a,1,1) = \frac{2}{27}(2a+1)(a^2+a+7),$$

then

$$\begin{aligned} g_6(a,1,1) &= \frac{1}{9}(2a+1)(10a^5 - 29a^4 + 16a^3 + 170a^2 - 322a + 155) \\ &= \frac{1}{9}(2a+1)(a-1)^2(10a^3 - 9a^2 - 12a + 155). \end{aligned}$$

We need to show that $10a^3 - 9a^2 - 12a + 155 \ge 0$ for $a \ge -1/2$. This is clearly true for $-1/2 \le a \le 0$. Also, for $a > 0$, we have

$$10a^3 - 9a^2 - 12a + 155 = 10a(a^2 - a + 1) + (a-11)^2 + 34 > 0.$$

The proof is completed. The equality holds for $-2a = b = c$ (or any cyclic permutation).
□

P 2.86. *If a,b,c are real numbers such that $ab+bc+ca = 3$, then*

$$20(a^6+b^6+c^6) + 43abc(a^3+b^3+c^3) \ge 189.$$

(Vasile Cîrtoaje, 2014)

Solution. Write the inequality in the homogeneous form $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c) = 20(a^6+b^6+c^6) + 43abc(a^3+b^3+c^3) - 7(ab+bc+ca)^3.$$

Since the highest coefficient of $f_6(a,b,c)$ is positive, namely

$$A = 20 \cdot 3 + 43 \cdot 3 = 189,$$

we will use the *highest coefficient cancellation method*. From

$$f_6(a,1,1) = (2a+1)(a-1)^2(10a^3+15a^2+44a+33),$$

it follows that

$$f_6(1,1,1) = 0, \quad f_6(-1/2,1,1) = 0.$$

Define the homogeneous function

$$P(a,b,c) = r + Bp^3 + Cpq, \qquad p = a+b+c, \ q = ab+bc+ca, \ r = abc,$$

such that $P(1,1,1) = P(-1/2,1,1) = 0$; that is,

$$P(a,b,c) = r + \frac{4}{27}p^3 - \frac{5}{9}pq,$$

hence

$$P(a,1,1) = \frac{27a+4(a+2)^3-15(a+2)(2a+1)}{27} = \frac{2(a-1)^2(2a+1)}{27}.$$

We will show that the following sharper inequality holds for $ab+bc+ca \ge 0$:

$$f_6(a,b,c) \ge 189P^2(a,b,c).$$

Let us denote

$$g_6(a,b,c) = f_6(a,b,c) - 189P^2(a,b,c).$$

Since the highest coefficient of $g_6(a,b,c)$ is zero, it suffices to prove that $g_6(a,1,1) \ge 0$ for all real a such that $2a+1 \ge 0$ (see Remark 3 from P 2.75). We have

$$g_6(a,1,1) = f_6(a,1,1) - 189P^2(a,1,1) = (2a+1)(a-1)^2 g(a),$$

where

$$g(a) = 10a^3+15a^2+44a+33 - \frac{28}{27}(a-1)^2(2a+1).$$

Since

$$g(a) \ge 10a^3+15a^2+44a+33-5(a-1)^2(2a+1) = 22(a+1)^2+8a^2+6 > 0,$$

we have $g_6(a,1,1) \ge 0$ for all $a \ge -1/2$. Thus, the proof is completed. The equality holds for $a=b=c=1$.

□

P 2.87. *If a,b,c are real numbers, then*

$$4\sum(a^2+bc)(a-b)(a-c)(a-3b)(a-3c) \ge 7(a-b)^2(b-c)^2(c-a)^2.$$

(Vasile Cîrtoaje, 2014)

Solution. Write the inequality as $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c) = 4f(a,b,c) - 7(a-b)^2(b-c)^2(c-a)^2,$$

$$f(a,b,c) = \sum (a^2+bc)(a-b)(a-c)(a-3b)(a-3c).$$

We have

$$f_6(a,1,1) = 4(a^2+1)(a-1)^2(a-3)^2.$$

Let

$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc.$$

Since $(a-b)(a-c) = a^2+2bc-q$ and $(a-3b)(a-3c) = a^2+12bc-3q$, $f(a,b,c)$ has the same highest coefficient A_0 as $g(a,b,c)$, where

$$g(a,b,c) = \sum (a^2+bc)(a^2+2bc)[a^2+12bc];$$

that is, according to Remark 2 from P 2.75,

$$A_0 = g(1,1,1) = 3 \cdot 2 \cdot 3 \cdot 13 = 234.$$

Therefore, $f_6(a,b,c)$ has the highest coefficient

$$A = 4A_0 - 7(-27) = 1125.$$

Since the highest coefficient A is positive, we will use the *highest coefficient cancellation method*. There are two cases to consider: $q \ge 0$ and $q < 0$.

Case 1: $q \ge 0$. Since

$$f_6(1,1,1) = f_6(3,1,1) = 0,$$

define the homogeneous function

$$P(a,b,c) = r + Bp^3 + Cpq$$

such that $P(1,1,1) = P(3,1,1) = 0$; that is,

$$P(a,b,c) = r + \frac{2}{45}p^3 - \frac{11}{45}pq,$$

hence

$$P(a,1,1) = \frac{45a + 2(a+2)^3 - 11(a+2)(2a+1)}{45} = \frac{2(a-1)^2(a-3)}{45}.$$

We will show that the following sharper inequality holds for $q \ge 0$:

$$f_6(a,b,c) \ge 1125P^2(a,b,c).$$

Let us denote

$$g_6(a,b,c) = f_6(a,b,c) - 1125P^2(a,b,c).$$

Since the highest coefficient of $g_6(a,b,c)$ is zero, it suffices to prove that $g_6(a,1,1) \geq 0$ for all real a such that $2a+1 \geq 0$ (see Remark 3 from P 2.75). We have

$$g_6(a,1,1) = f_6(a,1,1) - 1125P^2(a,1,1) = \frac{8(a-1)^2(a-3)^2(a+2)(2a+1)}{9} \geq 0.$$

Case 2: $q < 0$. Define the homogeneous polynomial

$$P(a,b,c) = r + Bp^3 - \left(3B + \frac{1}{9}\right)pq,$$

which satisfies $P(1,1,1) = 0$. We will show that there is a real number B such that the following sharper inequality holds

$$f_6(a,b,c) \geq 1125P^2(a,b,c).$$

Let us denote

$$g_6(a,b,c) = f_6(a,b,c) - 1125P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient equal to zero. Then, by Remark 3 from P 2.75, it suffices to prove that $g_6(a,1,1) \geq 0$ for $2a+1 < 0$. We have

$$g_6(a,1,1) = f_6(a,1,1) - 1125P^2(a,1,1),$$

where

$$P^2(a,1,1) = \left[a + B(a+2)^3 - \left(3B + \frac{1}{9}\right)(a+2)(2a+1)\right]^2.$$

Let us denote $g(a) = g_6(a,1,1)$. Since $g(-2) = 0$, we can have $g(a) \geq 0$ in the vicinity of $a = -2$ only if $g'(-2) = 0$, which involves $B = 8/135$. Using this value of B, we get

$$P^2(a,1,1) = \frac{4(a-1)^4(4a-7)^2}{25 \cdot 729},$$

$$g_6(a,1,1) = 4(a-1)^2\left[(a^2+1)(a-3)^2 - \frac{5}{81}(a-1)^2(4a-7)^2\right]$$
$$= \frac{4}{81}(a-1)^2(a+2)^2(a^2-50a+121) \geq 0.$$

The proof is completed. The equality holds for $a = b = c$, for $a/3 = b = c$ (or any cyclic permutation), and for $a = 0$ and $b + c = 0$ (or any cyclic permutation).

□

P 2.88. *Let a,b,c be real numbers such that $ab+bc+ca \ge 0$. For any real k, prove that*

$$\sum 4bc(a-b)(a-c)(a-kb)(a-kc)+(a-b)^2(b-c)^2(c-a)^2 \ge 0.$$

(Vasile Cîrtoaje, 2014)

Solution. Write the inequality as $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c) = 4f(a,b,c)+(a-b)^2(b-c)^2(c-a)^2,$$

$$f(a,b,c) = \sum bc(a-b)(a-c)(a-kb)(a-kc), \quad k \in \mathbb{R}.$$

Let

$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc.$$

Since $(a-b)(a-c) = a^2+2bc-q$ and $(a-kb)(a-kc) = a^2+(k+k^2)bc-kq$, $f(a,b,c)$ has the same highest coefficient A_0 as $P_1(a,b,c)$, where

$$P_1(a,b,c) = \sum bc(a^2+2bc)[a^2+(k+k^2)bc];$$

that is, according to Remark 2 from P 2.75,

$$A_0 = P_1(1,1,1) = 3(1+2)(1+k+k^2) = 9(1+k+k^2).$$

Therefore, $f_6(a,b,c)$ has the highest coefficient

$$A = 4A_0 - 27 = 9(2k+1)^2.$$

We have

$$f_6(a,1,1) = 4(a-1)^2(a-k)^2.$$

Consider first that $k=-1/2$, when $A=0$. By P 2.75, it suffices to prove that $f_6(a,1,1) \ge 0$ for all real a. Clearly, these conditions are fulfilled. Consider further that $k \ne -1/2$, when the highest coefficient A is positive. We will use the *highest coefficient cancellation method*. Since

$$f_6(1,1,1) = f_6(k,1,1) = 0,$$

define the homogeneous function

$$P(a,b,c) = r + Cpq + D\frac{q^2}{p}$$

such that $P(1,1,1) = P(k,1,1) = 0$; that is,

$$P(a,b,c) = r + \frac{pq}{3(2k+1)} - \frac{2(k+2)q^2}{3(2k+1)p}.$$

We will show that the following sharper inequality holds for $ab+bc+ca \ge 0$:

$$f_6(a,b,c) \ge 9(2k+1)^2 P^2(a,b,c).$$

Let us denote

$$g_6(a,b,c) = f_6(a,b,c) - 9(2k+1)^2 P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient $A_1 = 0$. Then, by Remark 4 from P 2.75, it suffices to prove that $g_6(a,1,1) \ge 0$ for all real a such that $2a+1 \ge 0$. We have

$$\begin{aligned} P(a,1,1) &= a + \frac{(a+2)(2a+1)}{3(2k+1)} - \frac{2(k+2)(2a+1)^2}{3(2k+1)(a+2)} \\ &= \frac{2(a-1)^2(a-k)}{3(2k+1)(a+2)}, \end{aligned}$$

then

$$\begin{aligned} g_6(a,1,1) &= f_6(a,1,1) - 9(2k+1)^2 P^2(a,1,1) \\ &= \frac{12(a-1)^2(a-k)^2(2a+1)}{(a+2)^2} \ge 0. \end{aligned}$$

The proof is completed. The equality holds for $a=b=c$, for $a/k=b=c$ (or any cyclic permutation) - if $k \ne 0$, and for $b=c=0$ (or any cyclic permutation).

□

P 2.89. *If a,b,c are real numbers, then*

$$\left[(a^2b+b^2c+c^2a)+(ab^2+bc^2+ca^2)\right]^2 \ge 4(ab+bc+ca)(a^2b^2+b^2c^2+c^2a^2).$$

First Solution. Consider the nontrivial case $ab+bc+ca \ge 0$, and write the inequality as $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c) = [(a^2b+b^2c+c^2a)+(ab^2+bc^2+ca^2)]^2 - 4(ab+bc+ca)(a^2b^2+b^2c^2+c^2a^2).$$

Since

$$(a^2b+b^2c+c^2a)+(ab^2+bc^2+ca^2) = (a+b+c)(ab+bc+ca) - 3abc,$$

f_6 has the highest coefficient $A = (-3)^2 = 9$. Since $A > 0$, we will use the *highest coefficient cancellation method*. Because

$$f(1,1,1) = f(0,1,1) = 0,$$

define the homogeneous function

$$P(a,b,c)=abc+C(a+b+c)(ab+bc+ca)+D\frac{(ab+bc+ca)^2}{a+b+c}$$

such that $P(1,1,1)=P(0,1,1)=0$; that is,

$$P(a,b,c)=abc+\frac{(a+b+c)(ab+bc+ca)}{3}-\frac{4(ab+bc+ca)^2}{3(a+b+c)}.$$

We will show that the following sharper inequality holds for $ab+bc+ca\ge 0$:

$$f_6(a,b,c)\ge 9P^2(a,b,c).$$

Let us denote

$$g_6(a,b,c)=f_6(a,b,c)-9P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient $A_1=0$. Then, by Remark 4 from P 2.75, it suffices to prove that $g_6(a,1,1)\ge 0$ for all real a such that $2a+1\ge 0$. We have

$$f_6(a,1,1)=4a^2(a-1)^2,$$

$$P(a,1,1)=a+\frac{(a+2)(2a+1)}{3}-\frac{4(2a+1)^2}{3(a+2)}=\frac{2a(a-1)^2}{3(a+2)},$$

hence

$$f_6(a,1,1)=f_6(a,1,1)-9P^2(a,1,1)=\frac{12a^2(a-1)^2(2a+1)}{(a+2)^2}\ge 0.$$

The proof is completed. The equality holds for $a=b=c$, for $a=0$ and $b=c$ (or any cyclic permutation), and for $b=c=0$ (or any cyclic permutation).

Second Solution (by *Nguyen Van Quy*). Since the inequality remains unchanged by replacing a,b,c with $-a,-b,-c$, we may assume that $a+b+c\ge 0$. In addition, consider the non-trivial case $ab+bc+ca>0$. Using the Cauchy-Schwarz inequality, we have

$$(a^2b+b^2c+c^2a)+(ab^2+bc^2+ca^2)+(a^3+b^3+c^3)=(a^2+b^2+c^2)(a+b+c)$$

$$=\sqrt{\left[a^4+b^4+c^4+2(a^2b^2+b^2c^2+c^2a^2)\right]\left[a^2+b^2+c^2+2(ab+bc+ca)\right]}$$

$$\ge\sqrt{(a^4+b^4+c^4)(a^2+b^2+c^2)}+2\sqrt{(a^2b^2+b^2c^2+c^2a^2)(ab+bc+ca)}.$$

Thus, it suffices to show that

$$\sqrt{(a^4+b^4+c^4)(a^2+b^2+c^2)}\ge a^3+b^3+c^3,$$

which follows also from the Cauchy-Schwarz inequality.

□

P 2.90. *If a,b,c are real numbers such that $a+b+c=3$, then*

$$\frac{(a-1)(a-25)}{a^2+23}+\frac{(b-1)(b-25)}{b^2+23}+\frac{(c-1)(c-25)}{c^2+23}\ge 0.$$

Solution. Let $p=a+b+c$. Write the inequality in the homogeneous form $f_6(a,b,c)\ge 0$, where

$$f_6(a,b,c)=\sum(3a-p)(3a-25p)(9b^2+23p^2)(9c^2+23p^2).$$

Since the highest coefficient of f_6 is positive, namely

$$A=3\cdot 9^3,$$

we use the *highest coefficient cancellation method*. Thus, we will prove that there exist two real numbers B and C such that $g_6(a,b,c)\ge 0$, where

$$g_6(a,b,c)=f_6(a,b,c)-A[abc+B(a+b+c)^3+C(a+b+c)(ab+bc+ca)]^2.$$

Since g_6 has the highest coefficient equal to zero, it suffices to show that $g_6(a,1,1)\ge 0$ for all real a. Notice that

$$f_6(a,1,1)=12(a-1)^2(7a+11)^2[23(a+2)^2+9]$$

and

$$g_6(a,1,1)=f_6(a,1,1)-3\cdot 9^3[a+B(a+2)^3+C(a+2)(2a+1)]^2.$$

Let us denote $g(a)=g_6(a,1,1)$. Since $g(-2)=0$, we can have $g(a)\ge 0$ in the vicinity of $a=-2$ only if $g'(-2)=0$; this involves $C=-13/9$, hence

$$g_6(a,1,1)=12(a-1)^2(7a+11)^2[23(a+2)^2+9]-27[9a+9B(a+2)^3-13(a+2)(2a+1)]^2.$$

There are two cases to consider: $5p^2+q\le 0$ and $5p^2+q\ge 0$.

Case 1: $5p^2+q\le 0$. By Remark 3 from the proof of P 2.75, we only need to show that there exist a real number B such that $g_6(a,1,1)\ge 0$ for all real a satisfying $5(a+2)^2+2a+1\le 0$; that is, for $a\in[-3,-7/5]$. From $g_6(-11/7,1,1)=0$, we get $B=28/9$, then

$$\begin{aligned}g_6(a,1,1)&=12(a-1)^2(7a+11)^2[23(a+2)^2+9]-108(7a+11)^2(2a^2+7a+9)^2\\&=-12(a+2)^2(7a+11)^2(13a^2+154a+157)\ge 0.\end{aligned}$$

Case 2: $5p^2+q\ge 0$. By Remark 3 from the proof of P 2.75, we only need to show that there exist a real number B such that $g_6(a,1,1)\ge 0$ for all real a satisfying $5(a+$

$2)^2+2a+1\ge 0$; that is, for $a\in(-\infty,-3]\cup[-7/5,\infty)$. From $g_6(1,1,1)=0$, we get $B=4/9$, then

$$\begin{aligned}g_6(a,1,1)&=12(a-1)^2(7a+11)^2[23(a+2)^2+9]-108(a-1)^4(2a+3)^2\\&=12(a+2)^2(a-1)^2(1091a^2+3650a+3035)\ge 0.\end{aligned}$$

The proof is completed. The equality holds for $a=b=c=1$, and for $a=-11$ and $b=c=7$ (or any cyclic permutation).

□

P 2.91. *If a,b,c are real numbers such that $abc\ne 0$, then*

$$\left(\frac{b+c}{a}\right)^2+\left(\frac{c+a}{b}\right)^2+\left(\frac{a+b}{c}\right)^2>2.$$

(Michael Rozenberg, 2014)

Solution. Assume that $a^2=\min\{a^2,b^2,c^2\}$. By the Cauchy-Schwarz inequality, we have

$$\left(\frac{c+a}{b}\right)^2+\left(\frac{a+b}{c}\right)^2\ge\frac{[(c+a)+(-a-b)]^2}{b^2+c^2}=\frac{(b-c)^2}{b^2+c^2}.$$

On the other hand,

$$\left(\frac{b+c}{a}\right)^2\ge\frac{(b+c)^2}{b^2+c^2}.$$

Therefore,

$$\left(\frac{b+c}{a}\right)^2+\left(\frac{c+a}{b}\right)^2+\left(\frac{a+b}{c}\right)^2\ge\frac{(b+c)^2}{b^2+c^2}+\frac{(b-c)^2}{b^2+c^2}=2.$$

The equality holds if and only if

$$\left(\frac{c+a}{b}\right)^2+\left(\frac{a+b}{c}\right)^2=\frac{(b-c)^2}{b^2+c^2}$$

and $b+c=0$. Since these relations involves $a=0$, we conclude that the inequality is strict (the equality does not hold).

□

P 2.92. *If a,b,c are real numbers, then*

(a) $$(a^2+1)(b^2+1)(c^2+1)\ge\frac{8}{3\sqrt3}\,|(a-b)(b-c)(c-a)|;$$

(b) $$(a^2-a+1)(b^2-b+1)(c^2-c+1)\ge|(a-b)(b-c)(c-a)|.$$

(Kwon Ji Mun, 2011)

Solution. (a) ***First Solution.*** Without loss of generality, assume that $a \le b \le c$, when

$$|(a-b)(b-c)(c-a)| = (a-b)(b-c)(c-a).$$

Denote

$$k = \frac{4}{3\sqrt{3}}$$

and write the inequality as

$$Aa^2 + 2Ba + C \ge 0,$$

where

$$A = (b^2+1)(c^2+1) + 2k(b-c),$$

$$B = -k(b^2-c^2),$$

$$C = (b^2+1)(c^2+1) + 2kbc(b-c).$$

Substituting $b = \dfrac{-x}{\sqrt{3}}$ and $c = \dfrac{y}{\sqrt{3}}$, by the Cauchy-Schwarz inequality, we get

$$9A = (x^2+1+2)(1+y^2+2) - 8(x+y) \ge (x+y+2)^2 - 8(x+y) = (x+y-2)^2 \ge 0.$$

We have $A = 0$ only for $b = -1/\sqrt{3}$, $c = 1/\sqrt{3}$, when $Aa^2 + Ba + C = 64/27$. Otherwise, for $A > 0$, it suffices to prove that $AC - B^2 \ge 0$. Let us denote

$$E = b - c, \quad F = bc + 1.$$

Since

$$(b^2+1)(c^2+1) = (b-c)^2 + (bc+1)^2 = E^2 + F^2,$$

$$(b+c)^2 = (b-c)^2 + 4(bc+1) - 4 = E^2 + 4F - 4,$$

$$(b^2-c^2)^2 = (b-c)^2(b+c)^2 = E^2(E^2+4F-4),$$

we have

$$A = E^2 + F^2 + 2kE, \quad B^2 = k^2E^2(E^2+4F-4), \quad C = E^2 + F^2 + 2kE(F-1),$$

and hence

$$AC - B^2 = (E^2+F^2+2kE)(E^2+F^2+2kEF-2kE) - k^2E^2(E^2+4F-4)$$

$$= (E^2+F^2)(E^2+F^2+2kEF) - k^2E^4 = \frac{1}{27}(E+\sqrt{3}F)^2(11E^2 - 2\sqrt{3}EF + 9F^2) \ge 0.$$

The equality holds for

$$b - c + \sqrt{3}(bc+1) = 0, \quad a + \frac{b+c}{1+3bc} = 0$$

(or any cyclic permutation).

Second Solution (by *Vo Quoc Ba Can*). Substituting $a = x\sqrt{3}$, $b = y\sqrt{3}$, $c = z\sqrt{3}$, the inequality becomes

$$(3x^2+1)(3y^2+1)(3z^2+1) \ge 8|(x-y)(y-z)(z-x)|.$$

It suffices to show that

$$E^2 \ge 64(x-y)^2(y-z)^2(z-x)^2,$$

where

$$E = (3x^2+1)(3y^2+1)(3z^2+1) = 27x^2y^2z^2 + 9\sum x^2y^2 + 3\sum x^2 + 1.$$

It it easy to check that the equality holds for $x = -1$, $y = 0$ and $z = 1$ (or any cyclic permutation), when

$$x+y+z = 0, \quad xy+yz+zx = -1, \quad xyz = 0.$$

From

$$\left(\sum xy + 1\right)^2 \ge 0,$$

we get

$$1 \ge -\sum x^2y^2 - 2xyx\sum x - 2\sum xy,$$

and from

$$(9xyz + \sum x)^2 \ge 0,$$

we get

$$81x^2y^2z^2 \ge -18xyz\sum x - \sum x^2 - 2\sum xy.$$

Therefore,

$$\begin{aligned}3E &\ge \left(-18xyz\sum x - \sum x^2 - 2\sum xy\right) + 27\sum x^2y^2 + 9\sum x^2\\ &+3\left(-\sum x^2y^2 - 2xyx\sum x - 2\sum xy\right)\\ &= 24\left(\sum x^2y^2 - xyz\sum x\right) + 8\left(\sum x^2 - \sum xy\right)\\ &= 12\sum x^2(y-z)^2 + \frac{4}{3}\sum(2x-y-z)^2.\end{aligned}$$

By the AM-GM inequality, we have

$$3E \ge 8\sqrt{\left[\sum x^2(y-z)^2\right]\left[\sum(2x-y-z)^2\right]}.$$

In addition, by the Cauchy-Schwarz inequality, we get

$$\begin{aligned} E^2 &\geq \frac{64}{9}\left[\sum x(y-z)(2x-y-z)\right]^2 \\ &= 64\left(\sum x^2y - \sum xy^2\right)^2 \\ &= 64(x-y)^2(y-z)^2(z-x)^2. \end{aligned}$$

(b) Write the inequality as

$$\left[\left(a-\frac{1}{2}\right)^2+\frac{3}{4}\right]\left[\left(b-\frac{1}{2}\right)^2+\frac{3}{4}\right]\left[\left(c-\frac{1}{2}\right)^2+\frac{3}{4}\right] \geq (a-b)(b-c)(c-a).$$

Using the substitution

$$a-\frac{1}{2}=\frac{\sqrt{3}}{2}x, \quad b-\frac{1}{2}=\frac{\sqrt{3}}{2}y, \quad c-\frac{1}{2}=\frac{\sqrt{3}}{2}z,$$

the inequality turns out into the inequality in (a). From the equality conditions in (a), namely

$$y-z+\sqrt{3}(yz+1)=0, \quad x+\frac{y+z}{1+3yz}=0,$$

we get the following equality conditions

$$b=\frac{c-1}{c}, \quad a=\frac{1}{1-c}$$

(or any cyclic permutation).

□

P 2.93. *If a,b,c are real numbers such that $a+b+c=3$, then*

$$(1-a+a^2)(1-b+b^2)(1-c+c^2) \geq 1.$$

Solution *(by Marian Tetiva).* Assume first that $a,b,c \geq 0$. Among the numbers a,b,c always there exist two (let b and c) which are either less than or equal to 1, or greater than or equal to 1. Then,

$$bc(b-1)(c-1) \geq 0,$$

hence

$$\begin{aligned} (1-b+b^2)(1-c+c^2) &= 1+(b^2-b)+(c^2-c)+(b^2-b)(c^2-c) \\ &\geq 1-b-c+b^2+c^2 \geq 1-(b+c)-\frac{1}{2}(b+c)^2 \\ &= 1-(3-a)-\frac{1}{2}(3-a)^2 = \frac{1}{2}(5-4a+a^2). \end{aligned}$$

Therefore, it suffices to show that

$$(1-a+a^2)(5-4a+a^2)\geq 2.$$

Indeed,

$$(1-a+a^2)(5-4a+a^2)-2=(a-1)^2(a^2-3a+3)\geq 0.$$

Assume now that $a\leq b\leq c$ and $a<0$. We have

$$c\geq \frac{b+c}{2}>\frac{a+b+c}{2}=\frac{3}{2}.$$

The desired inequality is true since

$$1-a+a^2>1,$$

$$1-b+b^2=\left(\frac{1}{2}-b\right)^2+\frac{3}{4}\geq \frac{3}{4},$$

$$1-c+c^2>1-c+\frac{3c}{2}=1+\frac{c}{2}>1+\frac{3}{4}=\frac{7}{4}.$$

The proof is completed. The equality holds for $a=b=c=1$.

□

P 2.94. *If a,b,c are real numbers such that $a+b+c=0$, then*

$$\frac{a(a-4)}{a^2+2}+\frac{b(b-4)}{b^2+2}+\frac{c(c-4)}{c^2+2}\geq 0.$$

Solution. Write the inequality as follows

$$\sum\left[\frac{a(a-4)}{a^2+2}+1\right]\geq 3,$$

$$\sum\frac{(a-1)^2}{a^2+2}\geq \frac{3}{2}.$$

From

$$a^2=(b+c)^2\leq 2(b^2+c^2),$$

we get

$$3a^2\leq 2(a^2+b^2+c^2).$$

Similarly,

$$3b^2\leq 2(a^2+b^2+c^2),\quad 3c^2\leq 2(a^2+b^2+c^2).$$

Therefore, we have

$$\sum \frac{(a-1)^2}{a^2+2} = \sum \frac{3(a-1)^2}{3a^2+6} \ge \sum \frac{3(a-1)^2}{2(a^2+b^2+c^2)+6}$$
$$= \frac{3}{2(a^2+b^2+c^2+3)} \sum (a-1)^2 = \frac{3}{2}.$$

Thus, the proof is completed. The equality holds for $a=b=c=0$, and also for $a=-2$ and $b=c=1$ (or any cyclic permutation).

□

P 2.95. *If a,b,c,d are real numbers, then*

$$(1-a+a^2)(1-b+b^2)(1-c+c^2)(1-d+d^2) \ge \left(\frac{1+abcd}{2}\right)^2.$$

(Vasile Cîrtoaje, 1992)

Solution. For $a=b=c=d$, the inequality can be written as

$$2(1-a+a^2)^2 \ge 1+a^4.$$

It is true, since

$$2(1-a+a^2)^2-1-a^4 = (1-a)^4 \ge 0.$$

Using this result, we get

$$4(1-a+a^2)^2(1-b+b^2)^2 \ge (1+a^4)(1+b^4) \ge (1+a^2b^2)^2.$$

Then, the desired inequality follows by multiplying the inequalities

$$2(1-a+a^2)(1-b+b^2) \ge 1+a^2b^2,$$

$$2(1-c+c^2)(1-d+d^2) \ge 1+c^2d^2,$$

$$(1+a^2b^2)(1+c^2d^2) \ge (1+abcd)^2.$$

The equality holds for $a=b=c=d=1$.

□

P 2.96. *Let a,b,c,d be real numbers such that $abcd>0$. Prove that*

$$\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right)\left(d+\frac{1}{d}\right) \ge (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$$

(Vasile Cîrtoaje, 2011)

First Solution. Write the inequality as $A \ge B$, where

$$A = (a^2+1)(b^2+1)(c^2+1)(d^2+1)$$
$$= (1+a^2c^2)(1+b^2d^2) + \sum a^2 + \sum a^2b^2 + \sum a^2b^2c^2,$$
$$B = (\sum a)(\sum abc) = 4abcd + \sum a^2(bc+cd+bd).$$

Then,

$$A-B = (1-abcd)^2 + (ac-bd)^2 + \frac{1}{2}\sum a^2(1-bc)^2 + \frac{1}{2}\sum a^2(1-cd)^2 + \sum a^2b^2 - \sum a^2bd,$$

and hence

$$A-B \ge \sum a^2b^2 - \sum a^2bd = \frac{1}{2}\sum a^2(b-d)^2 \ge 0.$$

The equality holds for $a=b=c=d=1$.

Second Solution. Since

$$(a+b)(b+c)(c+d)(d+a) - (a+b+c+d)(bcd+cda+dab+abc) = (ac-bd)^2 \ge 0,$$

it suffices to show that

$$(a^2+1)(b^2+1)(c^2+1)(d^2+1) \ge (a+b)(b+c)(c+d)(d+a).$$

By the Cauchy-Schwarz inequality, we have

$$(a^2+1)(1+b^2) \ge (a+b)^2,$$

$$(b^2+1)(1+c^2) \ge (b+c)^2,$$

$$(c^2+1)(1+d^2) \ge (c+d)^2,$$

$$(d^2+1)(1+a^2) \ge (d+a)^2.$$

Multiplying these inequalities, we get

$$(a^2+1)(b^2+1)(c^2+1)(d^2+1) \ge |(a+b)(b+c)(c+d)(d+a)|$$
$$\ge (a+b)(b+c)(c+d)(d+a).$$

□

P 2.97. *Let a, b, c, d be real numbers such that*

$$a+b+c+d=4, \qquad a^2+b^2+c^2+d^2=7.$$

Prove that

$$a^3+b^3+c^3+d^3 \le 16.$$

(Vasile Cîrtoaje, 2010)

First Solution. Assume that $a \le b \le c \le d$, and denote $s=a+b$ and $p=ab$, $s \le 2$, $4p \le s^2$. Since

$$2(a^3+b^3)=2(s^3-3ps),$$

$$c+d=4-s, \qquad c^2+d^2=7-(a^2+b^2)=7-s^2+2p,$$

$$2(c^3+d^3)=(c+d)[3(c^2+d^2)-(c+d)^2]=(4-s)(-4s^2+8s+5+6p),$$

we have

$$\begin{aligned}2(a^3+b^3+c^3+d^3-16) &= 12p(2-s)+6s^3-24s^2+27s-12\\ &\le 3s^2(2-s)+6s^3-24s^2+27s-12\\ &= 3(s-1)^2(s-4) \le 0.\end{aligned}$$

This completes the proof. The equality holds for $a=b=c=1/2$ and $d=5/2$ (or any cyclic permutation).

Second Solution (by *Vo Quoc Ba Can*). From

$$7=a^2+b^2+c^2+d^2 \ge a^2+\frac{1}{3}(b+c+d)^2=a^2+\frac{1}{3}(4-a)^2,$$

it follows that $a \in [\frac{-1}{2}, \frac{5}{2}]$. Similarly, we have $b, c, d \in [\frac{-1}{2}, \frac{5}{2}]$. On the other hand,

$$\begin{aligned}a^3+b^3+c^3+d^3 &= \frac{5}{2}\sum a^2+\sum\left(a^3-\frac{5}{2}a^2\right)\\ &= \frac{35}{2}-\frac{1}{2}\sum a^2(5-2a)\end{aligned}$$

and, by virtue of the Cauchy-Schwarz inequality,

$$\sum a^2(5-2a) \ge \frac{[\sum a(5-2a)]^2}{\sum(5-2a)}=\frac{(5\sum a-2\sum a^2)^2}{20-2\sum a}=3.$$

Therefore,

$$a^3+b^3+c^3+d^3 \le \frac{35}{2}-\frac{3}{2}=16.$$

Remark. In the same manner as in the second solution, we can prove the following generalization.

- *If $a_1, a_2, \ldots, a_n$ are real numbers such that*

$$a_1 + a_2 + \cdots + a_n = 2n, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n(n+3),$$

then

$$a_1^3 + a_2^3 + \cdots + a_n^3 \le n(n^2 + 3n + 4),$$

with equality for $a_1 = \ldots = a_{n-1} = 1$ and $a_n = n+1$ (or any cyclic permutation). □

P 2.98. *Let a, b, c, d be real numbers such that $a + b + c + d = 0$. Prove that*

$$12(a^4 + b^4 + c^4 + d^4) \le 7(a^2 + b^2 + c^2 + d^2)^2.$$

(Vasile Cîrtoaje, 2010)

Solution. Assume that $a^2 = \max\{b^2, c^2, d^2\}$ and denote

$$x = \sqrt{\frac{b^2 + c^2 + d^2}{3}}, \quad x^2 \le a^2.$$

From

$$x^2 = \frac{b^2 + c^2 + d^2}{3} \ge \left(\frac{b + c + d}{3}\right)^2 = \frac{a^2}{9},$$

we get $9x^2 \ge a^2$. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} b^4 + c^4 + d^4 &= (b^2 + c^2 + d^2)^2 - 2(b^2c^2 + c^2d^2 + d^2b^2) \\ &= 9x^4 - 2(b^2c^2 + c^2d^2 + d^2b^2) \\ &\le 9x^4 - \frac{2}{3}(bc + cd + db)^2 \\ &= 9x^4 - \frac{1}{6}[(b + c + d)^2 - b^2 - c^2 - d^2]^2 \\ &= 9x^4 - \frac{1}{6}(a^2 - 3x^2)^2 = \frac{45x^4 + 6a^2x^2 - a^4}{6}. \end{aligned}$$

and hence

$$a^4 + b^4 + c^4 + d^4 \le \frac{45x^4 + 6a^2x^2 + 5a^4}{6}.$$

Therefore, it suffices to prove that

$$2(45x^4 + 6a^2x^2 + 5a^4) \le 7(a^2 + 3x^2)^2,$$

which is equivalent to the obvious inequality

$$(x^2-a^2)(9x^2-a^2)\le 0.$$

The equality holds for $-a/3=b=c=d$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization.

- *If $a_1,a_2,\dots,a_n$ are real numbers such that*

$$a_1+a_2+\cdots+a_n=0,$$

then

$$\frac{(a_1^2+a_2^2+\cdots+a_n^2)^2}{a_1^4+a_2^4+\cdots+a_n^4}\ge\frac{n(n-1)}{n^2-3n+3},$$

with equality for $-a_1/(n-1)=a_2=\cdots=a_n$ (*or any cyclic permutation*).

□

P 2.99. *Let a,b,c,d be real numbers such that $a+b+c+d=0$. Prove that*

$$(a^2+b^2+c^2+d^2)^3\ge 3(a^3+b^3+c^3+d^3)^2.$$

(Vasile Cîrtoaje, 2011)

Solution. Applying the AM-GM inequality and the identity

$$(a+b+c)^3=a^3+b^3+c^3+3(a+b)(b+c)(c+a),$$

we have

$$\begin{aligned}(a^2+b^2+c^2+d^2)^3&=[a^2+b^2+c^2+(a+b+c)^2]^3\\&=[(a+b)^2+(b+c)^2+(c+a)^2]^3\\&\ge 27(a+b)^2(b+c)^2(c+a)^2\\&=3[(a+b+c)^3-a^3-b^3-c^3]^2\\&=3(a^3+b^3+c^3+d^3)^2.\end{aligned}$$

The equality holds for $a=b=c=-d/3$ (or any cyclic permutation).

Remark. The following generalization holds (*Vasile Cirtoaje*, 2011).

- *If $a_1,a_2,\dots,a_n$ are real numbers such that*

$$a_1+a_2+\cdots+a_n=0,$$

then

$$(a_1^2+a_2^2+\cdots+a_n^2)^3 \ge \frac{n(n-1)}{(n-2)^2}(a_1^3+a_2^3+\cdots+a_n^3)^2.$$

Moreover,

- *If $k \ge 3$ is an odd number, and $a_1, a_2, \ldots, a_n$ are real numbers such that*

$$a_1+a_2+\cdots+a_n=0,$$

then

$$(a_1^2+a_2^2+\cdots+a_n^2)^k \ge \frac{n^k(n-1)^{k-2}}{[(n-1)^{k-1}-1]^2}(a_1^k+a_2^k+\cdots+a_n^k)^2,$$

with equality for $a_1 = \ldots = a_{n-1} = -a_n/(n-1)$ (or any cyclic permutation).

□

P 2.100. *If a,b,c,d are real numbers such that $abcd=1$. Prove that*

$$(1+a^2)(1+b^2)(1+c^2)(1+d^2) \ge (a+b+c+d)^2.$$

(Pham Kim Hung, 2006)

Solution. Substituting a,b,c,d by $|a|,|b|,|c|,|d|$, respectively, the left side of the inequality remains unchanged, while the right side either remains unchanged or increases. Therefore, it suffices to prove the inequality only for $a,b,c,d \ge 0$. Among a,b,c,d there are two numbers less than or equal to 1, or greater than or equal to 1. Let b and d be these numbers; that is,

$$(1-b)(1-d) \ge 0.$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}(1+a^2)(1+b^2)(1+c^2)(1+d^2) &= (1+a^2+b^2+a^2b^2)(c^2+1+d^2+c^2d^2)\\ &\ge (c+a+bd+abcd)^2 = (c+a+bd+1)^2.\end{aligned}$$

So, it suffices to show that

$$c+a+bd+1 \ge a+b+c+d,$$

which is equivalent to $(1-b)(1-d) \ge 0$. The equality holds for $a=b=c=d=1$.

□

P 2.101. *Let a,b,c,d be real numbers such that*

$$a^2+b^2+c^2+d^2=4.$$

Prove that

$$(abc)^3+(bcd)^3+(cda)^3+(dab)^3\le 4.$$

(Vasile Cîrtoaje, 2004)

Solution. Substituting a,b,c,d by $|a|,|b|,|c|,|d|$, respectively, the hypothesis and the right side of the inequality remains unchanged, while the left side either remains unchanged or decreases. Therefore, it suffices to prove the inequality only for $a,b,c,d\ge 0$. Setting $x=a^2$, $y=b^2$, $z=c^2$ and $t=d^2$, we need to prove that

$$(xyz)^{3/2}+(yzt)^{3/2}+(ztx)^{3/2}+(txy)^{3/2}\le 4$$

for $x+y+z+t=4$. By the AM-GM inequality, we have

$$4\sqrt[4]{xyz}\le 1+x+y+z=5-t,$$

$$(xyz)^{3/2}=xyz\sqrt{xyz}\le\frac{xyz(5-t)^2}{16}.$$

Analogously,

$$(yzt)^{3/2}\le\frac{yzt(5-x)^2}{16},\quad (ztx)^{3/2}\le\frac{ztx(5-y)^2}{16},\quad (txy)^{3/2}\le\frac{txy(5-z)^2}{16}.$$

Therefore, it suffices to show that

$$xyz(5-t)^2+yzt(5-x)^2+ztx(5-y)^2+txy(5-z)^2\le 64,$$

which is equivalent to $E(x,y,z,t)\ge 0$, where

$$E(x,y,z,t)=36xyzt-25(xyz+yzt+ztx+txy)+64.$$

To prove this inequality, we use the mixing variable method.Without loss of generality, assume that $x\ge y\ge z\ge t$. Setting $u=\dfrac{x+y+z}{3}$, we have $3u+t=4$, $t\le u\le\dfrac{4}{3}$ and $u^3\ge xyz$. We will show that

$$E(x,y,z,t)\ge E(u,u,u,t)\ge 0.$$

The left inequality is equivalent to

$$25[(u^3-xyz)+t(3u^2-xy-yz-zx)]\ge 36t(u^3-xyz).$$

By Schur's inequality

$$(x+y+z)^3+9xyz \ge 4(x+y+z)(xy+yz+zx),$$

we get

$$9u^3+3xyz \ge 4u(xy+yz+zx),$$

and hence

$$3u^2-xy-yz-zx \ge \frac{3(u^3-xyz)}{4u} \ge 0.$$

Therefore, it suffices to prove that

$$25(1+\frac{3t}{4u}) \ge 36t.$$

Write this inequality in the homogeneous form

$$25(3u+t)(4u+3t) \ge 576ut,$$

or, equivalently,

$$75(4u^2+t^2) \ge 251ut.$$

This inequality is true, since

$$75(4u^2+t^2)-251ut \ge 75(4u^2+t^2-4ut) \ge 300u(u-t) \ge 0.$$

The right inequality $E(u,u,u,t) \ge 0$ holds, since

$$\begin{aligned} E(u,u,u,t) &= (36u^3-75u^2)t-25u^3+64 \\ &= 4(16-75u^2+86u^3-27u^4) \\ &= 4(1-u)^2(16+32u-27u^2) \\ &= 4(1-u)^2[4(4-u)+9u(4-3u)] \ge 0. \end{aligned}$$

This completes the proof. The equality holds for $a=b=c=d=1$.

□

P 2.102. *Let a, b, c, d be real numbers such that $a^2+b^2+c^2+d^2=1$. Prove that*

$$(1-a)^4+(1-b)^4+(1-c)^4+(1-d)^4 \ge a^4+b^4+c^4+d^4.$$

(Vasile Cîrtoaje, 2007)

Solution. The desired inequality follows by summing the inequalities

$$(1-a)^4+(1-b)^4 \ge c^4+d^4,$$

$$(1-c)^4+(1-d)^4 \ge a^4+b^4.$$

Since

$$(1-a)^4+(1-b)^4 \ge 2(1-a)^2(1-b)^2$$

and

$$c^4+d^4 \ge \frac{1}{2}(c^2+d^2)^2,$$

the former inequality holds if

$$2(1-a)(1-b) \ge c^2+d^2.$$

Indeed,

$$2(1-a)(1-b)-c^2-d^2 = 2(1-a)(1-b)+a^2+b^2-1 = (a+b-1)^2 \ge 0.$$

The equality holds for $a=b=c=d=\frac{1}{2}$.

□

P 2.103. *If $a,b,c,d \ge \frac{-1}{2}$ such that $a+b+c+d=4$, then*

$$\frac{1-a}{1-a+a^2}+\frac{1-b}{1-b+b^2}+\frac{1-c}{1-c+c^2}+\frac{1-d}{1-d+d^2} \ge 0.$$

(Vasile Cîrtoaje, 2014)

Solution *(by Nguyen Van Quy)*. Assume that $a \le b \le c \le d$ and consider two cases: $a>0$ and $a \le 0$.

Case 1: $a>0$. Write the inequality as

$$\frac{a^2}{1-a+a^2}+\frac{b^2}{1-b+b^2}+\frac{c^2}{1-c+c^2}+\frac{d^2}{1-d+d^2} \le 4.$$

We have

$$\frac{a^2}{1-a+a^2}+\frac{b^2}{1-b+b^2}+\frac{c^2}{1-c+c^2}+\frac{d^2}{1-d+d^2} \le \frac{a^2}{a}+\frac{b^2}{b}+\frac{c^2}{c}+\frac{d^2}{d}=4.$$

Case 2: $-1/2 \le a \le 0$. We can check that the equality holds for $a=-1/2$ and $b=c=d=3/2$ (or any cyclic permutation). Define the function

$$f(x)=\frac{1-x}{1-x+x^2}+k_1x+k_2, \quad x \ge \frac{-1}{2},$$

such that

$$f(3/2) = f'(3/2) = 0.$$

We get

$$k_1 = \frac{12}{49}, \quad k_2 = \frac{-4}{49},$$

when

$$f(x) = \frac{1-x}{1-x+x^2} + \frac{12x-4}{49} = \frac{(2x-3)^2(3x+5)}{49(1-x+x^2)}.$$

Since $f(x) \ge 0$ for $x \ge -1/2$, we have

$$\frac{1-x}{1-x+x^2} \ge \frac{4-12x}{49}.$$

Therefore,

$$\frac{1-b}{1-b+b^2} + \frac{1-c}{1-c+c^2} + \frac{1-d}{1-d+d^2} \ge \frac{12-12(b+c+d)}{49} = \frac{12(a-3)}{49}.$$

Thus, it suffices to show that

$$\frac{1-a}{1-a+a^2} + \frac{12(a-3)}{49} \ge 0.$$

Indeed,

$$\frac{1-a}{1-a+a^2} + \frac{12(a-3)}{49} = \frac{(2a+1)(6a^2-27a+13)}{49(1-a+a^2)} \ge 0.$$

The proof is completed. The equality holds for $a = b = c = d = 1$, and also for $a = -1/2$ and $b = c = d = 3/2$ (or any cyclic permutation).

□

P 2.104. *If $a, b, c, d, e \ge -3$ such that $a + b + c + d + e = 5$, then*

$$\frac{1-a}{1+a+a^2} + \frac{1-b}{1+b+b^2} + \frac{1-c}{1+c+c^2} + \frac{1-d}{1+d+d^2} + \frac{1-e}{1+e+e^2} \ge 0.$$

(Vasile Cîrtoaje, 2014)

Solution. Assume that $a \le b \le c \le d \le e$ and consider two cases: $a \ge 0$ and $a \le 0$.

Case 1: $a \ge 0$. For any $x \ge 0$, we have

$$\frac{1-x}{1+x+x^2} - \frac{1-x}{3} = \frac{(x-1)^2(x+2)}{3(1+x+x^2)} \ge 0.$$

Therefore, it suffices to show that

$$\frac{1-a}{3}+\frac{1-b}{3}+\frac{1-c}{3}+\frac{1-d}{3}+\frac{1-e}{3}\ge 0,$$

which is an identity.

Case 2: $-3\le a\le 0$. We can check that the equality holds for $a=3$ and $b=c=d=e=2$. Based on this, define the function

$$f(x)=\frac{1-x}{1+x+x^2}+k_1x+k_2, \quad x\ge -3,$$

such that

$$f(2)=f'(2)=0.$$

We get

$$k_1=\frac{2}{49}, \quad k_2=\frac{3}{49},$$

when

$$f(x)=\frac{1-x}{1+x+x^2}+\frac{2x+3}{49}=\frac{(x-2)^2(2x+13)}{49(1+x+x^2)}.$$

Since $f(x)\ge 0$ for $x\ge -3$, we have

$$\frac{1-x}{1+x+x^2}\ge\frac{-2x-3}{49}.$$

Thus, it suffices to show that

$$\frac{1-a}{1+a+a^2}-\frac{2b+3}{49}-\frac{2c+3}{49}-\frac{2d+3}{49}-\frac{2e+3}{49}\ge 0,$$

which is equivalent to

$$\frac{1-a}{1+a+a^2}-\frac{2(b+c+d+e)+12}{49}\ge 0,$$

$$\frac{1-a}{1+a+a^2}-\frac{2(5-a)+12}{49}\ge 0,$$

$$\frac{(a+3)(2a^2-26a+9)}{49(1+a+a^2)}\ge 0.$$

Clearly, the last inequality is true for $-3\le a\le 0$. The equality holds for $a=b=c=d=e=1$, and also for $a=-3$ and $b=c=d=e=2$ (or any cyclic permutation).

□

P 2.105. *Let a,b,c,d,e be real numbers such that $a+b+c+d+e=0$. Prove that*

$$30(a^4+b^4+c^4+d^4+e^4)\ge 7(a^2+b^2+c^2+d^2+e^2)^2.$$

(Vasile Cîrtoaje, 2010)

Solution. Write the inequality as $E(a,b,c,d,e)\ge 0$, where

$$E(a,b,c,d,e)=30(a^4+b^4+c^4+d^4+e^4)-7(a^2+b^2+c^2+d^2+e^2)^2.$$

Among the numbers a,b,c,d,e there exist three with the same sign. Let a,b,c be these numbers. We will show that

$$E(a,b,c,d,e)\ge E(a,b,c,x,x)\ge 0,$$

where

$$x=\frac{d+e}{2}=\frac{-(a+b+c)}{2}.$$

The inequality $E(a,b,c,d,e)\ge E(a,b,c,x,x)$ is equivalent to

$$30(d^4+e^4-2x^4)\ge 7(d^2+e^2-2x^2)(2a^2+2b^2+2c^2+d^2+e^2+2x^2).$$

Since

$$d^4+e^4-2x^4=\frac{(d-e)^2(7d^2+10de+7e^2)}{8}$$

and

$$d^2+e^2-2x^2=\frac{(d-e)^2}{2},$$

we need to show that

$$15(7d^2+10de+7e^2)\ge 14(2a^2+2b^2+2c^2+d^2+e^2+2x^2),$$

which reduces to

$$21(d^2+e^2)+34de\ge 7(a^2+b^2+c^2).$$

Since

$$a^2+b^2+c^2\le (a+b+c)^2=(d+e)^2,$$

it suffices to prove that

$$21(d^2+e^2)+34de\ge 7(d+e)^2,$$

which is equivalent to the obvious inequality

$$4(d^2+e^2)+10(d+e)^2\ge 0.$$

The inequality $E(a,b,c,x,x)\geq 0$, where $x=\dfrac{-(a+b+c)}{2}$, can be written as

$$23\sum a^4+2(\sum a)^4\geq 7(\sum a^2)(\sum a)^2+14\sum a^2b^2.$$

Since $(\sum a)^2\geq 3\sum ab$, it suffices to prove that

$$23\sum a^4+6(\sum ab)(\sum a)^2\geq 7(\sum a^2)(\sum a)^2+14\sum a^2b^2.$$

This is equivalent to

$$\sum a^4+abc\sum a\geq \frac{1}{2}\sum ab(a^2+b^2)+\sum a^2b^2,$$

which follows by summing Schur's inequality of degree four

$$\sum a^4+abc\sum a\geq \sum ab(a^2+b^2)$$

and the obvious inequality

$$\frac{1}{2}\sum ab(a^2+b^2)\geq \sum a^2b^2.$$

This completes the proof. The equality holds for $a=b=c=2$ and $d=e=-3$ (or any permutation thereof).

Remark. Notice that the following generalization holds (*Vasile Cîrtoaje*, 2010).

- *If n is an odd positive integer and $a_1,a_2,\ldots,a_n$ are real numbers such that*

$$a_1+a_2+\cdots+a_n=0,$$

then

$$\frac{(a_1^2+a_2^2+\cdots+a_n^2)^2}{a_1^4+a_2^4+\cdots+a_n^4}\leq \frac{n(n^2-1)}{n^2+3},$$

with equality when $(n+1)/2$ of $a_1,a_2,\ldots,a_n$ are equal to $(n-1)/2$ and the other numbers are equal to $-(n+1)/2$.

□

P 2.106. *Let $a_1,a_2,\ldots,a_n\geq -1$ such that $a_1+a_2+\cdots+a_n=0$. Prove that*

$$(n-2)(a_1^2+a_2^2+\cdots+a_n^2)\geq a_1^3+a_2^3+\cdots+a_n^3.$$

(Vasile Cîrtoaje, 2005)

Solution. Without loss of generality, assume that

$$a_1 \ge a_2 \ge \cdots \ge a_n.$$

Write the inequality as

$$\sum_{i=1}^{n} a_i f(a_i) \ge 0,$$

where

$$f(x) = (n-2)x - x^2.$$

Since

$$f(a_i) - f(a_{i+1}) = (a_i - a_{i+1})[n-2-(a_i + a_{i+1})]$$
$$\ge (a_i - a_{i+1})[n-2-(a_1+a_2)] = (a_i - a_{i+1})(n-2+a_3+\cdots+a_n)]$$
$$= (a_i - a_{i+1})[(1+a_3)+\cdots+(1+a_n)] \ge 0,$$

we have $a_1 \ge a_2 \ge \cdots \ge a_n$ and $f(a_1) \ge f(a_2) \ge \cdots \ge f(a_n)$. Therefore, by Chebyshev's inequality, we get

$$n\sum_{i=1}^{n} a_i f(a_i) \ge (a_1 + a_2 + \cdots + a_n)[f(a_1) + f(a_2) + \cdots + f(a_n)] = 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 0$, and for $a_1 = n-1$ and $a_2 = \cdots = a_n = -1$ (or any cyclic permutation).

□

P 2.107. *Let $a_1, a_2, \ldots, a_n \ge -1$ such that $a_1 + a_2 + \cdots + a_n = 0$. Prove that*

$$(n-2)(a_1^2 + a_2^2 + \cdots + a_n^2) + (n-1(a_1^3 + a_2^3 + \cdots + a_n^3) \ge 0.$$

(Vasile Cîrtoaje, 2005)

Solution. For the nontrivial case $a_1^2 + a_2^2 + \cdots + a_n^2 \ne 0$, write the inequality as

$$(n-1)\frac{S_3}{S_2} + n - 2 \ge 0,$$

where

$$S_2 = a_1^2 + a_2^2 + \cdots + a_n^2, \quad S_3 = a_1^3 + a_2^3 + \cdots + a_n^3.$$

Without loss of generality, assume that

$$a_1 \le a_2 \le \cdots \le a_n.$$

For any $p > 0$ such that $a_1 + p \ge 0$, by the Cauchy-Schwarz inequality, we have

$$\sum_{i=1}^{n}(a_i+1)^2(a_i+p) \ge \frac{[\sum_{i=1}^{n}(a_i+1)(a_i+p)]^2}{\sum_{i=1}^{n}(a_i+p)},$$

$$\sum_{i=1}^{n}a_i^3 + (p+2)\sum_{i=1}^{n}a_i^2 + np \ge \frac{(\sum_{i=1}^{n}a_i^2+np)^2}{np},$$

$$\frac{S_3}{S_2} \ge \frac{S_2}{np} - p.$$

Thus, it suffices to show that

$$S_2 + \frac{n(n-2)}{n-1}p \ge np^2.$$

Case 1: $S_2 \ge \dfrac{n}{n-1}$. Choosing $p = 1$, we have $a_1 + p \ge 0$, and

$$S_2 + \frac{n(n-2)}{n-1}p - np^2 = S_2 - \frac{n}{n-1} \ge 0.$$

Case 2: $0 < S_2 \le \dfrac{n}{n-1}$. We set

$$p = \sqrt{\frac{n-1}{n}S_2}.$$

Since

$$p^2 - a_1^2 = \frac{n-1}{n}S_2 - a_1^2 = \frac{n-1}{n}(a_2^2 + \cdots + a_n^2) - \frac{1}{n}a_1^2$$
$$\ge \frac{(a_2+\cdots+a_n)^2}{n} - \frac{a_1^2}{n} = 0,$$

we have $a_1 + p \ge 0$. In addition,

$$S_2 + \frac{n(n-2)}{n-1}p - np^2 = (n-2)\sqrt{S_2}\left(\sqrt{\frac{n}{n-1}} - \sqrt{S_2}\right) \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 0$, and for $a_1 = -1$ and $a_2 = \cdots = a_n = 1/(n-1)$ (or any cyclic permutation).

□

P 2.108. *Let $a_1, a_2, \ldots, a_n \ge n - 1 - \sqrt{n^2-n+1}$ be nonzero real numbers such that $a_1 + a_2 + \cdots + a_n = n$. Prove that*

$$\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2} \ge n.$$

(Vasile Cîrtoaje, 2010)

Solution. Without loss of generality, assume that $a_1 \le a_2 \le \cdots \le a_n$. For $a_1 > 0$, by the Cauchy-Schwarz inequality, we have

$$\frac{1}{a_1^2}+\frac{1}{a_2^2}+\cdots+\frac{1}{a_n^2} \ge \frac{1}{n}\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)^2$$

$$\ge \frac{1}{n}\left(\frac{n^2}{a_1+a_2+\cdots a_n}\right)^2 = n.$$

Further, consider that $a_1 < 0$, when there exists k, $1 \le k \le n-1$, such that

$$a_1 \le \cdots \le a_k < 0 < a_{k+1} \le \cdots \le a_n.$$

Let us denote $x = \dfrac{a_1+\cdots+a_k}{k}$ and $y = \dfrac{a_{k+1}+\cdots+a_n}{n-k}$. We have

$$-1 < n-1-\sqrt{n^2-n+1} \le x < 0, \quad y > 1, \quad kx+(n-k)y = n.$$

From $k \ge 1$ and $k(y-x) = n(y-1) > 0$, we get $y-x \le n(y-1)$, and hence

$$y \ge \frac{n-x}{n-1}.$$

In addition, from $n-1-\sqrt{n^2-n+1} \le x$, we get

$$n+2(n-1)x-x^2 \ge 0.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{1}{a_1^2}+\cdots+\frac{1}{a_k^2} \ge \frac{1}{k}\left(\frac{1}{-a_1}+\cdots+\frac{1}{-a_k}\right)^2 \ge \frac{1}{k}\left(\frac{k^2}{-a_1-\cdots-a_k}\right)^2 = \frac{k}{x^2}$$

and

$$\frac{1}{a_{k+1}^2}+\cdots+\frac{1}{a_n^2} \ge \frac{1}{n-k}\left(\frac{1}{a_{k+1}}+\cdots+\frac{1}{a_n}\right)^2 \ge \frac{1}{n-k}\left(\frac{(n-k)^2}{a_{k+1}+\cdots+a_n}\right)^2 = \frac{n-k}{y^2}.$$

Then, it suffices to prove that

$$\frac{k}{x^2}+\frac{n-k}{y^2} \ge n,$$

which is equivalent to

$$k(y-x)(y+x) \ge nx^2(y-1)(y+1).$$

Since $k(y-x) = n(y-1) > 0$, we need to show that $y+x \ge x^2(y+1)$, which is equivalent to

$$(1-x)(x+y+xy) \ge 0.$$

This is true, since $1-x>0$ and

$$x+y+xy \ge x+\frac{(1+x)(n-x)}{n-1}=\frac{n+2(n-1)x-x^2}{n-1} \ge 0.$$

The equality holds when $a_1=a_2=\cdots=a_n=1$, as well as when one of $a_1,a_2,\ldots,a_n$ is $n-1-\sqrt{n^2-n+1}$ and the others are $\dfrac{1+\sqrt{n^2-n+1}}{n-1}$.

□

P 2.109. *Let* $a_1,a_2,\ldots,a_n \le \dfrac{n}{n-2}$ *be real numbers such that*

$$a_1+a_2+\cdots+a_n=n.$$

If k is a positive integer, $k \ge 2$*, then*

$$a_1^k+a_2^k+\cdots+a_n^k \ge n.$$

(Vasile Cîrtoaje, 2012)

Solution. First we show that at most one of a_i is negative. Assume, for the sake of contradiction, that $a_{n-1}<0$ and $a_n<0$. Then,

$$a_{n-1}+a_n=n-(a_1+\cdots+a_{n-2}) \ge n-(n-2)\cdot\frac{n}{n-2}=0,$$

which is a contradiction. There are two cases to consider.
Case 1: $a_1,a_2,\ldots,a_n \ge 0$. The desired inequality is just Jensen's inequality applied to the convex function $f(x)=x^k$.
Case 2: $a_1,a_2,\cdots,a_{n-1} \ge 0$ and $a_n<0$. Let us denote

$$x=\frac{a_1+a_2+\cdots+a_{n-1}}{n-1}, \quad y=-a_n,$$

where

$$(n-1)x-y=n, \quad x \ge y>0.$$

The condition $x \ge y$ follows from

$$x-y=n-(n-2)x \ge 0.$$

By Jensen's inequality, we have

$$a_1^k+a_2^k+\cdots+a_{n-1}^k \ge (n-1)x^k.$$

In addition, $a_n^k \ge -y^k$. Thus, it suffices to show that

$$(n-1)x^k-y^k \ge n.$$

We will use the inequality
$$x^k - y^k > (x-y)^k,$$
which is equivalent to
$$1-\left(\frac{y}{x}\right)^k > \left(1-\frac{y}{x}\right)^k.$$
Indeed,
$$1-\left(\frac{y}{x}\right)^k - \left(1-\frac{y}{x}\right)^k > 1-\left(\frac{y}{x}\right)-\left(1-\frac{y}{x}\right) = 0.$$
Therefore, it suffices to show that
$$(n-2)x^k + (x-y)^k \ge n.$$
By Jensen's inequality, we have
$$(n-2)x^k + (x-y)^k \ge [(n-2)+1]\left[\frac{(n-2)x+(x-y)}{(n-2)+1}\right]^k = n\left(\frac{n}{n-1}\right)^{k-1} > n.$$
This completes the proof. The equality holds for $a_1 = a_2 = \cdots a_n = 1$.

□

P 2.110. *If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are real numbers such that*
$$a_1, a_2, \ldots, a_n \ge \frac{-(3n-2)}{n-2}, \quad a_1 + a_2 + \cdots + a_n = n,$$
then
$$\frac{1-a_1}{(1+a_1)^2} + \frac{1-a_2}{(1+a_2)^2} + \cdots + \frac{1-a_n}{(1+a_n)^2} \ge 0.$$
(Vasile Cîrtoaje, 2014)

Solution. Since the inequality holds for $n = 3$ (see P 2.25), consider further that $n \ge 4$. Assume that $a_1 \le a_2 \le \cdots \le a_n$ and consider two cases: $a_1 \ge 0$ and $a_1 \le 0$.

Case 1: $a_1 \ge 0$. For any $x \ge 0$, we have
$$\frac{1-x}{(1+x)^2} + \frac{x-1}{4} = \frac{(x-1)^2(x+3)}{4(1+x)^2} \ge 0.$$
Therefore, it suffices to show that
$$\frac{1-a_1}{4} + \frac{1-a_2}{4} + \cdots + \frac{1-a_n}{4} \ge 0,$$
which is an identity.

Case 2: $-(3n-2)/(n-2) \le a_1 \le 0$. We can check that the equality holds for

$$a_1 = \frac{-(3n-2)}{n-2}, \quad a_2 = a_3 = \cdots = a_n = \frac{n+2}{n-2}.$$

Based on this, define the function

$$f(x) = \frac{1-x}{(1+x)^2} + k_1 x + k_2, \quad x \ge \frac{-(3n-2)}{n-2},$$

such that

$$f\left(\frac{n+2}{n-2}\right) = f'\left(\frac{n+2}{n-2}\right) = 0.$$

We get

$$k_1 = \frac{(n-4)(n-2)^2}{4n^3}, \quad k_2 = \frac{(n-2)(-n^2+6n+8)}{4n^3},$$

$$f(x) = \frac{[(n-2)x-n-2]^2[(n-4)x+3n-4]}{4n^3(1+x)^2}.$$

Since $f(x) \ge 0$ for $n \ge 4$ and $x \ge -(3n-2)/(n-2)$, we have

$$\frac{1-x}{(1+x)^2} \ge -k_1 x - k_2.$$

Thus, it suffices to show that

$$\frac{1-a_1}{(1+a_1)^2} - k_1(a_2+a_3+\cdots+a_n) - (n-1)k_2 \ge 0,$$

which is equivalent to

$$\frac{1-a_1}{(1+a_1)^2} - k_1(n-a_1) - (n-1)k_2 \ge 0,$$

$$[(n-2)a_1+3n-2][(n-4)(n-2)a_1^2 - 2(n^2+4n-8)a_1 + n^2 - 2n + 8] \ge 0.$$

Clearly, the last inequality is true for $n \ge 4$ and $-(3n-2)/(n-2) \le a_1 \le 0$. This completes the proof. The equality holds for $a_1 = a_2 = \cdots a_n = 1$, and also for $x_1 = \frac{-(3n-2)}{n-2}$ and $x_2 = \cdots = x_n = \frac{n+2}{n-2}$ (or any cyclic permutation).

□

P 2.111. *Let $a_1, a_2, \ldots, a_n$ be real numbers. Prove that*

(a)
$$\frac{(a_1+a_2+\cdots+a_n)^2}{(a_1^2+1)(a_2^2+1)\cdots(a_n^2+1)} \le \frac{(n-1)^{n-1}}{n^{n-2}};$$

(b)
$$\frac{a_1+a_2+\cdots+a_n}{(a_1^2+1)(a_2^2+1)\cdots(a_n^2+1)} \le \frac{(2n-1)^{n-\frac{1}{2}}}{2^n n^{n-1}}.$$

(Vasile Cîrtoaje, 1994)

Solution. Let m be a positive integer ($m \ge n$), and let $a_i = \dfrac{x_i}{\sqrt{m-1}}$ for all i. Assume that

$$x_1^2 \le \cdots \le x_k^2 \le 1 \le x_{k+1}^2 \le \cdots \le x_n^2,$$

where $0 \le k \le n$. By Bernoulli's inequality and the Cauchy-Schwarz inequality, we have

$$\left(\frac{m-1}{m}\right)^n \prod_{i=1}^n (a_i^2+1) = \left(\frac{m-1}{m}\right)^n \prod_{i=1}^n \left(\frac{x_i^2}{m-1}+1\right)$$

$$= \prod_{i=1}^n \left(1+\frac{x_i^2-1}{m}\right) = \prod_{i=1}^k \left(1+\frac{x_i^2-1}{m}\right) \prod_{i=k+1}^n \left(1+\frac{x_i^2-1}{m}\right)$$

$$\ge \left(1+\sum_{i=1}^k \frac{x_i^2-1}{m}\right)\left(1+\sum_{i=k+1}^n \frac{x_i^2-1}{m}\right)$$

$$= \frac{1}{m^2}[x_1^2+\cdots+x_k^2+(m-n)+(n-k)][k+(m-n)+x_{k+1}^2+\cdots+x_n^2]$$

$$\ge \frac{1}{m^2}(x_1+\cdots+x_k+m-n+x_{k+1}+\cdots+x_n)^2$$

$$= \frac{1}{m^2}(m-n+x_1+x_2+\cdots+x_n)^2.$$

Therefore,

$$\left(\frac{m-1}{m}\right)^n \prod_{i=1}^n (a_i^2+1) \ge \frac{1}{m^2}(m-n+x_1+x_2+\cdots+x_n)^2,$$

and hence

$$\prod_{i=1}^n (a_i^2+1) \ge \frac{m^{n-2}}{(m-1)^{n-1}}\left(\frac{m-n}{\sqrt{m-1}}+a_1+a_2+\cdots+a_n\right)^2.$$

The equality occurs in this inequality for

$$a_1 = a_2 = \cdots = a_n = \frac{1}{\sqrt{m-1}}.$$

(a) Choosing $m = n$, we get the desired inequality. The equality holds for

$$a_1 = a_2 = \cdots = a_n = \frac{1}{\sqrt{n-1}}$$

or

$$a_1 = a_2 = \cdots = a_n = \frac{-1}{\sqrt{n-1}}.$$

(b) Since

$$\left(\frac{m-n}{\sqrt{m-1}}+a_1+a_2+\cdots+a_n\right)^2 \geq \frac{4(m-n)}{\sqrt{m-1}}(a_1+a_2+\cdots+a_n),$$

we get

$$\prod_{i=1}^{n}(a_i^2+1) \geq \frac{4m^{n-2}(m-n)}{(m-1)^{n-\frac{1}{2}}}(a_1+a_2+\cdots+a_n).$$

Choosing $m=2n$, we get the desired inequality. The equality holds for

$$a_1=a_2=\cdots=a_n=\frac{1}{\sqrt{2n-1}}.$$

□

Chapter 3

Symmetric Polynomial Inequalities in Nonnegative Variables

3.1 Applications

3.1. If a, b, c are positive real numbers, then

$$a^2 + b^2 + c^2 + 2abc + 1 \geq 2(ab + bc + ca).$$

3.2. Let a, b, c be nonnegative real numbers. If $0 \leq k \leq \sqrt{2}$, then

$$a^2 + b^2 + c^2 + kabc + 2k + 3 \geq (k + 2)(a + b + c).$$

3.3. If a, b, c are positive real numbers, then

$$abc(a + b + c) + 2(a^2 + b^2 + c^2) + 3 \geq 4(ab + bc + ca).$$

3.4. If a, b, c are positive real numbers, then

$$a(b^2 + c^2) + b(c^2 + a^2) + c(a^2 + b^2) + 3 \geq 3(ab + bc + ca).$$

3.5. If a, b, c are positive real numbers, then

$$\left(\frac{a^2 + b^2 + c^2}{3}\right)^3 \geq a^2b^2c^2 + (a - b)^2(b - c)^2(c - a)^2.$$

3.6. If a, b, c are positive real numbers, then

$$(a+b+c-3)(ab+bc+ca-3) \geq 3(abc-1)(a+b+c-ab-bc-ca).$$

3.7. If a, b, c are positive real numbers, then

(a) $$a^3+b^3+c^3+ab+bc+ca+9 \geq 5(a+b+c);$$

(b) $$a^3+b^3+c^3+4(ab+bc+ca)+18 \geq 11(a+b+c).$$

3.8. If a, b, c are positive real numbers, then

(a) $$a^3+b^3+c^3+abc+8 \geq 4(a+b+c);$$

(b) $$4(a^3+b^3+c^3)+15abc+54 \geq 27(a+b+c).$$

3.9. Let a, b, c be nonnegative real numbers such that

$$a+b+c=a^2+b^2+c^2.$$

Prove that

$$ab+bc+ca \geq a^2b^2+b^2c^2+c^2a^2.$$

3.10. If a, b, c are nonnegative real numbers, then

$$(a^2+2bc)(b^2+2ca)(c^2+2ab) \geq (ab+bc+ca)^3.$$

3.11. If a, b, c are nonnegative real numbers, then

$$(2a^2+bc)(2b^2+ca)(2c^2+ab) \geq (ab+bc+ca)^3.$$

3.12. Let a, b, c be nonnegative real numbers such that $a+b+c=2$. Prove that

(a) $$(a^2+b^2)(b^2+c^2)(c^2+a^2) \leq (a+b)(b+c)(c+a);$$

(b) $$(a^2+b^2)(b^2+c^2)(c^2+a^2) \leq 2.$$

3.13. Let a, b, c be nonnegative real numbers such that $a + b + c = 2$. Prove that

$$(a^3 + b^3)(b^3 + c^3)(c^3 + a^3) \le 2.$$

3.14. Let a, b, c be nonnegative real numbers such that $a^2 + b^2 + c^2 = 2$. Prove that

$$(a^3 + b^3)(b^3 + c^3)(c^3 + a^3) \le 2.$$

3.15. If a, b, c are nonnegative real numbers such that $a + b + c = 2$, then

$$(3a^2 - 2ab + 3b^2)(3b^2 - 2bc + 3c^2)(3c^2 - 2ca + 3a^2) \le 36.$$

3.16. Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. Prove that

$$(a^2 - 4ab + b^2)(b^2 - 4bc + c^2)(c^2 - 4ca + a^2) \le 3.$$

3.17. If a, b, c are positive real numbers such that $a + b + c = 3$, then

$$abc + \frac{12}{ab + bc + ca} \ge 5.$$

3.18. If a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 3$, then

$$5(a + b + c) + \frac{3}{abc} \ge 18.$$

3.19. If a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 3$, then

$$12 + 9abc \ge 7(ab + bc + ca).$$

3.20. If a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 3$, then

$$21 + 18abc \ge 13(ab + bc + ca).$$

3.21. If a, b, c are positive real numbers such that $a^2+b^2+c^2=3$, then

$$(2-ab)(2-bc)(2-ca) \geq 1.$$

3.22. Let a, b, c be positive real numbers such that $abc=1$. Prove that

$$\left(\frac{a+b+c}{3}\right)^5 \geq \frac{a^2+b^2+c^2}{3}.$$

3.23. If a, b, c are positive real numbers such that $abc=1$, then

$$a^3+b^3+c^3+a^{-3}+b^{-3}+c^{-3}+21 \geq 3(a+b+c)(a^{-1}+b^{-1}+c^{-1}).$$

3.24. If a, b, c are positive real numbers such that $abc=1$, then

$$a^2+b^2+c^2-ab-bc-ca \geq \frac{9}{4}(a+b+c-3).$$

3.25. If a, b, c are positive real numbers such that $abc=1$, then

$$a^2+b^2+c^2+a+b+c \geq 2(ab+bc+ca).$$

3.26. If a, b, c are positive real numbers such that $abc=1$, then

$$a^2+b^2+c^2+15(ab+bc+ca) \geq 16(a+b+c).$$

3.27. If a, b, c are positive real numbers such that $abc=1$, then

$$\frac{2}{a+b+c}+\frac{1}{3} \geq \frac{3}{ab+bc+ca}.$$

3.28. If a, b, c are positive real numbers such that $abc=1$, then

$$ab+bc+ca+\frac{6}{a+b+c} \geq 5.$$

3.29. If a, b, c are positive real numbers such that $abc = 1$, then

$$\sqrt[3]{(1+a)(1+b)(1+c)} \ge \sqrt[4]{4(1+a+b+c)}.$$

3.30. If a, b, c are positive real numbers, then

$$a^6 + b^6 + c^6 - 3a^2b^2c^2 \ge 18(a^2 - bc)(b^2 - ca)(c^2 - ab).$$

3.31. If a, b, c are positive real numbers such that $a + b + c = 3$, then

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \ge a^2 + b^2 + c^2.$$

3.32. If a, b, c are positive real numbers such that $ab + bc + ca = 3$, then

$$a^3 + b^3 + c^3 + 7abc \ge 10.$$

3.33. If a, b, c are nonnegative real numbers such that $a^3 + b^3 + c^3 = 3$, then

$$a^4b^4 + b^4c^4 + c^4a^4 \le 3.$$

3.34. If a, b, c are nonnegative real numbers, then

$$(a+1)^2(b+1)^2(c+1)^2 \ge 4(a+b+c)(ab+bc+ca) + 28abc.$$

3.35. If a, b, c are positive real numbers such that $a + b + c = 3$, then

$$1 + 8abc \ge 9\min\{a, b, c\}.$$

3.36. If a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 3$, then

$$1 + 4abc \ge 5\min\{a, b, c\}.$$

3.37. If a, b, c are positive real numbers such that $a+b+c=abc$, then

$$(1-a)(1-b)(1-c)+(\sqrt{3}-1)^3 \geq 0.$$

3.38. If a, b, c are nonnegative real numbers such that $a+b+c=2$, then

$$(a^2+bc)(b^2+ca)(c^2+ab) \leq 1.$$

3.39. If a, b, c are nonnegative real numbers, then

$$(8a^2+bc)(8b^2+ca)(8c^2+ab) \leq (a+b+c)^6.$$

3.40. If a, b, c are positive real numbers such that $a^2b^2+b^2c^2+c^2a^2=3$, then

$$a+b+c \geq abc+2.$$

3.41. Let a, b, c be nonnegative real numbers such that $a+b+c=5$. Prove that

$$(a^2+3)(b^2+3)(c^2+3) \geq 192.$$

3.42. If a, b, c are nonnegative real numbers, then

$$a^2+b^2+c^2+abc+2 \geq a+b+c+ab+bc+ca.$$

3.43. If a, b, c are nonnegative real numbers, then

$$\sum a^3(b+c)(a-b)(a-c) \geq 3(a-b)^2(b-c)^2(c-a)^2.$$

3.44. Find the greatest real number k such that

$$a+b+c+4abc \geq k(ab+bc+ca)$$

for all $a, b, c \in [0,1]$.

3.45. If $a,b,c \ge \frac{2}{3}$ such that $a+b+c=3$, then

$$a^2b^2+b^2c^2+c^2a^2 \ge ab+bc+ca.$$

3.46. If a,b,c are positive real numbers such that $a \le 1 \le b \le c$ and

$$a+b+c=3,$$

then

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ge a^2+b^2+c^2.$$

3.47. If a,b,c are positive real numbers such that $a \le 1 \le b \le c$ and

$$a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c},$$

then

$$a^2+b^2+c^2 \le \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}.$$

3.48. If a,b,c are positive real numbers such that

$$a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c},$$

then

$$(1-abc)\left(a^n+b^n+c^n-\frac{1}{a^n}-\frac{1}{b^n}-\frac{1}{c^n}\right) \ge 0$$

for any integer $n \ge 2$.

3.49. Let a,b,c be positive real numbers, and let

$$E(a,b,c)=a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b).$$

Prove that

(a) $\quad (a+b+c)E(a,b,c) \ge ab(a-b)^2+bc(b-c)^2+ca(c-a)^2;$

(b) $\quad 2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)E(a,b,c) \ge (a-b)^2+(b-c)^2+(c-a)^2.$

3.50. Let $a \ge b \ge c$ be nonnegative real numbers. Schur's inequalities of third and fourth degree state that

(a) $$a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b) \ge 0;$$

(b) $$a^2(a-b)(a-c)+b^2(b-c)(b-a)+c^2(c-a)(c-b) \ge 0.$$

Prove that (a) is sharper than (b) if

$$\sqrt{b}+\sqrt{c} \le \sqrt{a},$$

and (b) is sharper than (a) if

$$\sqrt{b}+\sqrt{c} \ge \sqrt{a}.$$

3.51. If a, b, c are nonnegative real numbers such that

$$(a+b)(b+c)(c+a)=8,$$

then

$$\sqrt{a}+\sqrt{b}+\sqrt{c} \ge ab+bc+ca.$$

3.52. If $a, b, c \in [1,\ 4+3\sqrt{2}]$, then

$$9(ab+bc+ca)(a^2+b^2+c^2) \ge (a+b+c)^4.$$

3.53. If a, b, c are nonnegative real numbers such that $a+b+c+abc=4$, then

(a) $$a^2+b^2+c^2+12 \ge 5(ab+bc+ca);$$

(b) $$3(a^2+b^2+c^2)+13(ab+bc+ca) \ge 48.$$

3.54. Let a, b, c be the lengths of the sides of a triangle. If $a^2+b^2+c^2=3$, then

$$ab+bc+ca \ge 1+2abc.$$

3.55. Let a, b, c be the lengths of the sides of a triangle. If $a^2+b^2+c^2=3$, then

$$a^2b^2+b^2c^2+c^2a^2 \ge ab+bc+ca.$$

3.56. Let a, b, c be the lengths of the sides of a triangle. If $a + b + c = 3$, then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{41}{6} \geq 3(a^2 + b^2 + c^2).$$

3.57. Let $a \leq b \leq c$ such that $a + b + c = p$ and $ab + bc + ca = q$, where p and q are fixed real numbers satisfying $p^2 \geq 3q$.

(a) If a, b, c are nonnegative real numbers, then the product $r = abc$ is maximal when $a = b$, and is minimal when $b = c$ or $a = 0$;

(b) If a, b, c are the lengths of the sides of a triangle (non-degenerate or degenerate), then the product $r = abc$ is maximal when $a = b \geq \frac{c}{2}$ or $a + b = c$, and is minimal when $b = c \geq a$.

3.58. If a, b, c are positive real numbers such that $a + b + c = 3$, then

$$\frac{9}{abc} + 16 \geq \frac{75}{ab + bc + ca}.$$

3.59. If a, b, c are positive real numbers such that $a + b + c = 3$, then

$$8\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + 9 \geq 10(a^2 + b^2 + c^2).$$

3.60. If a, b, c are positive real numbers such that $a + b + c = 3$, then

$$7(a^2 + b^2 + c^2) + 8(a^2b^2 + b^2c^2 + c^2a^2) + 4a^2b^2c^2 \geq 49.$$

3.61. If a, b, c are nonnegative real numbers, then

$$(a^3 + b^3 + c^3 + abc)^2 \geq 2(a^2 + b^2)(b^2 + c^2)(c^2 + a^2).$$

3.62. If a, b, c are nonnegative real numbers, then

$$[ab(a + b) + bc(b + c) + ca(c + a)]^2 \geq 4(ab + bc + ca)(a^2b^2 + b^2c^2 + c^2a^2).$$

3.63. Let a, b, c be nonnegative real numbers such that $ab + bc + ca = 3$. Prove that

$$4(a^3 + b^3 + c^3) + 7abc + 125 \geq 48(a + b + c).$$

3.64. If $a, b, c \in [0, 1]$, then

$$a\sqrt{a} + b\sqrt{b} + c\sqrt{c} + 4abc \geq 2(ab + bc + ca).$$

3.65. If $a, b, c \in [0, 1]$, then

$$a\sqrt{a} + b\sqrt{b} + c\sqrt{c} \geq \frac{3}{2}(ab + bc + ca - abc).$$

3.66. If $a, b, c \in [0, 1]$, then

$$3(a\sqrt{a} + b\sqrt{b} + c\sqrt{c}) + \frac{500}{81}abc \geq 5(ab + bc + ca).$$

3.67. Let a, b, c be the lengths of the sides of a triangle. If $a^2 + b^2 + c^2 = 3$, then

$$a + b + c \geq 2 + abc.$$

3.68. Let $f_n(a, b, c)$ be a symmetric homogeneous polynomial of degree $n \leq 5$. Prove that

(a) the inequality $f_n(a, b, c) \geq 0$ holds for all nonnegative real numbers a, b, c if and only if $f_n(a, 1, 1) \geq 0$ and $f_n(0, b, c) \geq 0$ for all nonnegative real numbers a, b, c;

(b) the inequality $f_n(a, b, c) \geq 0$ holds for all the lengths a, b, c of the sides of a non-degenerate or degenerate triangle if and only if $f_n(x, 1, 1) \geq 0$ for $0 \leq x \leq 2$, and $f_n(y + z, y, z) \geq 0$ for all $y, z \geq 0$.

3.69. If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then

$$4(a^4 + b^4 + c^4) + 45 \geq 19(a^2 + b^2 + c^2).$$

3.70. Let a,b,c be nonnegative real numbers. If $k \le 2$, then

$$\sum a(a-b)(a-c)(a-kb)(a-kc) \ge 0.$$

3.71. Let a,b,c be nonnegative real numbers. If $k \in \mathbf{R}$, then

$$\sum (b+c)(a-b)(a-c)(a-kb)(a-kc) \ge 0.$$

3.72. If a,b,c are nonnegative real numbers, then

$$\sum a(a-2b)(a-2c)(a-5b)(a-5c) \ge 0.$$

3.73. If a,b,c are the lengths of the sides of a triangle, then

$$a^4+b^4+c^4+9abc(a+b+c) \le 10(a^2b^2+b^2c^2+c^2a^2).$$

3.74. If a,b,c are the lengths of the sides of a triangle, then

$$3(a^4+b^4+c^4)+7abc(a+b+c) \le 5\sum ab(a^2+b^2).$$

3.75. If a,b,c are the lengths of the sides of a triangle, then

$$\frac{b^2+c^2-6bc}{a}+\frac{c^2+a^2-6ca}{b}+\frac{a^2+b^2-6ab}{c}+4(a+b+c) \le 0.$$

3.76. Let $f_6(a,b,c)$ be a sixth degree symmetric homogeneous polynomial written in the form

$$f_6(a,b,c) = Ar^2 + B(p,q)r + C(p,q), \quad A \le 0,$$

where

$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc.$$

Prove that

(a) the inequality $f_6(a,b,c) \ge 0$ holds for all nonnegative real numbers a,b,c if and only if $f_6(a,1,1) \ge 0$ and $f_6(0,b,c) \ge 0$ for all nonnegative real numbers a,b,c;

(b) the inequality $f_6(a,b,c) \ge 0$ holds for all lengths a,b,c of the sides of a non-degenerate or degenerate triangle if and only if $f_6(x,1,1) \ge 0$ for $0 \le x \le 2$, and $f_6(y+z,y,z) \ge 0$ for all $y,z \ge 0$.

3.77. If a,b,c are nonnegative real numbers, then

$$\sum a(b+c)(a-b)(a-c)(a-2b)(a-2c) \ge (a-b)^2(b-c)^2(c-a)^2.$$

3.78. Let a,b,c be nonnegative real numbers.

(a) If $2 \le k \le 6$, then

$$\sum a(a-b)(a-c)(a-kb)(a-kc) + \frac{4(k-2)(a-b)^2(b-c)^2(c-a)^2}{a+b+c} \ge 0;$$

(b) If $k \ge 6$, then

$$\sum a(a-b)(a-c)(a-kb)(a-kc) + \frac{(k+2)^2(a-b)^2(b-c)^2(c-a)^2}{4(a+b+c)} \ge 0.$$

3.79. If a,b,c are nonnegative real numbers, then

$$(3a^2+2ab+3b^2)(3b^2+2bc+3c^2)(3c^2+2ca+3a^2) \ge 8(a^2+3bc)(b^2+3ca)(c^2+3ab).$$

3.80. Let a,b,c be nonnegative real numbers such that $a+b+c=2$. If

$$\frac{-2}{3} \le k \le \frac{11}{8},$$

then

$$(a^2+kab+b^2)(b^2+kbc+c^2)(c^2+kca+a^2) \le k+2.$$

3.81. Let a,b,c be nonnegative real numbers such that $a+b+c=2$. Prove that

$$(2a^2+bc)(2b^2+ca)(2c^2+ab) \le 4.$$

3.82. Let a,b,c be nonnegative real numbers, no two of which are zero. Then,

$$\sum (a-b)(a-c)(a-2b)(a-2c) \ge \frac{5(a-b)^2(b-c)^2(c-a)^2}{ab+bc+ca}.$$

3.83. Let $a \le b \le c$ be positive real numbers such that

$$a+b+c=p, \quad abc=r,$$

where p and r are fixed positive numbers satisfying $p^3 \ge 27r$. Prove that

$$q=ab+bc+ca$$

is maximal when $b=c$, and is minimal when $a=b$.

3.84. If a,b,c are positive real numbers, then

$$(a+b+c-3)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-3\right)+abc+\frac{1}{abc} \ge 2.$$

3.85. If a,b,c are positive real numbers such that $abc=1$, then

(a) $$\frac{3}{7}\left(ab+bc+ca-\frac{2}{3}\right) \ge \sqrt{\frac{2}{3}(a+b+c)-1};$$

(b) $$ab+bc+ca-3 \ge \frac{46}{27}(\sqrt{a+b+c-2}-1).$$

3.86. If a,b,c are positive real numbers such that $abc=1$, then

$$ab+bc+ca+\frac{50}{a+b+c+5} \ge \frac{37}{4}.$$

3.87. Let a,b,c be positive real numbers.

(a) If $abc=2$, then

$$(a+b+c-3)^2+1 \ge \frac{a^2+b^2+c^2}{3};$$

(b) If $abc=\frac{1}{2}$, then

$$a^2+b^2+c^2+3(3-a-b-c)^2 \ge 3.$$

3.88. If a, b, c are positive real numbers such that $a + b + c = 3$, then

$$4\left(\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\right)+9abc \geq 21.$$

3.89. If a, b, c are nonnegative real numbers such that

$$ab + bc + ca = abc + 2,$$

then

$$a^2 + b^2 + c^2 + abc \geq 4.$$

3.90. If a, b, c are positive real numbers, then

$$\left(\frac{b+c}{a}-2-\sqrt{2}\right)^2+\left(\frac{c+a}{b}-2-\sqrt{2}\right)^2+\left(\frac{a+b}{c}-2-\sqrt{2}\right)^2 \geq 6.$$

3.91. If a, b, c are positive real numbers, then

$$2(a^3 + b^3 + c^3) + 9(ab + bc + ca) + 39 \geq 24(a + b + c).$$

3.92. If a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 3$, then

$$a^3 + b^3 + c^3 - 3 \geq |(a - b)(b - c)(c - a)|.$$

3.93. If a, b, c are nonnegative real numbers, then

$$a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2 \geq 2|a^3b + b^3c + c^3a - ab^3 - bc^3 - ca^3|.$$

3.94. If a, b, c are nonnegative real numbers, then

$$a^4 + b^4 + c^4 - abc(a + b + c) \geq 2\sqrt{2}\,|a^3b + b^3c + c^3a - ab^3 - bc^3 - ca^3|.$$

3.95. If $a, b, c \geq -5$ such that $a + b + c = 3$, then

$$\frac{1-a}{1+a+a^2}+\frac{1-b}{1+b+b^2}+\frac{1-c}{1+c+c^2} \geq 0.$$

3.96. Let $a, b, c \neq \frac{1}{k}$ be nonnegative real numbers such that $a+b+c=3$. If $k \geq \frac{4}{3}$, then

$$\frac{1-a}{(1-ka)^2}+\frac{1-b}{(1-kb)^2}+\frac{1-c}{(1-kc)^2} \geq 0.$$

3.97. Let a, b, c, d be nonnegative real numbers such that

$$a^2+b^2+c^2+d^2=1.$$

Prove that

$$(1-a)(1-b)(1-c)(1-d) \geq abcd.$$

3.98. Let a, b, c, d and x be positive real numbers such that

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}=\frac{4}{x^2}.$$

If $x \geq 2$, then

$$(a-1)(b-1)(c-1)(d-1) \geq (x-1)^4.$$

3.99. If a, b, c, d are positive real numbers, then

$$\frac{(1+a^3)(1+b^3)(1+c^3)(1+d^3)}{(1+a^2)(1+b^2)(1+c^2)(1+d^2)} \geq \frac{1+abcd}{2}.$$

3.100. Let a, b, c, d be positive real numbers such that $a+b+c+d=4$. Prove that

$$\left(a+\frac{1}{a}-1\right)\left(b+\frac{1}{b}-1\right)\left(c+\frac{1}{c}-1\right)\left(d+\frac{1}{d}-1\right)+3 \geq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}.$$

3.101. If a, b, c, d are nonnegative real numbers, then

$$4(a^3+b^3+c^3+d^3)+15(abc+bcd+cda+dab) \geq (a+b+c+d)^3.$$

3.102. Let a, b, c, d be positive real numbers such that

$$a+b+c+d=4.$$

Prove that

$$1+2(abc+bcd+cda+dab) \geq 9\min\{a, b, c, d\}.$$

3.103. Let a,b,c,d be nonnegative real numbers such that

$$a+b+c+d=4.$$

Prove that

$$5(a^2+b^2+c^2+d^2)\ge a^3+b^3+c^3+d^3+16.$$

3.104. Let a,b,c,d be nonnegative real numbers such that

$$a+b+c+d=4.$$

Prove that

$$3(a^2+b^2+c^2+d^2)+4abcd\ge 16.$$

3.105. Let a,b,c,d be nonnegative real numbers such that

$$a+b+c+d=4.$$

Prove that

$$27(abc+cd+cda+dab)\le 44abcd+64.$$

3.106. Let a,b,c,d be positive real numbers such that

$$a+b+c+d=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}.$$

Prove that

$$(1-abcd)\left(a^2+b^2+c^2+d^2-\frac{1}{a^2}-\frac{1}{b^2}-\frac{1}{c^2}-\frac{1}{d^2}\right)\ge 0.$$

3.107. Let a,b,c,d be positive real numbers such that

$$a+b+c+d=1.$$

Prove that

$$(1-a)(1-b)(1-c)(1-d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\ge\frac{81}{16}.$$

3.108. Let a,b,c,d be nonnegative real numbers such that

$$a+b+c+d=a^3+b^3+c^3+d^3=2.$$

Prove that

$$a^2+b^2+c^2+d^2 \geq \frac{7}{4}.$$

3.109. Let $a,b,c,d \in (0,4]$ such that $abcd=1$. Prove that

$$(1+2a)(1+2b)(1+2c)(1+2d) \geq (5-2a)(5-2b)(5-2c)(5-2d).$$

3.110. Let a,b,c,d and k be positive real numbers such that

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)=k.$$

If $16 \leq k \leq (1+\sqrt{10})^2$, then any three of a,b,c,d are the lengths of the sides of a triangle (non-degenerate or degenerate).

3.111. Let a,b,c,d and k be positive real numbers such that

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)=k.$$

If $16 \leq k \leq \frac{119}{6}$, then there exist three numbers of a,b,c,d which are the lengths of the sides of a triangle (non-degenerate or degenerate).

3.112. Let a,b,c,d and k be positive real numbers such that

$$(a+b+c+d)^2=k(a^2+b^2+c^2+d^2).$$

If $\frac{11}{3} \leq k \leq 4$, then any three of a,b,c,d are the lengths of the sides of a triangle (non-degenerate or degenerate).

3.113. Let a,b,c,d and k be positive real numbers such that

$$(a+b+c+d)^2=k(a^2+b^2+c^2+d^2).$$

If $\frac{49}{15} \leq k \leq 4$, then there exist three numbers of a,b,c,d which are the lengths of the sides of a triangle (non-degenerate or degenerate).

3.114. Let a,b,c,d,e be nonnegative real numbers.

(a) If $a+b+c=3(d+e)$, then

$$4(a^4+b^4+c^4+d^4+e^4)\ge(a^2+b^2+c^2+d^2+e^2)^2;$$

(b) If $a+b+c=d+e$, then

$$12(a^4+b^4+c^4+d^4+e^4)\le 7(a^2+b^2+c^2+d^2+e^2)^2.$$

3.115. Let a,b,c,d,e be nonnegative real numbers such that

$$a+b+c+d+e=5.$$

Prove that

$$a^4+b^4+c^4+d^4+e^4+150\le 31(a^2+b^2+c^2+d^2+e^2).$$

3.116. Let a,b,c,d,e be positive real numbers such that

$$a^2+b^2+c^2+d^2+e^2=5.$$

Prove that

$$abcde(a^4+b^4+c^4+d^4+e^4)\le 5.$$

3.117. Let a,b,c,d,e be positive real numbers such that

$$a+b+c+d+e=5.$$

Prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}+\frac{20}{a^2+b^2+c^2+d^2+e^2}\ge 9.$$

3.118. If $a,b,c,d,e\ge 1$, then

$$\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right)\left(d+\frac{1}{d}\right)\left(e+\frac{1}{e}\right)+68\ge$$

$$\ge 4(a+b+c+d+e)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right).$$

3.119. Let a, b, c and x, y, z be positive real numbers such that

$$(a+b+c)(x+y+z)=(a^2+b^2+c^2)(x^2+y^2+z^2)=4.$$

Prove that

$$abcxyz<\frac{1}{36}.$$

3.120. Let $a_1, a_2, \cdots, a_n$ $(n\geq 3)$ be positive real numbers such that

$$a_1+a_2+\cdots+a_n=a_1^2+a_2^2+\cdots+a_n^2=n-1.$$

Prove that

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\geq\frac{n^2(2n-3)}{2(n-1)(n-2)}.$$

3.121. Let $a_1, a_2, \cdots, a_n$ be positive real numbers such that $a_1+a_2+\cdots+a_n=n$. Prove that

$$n^2\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}-n\right)\geq 4(n-1)(a_1^2+a_2^2+\cdots+a_n^2-n).$$

3.122. Let $a_1, a_2, \cdots, a_n$ be nonnegative real numbers such that $a_1+a_2+\cdots+a_n=n$. Prove that

$$(n+1)(a_1^2+a_2^2+\cdots+a_n^2)\geq n^2+a_1^3+a_2^3+\cdots+a_n^3.$$

3.123. Let $a_1, a_2, \cdots, a_n$ be nonnegative real numbers such that $a_1+a_2+\cdots+a_n=n$. Prove that

$$(n-1)(a_1^3+a_2^3+\cdots+a_n^3)+n^2\geq(2n-1)(a_1^2+a_2^2+\cdots+a_n^2).$$

3.124. Let $a_1, a_2, \ldots, a_n$ $(n\geq 3)$ be positive real numbers such that

$$a_1+a_2+\cdots+a_n=1.$$

Prove that

$$\left(a_1+\frac{1}{a_1}-2\right)\left(a_2+\frac{1}{a_2}-2\right)\left(a_n+\frac{1}{a_n}-2\right)\geq\left(n+\frac{1}{n}-2\right)^n.$$

3.125. Let $a_1, a_2, \dots, a_n$ $(n \ge 3)$ be positive real numbers such that

$$a_1 + a_2 + \cdots + a_n = n.$$

Prove that

$$a_1^2 + a_2^2 + \cdots + a_n^2 - n \ge \frac{n}{n-1}(1 - a_1 a_2 \cdots a_n).$$

3.126. Let $a_1, a_2, \dots, a_n$ $(n \ge 3)$ be positive real numbers such that

$$a_1 \le a_2 \le \cdots \le a_n,$$

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) = k.$$

(a) If $n^2 \le k \le n^2 + \frac{i(n-i)}{2}$, $i \in \{2, 3, \cdots, n-1\}$, then a_{i-1}, a_i and a_{i+1} are the lengths of the sides of a non-degenerate or degenerate triangle;

(b) If $n^2 \le k \le \alpha_n$, where $\alpha_n = \frac{9n^2}{8}$ for even n, and $\alpha_n = \frac{9n^2 - 1}{8}$ for odd n, then there exist three numbers a_i which are the lengths of the sides of a non-degenerate or degenerate triangle.

3.127. Let $a_1, a_2, \dots, a_n$ $(n \ge 3)$ be positive real numbers such that

$$a_1 \le a_2 \le \cdots \le a_n,$$

$$(a_1 + a_2 + \cdots + a_n)^2 = k(a_1^2 + a_2^2 + \cdots + a_n^2).$$

(a) If $\frac{(2n-i)^2}{4n-3i} \le k \le n$, $i \in \{2, 3, \cdots, n-1\}$, then a_{i-1}, a_i and a_{i+1} are the lengths of the sides of a non-degenerate or degenerate triangle;

(b) If $\frac{8n+1}{9} \le k \le n$, then there exist three numbers a_i which are the lengths of the sides of a non-degenerate or degenerate triangle.

3.2 Solutions

P 3.1. *If a,b,c are positive real numbers, then*

$$a^2+b^2+c^2+2abc+1 \ge 2(ab+bc+ca).$$

(Darij Grinberg, 2004)

First Solution. Setting $a=x^3$, $b=y^3$ and $c=z^3$, where $x,y,z>0$, we need to prove that

$$x^6+y^6+z^6+2x^3y^3z^3+1 \ge 2(x^3y^3+y^3z^3+z^3x^3).$$

Using Schur's inequality and the AM-GM inequality, we have

$$x^6+y^6+z^6+3x^2y^2z^2 \ge \sum x^2y^2(x^2+y^2) \ge 2\sum x^3y^3.$$

Thus, it suffices to show that

$$2x^3y^3z^3-3x^2y^2z^2+1 \ge 0,$$

which is equivalent to

$$(xyz-1)^2(2xyz+1) \ge 0.$$

The equality holds for $a=b=c=1$.

Second Solution. Among the numbers $1-a$, $1-b$ and $1-c$ there are always two with the same sign; let us say $(1-b)(1-c)\ge 0$. Then, we have

$$a^2+b^2+c^2+2abc+1-2(ab+bc+ca)=$$
$$=(a-1)^2+(b-c)^2+2a+2abc-2a(b+c)$$
$$=(a-1)^2+(b-c)^2+2a(1-b)(1-c) \ge 0.$$

Remark. The following generalization holds:

- *Let a,b,c be positive real numbers. If $0\le k\le 1$, then*

$$a^2+b^2+c^2+2kabc+k \ge (k+1)(ab+bc+ca).$$

Since the both sides of the inequality are linear of k, it suffices to prove it for only $k=0$ and $k=1$. For $k=0$, the inequality reduces to the the known $a^2+b^2+c^2\ge ab+bc+ca$.

□

P 3.2. *Let a,b,c be nonnegative real numbers. If $0\le k\le\sqrt{2}$, then*

$$a^2+b^2+c^2+kabc+2k+3 \ge (k+2)(a+b+c).$$

Solution. Since the both sides of the inequality are linear of k, it suffices to prove the inequality for $k=0$ and $k=\sqrt{2}$. For $k=0$, the inequality reduces to

$$(a-1)^2+(b-1)^2+(c-1)^2 \ge 0.$$

Further, we consider that $k=\sqrt{2}$.

First Solution. Write the inequality as

$$(a-1)^2+(b-1)^2+(c-1)^2 \ge \sqrt{2}\,(a+b+c-2-abc).$$

Using the substitutions $x=a-1$, $y=b-1$ and $z=c-1$, we need to show that

$$x^2+y^2+z^2+\sqrt{2}\,(xyz+xy+yz+zx) \ge 0$$

for $x,y,z \ge -1$. Among the numbers x, y and z there are always two of them with the same sign; let us say $yz \ge 0$. Since

$$y^2+z^2 \ge \frac{1}{2}(y+z)^2$$

and

$$xyz+xy+yz+zx=(x+1)yz+x(y+z) \ge x(y+z),$$

it suffices to prove that

$$x^2+\frac{1}{2}(y+z)^2+\sqrt{2}\,x(y+z) \ge 0,$$

which is equivalent to

$$[x+\frac{1}{\sqrt{2}}(y+z)]^2 \ge 0.$$

Thus, the proof is completed. The equality holds for $a=b=c=1$. In addition, if $k=\sqrt{2}$, then the equality holds also for $a=0$ and $b=c=1+1/\sqrt{2}$ (or any cyclic permutation).

Second Solution. Let $a+b+c=p$. For the nontrivial case $p>0$, we write the original inequality in the homogeneous form

$$\frac{(a+b+c)(a^2+b^2+c^2)}{p}+kabc+\frac{(2k+3)(a+b+c)^3}{p^3} \ge \frac{(k+2)(a+b+c)^3}{p^2},$$

or

$$[p^2-(k+2)p+2k+3](a+b+c)^3+kp^3abc-2p^2(a+b+c)(ab+bc+ca) \ge 0.$$

Since $p^2-(k+2)p+2k+3>0$ for $k=\sqrt{2}$, using Schur's inequality

$$(a+b+c)^3 \ge -9abc+4(a+b+c)(ab+bc+ca),$$

it suffices to show that

$$2[p^2-2(k+2)p+2(2k+3)](a+b+c)(ab+bc+ca)+$$
$$+[kp^3-9p^2+9(k+2)p-9(2k+3)]abc\geq 0.$$

Since

$$p^2-2(k+2)p+2(2k+3)=(p-k-2)^2\geq 0$$

and

$$(a+b+c)(ab+bc+ca)\geq 9abc,$$

it is enough to prove that

$$18[p^2-2(k+2)p+2(2k+3)]+kp^3-9p^2+9(k+2)p-9(2k+3)\geq 0;$$

that is,

$$kp^3+9p^2-27(k+2)p+27(2k+3)\geq 0,$$
$$(kp+6k+9)(p-3)^2\geq 0.$$

□

P 3.3. *If a,b,c are positive real numbers, then*

$$abc(a+b+c)+2(a^2+b^2+c^2)+3\geq 4(ab+bc+ca).$$

First Solution. Applying the AM-GM inequality two times, we get

$$abc(a+b+c)+3\geq 2\sqrt{3abc(a+b+c)}\geq \frac{18abc}{a+b+c}.$$

Therefore, it suffices to prove that

$$a^2+b^2+c^2+\frac{18abc}{a+b+c}\geq 2(ab+bc+ca),$$

which is just Schur's inequality of third degree. The equality holds for $a=b=c=1$.

Second Solution. Applying the AM-GM, we get

$$abc(a+b+c)+3=(a^2bc+1)+(ab^2c+1)+(abc^2+1)\geq 2a\sqrt{bc}+2b\sqrt{ca}+2c\sqrt{ab}.$$

Thus, it suffices to prove that

$$a^2+b^2+c^2+a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}\geq 2(ab+bc+ca).$$

Substituting $x = \sqrt{a}$, $y = \sqrt{b}$, $z = \sqrt{c}$, where $x, y, z \ge 0$, we need to show that

$$x^4 + y^4 + z^4 + xyz(x + y + z) \ge 2(x^2y^2 + y^2z^2 + z^2x^2).$$

This inequality can be obtained by summing Schur's inequality of degree four

$$x^4 + y^4 + z^4 + xyz(x + y + z) \ge xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2)$$

to

$$xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2) \ge 2(x^2y^2 + y^2z^2 + z^2x^2),$$

which is equivalent to

$$xy(x - y)^2 + yz(y - z)^2 + zx(z - x)^2 \ge 0.$$

□

P 3.4. *If a, b, c are positive real numbers, then*

$$a(b^2 + c^2) + b(c^2 + a^2) + c(a^2 + b^2) + 3 \ge 3(ab + bc + ca).$$

Solution. Write the inequality as follows

$$(a + b + c)(ab + bc + ca) + 3 \ge 3(abc + ab + bc + ca).$$

$$(a + b + c - 3)(ab + bc + ca) + 3 \ge 3abc.$$

Using the known inequality

$$(a + b + c)(ab + bc + ca) \ge 9abc,$$

it suffices to show that

$$3(a + b + c - 3)(ab + bc + ca) + 9 \ge (a + b + c)(ab + bc + ca),$$

which is equivalent to

$$[2(a + b + c) - 9](ab + bc + ca) + 9 \ge 0.$$

For the nontrivial case $2(a + b + c) - 9 < 0$, using the known inequality

$$(a + b + c)^2 \ge 3(ab + bc + ca),$$

it is enough to show that

$$[2(a + b + c) - 9](a + b + c)^2 + 27 \ge 0.$$

This inequality is equivalent to the obvious inequality

$$(a + b + c - 3)^2[2(a + b + c) + 3] \ge 0.$$

The equality holds for $a = b = c = 1$.

□

P 3.5. *If a, b, c are positive real numbers, then*

$$\left(\frac{a^2+b^2+c^2}{3}\right)^3 \geq a^2b^2c^2+(a-b)^2(b-c)^2(c-a)^2.$$

(Vasile Cîrtoaje, 2011)

Solution *(by Vo Quoc Ba Can)*. Assume that $a = \min\{a, b, c\}$. By virtue of the AM-GM inequality, we have

$$\left(\frac{a^2+b^2+c^2}{3}\right)^3 = \left[\frac{(a^2+b^2+c^2-2bc)+bc+bc}{3}\right]^3 \geq (a^2+b^2+c^2-2bc)b^2c^2$$
$$= a^2b^2c^2+(b-c)^2b^2c^2.$$

Thus, it suffices to prove that

$$(b-c)^2b^2c^2 \geq (b-c)^2(b-a)^2(c-a)^2.$$

This is obvious, because $b^2 > (b-a)^2$ and $c^2 > (c-a)^2$. The equality occurs for $a = b = c$.

□

P 3.6. *If a, b, c are nonnegative real numbers, then*

$$[ab(a+b)+bc(b+c)+ca(c+a)]^2 \geq 4(ab+bc+ca)(a^2b^2+b^2c^2+c^2a^2).$$

(Vasile Cîrtoaje and Vo Quoc Ba Can, 2011)

First Solution. Assume that $a \geq b \geq c$. For the nontrivial case $b > 0$, by the AM-GM inequality, we have

$$4(ab+bc+ca)(a^2b^2+b^2c^2+c^2a^2) \leq \left[b(ab+bc+ca)+\frac{a^2b^2+b^2c^2+c^2a^2}{b}\right]^2.$$

Thus, it suffices to prove that

$$ab(a+b)+bc(b+c)+ca(c+a) \geq b(ab+bc+ca)+\frac{a^2b^2+b^2c^2+c^2a^2}{b}.$$

This inequality reduces to the obvious form

$$ac(a-b)(b-c) \geq 0.$$

The equality holds for $a = b = c$, for $b = c = 0$ (or any cyclic permutation), and for $a = 0$ and $b = c$ (or any cyclic permutation).

Second Solution. We well prove the stronger inequality

$$[ab(a+b)+bc(b+c)+ca(c+a)]^2 \geq 4(ab+bc+ca)(a^2b^2+b^2c^2+c^2a^2)+A,$$

where

$$A=(a-b)^2(b-c)^2(c-a)^2.$$

Let $p=a+b+c$, $q=ab+bc+ca$, $r=abc$. Since

$$(a-b)^2(b-c)^2(c-a)^2=-27r^2+2(9pq-2p^3)r+p^2q^2-4q^3,$$

we can write this inequality as

$$(pq-3r)^2 \geq 4q(q^2-2pr)-27r^2+2(9pq-2p^3)r+p^2q^2-4q^3,$$

which reduces to

$$r(p^3+9r-4pq) \geq 0.$$

This is true since $p^3+9r-4pq \geq 0$ (by the third degree Schur's inequality).

□

P 3.7. *If a,b,c are positive real numbers, then*

(a) $$a^3+b^3+c^3+ab+bc+ca+9 \geq 5(a+b+c);$$

(b) $$a^3+b^3+c^3+4(ab+bc+ca)+18 \geq 11(a+b+c).$$

(Vasile Cîrtoaje, 2010)

Solution. Let $p=a+b+c$ and $q=ab+bc+ca$. From

$$a(a-1)^2+b(b-1)^2+c(c-1)^2 \geq 0,$$

we get

$$a^3+b^3+c^3 \geq 2(a^2+b^2+c^2)-a-b-c=2p^2-p-4q.$$

(a) Using the result above and the known inequality $p^2 \geq 3q$, we have

$$\begin{aligned} a^3+b^3+c^3+ab+bc+ca+9-5(a+b+c) &\geq \\ \geq (2p^2-p-4q)+q+9-5p &= 2p^2-6p+9-3q \\ \geq 2p^2-6p+9-p^2 &= (p-3)^2 \geq 0. \end{aligned}$$

The equality holds for $a=b=c=1$.

(b) Using the result above, we have

$$\begin{aligned} a^3+b^3+c^3+4(ab+bc+ca)+18-11(a+b+2 &\geq \\ &\geq (2p^2-p-4q)+4q+18-11p = 2p^2-12p+18 \\ &= 2(p-3)^3 \geq 0. \end{aligned}$$

The equality holds for $a=b=c=1$.

□

P 3.8. *If a,b,c are positive real numbers, then*

(a) $$a^3+b^3+c^3+abc+8 \geq 4(a+b+c);$$

(b) $$4(a^3+b^3+c^3)+15abc+54 \geq 27(a+b+c).$$

Solution. Let $p=a+b+c$ and $q=ab+bc+ca$. By Schur's inequality of third degree, we have

$$p^3+9abc \geq 4pq,$$

$$abc \geq \frac{p(4q-p^2)}{9}.$$

a) We have

$$\begin{aligned} a^3+b^3+c^3+abc &= 4abc+p(p^2-3q) \\ &\geq \frac{4p(4q-p^2)}{9}+p(p^2-3q) \\ &= \frac{p(5p^2-11q)}{9}. \end{aligned}$$

Then, it suffices to prove that

$$\frac{p(5p^2-11q)}{9}+8 \geq 4p,$$

which is equivalent to

$$5p^3-36p+72 \geq 11pq.$$

Since $p^2 \geq 3q$, we have

$$3(5p^3-36p+72-11pq) \geq 3(5p^3-36p+72)-11p^2$$

$$=4(p^3-27p+54)=4(p-3)^2(p+6) \geq 0.$$

The equality holds for $a=b=c=1$.

(b) We have

$$\begin{aligned}4(a^3+b^3+c^3)+15abc &= 27abc+4p(p^2-3q)\\ &\ge 3p(4q-p^2)+4p(p^2-3q)=p^3.\end{aligned}$$

Then, it suffices to prove that

$$p^3+54\ge 27p,$$

which is equivalent to the obvious inequality

$$(p-3)^2(p+6)\ge 0.$$

The equality holds for $a=b=c=1$, and also for $a=0$ and $b=c=3/2$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization (*Vasile Cîrtoaje*, 2010):

- *Let a,b,c be nonnegative real numbers. If $0\le k\le 27/4$, then*

$$a^3+b^3+c^3+(k-3)abc+2k\ge k(a+b+c).$$

□

P 3.9. *Let a,b,c be nonnegative real numbers such that*

$$a+b+c=a^2+b^2+c^2.$$

Prove that

$$ab+bc+ca\ge a^2b^2+b^2c^2+c^2a^2.$$

(Vasile Cîrtoaje, 2006)

Solution *(by Michael Rozenberg).* From the hypothesis condition, by squaring, we get

$$a^4+b^4+c^4-a^2-b^2-c^2=2(ab+bc+ca-a^2b^2-b^2c^2-c^2a^2).$$

Therefore, we can write the required inequality as

$$a^4+b^4+c^4\ge a^2+b^2+c^2.$$

This inequality has the homogeneous form

$$(a+b+c)^2(a^4+b^4+c^4)\ge(a^2+b^2+c^2)^3,$$

which follows immediately from Hölder's inequality. The equality holds for $a=b=c=1$, for $a=b=c=0$, for $(a,b,c)=(0,1,1)$ (or any cyclic permutation), and for $(a,b,c)=(1,0,0)$ (or any cyclic permutation).

□

P 3.10. *If a, b, c are nonnegative real numbers, then*

$$(a^2 + 2bc)(b^2 + 2ca)(c^2 + 2ab) \geq (ab + bc + ca)^3.$$

(Vasile Cîrtoaje, 2006)

Solution. We have

$$(a^2 + 2bc)(b^2 + 2ca)(c^2 + 2ab) = 9a^2b^2c^2 + 2\sum a^3b^3 + 4abc\sum a^3$$

and

$$(ab + bc + ca)^3 = 6a^2b^2c^2 + \sum a^3b^3 + 3abc\sum ab(a + b).$$

So, we can rewrite the inequality as

$$3a^2b^2c^2 + \sum a^3b^3 + 4abc\sum a^3 \geq 3abc\sum ab(a + b).$$

Since $\sum a^3b^3 \geq 3a^2b^2c^2$ (by the AM-GM inequality), it suffices to prove that

$$6abc + 4\sum a^3 \geq 3\sum ab(a + b).$$

We can get this inequality by summing the inequalities

$$\frac{1}{3}\sum a^3 \geq abc$$

and

$$3abc + \sum a^3 \geq \sum ab(a + b).$$

The first inequality follows from the AM-GM inequality, while the second is just the third degree Schur's inequality. The equality holds when $a = b = c$, and also when two of a, b, c are zero.

Remark. Similarly, we can also prove the following inequality

$$(2a^2 + 7bc)(2b^2 + 7ca)(2c^2 + 7ab) \geq 27(ab + bc + ca)^3.$$

□

P 3.11. *If a, b, c are nonnegative real numbers, then*

$$(2a^2 + bc)(2b^2 + ca)(2c^2 + ab) \geq (ab + bc + ca)^3.$$

(Vasile Cîrtoaje, 2006)

First Solution. Since

$$(2a^2+bc)(2b^2+ca)(2c^2+ab)=9a^2b^2c^2+4\sum a^3b^3+2abc\sum a^3$$

and

$$(ab+bc+ca)^3=6a^2b^2c^2+\sum a^3b^3+3abc\sum ab(a+b),$$

the inequality is equivalent to

$$3a^2b^2c^2+3\sum a^3b^3+2abc\sum a^3\geq 3abc\sum ab(a+b).$$

We can get this inequality by summing

$$\frac{2}{3}abc\sum a^3\geq 2a^2b^2c^2$$

and

$$\sum a^3b^3+3a^2b^2c^2\geq abc\sum ab(a+b).$$

The first inequality follows from the AM-GM inequality, while the second is just the third degree Schur's inequality applied to the numbers ab, bc and ca. The equality holds when $a=b=c$, and also when two of a,b,c are zero.

Second Solution. By Hölder's inequality, we have

$$(a^2+bc+a^2)(b^2+b^2+ca)(ab+c^2+c^2)\geq(ab+bc+ca)^3,$$

from which the desired inequality follows.

Remark. Using the first method, we can also prove the following inequality

$$(5a^2+bc)(5b^2+ca)(5c^2+ab)\geq 8(ab+bc+ca)^3.$$

□

P 3.12. *Let a,b,c be nonnegative real numbers such that $a+b+c=2$. Prove that*

(a) $$(a^2+b^2)(b^2+c^2)(c^2+a^2)\leq(a+b)(b+c)(c+a);$$

(b) $$(a^2+b^2)(b^2+c^2)(c^2+a^2)\leq 2.$$

Solution. Assume that $a = \min\{a, b, c\}$. It is easy to check that the equality holds in both of the inequalities for $a = 0$ and $b = c = 1$.

(a) Since

$$a^2 + b^2 \le b(a + b)$$

and

$$c^2 + a^2 \le c(c + a),$$

it suffices to show that

$$bc(b^2 + c^2) \le b + c.$$

By virtue of the AM-GM inequality and the hypothesis $b + c \le 2$, we have

$$2bc(b^2 + c^2) \le \left[\frac{2bc + (b^2 + c^2)}{2}\right]^2 = \frac{(b + c)^4}{4} \le 2(b + c).$$

The equality holds for $a = 0$ and $b = c = 1$ (or any cyclic permutation).

(b) **First Solution.** Since

$$a^2 + b^2 \le b(a + b)$$

and

$$c^2 + a^2 \le c(a + c),$$

it suffices to show that

$$bc(a + b)(a + c)(b^2 + c^2) \le 2.$$

By the AM-GM inequality, we have

$$4bc(a + b)(a + c)(b^2 + c^2) \le \left[\frac{2b(a + c) + 2c(a + b) + (b^2 + c^2)}{3}\right]^3.$$

Therefore, we need to show that

$$b^2 + c^2 + 4bc + 2ab + 2ac \le 6.$$

This is true since

$$12 - 2(b^2 + c^2 + 4bc + 2ab + 2ac) = 3(a + b + c)^2 - 2(b^2 + c^2 + 4bc + 2ab + 2ac)$$

$$= 2a^2 + b^2 + c^2 - 2bc + 2ab + 2ac = 2a(a + b + c) + (b - c)^2 \ge 0.$$

The equality holds for $a = 0$ and $b = c = 1$ (or any cyclic permutation).

Second Solution. Let us denote

$$F(a, b, c) = (a^2 + b^2)(b^2 + c^2)(c^2 + a^2).$$

We will show that

$$F(a,b,c) \le F(0, b+a/2, c+a/2) \le 2.$$

The left inequality,

$$(a^2+b^2)(b^2+c^2)(c^2+a^2) \le (b+a/2)^2[(b+a/2)^2+(c+a/2)^2](c+a/2)^2,$$

is true since

$$a^2+b^2 \le (b+a/2)^2,$$

$$b^2+c^2 \le (b+a/2)^2+(c+a/2)^2,$$

$$c^2+a^2 \le (c+a/2)^2.$$

The right inequality holds if the original inequality holds for $a=0$; that is,

$$b^2c^2(b^2+c^2) \le 2$$

for $b+c=2$. Indeed, by virtue of the AM-GM inequality, we have

$$bc \le \left(\frac{b+c}{2}\right)^2 = 1,$$

$$2b^2c^2(b^2+c^2) \le 2bc(b^2+c^2) \le \left[\frac{2bc+(b^2+c^2)}{2}\right]^2$$

$$= \frac{(b+c)^4}{4} = 4.$$

□

P 3.13. *Let a,b,c be nonnegative real numbers such that $a+b+c=2$. Prove that*

$$(a^3+b^3)(b^3+c^3)(c^3+a^3) \le 2.$$

Solution. Due to symmetry, we may assume that $a=\min\{a,b,c\}$. It is easy to check that the equality holds for $a=0$ and $b=c=1$. Write the inequality as

$$\left[\prod(a+b)\right]\left[\prod(a^2-ab+b^2)\right] \le 2.$$

Since

$$\prod(a+b) \le (a+b+c)(ab+bc+ca) = 2(ab+bc+ca),$$

it suffices to show that

$$(ab+bc+ca)\prod(a^2-ab+b^2) \le 1.$$

Since

$$a^2 - ab + b^2 \le b^2$$

and

$$c^2 + ca + a^2 \le c^2,$$

it suffices to show that

$$b^2c^2(ab + bc + ca)(b^2 - bc + c^2) \le 1.$$

In virtue of the AM-GM inequality, we have

$$b^2c^2(ab + bc + ca)(b^2 - bc + c^2) \le \left[\frac{bc + bc + (ab + bc + ca) + (b^2 - bc + c^2)}{4}\right]^4.$$

Therefore, it remains to show that

$$b^2 + c^2 + 2bc + ab + ca \le 4.$$

This is true since

$$4 - (b^2 + c^2 + 2bc + ab + ca) = (a + b + c)^2 - (b^2 + c^2 + 2bc + ab + ca)$$

$$= a(a + b + c) \ge 0.$$

The equality holds for $a = 0$ and $b = c = 1$ (or any cyclic permutation).

□

P 3.14. *Let a, b, c be nonnegative real numbers such that $a^2 + b^2 + c^2 = 2$. Prove that*

$$(a^3 + b^3)(b^3 + c^3)(c^3 + a^3) \le 2.$$

(Vasile Cîrtoaje, 2011)

Solution. Let $x = a^2$, $y = b^2$, $z = c^2$, where $x + y + z = 2$. Since

$$(a^3 + b^3)^2 \le (a^2 + b^2)(a^4 + b^4) = (x + y)(x^2 + y^2),$$

it suffices to prove that

$$(x + y)(y + z)(z + x)(x^2 + y^2)(y^2 + z^2)(z^2 + x^2) \le 4.$$

Due to symmetry, we may assume that $x = \min\{x, y, z\}$. It is easy to check that the equality holds for $x = 0$ and $y = z = 1$. Since

$$(x + y)(y + z)(z + x) \le (x + y + z)(xy + yz + zx) = 2(xy + yz + zx)$$

and

$$x^2 + y^2 \le y(x+y), \quad z^2 + x^2 \le z(x+z),$$

it suffices to show that

$$yz(xy + yz + zx)(x+y)(x+z)(y^2+z^2) \le 2.$$

Write this inequality as

$$[2yz][2(xy+yz+zx)][2(x+y)(x+z)](y^2+z^2) \le 16.$$

By the AM-GM inequality, it suffices to show that

$$\left[\frac{2yz + 2(xy+yz+zx) + 2(x+y)(x+z) + (y^2+z^2)}{4}\right]^4 \le 16.$$

This inequality is equivalent to

$$2x^2 + y^2 + z^2 + 6yz + 4xy + 4zx \le 8,$$

$$2x^2 + y^2 + z^2 + 6yz + 4xy + 4zx \le 2(x+y+z)^2,$$

$$= (y-z)^2 \ge 0.$$

The equality holds for $a = 0$ and $b = c = 1$ (or any cyclic permutation).

□

P 3.15. *If a, b, c are nonnegative real numbers such that $a + b + c = 2$, then*

$$(3a^2 - 2ab + 3b^2)(3b^2 - 2bc + 3c^2)(3c^2 - 2ca + 3a^2) \le 36.$$

(Vasile Cîrtoaje, 2011)

Solution. Due to symmetry, assume that $a = \min\{a, b, c\}$. Under this assumption, we can check that the equality holds for $(a, b, c) = (0, 1, 1)$. Since

$$0 \le 3a^2 - 2ab + 3b^2 \le b(a + 3b)$$

and

$$0 < 3c^2 - 2ca + 3a^2 \le c(a + 3c),$$

it suffices to show that

$$bc(a+3b)(a+3c)(3b^2 - 2bc + 3c^2) \le 36.$$

Write this inequality as

$$[4b(a+3c)][4c(a+3b)][3(3b^2 - 2bc + 3c^2)] \le 12^3.$$

By virtue of the AM-GM inequality, it suffices to show that

$$\left[\frac{4b(a+3c)+4c(a+3b)+3(3b^2-2bc+3c^2)}{3}\right]^3 \le 12^3.$$

This is equivalent to

$$9(b+c)^2+4a(b+c) \le 36.$$

We have

$$36-9(b+c)^2-4a(b+c) = 9(a+b+c)^2-9(b+c)^2-4a(b+c)$$
$$= a(9a+14b+14c) \ge 0.$$

The equality holds for $a=0$ and $b=c=1$ (or any cyclic permutation).

Remark. Similarly, we can prove the following more general statement.

- *Let a,b,c be nonnegative real numbers. If $\frac{2}{3} \le k \le 2$, then*

$$(a^2-kab+b^2)(b^2-kbc+c^2)(c^2-kca+a^2) \le \frac{4}{27(2+k)^2}(a+b+c)^6,$$

with equality for $a=0$ and $\frac{b}{c}+\frac{c}{b}=1+\frac{3k}{2}$ (or any cyclic permutation).

Actually, this inequality holds for $\frac{2}{3} \le k \le 5$. □

P 3.16. *Let a,b,c be nonnegative real numbers such that $a+b+c=3$. Prove that*

$$(a^2-4ab+b^2)(b^2-4bc+c^2)(c^2-4ca+a^2) \le 3.$$

(Vasile Cîrtoaje, 2011)

Solution. Assume that $a \le b \le c$. It is easy to show that the equality holds for $a=0$ and $b^2+c^2=7bc$. If $c^2-4ca+a^2<0$, then $a^2-4ab+b^2<0$ and $b^2-4bc+c^2<0$, since $b \le c < (2+\sqrt{3})a$ and $c < (2+\sqrt{3})a \le (2+\sqrt{3})b$. Therefore, there two non-trivial cases when the left hand side of the inequality is nonnegative: when all three factors are nonnegative and when $a^2-4ab+b^2 \le 0$, $b^2-4bc+c^2 \le 0$, $c^2-4ca+a^2 \ge 0$.

Case 1: $a^2-4ab+b^2 \ge 0$, $b^2-4bc+c^2 \ge 0$, $c^2-4ca+a^2 \ge 0$. Since $a^2-4ab+b^2 \le b^2$ and $c^2-4ca+a^2 \le c^2$, it suffices to prove that

$$b^2c^2(b^2-4bc+c^2) \le 3.$$

In virtue of the AM-GM inequality, we have

$$b^2c^2(b^2-4bc+c^2) = \frac{1}{9}(3bc)(3bc)(b^2-4bc+c^2)$$

$$\leq \frac{1}{9}\left(\frac{3bc+3bc+(b^2-4bc+c^2)}{3}\right)^3$$
$$=3\left(\frac{b+c)}{3}\right)^6 \leq 3\left(\frac{a+b+c)}{3}\right)^6=3.$$

Case 2: $a^2-4ab+b^2\leq 0$, $b^2-4bc+c^2\leq 0$, $c^2-4ca+a^2\geq 0$. We have

$$a\geq (2-\sqrt{3})b.$$

Since

$$(4ab-a^2-b^2)(4bc-b^2-c^2)(c^2-4ca+a^2)\leq$$
$$\leq \left(\frac{(4ab-a^2-b^2)+(4bc-b^2-c^2)+(c^2-4ca+a^2)}{3}\right)^3$$
$$=\frac{8}{27}(2ab+2bc-2ca-b^2)^3,$$

it suffices to prove that

$$(2ab+2bc-2ca-b^2)^3\leq \frac{81}{8}=\frac{1}{72}(a+b+c)^6,$$

which is equivalent to

$$2\sqrt[3]{9}(2ab+2bc-2ca-b^2)\leq (a+b+c)^2.$$

Since $2\sqrt[3]{9}<\dfrac{21}{5}$, it suffices to show that

$$21(2ab+2bc-2ca-b^2)\leq 5(a+b+c)^2.$$

Write this inequality as $f(a)\geq 0$, where

$$f(a)=5a^2+4(13c-8b)a+26b^2+5c^2-32bc.$$

From $a\geq (2-\sqrt{3})b$, we get $4a>b$, $5a^2>\dfrac{b^2}{20}$, and hence

$$f(a)>\frac{b^2}{20}+(13c-8b)b+26b^2+5c^2-32bc=\frac{1}{20}(19b-10c)^2\geq 0.$$

This completes the proof. The equality holds for $a=0$ and $\dfrac{b}{c}+\dfrac{c}{b}=7$ (or any cyclic permutation).

□

P 3.17. *If a, b, c are positive real numbers such that $a + b + c = 3$, then*

$$abc + \frac{12}{ab + bc + ca} \geq 5.$$

Solution. By the third degree Schur's inequality

$$(a + b + c)^3 + 9abc \geq 4(a + b + c)(ab + bc + ca),$$

we get $3abc \geq 4(ab + bc + ca) - 9$. Thus, it suffices to prove that

$$4(ab + bc + ca) - 9 + \frac{36}{ab + bc + ca} \geq 15.$$

This inequality is equivalent to

$$(ab + bc + ca - 3)^2 \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 3.18. *If a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 3$, then*

$$5(a + b + c) + \frac{3}{abc} \geq 18.$$

Solution. Let $x = (a + b + c)/3$. From

$$2(ab + bc + ca) = (a + b + c)^2 - (a^2 + b^2 + c^2) = 3(3x^2 - 1) > 0,$$

we get $x > 1/\sqrt{3}$. By the known inequality

$$(ab + bc + ca)^2 \geq 3abc(a + b + c),$$

we get

$$\frac{1}{abc} \geq \frac{4x}{(3x^2 - 1)^2}.$$

Then, it suffices to prove that

$$5x + \frac{4x}{(3x^2 - 1)^2} \geq 6,$$

which is equivalent to

$$15x^5 - 18x^4 - 10x^3 + 12x^2 + 3x - 2 \geq 0,$$

$$(x-1)^2(15x^3+12x^2-x-2) \geq 0.$$

We still have to show that $15x^3+12x^2-x-2 \geq 0$. Since $x > 1/\sqrt{3}$, we get

$$15x^3+12x^2-x-2 > x^2(12-\frac{1}{x}-\frac{2}{x^2}) > x^2(12-\sqrt{3}-6) > 0.$$

The equality holds for $a = b = c = 1$.

□

P 3.19. *If a,b,c are positive real numbers such that $a^2+b^2+c^2=3$, then*

$$12+9abc \geq 7(ab+bc+ca).$$

(Vasile Cîrtoaje, 2005)

Solution. Let $x = (a+b+c)/3$. Since

$$2(ab+bc+ca) = (a+b+c)^2-(a^2+b^2+c^2) = 3(3x^2-1),$$

we can write the inequality as

$$5+2abc \geq 7x^2.$$

By Schur's inequality of degree three

$$(a+b+c)^3+9abc \geq 4(a+b+c)(ab+bc+ca),$$

we get

$$3x^3+abc \geq 2x(3x^2-1),$$

$$abc \geq 3x^3-2x.$$

Then,

$$5+2abc-7x^2 \geq 5+2(3x^3-2x)-7x^2 = (x-1)^2(6x+5) \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 3.20. *If a,b,c are positive real numbers such that $a^2+b^2+c^2=3$, then*

$$21+18abc \geq 13(ab+bc+ca).$$

(Vasile Cîrtoaje, 2005)

Solution. Let $p = a + b + c$ and $q = ab + bc + ca$. From

$$2q = (a + b + c)^2 - (a^2 + b^2 + c^2) = p^2 - 3 > 0,$$

we get $p > \sqrt{3}$. In addition, from Schur's inequality of degree four, written in the form

$$abc \geq \frac{(p^2 - q)(4q - p^2)}{6p},$$

we get

$$abc \geq \frac{(p^2 + 3)(p^2 - 6)}{12p}.$$

Therefore,

$$21 + 18abc - 13(ab + bc + ca) \geq 21 + \frac{3(p^2 + 3)(p^2 - 6)}{2p} - \frac{13(p^2 - 3)}{2}$$
$$= \frac{(p - 3)^2(3p^2 + 5p - 6)}{2p} \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 3.21. *If a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 3$, then*

$$(2 - ab)(2 - bc)(2 - ca) \geq 1.$$

(Vasile Cîrtoaje, 2005)

First Solution. Let $p = a + b + c$. From $3(a^2 + b^2 + c^2) \geq (a + b + c)^2$, we get $p \leq 3$. Since

$$(2 - ab)(2 - bc)(2 - ca) = 8 - 4(ab + bc + ca) + 2abc(a + b + c) - a^2b^2c^2$$
$$= 8 - 2(p^2 - 3) + 2abcp - a^2b^2c^2$$
$$= 14 - p^2 - (p - abc)^2,$$

we can write the inequality as

$$13 - p^2 - (p - abc)^2 \geq 0.$$

Clearly,

$$3(p - abc) = (a^2 + b^2 + c^2)(a + b + c) - 3abc > 0.$$

By Schur's inequality

$$(a + b + c)^3 + 9abc \geq 4(a + b + c)(ab + bc + ca),$$

we get

$$p^3+9abc \ge 2p(p^2-3),$$

$$abc \ge \frac{p(p^2-6)}{9}.$$

Since

$$0 < p-abc \le p-\frac{p(p^2-6)}{9} = \frac{p(15-p^2)}{9},$$

it suffices to prove that

$$13-p^2-\frac{p^2(15-p^2)^2}{81} \ge 0.$$

Setting $p=3\sqrt{x}$, $0<x\le 1$, this inequality becomes

$$13-34x+30x^2-9x^3 \ge 0.$$

It is true because

$$\begin{aligned}13-34x+30x^2-9x^3 &= (1-x)(13-21x+9x^2)\\ &= (1-x)[1+3(1-x)(4-3x)] \ge 0.\end{aligned}$$

The equality holds for $a=b=c=1$.

Second Solution. We use the mixing variable method. Assume that $a \le 1$ and show that

$$(2-ab)(2-bc)(2-ca) \ge (2-x^2)(2-ax)^2 \ge 1,$$

where $x=\sqrt{\dfrac{b^2+c^2}{2}}=\sqrt{\dfrac{3-a^2}{2}}$. Since $2-bc \ge 2-\dfrac{1}{2}(b^2+c^2) \ge 2-\dfrac{3}{2} > 0$ and, similarly, $2-ca>0$, $2-ab>0$, we can prove the left inequality by multiplying the inequalities

$$2-bc \ge 2-x^2$$

and

$$(2-ca)(2-ab) \ge (2-ax)^2.$$

The last inequality is true because

$$\begin{aligned}(2-ca)(2-ab)-(2-ax)^2 &= 2a(2x-b-c)-a^2(x^2-bc)\\ &= \frac{2a(b-c)^2}{2x+b+c}-\frac{a^2(b-c)^2}{2}\\ &= \frac{a(b-c)^2[4-a(2x+b+c)]}{2(2x+b+c)}\end{aligned}$$

and

$$4-a(2x+b+c)\geq 4(1-ax)=2(2-a\sqrt{6-2a^2})$$
$$=\frac{4(1-a^2)(2-a^2)}{2+a\sqrt{6-2a^2}}\geq 0.$$

The right inequality $(2-x^2)(2-ax)^2\geq 1$ is equivalent to

$$(1+a^2)(2-ax)^2\geq 2.$$

Since $2(1+a^2)\geq(1+a)^2$, it suffices to show that

$$(1+a)(2-ax)\geq 2.$$

Indeed,

$$(1+a)(2-ax)-2=a(2-x-ax)=\frac{a(a^4+2a^3-2a^2-6a+5)}{2(2+x+ax)}$$
$$=\frac{a(a-1)^2(a^2+4a+5)}{2(2+x+ax)}\geq 0.$$

□

P 3.22. *Let a,b,c be positive real numbers such that $abc=1$. Prove that*

$$\left(\frac{a+b+c}{3}\right)^5\geq\frac{a^2+b^2+c^2}{3}.$$

First Solution. Write the inequality in the homogeneous form

$$(a+b+c)^5\geq 81abc(a^2+b^2+c^2).$$

Using to the known inequality

$$(ab+bc+ca)^2\geq 3abc(a+b+c),$$

it suffices to show that

$$(a+b+c)^6\geq 27(ab+bc+ca)^2(a^2+b^2+c^2).$$

Setting $p=a+b+c$ and $q=ab+bc+ca$, we have

$$(a+b+c)^6-27(ab+bc+ca)^2(a^2+b^2+c^2)=p^6-27q^2(p^2-2q)$$
$$=(p^2-3q)^2(p^2+6q)\geq 0.$$

The equality occurs for $a=b=c=1$.

Second Solution. Use the mixing variable method. We show that

$$E(a,b,c) \ge E(a, \frac{b+c}{2}, \frac{b+c}{2}) \ge 0,$$

where

$$E(a,b,c) = (a+b+c)^5 - 81abc(a^2+b^2+c^2).$$

Indeed, we have

$$\begin{aligned} &\frac{1}{81}[E(a,b,c) - E(a, \frac{b+c}{2}, \frac{b+c}{2})] = \\ &\qquad = a^3[(\frac{b+c}{2})^2 - bc] + a[2(\frac{b+c}{2})^2 - bc(b^2+c^2)] \\ &\qquad = \frac{1}{4}a^3(b-c)^2 + \frac{1}{8}a(b-c)^4 \ge 0 \end{aligned}$$

and

$$\begin{aligned} &E(a, \frac{b+c}{2}, \frac{b+c}{2}) = \\ &\quad = (a+b+c)^5 - \frac{81}{8}a(b+c)^2[2a^2+(b+c)^2] \\ &\quad = \frac{1}{8}(2a-b-c)^2[2a^3+12a^2(b+c)-9a(b+c)^2+8(b+c)^3] \ge 0, \end{aligned}$$

since

$$\begin{aligned} 2a^3+12a^2(b+c)-9a(b+c)^2+8(b+c)^3] &> \\ > 6a^2(b+c) - 12a(b+c)^2 + 6(b+c)^3 & \\ = 6(b+c)(a-b-c)^2 \ge 0. & \end{aligned}$$

□

P 3.23. *If a,b,c are positive real numbers such that $abc=1$, then*

$$a^3+b^3+c^3+a^{-3}+b^{-3}+c^{-3}+21 \ge 3(a+b+c)(a^{-1}+b^{-1}+c^{-1}).$$

Solution. Since

$$a^3+b^3+c^3+a^{-3}+b^{-3}+c^{-3}+3 = \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)$$

and

$$(a+b+c)(a^{-1}+b^{-1}+c^{-1}) = \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)+3,$$

we can write the desired inequality in the homogeneous form

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)+9 \ge 3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+3\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right),$$

or

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right)\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}-3\right) \ge 0.$$

This is true since, by the AM-GM inequality,

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge 3, \quad \frac{b}{a}+\frac{c}{b}+\frac{a}{c} \ge 3.$$

The equality holds for $a=b=c=1$.

□

P 3.24. *If a,b,c are positive real numbers such that $abc=1$, then*

$$a^2+b^2+c^2-ab-bc-ca \ge \frac{9}{4}(a+b+c-3).$$

Solution. Write the inequality in the form

$$3(4x^2-3x+3) \ge 4(ab+bc+ca),$$

where $x=(a+b+c)/3$. By the AM-GM inequality, we have $x \ge \sqrt[3]{abc}=1$. The third degree Schur's inequality states that

$$(a+b+c)^3+9abc \ge 4(a+b+c)(ab+bc+ca),$$

which is equivalent to

$$4(ab+bc+ca) \le \frac{3(3x^3+1)}{x}.$$

Therefore, it suffices to show that

$$3(4x^2-3x+3) \ge \frac{3(3x^3+1)}{x}.$$

This inequality reduces to $(x-1)^3 \ge 0$, which is obviously true for $x \ge 1$. The equality holds for $a=b=c=1$.

□

P 3.25. *If a, b, c are positive real numbers such that $abc = 1$, then*

$$a^2 + b^2 + c^2 + a + b + c \geq 2(ab + bc + ca).$$

Solution. Let $p = a + b + c$ and $q = ab + bc + ca$. By virtue of the AM-GM inequality, we have $p \geq 3\sqrt[3]{abc} = 3$, and by Schur's inequality

$$p^3 + 9abc \geq 4pq,$$

we get

$$4q \leq \frac{p^3 + 9}{p}.$$

Therefore,

$$a^2 + b^2 + c^2 + a + b + c - 2(ab + bc + ca) = p^2 + p - 4q \geq p^2 + p - \frac{p^3 + 9}{p}$$
$$= \frac{(p-3)(p+3)}{p} \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 3.26. *If a, b, c are positive real numbers such that $abc = 1$, then*

$$a^2 + b^2 + c^2 + 15(ab + bc + ca) \geq 16(a + b + c).$$

Solution. Write the inequality as $F(a, b, c) \geq 0$, where

$$F(a, b, c) = a^2 + b^2 + c^2 + 15\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 16(a + b + c).$$

Assume that $a \geq b \geq c$, and denote

$$t = \sqrt{bc}, \quad 0 < t \leq 1, \quad at^2 = 1.$$

We will show that

$$F(a, b, c) \geq F(a, t, t) \geq 0.$$

Since

$$F(a, b, c) - F(a, t, t) = b^2 + c^2 - 2t^2 + 15\left(\frac{1}{b} + \frac{1}{c} - \frac{2}{t}\right) - 16(b + c - 2t)$$

$$= (b-c)^2 + 15\left(\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{c}}\right)^2 - 16(\sqrt{b}-\sqrt{c})^2$$

$$= (\sqrt{b}-\sqrt{c})^2\left[(\sqrt{b}+\sqrt{c})^2 + \frac{15}{bc} - 16\right] \ge (\sqrt{b}-\sqrt{c})^2\left(4\sqrt{bc} + \frac{15}{bc} - 16\right),$$

it suffices to show that

$$4t + \frac{15}{t^2} - 16 \ge 0.$$

Indeed,

$$4t + \frac{15}{t^2} - 16 > t + \frac{15}{t} - 16 = \frac{(1-t)(15-t)}{t} \ge 0.$$

The inequality $F(a,t,t) \ge 0$ is equivalent to

$$(t-1)^2(17t^4 + 2t^3 - 13t^2 + 2t + 1) \ge 0.$$

We have

$$17t^4 + 2t^3 - 13t^2 + 2t + 1 = (2t-1)^4 + t(t^3 + 34t^2 - 37t + 10)$$

$$= (2t-1)^4 + \frac{t}{4}[t(2t-1)^2 + 140t^2 - 149t + 40] > 0$$

since $D = 149^2 - 4 \cdot 140 \cdot 40 = -199$. The equality holds for $a = b = c = 1$.

□

P 3.27. *If a, b, c are positive real numbers such that $abc = 1$, then*

$$\frac{2}{a+b+c} + \frac{1}{3} \ge \frac{3}{ab+bc+ca}.$$

Solution. Let $x = (ab+bc+ca)/3$. By virtue of the AM-GM inequality, we have

$$x \ge \sqrt[3]{ab \cdot bc \cdot ca} = 1.$$

The third degree Schur's inequality applied to ab, bc, ca, states that

$$(ab+bc+ca)^3 + 9a^2b^2c^2 \ge 4abc(a+b+c)(ab+bc+ca),$$

which is equivalent to

$$\frac{3}{a+b+c} \ge \frac{4x}{3x^3+1}.$$

Therefore,

$$3\left(\frac{2}{a+b+c} + \frac{1}{3} - \frac{3}{ab+bc+ca}\right) \ge \frac{8x}{3x^3+1} + 1 - \frac{3}{x}$$

$$= \frac{3x^4 - 9x^3 + 8x^2 + x - 3}{x(3x^3 + 1)} = \frac{(x-1)(3x^3 - 6x^2 + 2x + 3)}{x(3x^3 + 1)}.$$

Since $x \ge 1$, we need to show that $3x^3 - 6x^2 + 2x + 3 \ge 0$. For $x \ge 2$, we have

$$3x^3 - 6x^2 + 2x + 3 > 3x^3 - 6x^2 = 3x^2(x-2) \ge 0,$$

and for $1 \le x < 2$, we have

$$3x^3 - 6x^2 + 2x + 3 = 3x(x-1)^2 + 3 - x > 0.$$

The equality holds for $a = b = c = 1$.

□

P 3.28. *If a, b, c are positive real numbers such that $abc = 1$, then*

$$ab + bc + ca + \frac{6}{a + b + c} \ge 5.$$

(Vasile Cîrtoaje, 2005)

First Solution. Denoting $x = (ab + bc + ca)/3$, the inequality can be written as

$$(a + b + c)(3x - 5) + 6 \ge 0.$$

In virtue of the AM-GM inequality, we get $x \ge 1$. Since the inequality holds for $x \ge 5/3$, consider next that $1 \le x < 5/3$. Applying the third degree Schur's inequality to the numbers ab, bc and ca, we have

$$(ab + bc + ca)^3 + 9a^2b^2c^2 \ge 4abc(a + b + c)(ab + bc + ca),$$

which is equivalent to

$$a + b + c \le \frac{3(3x^3 + 1)}{4x}.$$

Having in view that $3x - 5 < 0$, it suffices to prove that

$$\frac{3(3x^3 + 1)(3x - 5)}{4x} + 6 \ge 0.$$

This inequality is equivalent to

$$9x^4 - 15x^3 + 11x - 5 \ge 0,$$

$$(x - 1)(9x^3 - 6x^2 - 6x + 5) \ge 0.$$

Since

$$9x^3-6x^2-6x+5>9x^3-6x^2-6x+3=3(x-1)(3x^2+x-1)\ge 0,$$

the conclusion follows. The equality holds for $a=b=c=1$.

Second Solution (by *Vo Quoc Ba Can*). Among $a-1,\ b-1,\ c-1$ there are two with the same sign. Due to symmetry, assume that $(b-1)(c-1)\ge 0$; that is, $b+c\le 1+bc$. Then,

$$\frac{6}{a+b+c}\ge\frac{6}{a+1+bc}=\frac{6a}{a^2+a+1}.$$

On the other hand, using the AM-GM inequality yields

$$ab+bc+ca=a(b+c)+bc\ge 2a\sqrt{bc}+bc=2\sqrt{a}+\frac{1}{a}.$$

Therefore, it suffices to prove that

$$2\sqrt{a}+\frac{1}{a}+\frac{6a}{a^2+a+1}\ge 5.$$

Setting $\sqrt{a}=x$, this inequality becomes as follows

$$2x+\frac{1}{x^2}+\frac{6x^2}{x^4+x^2+1}\ge 5,$$

$$2x+\frac{1}{x^2}-3\ge 2-\frac{6x^2}{x^4+x^2+1},$$

$$\frac{(x-1)^2(2x+1)}{x^2}\ge\frac{2(x^2-1)^2}{x^4+x^2+1},$$

$$(x-1)^2(2x^5-x^4-2x^3-x^2+2x+1)\ge 0,$$

$$(x-1)^2[x(x-1)^2(2x^2+3x+2)+1]\ge 0.$$

□

P 3.29. *If a,b,c are positive real numbers such that $abc=1$, then*

$$\sqrt[3]{(1+a)(1+b)(1+c)}\ge\sqrt[4]{4(1+a+b+c)}.$$

(Pham Huu Duc, 2008)

Solution. Since

$$(1+a)(1+b)(1+c) = (1+a+b+c) + (1+ab+bc+ca) \geq 2\sqrt{(1+a+b+c)(1+ab+bc+ca)},$$

it suffices to prove that

$$(1+ab+bc+ca)^2 \geq 4(1+a+b+c),$$

which is equivalent to

$$(1+q)^2 \geq 4(1+p),$$

where $p = a+b+c$, $q = ab+bc+ca$. Setting $x = bc$, $y = ca$, $z = ab$ in Schur's inequality

$$(x+y+z)^3 + 9xyz \geq 4(x+y+z)(xy+yz+zx),$$

we get

$$q^3 + 9 \geq 4pq.$$

Since

$$(1+q)^2 - 4(1+p) \geq (1+q)^2 - 4 - \frac{q^3+9}{q} = \frac{(q-3)(2q+3)}{q},$$

it suffices to show that $q \geq 3$. Indeed, by the AM-GM inequality, we have

$$q = ab+bc+ca \geq 3\sqrt[3]{a^2b^2c^2} = 3.$$

The equality holds for $a = b = c = 1$

□

P 3.30. *If a, b, c are positive real numbers, then*

$$a^6 + b^6 + c^6 - 3a^2b^2c^2 \geq 18(a^2 - bc)(b^2 - ca)(c^2 - ab).$$

(Vasile Cîrtoaje, 2007)

Solution. Due to homogeneity, we may assume that $abc = 1$, when the inequality can be written as

$$a^6 + b^6 + c^6 - 3 \geq 18(a^3 + b^3 + c^3 - a^3b^3 - b^3c^3 - c^3a^3).$$

Substituting a^3, b^3, c^3 by a, b, c, respectively, we need to show that $abc = 1$ implies $F(a,b,c) \geq 0$, where

$$F(a,b,c) = a^2 + b^2 + c^2 - 3 - 18(a + b + c - ab - bc - ca).$$

To do this, we use the mixing variable method. Without loss of generality, assume that $a \geq 1$. We claim that

$$F(a,b,c) \geq F(a,\sqrt{bc},\sqrt{bc}) \geq 0.$$

We have

$$\begin{aligned} F(a,b,c) - F(a,\sqrt{bc},\sqrt{bc}) &= (b-c)^2 - 18(\sqrt{b} - \sqrt{c})^2 + 18a(\sqrt{b} - \sqrt{c})^2 \\ &= (b-c)^2 + 18(a-1)(\sqrt{b} - \sqrt{c})^2 \geq 0. \end{aligned}$$

Also, putting $\sqrt{bc} = t$, we have

$$\begin{aligned} F(a,\sqrt{bc},\sqrt{bc}) = F(\frac{1}{t^2},t,t) &= \frac{1}{t^4} + 20t^2 - 3 - \frac{18}{t^2} - 36t + \frac{36}{t} \\ &= \frac{(t-1)^2(2t-1)^2(t+1)(5t+1)}{t^4} \geq 0. \end{aligned}$$

The equality holds for $a = b = c$, and for $a/2 = b = c$ (or any cyclic permutation).

□

P 3.31. *If a,b,c are positive real numbers such that $a + b + c = 3$, then*

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq a^2 + b^2 + c^2.$$

(Vasile Cîrtoaje, 2006)

First Solution. Since

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca},$$

it suffices to prove that

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \geq a^2 + b^2 + c^2,$$

which is equivalent to

$$abc(a^2 + b^2 + c^2) \leq 3.$$

Let $x = (ab + bc + ca)/3$. From the known inequality

$$(ab + bc + ca)^2 \geq 3abc(a + b + c),$$

we get $abc \le x^2$. On the other hand, we have

$$a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ca)=9-6x.$$

Then,

$$abc(a^2+b^2+c^2)-3 \le x^2(9-6x)-3=-3(x-1)^2(2x+1) \le 0.$$

The equality holds for $a=b=c=1$.

Second Solution. Since $a+b+c=3$, we can write the inequality as

$$\sum(\frac{1}{a^2}-a^2+4a-4) \ge 0,$$

which is equivalent to

$$\sum \frac{(1-a)^2(1+2a-a^2)}{a^2} \ge 0.$$

Without loss of generality, assume that $a=\max\{a,b,c\}$. We have two cases to consider.
Case 1: $a \le 1+\sqrt{2}$. Since $a,b,c \le 1+\sqrt{2}$, we have

$$1+2a-a^2 \ge 0, \quad 1+2b-b^2 \ge 0, \quad 1+2c-c^2 \ge 0.$$

Thus, the conclusion follows.
Case 2: $a > 1+\sqrt{2}$. Since $b+c=3-a<2-\sqrt{2}<\frac{2}{3}$, we have

$$bc \le \frac{1}{4}(b+c)^2 < \frac{1}{9},$$

and hence

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} > \frac{1}{b^2}+\frac{1}{c^2} \ge \frac{2}{bc} > 18 > (a+b+c)^2 > a^2+b^2+c^2.$$

□

P 3.32. *If a,b,c are positive real numbers such that $ab+bc+ca=3$, then*

$$a^3+b^3+c^3+7abc \ge 10.$$

(Vasile Cîrtoaje, 2005)

Solution. Let $x = (a+b+c)/3$. By the well-known inequality

$$(a+b+c)^2 \geq 3(ab+bc+ca),$$

we get $x \geq 1$. Since

$$a^3+b^3+c^3 = 3abc+(a+b+c)^3-3(a+b+c)(ab+bc+ca)$$
$$= 3abc+27x^3-27x,$$

we can write the inequality as

$$10abc+27x^3-27x-10 \geq 0.$$

For $x \geq \frac{4}{3}$, this inequality is true, since

$$27x^3-27x-10 = 27x(x^2-1)-10 \geq 36(\frac{16}{9}-1)-10 = 18.$$

For $1 \leq x \leq \frac{4}{3}$, we use Schur's inequality

$$(a+b+c)^3+9abc \geq 4(a+b+c)(ab+bc+ca),$$

which is equivalent to

$$abc+3x^3-4x \geq 0.$$

Therefore,

$$10abc+27x^3-27x-10 \geq 10(-3x^3+4x)+27x^3-27x-10$$
$$= (x-1)[4-3x+3(2-x^2)] \geq 0.$$

The equality holds for $a=b=c=1$.

□

P 3.33. *If a,b,c are nonnegative real numbers such that $a^3+b^3+c^3=3$, then*

$$a^4b^4+b^4c^4+c^4a^4 \leq 3.$$

(Vasile Cîrtoaje, 2003)

Solution. By virtue of the AM-GM inequality, we have

$$bc \leq \frac{b^3+c^3+1}{3} = \frac{4-a^3}{3}.$$

Then, we have

$$b^4c^4 \le \frac{4b^3c^3 - a^3b^3c^3}{3}.$$

Similarly,

$$c^4a^4 \le \frac{4c^3a^3 - a^3b^3c^3}{3}, \quad a^4b^4 \le \frac{4a^3b^3 - a^3b^3c^3}{3}.$$

Summing these inequalities, we obtain

$$a^4b^4 + b^4c^4 + c^4a^4 \le \frac{4(a^3b^3 + b^3c^3 + c^3a^3)}{3} - a^3b^3c^3.$$

Thus, using the substitutions $x = a^3$, $y = b^3$, $z = c^3$, it suffices to prove that

$$4(xy + yz + zx) \le 3xyz + 9,$$

where x, y, z are nonnegative real numbers satisfying $x + y + z = 3$. This follows immediately from Schur's inequality

$$4(x + y + z)(xy + yz + zx) \le 9xyz + (x + y + z)^3.$$

The equality holds for $a = b = c = 1$.

Remark 1. Using the contradiction method, it is easy to prove the reverse statement.

- *If a, b, c are nonnegative real numbers such that $a^4b^4 + b^4c^4 + c^4a^4 = 3$, then*

$$a^3 + b^3 + c^3 \ge 3.$$

Remark 2. The inequality in P 3.33 is a particular case of the following more general statement (*Vasile Cîrtoaje*, 2003).

- *Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. If $0 < k \le k_0$, where*

$$k_0 = \frac{\ln 3}{\ln 9 - \ln 4} \approx 1.355,$$

then

$$a^kb^k + b^kc^k + c^ka^k \le 3.$$

□

P 3.34. *If a, b, c are nonnegative real numbers, then*

$$(a+1)^2(b+1)^2(c+1)^2 \ge 4(a + b + c)(ab + bc + ca) + 28abc.$$

(Vasile Cîrtoaje, 2011)

Solution. By the AM-GM inequality, we have

$$(a+1)(b+1)(c+1) = (abc+1)+(a+b+c)+(ab+bc+ca)$$

$$\geq 2\sqrt{abc}+2\sqrt{(a+b+c)(ab+bc+ca)}.$$

Thus, it suffices to prove that

$$[\sqrt{abc}+\sqrt{(a+b+c)(ab+bc+ca)}]^2 \geq (a+b+c)(ab+bc+ca)+7abc,$$

which can be written as

$$\sqrt{abc(a+b+c)(ab+bc+ca)} \geq 3abc.$$

This is true since

$$(a+b+c)(ab+bc+ca)-9abc = a(b-c)^2+b(c-a)^2+c(a-b)^2 \geq 0.$$

The equality holds for $a=b=c=1$.

□

P 3.35. *If a,b,c are positive real numbers such that $a+b+c=3$, then*

$$1+8abc \geq 9\min\{a,b,c\}.$$

(Vasile Cîrtoaje, 2007)

Solution. Without loss of generality, assume that $a=\min\{a,b,c\}$, $a \leq 1$. The inequality becomes

$$1+8abc \geq 9a.$$

From $(a-b)(a-c) \geq 0$, we get

$$bc \geq a(b+c)-a^2 = a(3-a)-a^2 = a(3-2a^2).$$

Therefore,

$$1+8abc-9a \geq 1+8a^2(3-2a^2)-9a = (1-a)(1-4a)^2 \geq 0.$$

The equality holds for $a=b=c=1$, and also for $(a,b,c)=\left(\frac{1}{4},\frac{1}{4},\frac{5}{2}\right)$ or any cyclic permutation.

□

P 3.36. *If a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 3$, then*

$$1 + 4abc \ge 5\min\{a, b, c\}.$$

(Vasile Cîrtoaje, 2007)

Solution. Without loss of generality, assume that $a = \min\{a, b, c\}$, $a \le 1$. The inequality can be written as

$$1 + 4abc \ge 5a.$$

From $(a^2 - b^2)(a^2 - c^2) \ge 0$, we get

$$bc \ge a\sqrt{b^2 + c^2 - a^2} = a\sqrt{3 - 2a^2}.$$

Therefore, it suffices to prove that

$$4a^2\sqrt{3 - 2a^2} \ge 5a - 1.$$

We consider two cases.

Case 1: $0 < a \le 1/2$. Since $\sqrt{3 - 2a^2} \ge \sqrt{5/2} > 25/16$, it is enough to show that

$$\frac{25}{4}a^2 \ge 5a - 1.$$

This inequality is equivalent to $(5a - 2)^2 \ge 0$.

Case 2: $1/2 < a \le 1$. By squaring, the inequality can be restated as

$$32a^6 - 48a^4 + 25a^2 - 10a + 1 \le 0,$$

or

$$(1 - a)(32a^5 + 32a^4 - 16a^3 - 16a^2 + 9a - 1) \ge 0.$$

It is true, since

$$32a^5 + 32a^4 - 16a^3 - 16a^2 + 9a - 1 =$$

$$= (2a - 1)(16a^4 + 24a^3 + 4a^2 - 6a + 1) + a$$

$$> (2a - 1)(8a^3 + 4a^2 - 6a + 1) = (2a - 1)^2(4a^2 + 4a - 1) > 0.$$

The equality holds for $a = b = c = 1$.

□

P 3.37. *If a, b, c are positive real numbers such that $a + b + c = abc$, then*

$$(1 - a)(1 - b)(1 - c) + (\sqrt{3} - 1)^3 \ge 0.$$

Solution. Without loss of generality, assume that $a \geq b \geq c$. If the product $(1-a)(1-b)(1-c)$ is nonnegative, the inequality is trivial. Otherwise, this product is positive for either $a > 1 > b \geq c$ or $a \geq b \geq c > 1$. Since $a > 1 > b \geq c$ involves

$$0 = a + b + c - abc > a(1 - bc) > 0,$$

which is a contradiction, it suffices to consider only the case $a \geq b \geq c > 1$. Setting $x = a - 1$, $y = b - 1$, $z = c - 1$, we need to show that

$$xyz \leq (\sqrt{3} - 1)^3,$$

where x, y, z are positive real numbers satisfying $xy + yz + zx + xyz = 2$. Let $t = \sqrt[3]{xyz}$, $t > 0$. By the AM-GM inequality, we have

$$2 = xy + yz + zx + xyz \geq 3\sqrt[3]{x^2y^2z^2} + xyz = 3t^2 + t^3,$$

and hence

$$t^3 + 3t^2 - 2 \leq 0,$$

$$(t + 1)(t^2 + 2t - 2) \leq 0,$$

$$t^2 + 2t - 2 \leq 0,$$

$$t \leq \sqrt{3} - 1,$$

$$xyz \leq (\sqrt{3} - 1)^3.$$

The equality holds for $a = b = c = \sqrt{3}$.

□

P 3.38. *If a, b, c are nonnegative real numbers such that $a + b + c = 2$, then*

$$(a^2 + bc)(b^2 + ca)(c^2 + ab) \leq 1.$$

(Vasile Cîrtoaje, 2005)

Solution. Without loss of generality, assume that $a \geq b \geq c$. Since

$$a^2 + bc \leq (a + \frac{c}{2})^2$$

and

$$(b^2 + ca)(c^2 + ab) \leq \frac{1}{4}(b^2 + ca + c^2 + ab)^2,$$

it suffices to show that

$$(2a + c)(b^2 + c^2 + ab + ac) \leq 4.$$

Let

$$E(a,b,c) = (2a+c)(b^2+c^2+ab+ac).$$

We will show that

$$E(a,b,c) \le E(a,b+c,0) \le 4.$$

Indeed,

$$E(a,b,c) - E(a,b+c,0) = c(b^2+c^2+ac-3ab) \le 0$$

and

$$\begin{aligned} E(a,b+c,0) - 4 &= 2a(a+b+c)(b+c) - 4 \\ &= 4a(2-a) - 4 = -4(a-1)^2 \le 0. \end{aligned}$$

The equality occurs for $(a,b,c) = (1,1,0)$ or any cyclic permutation.

□

P 3.39. *If a,b,c are nonnegative real numbers, then*

$$(8a^2+bc)(8b^2+ca)(8c^2+ab) \le (a+b+c)^6.$$

Solution. We use the mixing variable method. Without loss of generality, assume that $a \le b \le c$. Let $x = (b+c)/2$, $x \ge a$, and

$$E(a,b,c) = (8a^2+bc)(8b^2+ca)(8c^2+ab) - (a+b+c)^6.$$

We will prove that

$$E(a,b,c) \le E(a,x,x) \le 0.$$

The left inequality is equivalent to

$$(8a^2+x^2)(8x^2+ax)^2 \ge (8a^2+bc)(8b^2+ca)(8c^2+ab),$$

which follows by multiplying the obvious inequality

$$8a^2+x^2 \ge 8a^2+bc$$

and

$$(8x^2+ax)^2 \ge (8b^2+ca)(8c^2+ab).$$

Write the last inequality as

$$64(x^4-b^2c^2) + a^2(x^2-bc) - 8a(b^3+c^3-2x^3) \ge 0.$$

Since

$$b^3+c^3-2x^3=\frac{3(b+c)(b-c)^2}{4}=6x(x^2-bc)\geq 0,$$

we need to show that

$$64(x^2+bc)+a^2-48ax\geq 0.$$

This is true, since

$$64(x^2+bc)+a^2-48ax\geq 64x^2-48ax\geq 48x(x-a)\geq 0.$$

The right inequality $E(a,x,x)\leq 0$ is equivalent to

$$(8a^2+x^2)(8x^2+ax)^2-(a+2x)^6\leq 0,$$

$$176x^5-273ax^4+32a^2x^3+52a^3x^2+12a^4x+a^5\geq 0,$$

$$(x-a)^2(176x^3+79ax^2+14a^2x+a^3)\geq 0,$$

the last being clearly true. The equality holds for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 3.40. *If a,b,c are positive real numbers such that $a^2b^2+b^2c^2+c^2a^2=3$, then*

$$a+b+c\geq abc+2.$$

(Vasile Cîrtoaje, 2006)

Solution. Without loss of generality, assume that $a\geq b\geq c$. From $a^2b^2+b^2c^2+c^2a^2=3$, it follows that $1\leq ab<\sqrt{3}$ and

$$c=\sqrt{\frac{3-a^2b^2}{a^2+b^2}}\leq\sqrt{\frac{3-a^2b^2}{2ab}}.$$

We have

$$\begin{aligned}a+b+c-abc-2&=a+b-2-(ab-1)c\\&\geq 2\sqrt{ab}-2-(ab-1)\sqrt{\frac{3-a^2b^2}{2ab}}\\&=(\sqrt{ab}-1)\left[2-(\sqrt{ab}+1)\sqrt{\frac{3-a^2b^2}{2ab}}\right].\end{aligned}$$

So, we need to prove that

$$2\geq(\sqrt{ab}+1)\sqrt{\frac{3-a^2b^2}{2ab}}.$$

This inequality is true, since

$$(\sqrt{ab}+1)\sqrt{\frac{3-a^2b^2}{2ab}}-2 \le (\sqrt{ab}+\sqrt{ab})\sqrt{\frac{3-a^2b^2}{2ab}}-2$$
$$=\sqrt{6-2a^2b^2}-2$$
$$=\frac{2(1-a^2b^2)}{\sqrt{6-2a^2b^2}+2}\le 0.$$

The equality holds for $a=b=c=1$.

□

P 3.41. *Let a,b,c be nonnegative real numbers such that $a+b+c=5$. Prove that*

$$(a^2+3)(b^2+3)(c^2+3)\ge 192.$$

First Solution. Without loss of generality, assume that $a=min\{a,b,c\}$, $a\le\dfrac{5}{3}$. By virtue of the Cauchy-Schwarz inequality, we have

$$(b^2+3)(c^2+3)=(b^2+3)(3+c^2)\ge 3(b+c)^2=3(5-a)^2.$$

Therefore, it suffices to sow that

$$(a^2+3)(5-a)^2\ge 64$$

for $0\le a\le\dfrac{5}{3}$. Indeed,

$$(a^2+3)(5-a)^2-64=(a-1)^2(a^2-8a+11)\ge 0,$$

since

$$a^2-8a+11=\left(\frac{5}{3}-a\right)\left(\frac{19}{3}-a\right)+\frac{4}{9}>0.$$

The equality holds for $a=3$ and $b=c=1$ (or any cyclic permutation).

Second Solution. Without loss of generality, assume that $a=max\{a,b,c\}$. First, we show that

$$(b^2+3)(c^2+3)\ge (x^2+3)^2,$$

where $x=\dfrac{b+c}{2}$, $0\le x\le\dfrac{5}{3}$. This inequality is equivalent to

$$(b-c)^2(6-bc-x^2)\ge 0,$$

which is true because

$$6-bc-x^2 \geq 2(3-x^2) > 0.$$

Thus, it suffices to prove that

$$(a^2+3)(x^2+3)^2 \geq 192,$$

which is equivalent to

$$[(5-2x)^2+3](x^2+3)^2 \geq 192,$$
$$(x^2-5x+7)(x^2+3)^2 \geq 48,$$
$$(x-1)^2(x^4-3x^3+6x^2-15x+15) \geq 0.$$

This inequality is true since

$$x^4-3x^3+6x^2-15x+15 = x^2\left(x-\frac{3}{2}\right)^2+15\left(\frac{x}{2}-1\right)^2 > 0.$$

□

P 3.42. *If a, b, c are nonnegative real numbers, then*

$$a^2+b^2+c^2+abc+2 \geq a+b+c+ab+bc+ca.$$

(Michael Rozenberg, 2012)

Solution. Among the numbers $1-a$, $1-b$ and $1-c$ there are always two with the same sign; let us say $(1-b)(1-c) \geq 0$. Thus, it suffices to show that

$$a^2+b^2+c^2+a(b+c-1)+2 \geq a+b+c+ab+bc+ca,$$

which is equivalent to

$$a^2-2a+b^2+c^2-bc-(b+c)+2 \geq 0.$$

Since

$$b^2+c^2-bc \geq \frac{1}{4}(b+c)^2,$$

it suffices to show that

$$a^2-2a+\frac{1}{4}(b+c)^2-(b+c)+2 \geq 0,$$

which can be written in the obvious form

$$(a-1)^2+\left(\frac{b+c}{2}-1\right)^2 \geq 0.$$

The equality holds for $a=b=c=1$.

□

P 3.43. *If a,b,c are nonnegative real numbers, then*

$$\sum a^3(b+c)(a-b)(a-c) \ge 3(a-b)^2(b-c)^2(c-a)^2.$$

Solution. Without loss of generality, assume that $a = \min\{a,b,c\}$. Since

$$a^3(b+c)(a-b)(a-c) \ge 0$$

and

$$b^3(c+a)(b-c)(b-a) + c^3(a+b)(c-a)(c-b) =$$

$$(b-c)[bc(b^3-c^3) + (b-c)(b^3+c^3)a - (b^3-c^3)a^2]$$

$$= (b-c)^2[(b^2+bc+c^2)(bc-a^2) + (b^3+c^3)a]$$

$$\ge (b-c)^2(b^2+bc+c^2)(bc-a^2),$$

it suffices to show that

$$(b^2+bc+c^2)(bc-a^2) \ge 3(a-b)^2(c-a)^2.$$

Since

$$bc-a^2 = (a-b)(a-c) + a(b+c-2a) \ge (a-b)(a-c),$$

it suffices to show that

$$b^2+bc+c^2 \ge 3(a-b)(a-c),$$

which is equivalent to the obvious inequality

$$(b-c)^2 + 3a(b+c-a) \ge 0.$$

The equality holds for $a=b=c$, for $a=0$ and $b=c$ (or any cyclic permutation), and for $b=c=0$ (or any cyclic permutation).

□

P 3.44. *Find the greatest real number k such that*

$$a+b+c+4abc \ge k(ab+bc+ca)$$

for all $a,b,c \in [0,1]$.

Solution. Setting $a = b = c = 1$, we get $k \le 7/3$, but setting $a = 0$ and $b = c = 1$, we get $k \le 2$. So, we claim that $k = 2$ is the greatest real number k. To prove this, we only need to show that

$$a + b + c + 4abc \ge 2(ab + bc + ca)$$

for all $a, b, c \in [0, 1]$. Write the inequality as

$$a(1 + 4bc - 2b - 2c) + b + c - 2bc \ge 0.$$

Since $b + c - 2bc = b(1 - c) + c(1 - b) \ge 0$, the inequality is clearly true for $1 + 4bc - 2b - 2c \ge 0$. Consider further that $1 + 4bc - 2b - 2c < 0$. It suffices to show that

$$(1 + 4bc - 2b - 2c) + b + c - 2bc \ge 0,$$

which is equivalent to the obvious inequality

$$bc + (1 - b)(1 - c) \ge 0.$$

Thus, the proof is completed. For $k = 2$, the equality holds when $a = b = c = 0$, and when one of a, b, c is zero and the others are 1.

Remark. From the proof above it follows that the following stronger inequality holds for all $a, b, c \in [0, 1]$:

$$a + b + c + 3abc \ge 2(ab + bc + ca),$$

with equality when $a = b = c = 0$, when $a = b = c = 1$, and when one of a, b, c is zero and the others are 1.

□

P 3.45. *If $a, b, c \ge \frac{2}{3}$ such that $a + b + c = 3$, then*

$$a^2b^2 + b^2c^2 + c^2a^2 \ge ab + bc + ca.$$

Solution. We use the mixing variable method. Assume that $a = \max\{a, b, c\}$ and denote $x = (b + c)/2$. From $a, b, c \ge 2/3$ and $a + b + c = 3$, it follows that $2/3 \le x \le 1$. We will show that

$$E(a, b, c) \ge E(a, x, x) \ge 0,$$

where

$$E(a, b, c) = a^2b^2 + b^2c^2 + c^2a^2 - ab - bc - ca.$$

We have

$$\begin{aligned} E(a,b,c)-E(a,x,x) &= a^2(b^2+c^2-2x^2)-(x^4-b^2c^2)+(x^2-bc) \\ &= (x^2-bc)(2a^2-x^2-bc+1) \\ &= \frac{1}{4}(b-c)^2[a^2+(a^2-bc)+(1-x^2)] \ge 0 \end{aligned}$$

and

$$E(a,x,x) = 2a^2x^2+x^4-2ax-x^2.$$

Since $a+2x=3$, we get

$$\begin{aligned} 9E(a,x,x) &= 18a^2x^2+9x^4-(2ax+x^2)(a+2x)^2 \\ &= x(5x^3-12ax^2+9a^2x-2a^3) = x(x-a)^2(5x-2a) \\ &= 3x(x-a)^2(3x-2) \ge 0. \end{aligned}$$

The equality holds for $a=b=c=1$, and also for $a=5/3$, $b=2/3$, $c=2/3$ (or any cyclic permutation).

□

P 3.46. *If a,b,c are positive real numbers such that $a \le 1 \le b \le c$ and*

$$a+b+c=3,$$

then

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ge a^2+b^2+c^2.$$

Solution. Let

$$F(a,b,c) = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}-a^2-b^2-c^2.$$

We will show that

$$F(a,b,c) \ge F(a,1,b+c-1) \ge 0.$$

The left inequality is true, since

$$\begin{aligned} &F(a,b,c)-F(a,1,b+c-1) = \\ &= \left(\frac{1}{b}+\frac{1}{c}-1-\frac{1}{b+c-1}\right)+1+(b+c-1)^2-b^2-c^2 \\ &= (b+c)\left(\frac{1}{bc}-\frac{1}{b+c-1}\right)+2(b-1)(c-1) \\ &= (b-1)(c-1)\left[2-\frac{b+c}{bc(b+c-1)}\right] \end{aligned}$$

and

$$\begin{aligned}2bc(b+c-1)-b-c &= (2bc-1)(b+c)-2bc \\ &\ge 2(2bc-1)\sqrt{bc}-2bc \\ &= 2\sqrt{bc}(\sqrt{bc}-1)(2\sqrt{bc}+1)\ge 0.\end{aligned}$$

The right inequality $F(a,1,b+c-1)\ge 0$ is equivalent to $F(a,1,x)\ge 0$, where $x>0$ and $x+a=2$. We have

$$F(a,1,x)=\frac{1}{a}+\frac{1}{x}-a^2-x^2=\frac{(x+a)^4}{8ax}-a^2-x^2=\frac{(x-a)^4}{8ax}\ge 0.$$

The equality holds for $a=b=c=1$.

□

P 3.47. *If a,b,c are positive real numbers such that $a\le 1\le b\le c$ and*

$$a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c},$$

then

$$a^2+b^2+c^2\le \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}.$$

(Vasile Cîrtoaje, 2008)

Solution. Write the inequality as

$$b^2-\frac{1}{b^2}\le (a^2+c^2)\left(\frac{1}{a^2c^2}-1\right).$$

From $a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$, we have

$$b-\frac{1}{b}=(a+c)\left(\frac{1}{ac}-1\right)\ge 0.$$

Thus, the desired inequality holds if

$$(a+c)(b+\frac{1}{b})\le (a^2+c^2)(\frac{1}{ac}+1).$$

On the other hand, from $(b-c)(1-\frac{1}{bc})\le 0$, one gets

$$b+\frac{1}{b}\ge c+\frac{1}{c}.$$

Then, it suffices to prove that

$$(a+c)(c+\frac{1}{c}) \le (a^2+c^2)(\frac{1}{ac}+1),$$

which is equivalent to

$$c(1-a^2)(a-c) \le 0.$$

Since this is obvious, the proof is completed. The equality holds for $b=1$ and $ac=1$.

□

P 3.48. *If a,b,c are positive real numbers such that*

$$a+b+c = \frac{1}{a}+\frac{1}{b}+\frac{1}{c},$$

then

$$(1-abc)\left(a^n+b^n+c^n-\frac{1}{a^n}-\frac{1}{b^n}-\frac{1}{c^n}\right) \ge 0$$

for any integer $n \ge 2$.

(Vasile Cîrtoaje, 2007)

Solution. Since the statement remains unchanged by substituting a, b, c with $1/a$, $1/b$, $1/c$, respectively, it suffices to prove that

$$a^n+b^n+c^n-\frac{1}{a^n}-\frac{1}{b^n}-\frac{1}{c^n} \le 0$$

for $abc \ge 1$ and $a+b+c = 1/a+1/b+1/c$. It is easy to check that $a+b+c = 1/a+1/b+1/c$ is equivalent to

$$(ab-1)(bc-1)(ca-1) = a^2b^2c^2-1,$$

and the desired inequality is equivalent to

$$(a^nb^n-1)(b^nc^n-1)(c^na^n-1) \ge a^{2n}b^{2n}c^{2n}-1.$$

Setting $x=bc$, $y=ca$, $z=ab$, we need to show that

$$(x-1)(y-1)(z-1) = xyz-1 \ge 0$$

involves

$$(x^n-1)(y^n-1)(z^n-1) \ge x^ny^nz^n-1.$$

This inequality holds if

$$(x^{n-1}+x^{n-2}+\cdots+1)(y^{n-1}+y^{n-2}+\cdots+1)(z^{n-1}+z^{n-2}+\cdots+1) \ge$$
$$\ge x^{n-1}y^{n-1}z^{n-1}+x^{n-2}y^{n-2}z^{n-2}+\cdots+1.$$

Since the last inequality is clearly true, the proof is completed. The equality occurs for $a=bc=1$, or $b=ca=1$, or $c=ab=1$.

□

P 3.49. *Let a, b, c be positive real numbers, and let*

$$E(a,b,c) = a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b).$$

Prove that

(a) $$(a+b+c)E(a,b,c) \geq ab(a-b)^2 + bc(b-c)^2 + ca(c-a)^2;$$

(b) $$2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)E(a,b,c) \geq (a-b)^2 + (b-c)^2 + (c-a)^2.$$

Solution. (a) Using Schur's inequality of degree four

$$\sum a^2(a-b)(a-c) \geq 0,$$

we have

$$\begin{aligned}(a+b+c)E(a,b,c) &= \sum a^2(a-b)(a-c) + \sum a(b+c)(a-b)(a-c) \\ &\geq \sum a(b+c)(a-b)(a-c) \\ &= \sum ab(a-b)(a-c) + \sum ac(a-b)(a-c) \\ &= \sum ab(a-b)(a-c) + \sum ba(b-c)(b-a) \\ &= \sum ab(a-b)^2 \geq 0.\end{aligned}$$

The equality holds for $a = b = c$. If a, b, c are nonnegative real numbers, then the equality also holds for $a = 0$ and $b = c$ (or any cyclic permutation).

(b) Since

$$\begin{aligned}&(ab+bc+ca)E(a,b,c) = \\ &= abc\sum(a-b)(a-c) + \sum(a^2b+a^2c)(a-b)(a-c) \\ &= \frac{1}{2}abc\sum(a-b)^2 + \sum[a^2b(a-b)(a-c) + b^2a(b-c)(b-a)] \\ &= \frac{1}{2}abc\sum(a-b)^2 + \sum ab(a-b)^2(a+b-c),\end{aligned}$$

the required inequality is equivalent to

$$\sum ab(a-b)^2(a+b-c) \geq 0.$$

Without loss of generality, assume that $a \geq b \geq c$. Then,

$$\begin{aligned}\sum ab(a-b)^2(a+b-c) &\geq bc(b-c)^2(b+c-a) + ac(a-c)^2(a+c-b) \\ &\geq bc(b-c)^2(b+c-a) + ac(b-c)^2(a+c-b) \\ &= c(b-c)^2[(a-b)^2 + c(a+b)] \geq 0.\end{aligned}$$

The equality holds for $a = b = c$.

□

P 3.50. *Let $a \geq b \geq c$ be nonnegative real numbers. Schur's inequalities of third and fourth degree state that*

(a) $$a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b) \geq 0;$$

(b) $$a^2(a-b)(a-c)+b^2(b-c)(b-a)+c^2(c-a)(c-b) \geq 0.$$

Prove that (a) is sharper than (b) if

$$\sqrt{b}+\sqrt{c} \leq \sqrt{a},$$

and (b) is sharper than (a) if

$$\sqrt{b}+\sqrt{c} \geq \sqrt{a}.$$

(Vasile Cîrtoaje, 2005)

Solution. Let $p = a+b+c$ and $q = ab+bc+ca$. If we rewrite Schur's inequalities above as

$$abc \geq f(p,q)$$

and

$$abc \geq g(p,q),$$

respectively, then (a) is sharper than (b) if $f(p,q) \geq g(p,q)$, while (b) is sharper than (a) if $g(p,q) \geq f(p,q)$. Therefore, we need to show that

$$(\sqrt{b}+\sqrt{c}-\sqrt{a})[g(p,q)-f(p,q)] \geq 0.$$

From the known relation

$$4q-p^2 = (\sqrt{a}+\sqrt{b}+\sqrt{c})(\sqrt{b}+\sqrt{c}-\sqrt{a})(\sqrt{c}+\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b}-\sqrt{c}),$$

it follows that $4q-p^2$ and $\sqrt{b}+\sqrt{c}-\sqrt{a}$ has the same sign. Therefore, it suffices to prove that

$$(4q-p^2)[g(p,q)-f(p,q)] \geq 0.$$

In order to find $f(p,q)$, write the inequality in (a) as follows

$$a^3+b^3+c^3+3abc \geq ab(a+b)+bc(b+c)+ca(c+a)$$
$$(a+b+c)^3+9abc \geq 4(a+b+c)(ab+bc+ca),$$

from which

$$f(p,q) = \frac{p(4q-p^2)}{9}.$$

Analogously, write the inequality in (b) as

$$a^4 + b^4 + c^4 + abc(a+b+c) \geq ab(a^2+b^2) + bc(b^2+c^2) + ca(c^2+a^2),$$

$$a^4 + b^4 + c^4 + 2abc(a+b+c) \geq (ab+bc+ca)(a^2+b^2+c^2).$$

Since

$$a^2 + b^2 + c^2 = p^2 - 2q$$

and

$$\begin{aligned} a^4 + b^4 + c^4 &= (a^2+b^2+c^2)^2 - 2(a^2b^2 + b^2c^2 + c^2a^2) \\ &= (p^2-2q)^2 - 2q^2 + 4abcp, \end{aligned}$$

we get

$$g(p,q) = \frac{(p^2-q)(4q-p^2)}{6p}.$$

Therefore, we have

$$g(p,q) - f(p,q) = \frac{(p^2-3q)(4q-p^2)}{18p},$$

and hence

$$(4q-p^2)[g(p,q) - f(p,q)] = \frac{(p^2-3q)(4q-p^2)^2}{18p} \geq 0.$$

Remark. If a, b, c are the lengths of the sides of a triangle, then Schur's inequality of degree four is stronger than Schur's inequality of degree three.

□

P 3.51. *If a, b, c are nonnegative real numbers such that*

$$(a+b)(b+c)(c+a) = 8,$$

then

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \geq ab + bc + ca.$$

(Vasile Cîrtoaje, 2010)

First Solution. Assume that $a \ge b \ge c$, and write the inequality in the equivalent homogeneous forms

$$(\sqrt{a}+\sqrt{b}+\sqrt{c})\sqrt{(a+b)(b+c)(c+a)} \ge 2\sqrt{2}(ab+bc+ca),$$

$$\sum \sqrt{a(b+c)}[\sqrt{(a+b)(a+c)}-\sqrt{2a(b+c)}] \ge 0,$$

$$\sum \frac{(a-b)(a-c)\sqrt{a(b+c)}}{\sqrt{(a+b)(a+c)}+\sqrt{2a(b+c)}} \ge 0.$$

Since $(c-a)(c-b) \ge 0$, it suffices to prove that

$$\frac{(a-b)(a-c)\sqrt{a(b+c)}}{\sqrt{(a+b)(a+c)}+\sqrt{2a(b+c)}} + \frac{(b-c)(b-a)\sqrt{b(c+a)}}{\sqrt{(b+c)(b+a)}+\sqrt{2b(c+a)}} \ge 0,$$

which is true if

$$\frac{(a-c)\sqrt{a(b+c)}}{\sqrt{(a+b)(a+c)}+\sqrt{2a(b+c)}} \ge \frac{(b-c)\sqrt{b(c+a)}}{\sqrt{(b+c)(b+a)}+\sqrt{2b(c+a)}}.$$

Since $\sqrt{a} \ge \sqrt{b}$,

$$\sqrt{(a+b)(a+c)} \ge \sqrt{2a(b+c)}$$

and

$$\sqrt{(b+c)(b+a)} \le \sqrt{2b(a+c)},$$

it suffices to show that

$$\frac{(a-c)\sqrt{b+c}}{\sqrt{(a+b)(a+c)}} \ge \frac{(b-c)\sqrt{c+a}}{\sqrt{(b+c)(b+a)}}.$$

This is equivalent to the obvious inequality $c(a-b) \ge 0$. The equality holds for $a=b=c=1$, and for $a=0$ and $b=c=\sqrt[3]{4}$ (or any cyclic permutation).

Second Solution. Let $p=a+b+c$ and $q=ab+bc+ca$. By squaring, the inequality becomes

$$p+2(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}) \ge q^2.$$

Since

$$\sqrt{ab} \ge \frac{2ab}{a+b} = \frac{ab(b+c)(c+a)}{4} = \frac{ab(q+c^2)}{4},$$

we have

$$2(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}) \ge \frac{q(ab+bc+ca)+abc(a+b+c)}{2} = \frac{q^2+abcp}{2}.$$

Using this result, it suffices to prove that

$$p+\frac{q^2+abcp}{2}\ge q^2,$$

or

$$p(2+abc)\ge q^2.$$

According to the hypothesis $pq-abc=8$, we can write this inequality in the homogeneous forms

$$p\left(\frac{pq-abc}{4}+abc\right)\ge q^2,$$

$$p^2q+3abcp\ge 4q^2.$$

Since $p^2\ge 3q$ and $p^3+9abc\ge 4pq$ (Schur's inequality), we have

$$p(p^2q+3abcp-4q^2)\ge q(p^3+9abc-4pq)\ge 0.$$

□

P 3.52. *If $a,b,c\in[1,\ 4+3\sqrt{2}]$, then*

$$9(ab+bc+ca)(a^2+b^2+c^2)\ge(a+b+c)^4.$$

(Vasile Cîrtoaje, 2005)

Solution. Let $A=a^2+b^2+c^2$ and $B=ab+bc+ca$. Since

$$9(ab+bc+ca)(a^2+b^2+c^2)-(a+b+c)^4=9AB-(A+2B)^2$$

$$=(A-B)(4B-A)$$

and

$$2(A-B)=(a-b)^2+(b-c)^2+(c-a)^2\ge 0,$$

we need to show that $4B-A\ge 0$; that is, to show that $E(a,b,c)\le 0$, where

$$E(a,b,c)=a^2+b^2+c^2-4(ab+bc+ca).$$

We claim that $E(a,b,c)$ is maximal for $a,b,c\in\{1,\ w\}$, where $w=4+3\sqrt{2}$. For the sake of contradiction, assume that there exists a triple (a,b,c) with $a\in(1,\ w)$ such that

$$E(a,b,c)\ge\max\{E(1,b,c),E(w,b,c)\}.$$

From

$$E(a,b,c)-E(1,b,c)=(a-1)(a+1-4b-4c)\ge 0,$$

we get
$$4(b+c)-a \le 1,$$
and from
$$E(a,b,c)-E(w,b,c)=(a-w)(a+w-4b-4c)\ge 0,$$
we get
$$4(b+c)-a \ge w.$$
These results involve $w \le 1$, which is false. Therefore, since $E(a,b,c)$ is symmetric, we have
$$E(a,b,c) \le \max\{E(1,1,1),E(1,1,w),E(1,w,w),E(w,w,w)\}$$
$$=\max\{-9,w^2-8w-2,1-2w^2-8w,-9w^2\}=w^2-8w-2=0.$$
This completes the proof. The equality holds for $a=b=c$, and also for $a=b=1$ and $c=4+3\sqrt{2}$ (or any cyclic permutation).

□

P 3.53. *If a,b,c are nonnegative real numbers such that $a+b+c+abc=4$, then*

(a) $$a^2+b^2+c^2+12 \ge 5(ab+bc+ca);$$

(b) $$3(a^2+b^2+c^2)+13(ab+bc+ca)\ge 48.$$

Solution. Let $p=a+b+c$, $q=ab+bc+ca$, $r=abc$.

(a) We need to show that $p^2+12\ge 7q$ for $p+r=4$. By Schur's inequality of degree three, we have $p^3+9r\ge 4pq$. Therefore, we get
$$4p(p^2+12-7q)\ge 4p^3+48p-7(p^3+9r)=-3(p^3-37p+84)$$
$$=3(p-3)(4-p)(7+p).$$
Since $4-p=r\ge 0$, to complete the proof, we need to show that $p\ge 3$. By virtue of the AM-GM inequality, we get
$$p^3\ge 27r,$$
$$p^3\ge 27(4-p),$$
$$(p-3)(p^2+3p+36)\ge 0,$$
$$p\ge 3.$$
The equality holds for $a=b=c=1$, and for $a=0$ and $b=c=2$ (or any cyclic permutation).

(b) We need to show that $3p^2+7q \geq 48$ for $p+r=4$. Using the known inequality $pq \geq 9r$, we get

$$p(3p^2+7q-48) \geq 3(p^3+21r-16p) = 3(p^3-37p+84)$$
$$= 3(p-3)(4-p)(7+p) \geq 0.$$

The equality holds for $a=b=c=1$.

□

P 3.54. *Let a,b,c be the lengths of the sides of a triangle. If $a^2+b^2+c^2=3$, then*

$$ab+bc+ca \geq 1+2abc.$$

(Vasile Cîrtoaje, 2005)

Solution. Write the inequality such that the left-hand side and right-hand side are homogeneous expressions

$$3(ab+bc+ca)-a^2-b^2-c^2 \geq 6abc.$$

From

$$3(a^2+b^2+c^2) \geq (a+b+c)^2,$$

we get $a+b+c \leq 3$. Therefore, it suffices to prove the homogeneous inequality

$$(a+b+c)[3(ab+bc+ca)-a^2-b^2-c^2] \geq 18abc.$$

This is equivalent to

$$2ab(a+b)+2bc(b+c)+2ca(c+a) \geq a^3+b^3+c^3+9abc.$$

Using the known substitutions $a=y+z$, $b=z+x$, $c=x+y$ $(x,y,z \geq 0)$, the inequality can be written as

$$x^3+y^3+z^3+3xyz \geq xy(x+y)+yz(y+z)+zx(z+x),$$

which is just the third degree Schur's inequality. The equality holds for an equilateral triangle.

□

P 3.55. *Let a,b,c be the lengths of the sides of a triangle. If $a^2+b^2+c^2=3$, then*

$$a^2b^2+b^2c^2+c^2a^2 \geq ab+bc+ca.$$

Solution. Write the inequality as follows:

$$9(a^2b^2 + b^2c^2 + c^2a^2) \ge (ab + bc + ca)(a + b + c)^2;$$

$$3[3(a^2b^2 + b^2c^2 + c^2a^2) - (ab + bc + ca)^2] \ge$$

$$\ge (ab + bc + ca)[(a + b + c)^2 - 3(ab + bc + ca)];$$

$$6[a^2(b - c)^2 + b^2(c - a)^2 + c^2(a - b)^2] \ge$$

$$\ge (ab + bc + ca)[(b - c)^2 + (c - a)^2 + (a - b)^2] \ge 0;$$

$$\sum S_a(b - c)^2 \ge 0,$$

where

$$S_a = 6a^2 - ab - bc - ca.$$

Without loss of generality, assume that $a \ge b \ge c$. It suffices to show that

$$S_b(a - c)^2 + S_c(a - b)^2 \ge 0.$$

Since

$$(a - c)^2 \ge (a - b)^2,$$

$$S_b = 6b^2 - bc - a(b + c) \ge 6b^2 - bc - (b + c)^2 > 0$$

and

$$\begin{aligned} S_b + S_c &= 6(b^2 + c^2) - 2bc - 2a(b + c) \ge 6(b^2 + c^2) - 2bc - 2(b + c)^2 \\ &= 4(b - c)^2 + 2bc > 0, \end{aligned}$$

we get

$$S_b(a - c)^2 + S_c(a - b)^2 \ge (S_b + S_c)(a - b)^2 \ge 0.$$

The equality holds for an equilateral triangle.

□

P 3.56. *Let a, b, c be the lengths of the sides of a triangle. If $a + b + c = 3$, then*

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{41}{6} \ge 3(a^2 + b^2 + c^2).$$

(Vasile Cîrtoaje, 2010)

Solution *(by Vo Quoc Ba Can)*. Using the substitutions

$$a=\frac{y+z}{2},\quad b=\frac{z+x}{2},\quad c=\frac{x+y}{2},$$

where $x,y,z\geq 0$ such that $x+y+z=3$, the inequality becomes as follow

$$\frac{1}{y+z}+\frac{1}{z+x}+\frac{1}{x+y}+\frac{41}{12}\geq\frac{3}{8}[(y+z)^2+(z+x)^2+(x+y)^2],$$

$$\sum\frac{x+y+z}{y+z}+\frac{41}{4}\geq\frac{9}{4}\left(\sum x^2+\sum xy\right),$$

$$\sum\frac{x}{y+z}+3+\frac{41}{4}\geq\frac{9}{4}\left(9-\sum xy\right),$$

$$\sum\frac{x}{y+z}\geq 7-\frac{9}{4}\sum xy.$$

Let us denote $t=xy+yz+zx$. Since

$$\sum\frac{x}{y+z}=\frac{1}{t}\sum\frac{x(xy+yz+zx)}{y+z}\geq\frac{1}{t}\sum\frac{x(xy+zx)}{y+z}$$
$$=\frac{1}{t}\sum x^2=\frac{9-2t}{t},$$

it suffices to show that

$$\frac{9-2t}{t}\geq 7-\frac{9}{4}t,$$

which is equivalent to

$$(t-2)^2\geq 0.$$

The equality holds for a degenerate triangle having $a=3/2$, $b=1$, $c=1/2$ (or any permutation thereof).

□

P 3.57. *Let $a\leq b\leq c$ such that $a+b+c=p$ and $ab+bc+ca=q$, where p and q are fixed nonnegative numbers satisfying $p^2\geq 3q$.*

(a) If a,b,c are nonnegative real numbers, then the product $r=abc$ is maximal when $a=b$, and is minimal when $b=c$ or $a=0$;

(b) If a,b,c are the lengths of the sides of a triangle (non-degenerate or degenerate), then the product $r=abc$ is maximal when $a=b\geq\frac{c}{2}$ or $a+b=c$, and is minimal when $b=c\geq a$.

(Vasile Cîrtoaje, 2005)

Solution. (a) The proof is similar to the first proof of problem P 2.53. We have here

$$a_1 = \begin{cases} \dfrac{p-2\sqrt{p^2-3q}}{3}, & 3q \le p^2 \le 4q \\ 0, & p^2 \ge 4q \end{cases}.$$

Therefore, a attains its minimum value a_1 when $b = c$ (if $p^2 \le 4q$) or $a = 0$ (if $p^2 \ge 4q$). This means that r is minimal when $b = c$ or $a = 0$.

(b) Using the known substitutions $a = y+z$, $b = z+x$, $c = x+y$, where $x \ge y \ge z \ge 0$, from

$$a+b+c = 2(x+y+z),$$
$$ab+bc+ca = (x+y+z)^2 + xy+yz+zx,$$
$$abc = (x+y+z)(xy+yz+zx) - xyz,$$

it follows that

$$x+y+z = \frac{p}{2}, \quad xy+yz+zx = q - \frac{p^2}{4},$$

and hence

$$abc = \frac{p}{2}\left(q - \frac{p^2}{4}\right) - xyz.$$

Therefore, the product abc is maximal when xyz is minimal; that is, according to (a), when $x = y$ or $z = 0$, which is equivalent to $a = b \ge c/2$ or $a + b = c$. Also, the product abc is minimal when xyz is maximal; that is, according to (a), when $y = z$, which is equivalent to $b = c \ge a$.

Remark. Using the result in (a), we can prove by the contradiction method (as in Remark of P 2.53) the following generalization:

- *Let $a_1, a_2, \ldots, a_n$ be nonnegative numbers such that*

$$a_1 + a_2 + \cdots + a_n = p, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = p_1,$$

where p and p_1 are fixed real numbers satisfying $p^2 \le np_1$. Then, the product

$$r = a_1 a_2 \cdots a_n$$

is maximal when $n-1$ numbers of $a_1, a_2, \ldots, a_n$ are equal, and is minimal when one of $a_1, a_2, \ldots, a_n$ is zero or $n-1$ numbers of $a_1, a_2, \ldots, a_n$ are equal.

□

P 3.58. *If a, b, c are positive real numbers such that $a+b+c=3$, then*

$$\frac{9}{abc}+16 \geq \frac{75}{ab+bc+ca}.$$

(Vasile Cîrtoaje, 2005)

Solution. Let $q = ab+bc+ca$. For fixed q, the product abc is maximal when two of a, b, c are equal - see P 3.57-(a). Therefore, it suffices to prove the inequality for $b = a$ and $c = 3-2a$, $a < 3/2$. We have

$$\begin{aligned}\frac{9}{abc}+16-\frac{75}{ab+bc+ca} &= \frac{9}{a^2c}+16-\frac{75}{a(a+2c)}\\ &= \frac{9}{a^2(3-2a)}+16-\frac{25}{a(2-a)}\\ &= \frac{2(16a^4-56a^3+73a^2-42a+9)}{a^2(3-2a)(2-a)}\\ &= \frac{2(a-1)^2(4a-3)^2}{a^2(3-2a)(2-a)} \geq 0,\end{aligned}$$

as desired. The equality holds for $(a, b, c) = (1, 1, 1)$, and also for $(a, b, c) = \left(\frac{3}{4}, \frac{3}{4}, \frac{3}{2}\right)$ or any cyclic permutation.

□

P 3.59. *If a, b, c are positive real numbers such that $a+b+c=3$, then*

$$8\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+9 \geq 10(a^2+b^2+c^2).$$

(Vasile Cîrtoaje, 2006)

First Solution. Putting $q = ab+bc+ca$, we can write the inequality as

$$\frac{8q}{abc}+20q \geq 81.$$

By P 3.57-(a), the product abc is maximal for fixed q when two of a, b, c are equal. Therefore, it suffices to prove the inequality for $b = a$ and $c = 3-2a$, $a < 3/2$. We have

$$\begin{aligned}\frac{8q}{abc}+20q-81 &= \frac{24a(2-a)}{a^2(3-2a)}+60a(2-a)-81\\ &= \frac{3a(40a^4-140a^3+174a^2-89a+16)}{a^2(3-2a)}\\ &= \frac{3a(2a-1)^2(10a^2-25a+16)}{a^2(3-2a)}.\end{aligned}$$

Since

$$10a^2-25a+16=10\left(a-\frac{5}{4}\right)^2+\frac{3}{8}>0,$$

the proof is completed. The equality holds for $(a,b,c)=\left(\frac{1}{2},\frac{1}{2},2\right)$ or any cyclic permutation.

Second Solution (by *Vo Quoc Ba Can*). It is easy to check that the equality holds when two of a,b,c are $1/2$. Then, let us define

$$f(x)=\frac{8}{x}-10x^2-\alpha x-\beta,$$

such that $(2x-1)^2$ divides $f(x)$. From $f(1/2)=0$, we get $\alpha+2\beta=27$. Therefore,

$$f(x)=\frac{8}{x}-10x^2-(27-2\beta)x-\beta=\frac{(1-2x)h(x)}{x},$$

where $h(x)=5x^2-(\beta-16)x+8$. From $h(1/2)=0$, we get $\beta=69/2$, and hence

$$h(x)=\frac{(1-2x)(16-5x)}{2},$$

$$f(x)=\frac{(1-2x)^2(16-5x)}{2x}.$$

Thus, we can write the inequality in the form

$$f(a)+f(b)+f(c)\ge 27.$$

Assume now that $a=\max\{a,b,c\}$, $a\ge 1$, and rewrite the inequality as

$$\frac{(1-2b)^2(16-5b)}{b}+\frac{(1-2c)^2(16-5c)}{c}\ge\frac{4(a-2)^2(5a-1)}{a}.$$

Since $16-5b>0$ and $16-5c>0$, the Cauchy-Schwarz inequality yields

$$\frac{(1-2b)^2(16-5b)}{b}+\frac{(1-2c)^2(16-5c)}{c}\ge\frac{(1-2b+1-2c)^2}{\dfrac{b}{16-5b}+\dfrac{c}{16-5c}}$$
$$=\frac{4(a-2)^2}{\dfrac{b}{16-5b}+\dfrac{c}{16-5c}}.$$

Therefore, it suffices to prove that

$$\frac{1}{\dfrac{b}{16-5b}+\dfrac{c}{16-5c}}\ge\frac{5a-1}{a},$$

which is equivalent to

$$\frac{a}{5a-1} \geq \frac{b}{16-5b} + \frac{c}{16-5c}.$$

Indeed,

$$\frac{a}{5a-1} - \frac{b}{16-5b} - \frac{c}{16-5c} \geq \frac{a}{5a-1} - \frac{b+c}{16-5a}$$
$$= \frac{3}{(5a-1)(16-5a)} > 0.$$

Remark. Using the second method, we can prove the following more general inequality.

- *If $x_1, x_2, \cdots, x_n$ are positive real numbers such that*

$$x_1 + x_2 + \cdots + x_n = n,$$

then

$$(n+1)^2\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right) \geq n(n^2-3n-6) + 4(n+2)(x_1^2 + x_2^2 + \cdots + x_n^2),$$

with equality for $x_1 = (n+1)/2$ and $x_2 = \cdots = x_n = 1/2$ (or any cyclic permutation).

□

P 3.60. *If a, b, c are positive real numbers such that $a + b + c = 3$, then*

$$7(a^2+b^2+c^2) + 8(a^2b^2+b^2c^2+c^2a^2) + 4a^2b^2c^2 \geq 49.$$

Solution. Let $q = ab + bc + ca$. Since

$$a^2+b^2+c^2 = 9-2q,$$

$$a^2b^2+b^2c^2+c^2a^2 = q^2-6abc,$$

we can rewrite the inequality as

$$2(6-abc)^2 + 4q^2 - 7q - 65 \geq 0.$$

Since

$$abc \leq \left(\frac{a+b+c}{3}\right)^3 = 1,$$

we have $6-abc>0$. By P 3.57-(a), the product abc is maximal for fixed q when two of a,b,c are equal. Therefore, it suffices to prove the inequality for $a=b$. Since $q=a^2+2ac=3a(2-a)$ and $abc=a^2(3-2a)$, we have

$$2(6-abc)^2+4q^2-7q-65=8a^6-24a^5+54a^4-96a^3+83a^2-42a+7)$$
$$=(a-1)^2(2a-1)^2(2a^2+7\geq 0.$$

The equality holds for $a=b=c=1$, and for $(a,b,c)=\left(\frac{1}{2},\frac{1}{2},2\right)$ or any cyclic permutation.

□

P 3.61. *If a,b,c are nonnegative real numbers, then*

$$(a^3+b^3+c^3+abc)^2\geq 2(a^2+b^2)(b^2+c^2)(c^2+a^2).$$

(Aleksandar Bulj, 2011)

First Solution. Let $p=a+b+c$, $q=ab+bc+ca$, $r=abc$. Using the identities

$$a^3+b^3+c^3=3r+p^3-3pq$$

and

$$(a^2+b^2)(b^2+c^2)(c^2+a^2)=(a^2+b^2+c^2)(a^2b^2+b^2c^2+c^2a^2)-a^2b^2c^2$$
$$=(p^2-2q)(q^2-2pr)-r^2,$$

we can write the required inequality as $f_6(a,b,c)\geq 0$, where

$$f_6(a,b,c)=18r^2+4p(3p^2-8q)r+p^6-6p^4q+7p^2q^2+4q^3.$$

Since

$$3p^2-8q=3(p^2-3q)+q\geq 0,$$

for fixed p and q, f_6 is an increasing function of r. Therefore, it suffices to prove the inequality $f_6(a,b,c)\geq 0$ for the case when r is minimal; that is, when $a=0$ or $b=c$ (see P 3.57). In this cases, the original inequality becomes respectively

$$(b-c)^2(b^4+2b^3c+b^2c^2+2bc^3+c^4)\geq 0$$

and

$$a(a-b)^2(a^3+2a^2b+ab^2+4b^3)\geq 0.$$

The equality holds for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

Second Solution. Without loss of generality, assume that $a = \min\{a, b, c\}$. From

$$a(a-b)(a-c) \geq 0,$$

we get

$$a^3 + abc \geq a^2(b+c),$$

and hence

$$a^3 + b^3 + c^3 + abc \geq a^2(b+c) + b^3 + c^3 = (b+c)(a^2 + b^2 + c^2 - bc).$$

On the other hand,

$$2(a^2+b^2)(c^2+a^2) \leq \frac{1}{2}(2a^2 + b^2 + c^2)^2.$$

Therefore, it suffices to prove that

$$2(b+c)^2(a^2+b^2+c^2-bc)^2 \geq (b^2+c^2)(2a^2+b^2+c^2)^2.$$

We can obtain this inequality by multiplying the obvious inequality

$$2(a^2+b^2+c^2-bc) \geq 2a^2+b^2+c^2$$

and

$$(b+c)^2(a^2+b^2+c^2-bc) \geq (b^2+c^2)(2a^2+b^2+c^2).$$

The last inequality is equivalent to

$$(b-c)^2(bc-a^2) \geq 0,$$

which is also true.

Remark. Using the first method, we can prove the following stronger inequality (*Vasile Cîrtoaje*, 2011).

- *If a, b, c are nonnegative real numbers then*

$$(a^3+b^3+c^3+abc)^2 \geq 2(a^2+b^2)(b^2+c^2)(c^2+a^2) + 7(a-b)^2(b-c)^2(c-a)^2.$$

Since

$$(a-b)^2(b-c)^2(c-a)^2 = -27r^2 + 2(9pq - 2p^3)r + p^2q^2 - 4q^3,$$

we can write this inequality as $f_6(a, b, c) \geq 0$, where

$$f_6(a, b, c) = 207r^2 + 2(20p^3 - 79pq)r + p^6 - 6p^4q + 32q^3.$$

We will show that for fixed p and q, f_6 is an increasing function of r. From

$$f_6 = 27r^2 + 2pqr + 20g(r) + p^6 - 6p^4q + 32q^3,$$

where
$$g(r) = 9r^2 + 2(p^3 - 4pq)r,$$
it suffices to show that $g(r)$ is increasing. Indeed, for $r_1 \geq r_2 \geq 0$, by the third degree Schur's inequality, we have

$$\begin{aligned} g(r_1) - g(r_2) &= (r_1 - r_2)[9(r_1 + r_2) + 2p^3 - 8pq] \\ &\geq 2(r_1 - r_2)(9r_2 + p^3 - 4pq) \geq 0. \end{aligned}$$

Therefore, it suffices to prove the inequality $f_6(a, b, c) \geq 0$ for the case when r is minimal; that is, when $a = 0$ or $b = c$ (see P 3.57). In this cases, the original inequality becomes respectively
$$(b - c)^4(b^2 + 4bc + c^2) \geq 0$$
and
$$a(a - b)^2(a^3 + 2a^2b + ab^2 + 4b^3) \geq 0.$$

□

P 3.62. *If a, b, c are positive real numbers, then*

$$(a + b + c - 3)(ab + bc + ca - 3) \geq 3(abc - 1)(a + b + c - ab - bc - ca).$$

(Vasile Cîrtoaje, 2011)

Solution. Setting

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc,$$

the inequality becomes

$$(p - 3)(q - 3) \geq 3(r - 1)(p - q),$$

or
$$3r(q - p) + pq - 6q + 9 \geq 0.$$

First Solution. For fixed p and q, the linear function $f(r) = 3r(q - p) + pq - 6q + 9$ is minimal when r is either minimal or maximal. Thus, according to P 3.57-(a), we need only to prove that $f(r) \geq 0$ for $a = 0$ and for $b = c$.

For $a = 0$, we need to show that

$$(b + c)bc - 6bc + 9 \geq 0.$$

Indeed, putting $x = \sqrt{bc}$, we have

$$(b+c)bc - 6bc + 9 \ge 2x^3 - 6x^2 + 9 = 2(x+1)(x-2)^2 + 1 > 0.$$

For $b = c$, since $p = a + 2b$, $q = 2ab + b^2$ and $r = ab^2$, we need to show that

$$3ab^2(2ab + b^2 - a - 2b) + (a + 2b - 6)(2ab + b^2) + 9 \ge 0;$$

that is,

$$Aa^2 + Ba + C \ge 0,$$

where

$$A = b(6b^2 - 3b + 2), \quad B = b(3b^3 - 6b^2 + 5b - 12), \quad C = 2b^3 - 6b^2 + 9.$$

Consider two cases.

Case 1: $b \ge 12/5$. Since $A > 0$, $B = 3b^2(b-2) + b(5b-12) > 0$, $C > 0$, we have $Aa^2 + Ba + C > 0$.

Case 2: $0 < b < 12/5$. Since

$$Aa^2 + Ba + C \ge (Aa^2 + C) + Ba \ge a(2\sqrt{AC} + B),$$

we need to show that $4AC \ge B^2$, which is equivalent to each of the inequalities

$$4b(6b^2 - 3b + 2)(2b^3 - 6b^2 + 9) \ge b^2(3b^3 - 6b^2 + 5b - 12)^2,$$

$$b(b-1)^4(8 + 4b - b^3) \ge 0.$$

This inequality is true since

$$8 + 4b - b^3 = 8 + 4b - 3b^2 + b^2(3-b) > 8 + 4b - 3b^2 > 0.$$

The equality holds for $a = b = c = 1$.

Second Solution. Consider the following two cases.

Case 1: $p \ge q$. We have

$$3r(q-p) + pq - 6q + 9 = (q - 3r)(p - q) + (q-3)^2 \ge 0,$$

since

$$q - 3r \ge q - \frac{q^2}{p} \ge 0.$$

Case 2: $p \le q$. For $p \ge 6$, we have

$$3r(q-p) + pq - 6q + 9 = 3r(q-p) + q(p-6) + 9 > 0.$$

Consider further that $p < 6$. From $p^2 \geq 3q \geq 3p$, we get $p \geq 3$, and from Schur's inequality

$$p^3 + 9r \geq 4pq,$$

we get

$$p^3 + p^2 r \geq 4pq,$$

$$p^2 + pr \geq 4q.$$

Using this result, we have

$$\begin{aligned} p[3r(q-p)+pq-6q+9)] &\geq 3(4q-p^2)(q-p)+p(pq-6q+9) \\ &= 12q^2 - 2p(p+9)q + 3p(p^2+3) \\ &= 12\left(q - \frac{p^2+9p}{12}\right)^2 + \frac{p(12-p)(p-3)^2}{12} \geq 0. \end{aligned}$$

□

P 3.63. *Let a, b, c be nonnegative real numbers such that $ab + bc + ca = 3$. Prove that*

$$4(a^3 + b^3 + c^3) + 7abc + 125 \geq 48(a + b + c).$$

(Vasile Cîrtoaje, 2011)

Solution. Since

$$a^3 + b^3 + c^3 = 3abc + (a+b+c)^3 - 9(a+b+c),$$

we can write the inequality as

$$19abc + 4(a+b+c)^3 - 84(a+b+c) + 125 \geq 0.$$

As it is known, for fixed $a + b + c$, the product abc is minimal when $a = 0$ or $b = c$ (see P 3.57). Therefore it suffices to consider these cases.

Case 1: $a = 0$. We need to show that $bc = 3$ yields

$$4(b^3 + c^3) + 125 \geq 48(b + c).$$

Since

$$b^3 + c^3 = (b+c)^3 - 3bc(b+c) = (b+c)^3 - 9(b+c)$$

and

$$b + c \geq 2\sqrt{bc} = 2\sqrt{3},$$

we have

$$4(b^3+c^3)+125-48(b+c)=4(b+c)^3-84(b+c)+125>0.$$

Case 2: $b=c$. We need to show that $2ab+b^2=3$ yields

$$4(a^3+2b^3)+7ab^2+125\geq 48(a+2b).$$

This inequality is equivalent to

$$8b^6-114b^4+250b^3-171b^2+27\geq 0,$$

$$(b-1)^2(2b-3)^2(2b^2+10b+3)\geq 0.$$

Since the last inequality is true, the proof is completed.
The equality holds for $a=b=c$, and also for $a=1/4$ and $b=c=3/2$ (or any cyclic permutation).

□

P 3.64. *If* $a,b,c\in[0,1]$, *then*

$$a\sqrt{a}+b\sqrt{b}+c\sqrt{c}+4abc\geq 2(ab+bc+ca).$$

(Vasile Cîrtoaje, 2012)

Solution. This inequality is equivalent to

$$x^3+y^3+z^3+4x^2y^2z^2\geq 2(x^2y^2+y^2z^2+z^2x^2),$$

where $x,y,z\in[0,1]$. In addition, using the substitutions $p=x+y+z$ and $q=xy+yz+zx$, we can rewrite this inequality as

$$4x^2y^2z^2+(3+4p)xyz+p^3-3pq-2q^2\geq 0.$$

As it is known, for fixed p and q, the product xyz is minimal when $x=0$ or $y=z$ (see P 3.57). Therefore it suffices to consider these cases.

Case 1: $x=0$. We need to show that $y^3+z^3\geq 2y^2z^2$. Indeed, we have

$$y^3+z^3-2y^2z^2\geq y^4+z^4-2y^2z^2=(y^2-z^2)^2\geq 0.$$

Case 2: $y=z$. We need to show that $f(x)\geq 0$ for $x\in[0,1]$, where

$$f(x)=x^3-4x^2y^2(1-y^2)+2y^3(1-y),\quad y\in[0,1].$$

For $y=0$ and $y=1$, we have $f(x)=x^3\geq 0$. Consider further that $y\in(0,1)$. From

$$f'(x)=x[3x-8y^2(1-y^2)],$$

it follows that $f(x)$is decreasing on $[0,x_1]$ and increasing on $[x_1,1]$, where

$$x_1=\frac{8y^2(1-y^2)}{3},\quad x_1\in\left(0,\frac{2}{3}\right].$$

Therefore, it remains to show that $f(x_1)\geq 0$, which is equivalent to

$$128y^3(1-y)^2(1+y)^3\leq 27.$$

Since

$$y^2(1-y^2)\leq\frac{1}{4},$$

it suffices to show that

$$32y(1-y)(1+y)^2\leq 27.$$

Using the AM-GM inequality, we get

$$32y(1-y)(1+y)^2=1024\cdot\frac{y}{2}(1-y)\left(\frac{1+y}{4}\right)^2$$
$$\leq 1024\left[\frac{\frac{y}{2}+(1-y)+2\cdot\frac{1+y}{4}}{4}\right]^4=\frac{81}{4}<27.$$

The equality holds for $a=0$ and $b=c=1$ (or any cyclic permutation), and also for $a=b=c=0$.

□

P 3.65. *If $a,b,c\in[0,1]$, then*

$$a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\geq\frac{3}{2}(ab+bc+ca-abc).$$

(Vasile Cîrtoaje, 2012)

Solution. This inequality is equivalent to

$$2(x^3+y^3+z^3\geq 3(x^2y^2+y^2z^2+z^2x^2-x^2y^2z^2),$$

where $x,y,z\in[0,1]$. In addition, using the substitutions $p=x+y+z$ and $q=xy+yz+zx$, we can rewrite this inequality as

$$3x^2y^2z^2+3(2+3p)xyz+2p^3-6pq-3q^2\geq 0.$$

As it is known, for fixed p and q, the product xyz is minimal when $x=0$ or $y=z$ (see P 3.57). Therefore it suffices to consider these cases.

Case 1: $x=0$. We need to show that $2(y^3+z^3)\geq 3y^2z^2$. Indeed, we have

$$2(y^3+z^3)-3y^2z^2\geq 2(y^4+z^4)-4y^2z^2=2(y^2-z^2)^2\geq 0.$$

Case 2: $y=z$. We need to show that $f(x)\geq 0$ for $x\in[0,1]$, where

$$f(x)=2x^3-3x^2y^2(2-y^2)+y^3(4-3y),\quad y\in[0,1].$$

For $y=0$, we have $f(x)=x^3\geq 0$. Consider further that $y\in(0,1]$. From

$$f'(x)=6x[x-y^2(2-y^2)],$$

it follows that $f(x)$is decreasing on $[0,x_1]$ and increasing on $[x_1,1]$, where

$$x_1=y^2(2-y^2),\quad x_1\in(0,1].$$

Therefore, we only need to show that $f(x_1)\geq 0$, which is equivalent to

$$y^3(2-y^2)^3\leq 4-3y.$$

Indeed,

$$y^3(2-y^2)^3-(4-3y)\leq y(2-y^2)^2-(4-3y)=(y-1)^2(y^3+2y^2-y-4)\leq 0.$$

The equality holds for $a=b=c=1$, and also for $a=b=c=0$.

□

P 3.66. *If $a,b,c\in[0,1]$, then*

$$3(a\sqrt{a}+b\sqrt{b}+c\sqrt{c})+\frac{500}{81}abc\geq 5(ab+bc+ca).$$

(Vasile Cîrtoaje, 2012)

Solution. This inequality is equivalent to

$$3(x^3+y^3+z^3)+\frac{500}{81}x^2y^2z^2\geq 5(x^2y^2+y^2z^2+z^2x^2),$$

where $x,y,z\in[0,1]$. In addition, using the substitutions $p=x+y+z$ and $q=xy+yz+zx$, we can rewrite this inequality as

$$\frac{500}{81}x^2y^2z^2+3(3+5p)xyz+3p^3-9pq-5q^2\geq 0.$$

As it is known, for fixed p and q, the product xyz is minimal when $x=0$ or $y=z$ (see P 3.57). Therefore it suffices to consider these cases.

Case 1: $x=0$. We need to show that $3(y^3+z^3)\ge 5y^2z^2$. Indeed, we have

$$3(y^3+z^3)-5y^2z^2\ge 3(y^4+z^4)-6y^2z^2=3(y^2-z^2)^2\ge 0.$$

Case 2: $y=z$. We need to show that $f(x)\ge 0$ for $x\in[0,1]$, where

$$f(x)=3x^3-10x^2y^2\left(1-\frac{50}{81}y^2\right)+y^3(6-5y),\quad y\in[0,1].$$

For $y=0$, we have $f(x)=3x^3\ge 0$. Consider further that $y\in(0,1]$. From

$$f'(x)=x\left[9x-20y^2\left(1-\frac{50}{81}y^2\right)\right],$$

it follows that $f(x)$is decreasing on $[0,x_1]$ and increasing on $[x_1,1]$, where

$$x_1=\frac{20}{9}y^2\left(1-\frac{50}{81}y^2\right),\quad x_1\in(0,1).$$

Therefore, it remains to show that $f(x_1)\ge 0$, which is equivalent to

$$\frac{4000}{243}y^3\left(1-\frac{50}{81}y^2\right)^3\le 6-5y.$$

Substituting

$$y=\frac{9t}{10},\quad 0<t\le\frac{10}{9},$$

this inequality can be written as

$$t^3(2-t^2)^3\le 4-3t,$$

Indeed,

$$t^3(2-t^2)^3-(4-3t)\le t(2-t^2)^2-(4-3t)=(t-1)^2(t^3+2t^2-t-4)\le 0.$$

The equality holds for $a=b=c=0$, and also for $a=b=c=81/100$.

□

P 3.67. *Let a,b,c be the lengths of the sides of a triangle. If $a^2+b^2+c^2=3$, then*

$$a+b+c\ge 2+abc.$$

(Vasile Cîrtoaje, 2005)

First Solution. Let $p = a+b+c$ and $q = ab+bc+ca$. We need to show that $p^2-2q = 3$ involves $p \geq 2 + abc$. According to P 3.57-(b), for fixed p and q, the product abc is maximal when $c/2 \leq a = b \leq c$ or $a + b = c$. Therefore, it suffices to consider only these two cases.

Case 1: $c/2 \leq a = b \leq c$. From

$$3 = 2b^2 + c^2 \geq 3b^2,$$

we get $b \leq 1$. In addition, from

$$3 = 2b^2 + c^2 \leq 2b^2 + 4b^2 = 6b^2,$$

it follows that $2b^2 \geq 1$. Therefore, we need to prove that $2b^2 + c^2 = 3$, $b \leq 1$, $2b^2 \geq 1$ involve $2b + c \geq 2 + b^2c$. Since

$$2b + c - 2 - b^2c = (1 - b)(c + bc - 2),$$

it suffices to show that $c(1 + b) \geq 2$. This is true, since

$$c^2(1 + b)^2 - 4 = (3 - 2b^2)(1 + b)^2 - 4$$

$$= -1 + 6b + b^2 - 4b^3 - 2b^4 = (1 - b)(-1 + 5b + 6b^2 + 2b^3) \geq 0.$$

Case 2: $a + b = c$. From $a^2 + b^2 + c^2 = 3$, we get $2ab = 2c^2 - 3$, $c^2 \geq 3/2$. In addition, from $4ab \leq c^2$, we get $c^2 \leq 2$, and hence $3/2 \leq c^2 \leq 2$. Since

$$a + b + c - 2 - abc = 2c - 2 - c(c^2 - \frac{3}{2}) = \frac{-2c^3 + 7c - 4}{2},$$

we need to show that

$$2c^3 - 7c + 4 \leq 0.$$

From

$$(c^2 - 2)(2c^2 - 3) \leq 0,$$

we get $2c^4 - 7c^2 \leq -6$. Therefore,

$$c(2c^3 - 7c + 4) \leq -6 + 4c < 0.$$

This completes the proof. The equality holds for $a = b = c = 1$.

Second Solution. Assume that $a \geq b \geq c$. From

$$3 = a^2 + b^2 + c^2 \geq a^2 + \frac{1}{2}(b + c)^2 \geq \frac{3}{2}a^2,$$

it follows that $a \leq \sqrt{2}$. Let

$$E(a, b, c) = a + b + c - 2 - abc$$

and

$$t = \sqrt{\frac{b^2 + c^2}{2}}, \quad t \le 1 \le a.$$

We will show that

$$E(a, b, c) \ge E(a, t, t) \ge 0.$$

We have

$$E(a,b,c) - E(a,t,t) = a(t^2 - bc) - (2t - b - c) = \frac{a(b-c)^2}{2} - \frac{(b-c)^2}{2t + b + c}$$
$$= \frac{(b-c)^2}{2}\left(\frac{3a}{a^2 + 2t^2} - \frac{2}{2t + b + c}\right)$$
$$= \frac{(b-c)^2[2t(3a - 2t) + a(3b + 3c - 2a)]}{2(a^2 + 2t^2)(2t + b + c)} \ge 0,$$

because $3a - 2t > 2(a - t) \ge 0$ and $3(b + c) - 2a > 2(b + c - a) \ge 0$. On the other hand, from

$$E(a,t,t) = a + 2t - 2 - at^2 = (1 - t)(a + at - 2),$$

it follows that $E(a,t,t) \ge 0$ if $at \ge 2 - a$; that is,

$$a\sqrt{\frac{3 - a^2}{2}} \ge 2 - a.$$

By squaring, the inequality can be restated as

$$(a - 1)(8 - a^2 - a^3) \ge 0.$$

This is true, since $1 \le a \le \sqrt{2}$ implies $a - 1 \ge 0$ and $8 - a^2 - a^3 \ge 8 - 2 - 2\sqrt{2} > 0$.

□

P 3.68. *Let $f_n(a, b, c)$ be a symmetric homogeneous polynomial of degree $n \le 5$. Prove that*

(a) the inequality $f_n(a, b, c) \ge 0$ holds for all nonnegative real numbers a, b, c if and only if $f_n(a, 1, 1) \ge 0$ and $f_n(0, b, c) \ge 0$ for all $a, b, c \ge 0$;

(b) the inequality $f_n(a, b, c) \ge 0$ holds for all lengths a, b, c of the sides of a non-degenerate or degenerate triangle if and only if $f_n(x, 1, 1) \ge 0$ for $0 \le x \le 2$, and $f_n(y + z, y, z) \ge 0$ for all $y, z \ge 0$.

(Vasile Cîrtoaje, 2005)

Solution. Let $p = a + b + c$, $q = ab + bc + ca$, $r = abc$. Any symmetric homogeneous polynomial $f_n(a, b, c)$ of degree $n \le 5$ can be written as

$$f_n(a, b, c) = A_n(p, q)r + B_n(p, q),$$

where $A_n(p, q)$ and $B_n(p, q)$ are polynomial functions. For fixed p and q, the linear function $g_n(r) = A_n(p, q)r + B_n(p, q)$ is minimal when r is either minimal or maximal.

(a) By P 3.57-(a), for fixed p and q, the product r is minimal and maximal when two of a, b, c are equal or one of a, b, c is 0. Due to symmetry and homogeneity, the conclusion follows.

(b) By P 3.57-(b), for fixed p and q, the product r is minimal and maximal when two of a, b, c are equal or one of a,b,c is the sum of the others. Due to symmetry and homogeneity, the conclusion follows.

Remark. Similarly, we can prove the following statement, which does not involve the homogeneity property.

- *Let $f_n(a, b, c)$ be a symmetric polynomial function of degree $n \le 5$. The inequality $f_n(a, b, c) \ge 0$ holds for all nonnegative real numbers a, b, c if and only if it holds for $a = 0$ and for $b = c$.*

□

P 3.69. *If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then*

$$4(a^4 + b^4 + c^4) + 45 \ge 19(a^2 + b^2 + c^2).$$

(Vasile Cîrtoaje, 2009)

First Solution. We use the mixing variable method. Write the inequality as $F(a, b, c) \ge 0$, where

$$F(a, b, c) = 4(a^4 + b^4 + c^4) + 45 - 19(a^2 + b^2 + c^2).$$

Due to symmetry, we may assume that $a \le b \le c$. Let us denote $x = (b + c)/2$, $1 \le x \le 3/2$. We will show that

$$F(a, b, c) \ge F(a, x, x) \ge 0.$$

We have

$$F(a, b, c) - F(a, x, x) = 4(b^4 + b^4 - 2x^4) - 19(b^2 + c^2 - 2x^2)$$

$$= 4[(b^2 + c^2)^2 - 4x^4] + 8(x^4 - b^2c^2) - 19(b^2 + c^2 - 2x^2)$$

$$= (b^2 + c^2 - 2x^2)[4(b^2 + c^2 + 2x^2) - 19] + 8(x^2 - bc)(x^2 + bc).$$

Since

$$b^2+c^2-2x^2=2(x^2-bc)=\frac{1}{2}(b-c)^2,$$

we get

$$F(a,b,c)-F(a,x,x)=\frac{1}{2}(b-c)^2[4(b^2+c^2+2x^2)-19+4(x^2+bc)]$$
$$=\frac{1}{2}(b-c)^2[4(x^2-bc)+24x^2-19]\ge 0.$$

Also,

$$F(a,x,x)=F(3-2x,x,x)=6(x-1)^2(3-2x)(11-6x)\ge 0.$$

This completes the proof. The equality holds for $a=b=c=1$, and for $a=0$ and $b=c=3/2$ (or any cyclic permutation).

Second Solution. Write the inequality in the homogeneous form $f_4(a,b,c)\ge 0$, where

$$f_4(a,b,c)=36(a^4+b^4+c^4)+5(a+b+c)^4-19(a^2+b^2+c^2)(a+b+c)^2.$$

According to P 3.68-(a), it suffices to prove that $f_4(a,1,1)\ge 0$ and $f_4(0,b,c)\ge 0$ for all $a,b,c\ge 0$. We have

$$f_4(a,1,1)=2a(11a^3-18a^2+3a+4)=2a(a-1)^2(11a+4)\ge 0,$$

$$f_4(0,b,c)=2(b-c)^2(11b^2+11c^2+13bc).$$

Remark. Similarly, we can prove the following more general statement (*Vasile Cîrtoaje* and *Le Huu Dien Khue*, 2008).

- *Let α,β,γ be real numbers such that*

$$1+\alpha+\beta=2\gamma.$$

The inequality

$$\sum a^4+\alpha\sum a^2b^2+\beta abc\sum a\ge\gamma\sum ab(a^2+b^2)$$

holds for all $a,b,c\ge 0$ if and only if

$$\alpha\ge(\gamma-1)\max\{2,\gamma+1\}.$$

□

P 3.70. *Let a,b,c be nonnegative real numbers. If $k\le 2$, then*

$$\sum a(a-b)(a-c)(a-kb)(a-kc)\ge 0.$$

(Vasile Cîrtoaje, 2008)

Solution. Let us denote by $f_5(a,b,c)$ the left hand side. By P 3.68-(a), it suffices to show that $f_5(a,1,1) \geq 0$ and $f_5(0,b,c) \geq 0$ for all $a,b,c \geq 0$. Indeed, we have

$$f_5(a,1,1) = a(a-1)^2(a-k)^2 \geq 0$$

and

$$f_5(0,b,c) = (b+c)(b-c)^2(b^2-kbc+c^2) \geq 0.$$

The equality holds for $a = b = c$, for $a = 0$ and $b = c$ (or any cyclic permutation), and for $a/k = b = c$, $k > 0$ (or any cyclic permutation).

□

P 3.71. *Let a,b,c be nonnegative real numbers. If $k \in \mathbf{R}$, then*

$$\sum (b+c)(a-b)(a-c)(a-kb)(a-kc) \geq 0.$$

(Vasile Cîrtoaje, 2008)

Solution. Let us denote by $f_5(a,b,c)$ the left hand side. By P 3.68-(a), it suffices to show that $f_5(a,1,1) \geq 0$ and $f_5(0,b,c) \geq 0$ for all $a,b,c \geq 0$. Indeed, we have

$$f_5(a,1,1) = 2(a-1)^2(a-k)^2 \geq 0$$

and

$$f_5(0,b,c) = k^2(b+c)b^2c^2 + bc(b+c)(b-c)^2 \geq 0.$$

The equality holds for $a = b = c$, for $b = c = 0$ (or any cyclic permutation), and for $a/k = b = c$, $k > 0$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- *Let a,b,c be nonnegative real numbers. If $m \geq 0$ and $m(k-2) \leq 1$, then*

$$\sum (ma+b+c)(a-b)(a-c)(a-kb)(a-kc) \geq 0.$$

□

P 3.72. *If a,b,c are nonnegative real numbers, then*

$$\sum a(a-2b)(a-2c)(a-5b)(a-5c) \geq 0.$$

(Vasile Cîrtoaje, 2008)

Solution. Let $f_5(a,b,c)=\sum a(a-2b)(a-2c)(a-5b)(a-5c)$. By P 3.68-(a), it suffices to show that $f_5(a,1,1)\geq 0$ and $f_5(0,b,c)\geq 0$ for all $a,b,c\geq 0$. Indeed, we have

$$\begin{aligned}f_5(a,1,1)&=a^3(a-7)^2+20a^3-60a^2+44a+8\\&\geq 20a^3-60a^2+44a+8,\end{aligned}$$

since

$$20a^3-60a^2+44a+8>20a^2(a-3)\geq 0$$

for $a\geq 3$, and

$$20a^3-60a^2+44a+8=5(2a-3)^2+8-a\geq 8-a\geq 0$$

for $a\leq 8$. Also,

$$f_5(0,b,c)=(b+c)(b^2-4bc+c^2)^2\geq 0.$$

The equality holds for $a=0$ and $b^2-4bc+c^2=0$ (or any cyclic permutation).

□

P 3.73. *If a,b,c are the lengths of the side of a triangle, then*

$$a^4+b^4+c^4+9abc(a+b+c)\leq 10(a^2b^2+b^2c^2+c^2a^2).$$

First Solution. Let

$$f_4(a,b,c)=10(a^2b^2+b^2c^2+c^2a^2)-a^4-b^4-c^4-9abc(a+b+c).$$

By P 3.68-(b), it suffices to show that $f_4(x,1,1)\geq 0$ for $0\leq x\leq 2$ and $f_4(y+z,y,z)\geq 0$ for $y,z\geq 0$. Since

$$f_4(x,1,1)=8-18x+11x^2-x^4=(2-x)(4+x)(1-x)^2\geq 0$$

and

$$\begin{aligned}f_4(y+z,y,z)&=8(y^2+z^2)^2-2yz(y^2+z^2)-28y^2z^2\\&=2(y-z)^2(4y^2+4z^2+7yz)\geq 0,\end{aligned}$$

the proof is completed. The equality holds for an equilateral triangle and for a degenerate triangle with $a/2=b=c$ (or any cyclic permutation).

Second Solution. We use the sum-of-squares method (SOS method). Write the inequality as follows

$$9(\sum b^2c^2-abc\sum a)-(\sum a^4-\sum b^2c^2)\geq 0,$$

$$9\sum a^2(b-c)^2-\sum(b^2-c^2)^2\geq 0,$$
$$\sum(b-c)^2(3a-b-c)(3a+b+c)\geq 0.$$

Without loss of generality, assume that $a\geq b\geq c$. Since

$$(b-c)^2(3a-b-c)(3a+b+c)\geq 0,$$

it suffices to show that

$$(c-a)^2(3b-c-a)(3b+c+a)+(a-b)^2(3c-a-b)(3c+a+b)\geq 0.$$

Since

$$3b-c-a\geq 2b-a\geq b+c-a\geq 0$$

and $(c-a)^2\geq(a-b)^2)$, it is enough to prove that

$$(3b-c-a)(3b+c+a)+(3c-a-b)(3c+a+b)\geq 0.$$

We have

$$(3b+c+a)-(3c+a+b)=2(b-c)\geq 0,$$

and hence

$$\begin{aligned}(3b-c-a)(3b+c+a)+(3c-a-b)(3c+a+b)&\geq\\ \geq(3b-c-a)(3c+a+b)+(3c-a-b)(3c+a+b)&\\ =2(b+c-a)(3c+a+b)\geq 0.&\end{aligned}$$

□

P 3.74. *If a,b,c are the lengths of the sides of a triangle, then*

$$3(a^4+b^4+c^4)+7abc(a+b+c)\leq 5\sum ab(a^2+b^2).$$

Solution. Let

$$f_4(a,b,c)=5\sum ab(a^2+b^2)-3(a^4+b^4+c^4)-7abc(a+b+c).$$

By P 3.68-(b), it suffices to show that $f_4(x,1,1)\geq 0$ for $0\leq x\leq 2$ and $f_4(y+z,y,z)\geq 0$ for $y,z\geq 0$. Since

$$f_4(x,1,1)=4-4x-7x^2+10x^3-3x^4=(2-x)(2+3x)(1-x)^2\geq 0$$

and

$$f_4(y+z,y,z) = 4(y^2+z^2)^2 + 4yz(y^2+z^2) - 24y^2z^2$$
$$= 4(y-z)^2(y^2+z^2+3yz) \ge 0,$$

the proof is completed. The equality holds for an equilateral triangle and for a degenerate triangle with $a/2 = b = c$ (or any cyclic permutation).

□

P 3.75. *If a,b,c are the lengths of the sides of a triangle, then*

$$\frac{b^2+c^2-6bc}{a} + \frac{c^2+a^2-6ca}{b} + \frac{a^2+b^2-6ab}{c} + 4(a+b+c) \le 0.$$

(Vasile Cîrtoaje, 2005)

First Solution. Write the inequality as $f_4(a,b,c) \ge 0$, where

$$f_4(a,b,c) = \sum bc(6bc-b^2-c^2) - 4abc(a+b+c).$$

By P 3.68-(b), it suffices to show that $f_4(x,1,1) \ge 0$ for $0 \le x \le 2$ and $f_4(y+z,y,z) \ge 0$ for $y,z \ge 0$. Since

$$f_4(x,1,1) = 2(2-5x+4x^2-x^3) = 2(1-x)^2(2-x) \ge 0$$

and

$$f_4(y+z,y,z) = 4(y^2+z^2)^2 - 2yz(y^2+z^2) - 12y^2z^2$$
$$= 2(y-z)^2(2y^2+3yz+2z^2) \ge 0,$$

the proof is completed. The equality holds for an equilateral triangle and for a degenerate triangle with $a/2 = b = c$ (or any cyclic permutation).

Second Solution. We use the SOS method. Write the inequality as follows:

$$\sum bc(b^2+c^2-6bc) + 4abc\sum a \le 0,$$

$$\sum bc(b^2+c^2-2bc) - 4(\sum b^2c^2 - abc\sum a) \le 0,$$

$$\sum bc(b-c)^2 - 2\sum a^2(b-c)^2 \le 0,$$

$$\sum (b-c)^2(2a^2-bc) \ge 0.$$

Without loss of generality, assume that $a \geq b \geq c$. Since $(b-c)^2(2a^2-bc) \geq 0$, it suffices to prove that

$$(c-a)^2(2b^2-ca)+(a-b)^2(2c^2-ab) \geq 0.$$

Since

$$2b^2-ca \geq 2b^2-c(b+c)=(b-c)(2b+c) \geq 0$$

and $(c-a)^2 \geq (a-b)^2$, it is enough to show that

$$(2b^2-ca)+(2c^2-ab) \geq 0.$$

Indeed,

$$(2b^2-ca)+(2c^2-ab)=(b-c)^2+(b+c)(b+c-a) \geq 0.$$

□

P 3.76. *Let $f_6(a,b,c)$ be a sixth degree symmetric homogeneous polynomial written in the form*

$$f_6(a,b,c)=Ar^2+B(p,q)r+C(p,q), \quad A \leq 0,$$

where

$$p=a+b+c, \quad q=ab+bc+ca, \quad r=abc.$$

Prove that

(a) the inequality $f_6(a,b,c) \geq 0$ holds for all nonnegative real numbers a,b,c if and only if $f_6(a,1,1) \geq 0$ and $f_6(0,b,c) \geq 0$ for all $a,b,c \geq 0$;

(b) the inequality $f_6(a,b,c) \geq 0$ holds for all for all lengths a,b,c of the sides of a non-degenerate or degenerate triangle if and only if $f_6(x,1,1) \geq 0$ for $0 \leq x \leq 2$, and $f_6(y+z,y,z) \geq 0$ for all $y,z \geq 0$.

(Vasile Cîrtoaje, 2006)

Solution. For fixed p and q, the function f defined by

$$f(r)=Ar^2+B(p,q)r+C(p,q)$$

is a concave quadratic function of r. Therefore, $f(r)$ is minimal when r is minimal or maximal. According to P 3.57, the conclusion follows. As we have shown in the proof of P 2.75, A is called the *highest coefficient* of $f_6(a,b,c)$.

Remark 1. We can extend the part (a) of P 3.76 as follows:

(a1) *For $A \leq 0$, the inequality $f_6(a,b,c) \geq 0$ holds for all nonnegative real numbers a,b,c satisfying $p^2 \leq 4q$ if and only if $f_6(a,1,1) \geq 0$ for all $0 \leq a \leq 4$;.*

(a2) *For $A \le 0$, the inequality $f_6(a,b,c) \ge 0$ holds for all nonnegative real numbers a,b,c satisfying $p^2 > 4q$ if and only if $f_6(a,1,1) \ge 0$ for all $a > 4$ and $f_6(0,b,c) \ge 0$ for all $b,c \ge 0$.*

Notice that the restriction $0 \le a \le 4$ in (a1) follows by setting $b = c = 1$ in $p^2 \le 4q$. In addition, since $a = 0$ and $p^2 \le 4q$ involve $b = c$, the condition $f_6(0,b,c) \ge 0$ is not necessary in (a1) because it is equivalent to $f_6(0,1,1) \ge 0$, which follows from $f_6(a,1,1) \ge 0$ for all $0 \le a \le 4$. Also, the restriction $a > 4$ in (a2) follows by setting $b = c = 1$ in $p^2 > 4q$.

Remark 2. The statement in P 3.76 and its extension in Remark 1 are also valid in the more general case when $f_6(a,b,c)$ is a symmetric homogeneous function of the form

$$f_6(a,b,c) = Ar^2 + B(p,q)r + C(p,q),$$

where $B(p,q)$ and $C(p,q)$ are rational functions.

□

P 3.77. *If a,b,c are nonnegative real numbers, then*

$$\sum a(b+c)(a-b)(a-c)(a-2b)(a-2c) \ge (a-b)^2(b-c)^2(c-a)^2.$$

(Vasile Cîrtoaje, 2008)

Solution. Let $p = a+b+c$, $q = ab+bc+ca$, $r = abc$, and

$$f_6(a,b,c) = f(a,b,c) - (a-b)^2(b-c)^2(c-a)^2,$$

where

$$f(a,b,c) = \sum a(b+c)(a-b)(a-c)(a-2b)(a-2c).$$

Since

$$\sum a(b+c)(a-b)(a-c)(a-2b)(a-2c) =$$

$$= \sum a(p-a)(a^2+2bc-q)(a^2+6bc-2q),$$

$f(a,b,c)$ has the same highest coefficient A_0 as

$$P_1(a,b,c) = -\sum a^2(a^2+2bc)(a^2+6bc);$$

that is, according to Remark 2 from P 2.75, $A_0 = P_1(1,1,1) = -3(1+2)(1+6) = -63$. Then, $f_6(a,b,c)$ has the highest coefficient

$$A = A_0 + 27 = -36.$$

Since $A < 0$, according to P 3.76-(a), it suffices to prove that $f_6(a,1,1) \geq 0$ and $f_6(0,b,c) \geq 0$ for all $a,b,c \geq$. Indeed, we have

$$f_6(a,1,1) = 2a(a-1)^2(a-2)^2 \geq 0$$

and

$$f_6(0,b,c) = bc(b-c)^4 \geq 0.$$

The equality holds for $a=b=c$, for $a=0$ and $b=c$ (or any cyclic permutation), and for $a/2=b=c$ (or any cyclic permutation).

□

P 3.78. *Let a,b,c be nonnegative real numbers.*

(a) If $2 \leq k \leq 6$, then

$$\sum a(a-b)(a-c)(a-kb)(a-kc) + \frac{4(k-2)(a-b)^2(b-c)^2(c-a)^2}{a+b+c} \geq 0;$$

(b) If $k \geq 6$, then

$$\sum a(a-b)(a-c)(a-kb)(a-kc) + \frac{(k+2)^2(a-b)^2(b-c)^2(c-a)^2}{4(a+b+c)} \geq 0.$$

(Vasile Cîrtoaje, 2009)

Solution. a) We need to prove that $f_6(a,b,c) \geq 0$, where

$$f_6(a,b,c) = (a+b+c)\sum a(a-b)(a-c)(a-kb)(a-kc)$$

$$+4(k-2)(a-b)^2(b-c)^2(c-a)^2.$$

Since $f_6(a,b,c)$ has the same highest coefficient as

$$4(k-2)(a-b)^2(b-c)^2(c-a)^2$$

and $(a-b)^2(b-c)^2(c-a)^2$ has the highest coefficient -27, it follows that $f_6(a,b,c)$ has the highest coefficient

$$A = -108(k-2).$$

Since $A \leq 0$, according to P 3.76-(a), it suffices to prove that $f_6(a,1,1) \geq 0$ and $f_6(0,b,c) \geq 0$ for all $a,b,c \geq 0$. Indeed, we have

$$f_6(a,1,1) = a(a+2)(a-1)^2(a-k)^2 \geq 0$$

and

$$f_6(0,b,c)=(b-c)^6+(6-k)bc(b-c)^4 \ge 0.$$

The equality holds for $a=b=c$, for $a=0$ and $b=c$ (or any cyclic permutation, and for and for $a/k=b=c$ (or any cyclic permutation).

(b) We need to prove that $f_6(a,b,c)\ge 0$, where

$$f_6(a,b,c)=4(a+b+c)\sum a(a-b)(a-c)(a-kb)(a-kc)$$

$$+(k+2)^2(a-b)^2(b-c)^2(c-a)^2.$$

Since $f_6(a,b,c)$ has the same highest coefficient as

$$(k+2)^2(a-b)^2(b-c)^2(c-a)^2$$

and $(a-b)^2(b-c)^2(c-a)^2$ has the highest coefficient -27, it follows that $f_6(a,b,c)$ has the highest coefficient

$$A=-27(k+2)^2<0.$$

According to P 3.76-(a), it suffices to prove that $f_6(a,1,1)\ge 0$ and $f_6(0,b,c)\ge 0$ for all $a,b,c\ge 0$. Indeed, we have

$$f_6(a,1,1)=4a(a+2)(a-1)^2(a-k)^2\ge 0$$

and

$$f_6(0,b,c)=(b-c)^2[2(b^2+c^2)-(k-2)bc]^2\ge 0.$$

The equality holds for $a=b=c$, for $a=0$ and $b=c$ (or any cyclic permutation, and for and for $a/k=b=c$ (or any cyclic permutation), and for $a=0$ and $\frac{b}{c}+\frac{c}{b}=\frac{k-2}{2}$ or any cyclic permutation.

□

P 3.79. *If a,b,c are nonnegative real numbers, then*

$$(3a^2+2ab+3b^2)(3b^2+2bc+3c^2)(3c^2+2ca+3a^2)\ge 8(a^2+3bc)(b^2+3ca)(c^2+3ab).$$

Solution. Let $p=a+b+c$, $q=ab+bc+ca$ and

$$f(a,b,c)=(3a^2+2ab+3b^2)(3b^2+2bc+3c^2)(3c^2+2ca+3a^2).$$

We need to prove that $f_6(a,b,c)\ge 0$, where

$$f_6(a,b,c)=f(a,b,c)-8(a^2+3bc)(b^2+3ca)(c^2+3ab).$$

Since

$$f(a,b,c)=(3p^2-6q+2ab-3c^2)(3p^2-6q+2bc-3a^2)(3p^2-6q+2ca-3b^2),$$

$f_6(a,b,c)$ has the same highest coefficient A as $P_2(a,b,c)$, where

$$P_2(a,b,c)=(2ab-3c^2)(2bc-3a^2)(2ca-3b^2)-8(a^2+3bc)(b^2+3ca)(c^2+3ab);$$

that is, according to Remark 2 from P 2.75,

$$A=P_2(1,1,1)=(2-3)^3-8(1+3)^3<0.$$

Then, by P 3.76-(a), it suffices to prove that $f_6(a,1,1)\ge 0$ and $f_6(0,b,c)\ge 0$ for all nonnegative real a,b,c. Indeed,

$$f_6(a,1,1)=8(3a^2+2a+3)^2-8(a^2+3)(3a+1)^2=48(a+1)(a-1)^2\ge 0,$$

$$f_6(0,b,c)=3b^2c^2(9b^2-2bc+9c^2)\ge 0.$$

The equality holds for $a=b=c$.

□

P 3.80. *Let a,b,c be nonnegative real numbers such that $a+b+c=2$. If*

$$\frac{-2}{3}\le k\le\frac{11}{8},$$

then

$$(a^2+kab+b^2)(b^2+kbc+c^2)(c^2+kca+a^2)\le k+2.$$

(Vasile Cîrtoaje, 2011)

Solution. We need to prove that $f_6(a,b,c)\ge 0$, where

$$f_6(a,b,c)=(k+2)(a+b+c)^6-64(a^2+kab+b^2)(b^2+kbc+c^2)(c^2+kca+a^2).$$

Since $f_6(a,b,c)$ has the same highest coefficient A as $P_2(a,b,c)$, where

$$P_2(a,b,c)=-64(kab-c^2)(kbc-a^2)(kca-b^2),$$

according to Remark 2 from P 2.75, we have

$$A=P_2(1,1,1)=64(1-k)^3.$$

Also,

$$f_6(a,1,1)=(k+2)a[(a-1)^2+11-8k][a^3+14a^2+(8k+12)a+16]$$

and

$$\begin{aligned} f_6(0,b,c) &= (k+2)(b+c)^6 - 64b^2c^2(b^2+kbc+c^2) \\ &= (b-c)^2\left[(k+2)(b^2+c^2)^2 + 8(k+2)bc(b^2+c^2) + 4(7k-2)b^2c^2\right]. \end{aligned}$$

Case 1: $1 \le k \le \dfrac{11}{8}$. Since $A \le 0$, according to P 3.76-(a), it suffices to prove that $f_6(a,1,1) \ge 0$ and $f_6(0,b,c) \ge 0$ for all $a,b,c \ge 0$. Clearly, these conditions are satisfied. The equality holds for $a = 0$ and $b = c = 1$ (or any cyclic permutation). If $k = 11/8$, then the equality holds also for $a = b = c = 2/3$.

Case 2: $\dfrac{-2}{3} \le k < 1$. Since $A > 0$, we will use the *highest coefficient cancellation method*. We will prove the sharper inequality $g_6(a,b,c) \ge 0$, where

$$g_6(a,b,c) = f_6(a,b,c) - 64(1-k)^3a^2b^2c^2.$$

Since $g_6(a,b,c)$ has the highest coefficient $A_1 = 0$, it suffices to show that $g_6(a,1,1) \ge 0$ and $g_6(0,b,c) \ge 0$ for all $a,b,c \ge 0$ (see P 3.76). The inequality $g_6(a,1,1) \ge 0$ is true if

$$(k+2)[(a-1)^2 + 11 - 8k][a^3 + 14a^2 + (8k+12)a + 16] \ge 64(1-k)^3a.$$

It suffices to show that

$$(k+2)(11-8k)(6a^2 + (8k+12)a + 16] \ge 64(1-k)^3a.$$

In addition, since $11 - 8k > 8(1-k)$, we only need to show that

$$(k+2)[3a^2 + (4k+6)a + 8] \ge 4(1-k)^2a.$$

Since

$$3a^2 + (4k+6)a + 8 \ge 3(2a-1) + (4k+6)a + 8 = (4k+12)a + 5 > 4(k+3)a,$$

it is enough to show that

$$(k+2)(k+3) \ge (1-k)^2.$$

Indeed,

$$(k+2)(k+3) - (1-k)^2 = 7k + 5 = 7\left(k + \frac{2}{3}\right) + \frac{1}{3} > 0.$$

The inequality $g_6(0,b,c) \ge 0$ is also true since

$$\begin{aligned} g_6(0,b,c) &\ge (b-c)^2[4(k+2)b^2c^2 + 16(k+2)b^2c^2 + 4(7k-2)b^2c^2] \\ &= 16(3k+2)b^2c^2 \ge 0. \end{aligned}$$

Thus, the proof is completed. The equality holds for $a = 0$ and $b = c = 1$ (or any cyclic permutation). If $k = 11/8$, then the equality holds also for $a = b = c = 2/3$.

□

P 3.81. *Let a, b, c be nonnegative real numbers such that $a+b+c=2$. Prove that*

$$(2a^2+bc)(2b^2+ca)(2c^2+ab) \le 4.$$

Solution. Write the inequality in the homogeneous form $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c) = (a+b+c)^6 - 16(2a^2+bc)(2b^2+ca)(2c^2+ab).$$

Since $f_6(a,b,c)$ has the same highest coefficient A as $P_2(a,b,c)$, where

$$P_2(a,b,c) = -16(2a^2+bc)(2b^2+ca)(2c^2+ab),$$

according to Remark 2 from P 2.75, we have

$$A = P_2(1,1,1) = -432.$$

Since $A < 0$, according to P 3.76-(a), it suffices to prove that $f_6(a,1,1) \ge 0$ and $f_6(0,b,c) \ge 0$ for all $a,b,c \ge 0$. We have

$$f_6(a,1,1) = a(a+2)^2(a^3+8a^2-8a+32) = a(a+2)^2[a^3+4a^2+28+4(a-1)^2] \ge 0,$$

$$f_6(0,b,c) = (b+c)^6 - 64b^3c^3 \ge 0.$$

The equality holds for $a=0$ and $b=c=1$ (or any cyclic permutation).

□

P 3.82. *Let a, b, c be nonnegative real numbers, no two of which are zero. Then,*

$$\sum (a-b)(a-c)(a-2b)(a-2c) \ge \frac{5(a-b)^2(b-c)^2(c-a)^2}{ab+bc+ca}.$$

(Vasile Cîrtoaje, 2010)

Solution. Denote

$$p = a+b+c, \quad q = ab+bc+ca,$$

and write the inequality as $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c) = q\sum (a-b)(a-c)(a-2b)(a-2c) - 5(a-b)^2(b-c)^2(c-a)^2.$$

Clearly, $f_6(a,b,c)$ has the highest coefficient

$$A = (-5)(-27) = 135.$$

Since $A > 0$, we will use the *highest coefficient cancellation method*. We have

$$f_6(a,1,1) = (2a+1)(a-1)^2(a-2)^2, \quad f_6(0,b,c) = bc[(b+c)^2 - 6bc]^2.$$

Consider two cases: $p^2 \le 4q$ and $p^2 > 4q$.

Case 1: $p^2 \le 4q$. Since

$$f_6(1,1,1) = 0, \quad f_6(2,1,1) = 0,$$

we define the symmetric homogeneous polynomial of degree three

$$P(a,b,c) = abc + B(a+b+c)^3 + C(a+b+c)(ab+bc+ca)$$

such that $P(1,1,1) = 0$ and $P(2,1,1) = 0$. We get $B = 1/18$ and $C = -5/18$, hence

$$P(a,b,c) = abc + \frac{1}{18}(a+b+c)^3 - \frac{5}{18}(a+b+c)(ab+bc+ca).$$

Consider now the sharper inequality $g_6(a,b,c) \ge 0$, where

$$g_6(a,b,c) = f_6(a,b,c) - 135P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient $A = 0$. Then, according to Remark 1 from the proof of P 3.76, it suffices to prove that $g_6(a,1,1) \ge 0$ for $0 \le a \le 4$. We have

$$P(a,1,1) = \frac{1}{18}(a-1)^2(a-2),$$

hence

$$g_6(a,1,1) = f_6(a,1,1) - 135P^2(a,1,1) = \frac{1}{12}(a-1)^2(a-2)^2(7+34a-5a^2) \ge 0.$$

Case 2: $p^2 > 4q$. Define the symmetric homogeneous function

$$R(a,b,c) = abc + C(a+b+c)(ab+bc+ca) - (9C+1)\frac{(ab+bc+ca)^2}{3(a+b+c)},$$

which satisfies $R(1,1,1) = 0$, and consider the sharper inequality $g_6(a,b,c) \ge 0$, where

$$g_6(a,b,c) = f_6(a,b,c) - 135R^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient $A = 0$. Then, according to Remark 2 from the proof of P 3.76, it suffices to prove that $g_6(a,1,1) \ge 0$ for $a > 4$, and $g_6(0,b,c) \ge 0$ for all $b,c \ge 0$. We have

$$R(a,1,1) = \frac{(a-1)^2[3C(2a+1)-1]}{3(a+2)}, \quad R(0,b,c) = \frac{bc[3C(b+c)^2 - (9C+1)bc]}{3(b+c)}.$$

The inequality $g_6(0,b,c) \ge 0$ holds for all $b,c \ge 0$ only if

$$\frac{9C+1}{3C} = 6;$$

that is, $C = 1/9$. For this value of C, we have

$$R(a,1,1) = \frac{2(a-1)^3}{9(a+2)}, \quad R(0,b,c) = \frac{bc[(b+c)^2-6bc]}{9(b+c)},$$

hence

$$g_6(a,1,1) = f_6(a,1,1) - 135R^2(a,1,1) = \frac{(a-1)^2}{3(a+2)^2}g(a),$$

where

$$g(a) = 3(2a+1)(a^2-4)^2 - 20(a-1)^4,$$

and

$$g_6(0,b,c) = f_6(0,b,c) - 135R^2(0,b,c) = \frac{bc(3b^2+bc+3c^2)[(b+c)^2-6bc]^2}{3(b+c)^2} \ge 0.$$

To complete the proof, we need to show that $g(a) \ge 0$ for $a > 4$. This is true since $a^2 - 4 > (a-1)^2$ and $3(2a+1) > 20$.

The equality holds for $a = b = c$, for $a/2 = b = c$ (or any cyclic permutation), and for $a = 0$ and $b/c + c/b = 4$ (or any cyclic permutation).

□

P 3.83. *Let $a \le b \le c$ be positive real numbers such that*

$$a+b+c = p, \quad abc = r,$$

where p and r are fixed positive numbers satisfying $p^3 \ge 27r$. Prove that

$$q = ab+bc+ca$$

is maximal when $b = c$, and is minimal when $a = b$.

Solution. First, we show that $b \in [b_1,\ b_2]$, where b_1 and b_2 are the positive roots of the equation

$$2x^3 - px^2 + r = 0.$$

Let $f(x) = 2x^3 - px^2 + r$, $x \ge 0$. From $f'(x) = 2x(3x-p)$, it follows that $f(x)$ is decreasing for $x \le p/3$ and increasing for $x \ge p/3$. Since $f(0) > 0$, $f(p/3) =$

$(27r - p^3)/27 \le 0$ and $f(p) = p^3 + r > 0$, there exists two positive numbers $b_1 \le b_2$ such that $f(b_1) = f(b_2) = 0$ and $f(x) \le 0$ for $x \in [b_1,\ b_2]$. From

$$\frac{f(b)}{b} = b^2 - b(p-b) + \frac{r}{b} = b^2 - b(a+c) + bc = (b-a)(b-c) \le 0,$$

it follows that $b \in [b_1,\ b_2]$. In addition, we have $b = b_1$ when $b = a$, and $b = b_2$ when $b = c$. On the other hand, from

$$q = b(a+c) + ac = b(p-b) + \frac{r}{b},$$

we get

$$q(b) = pb - b^2 + \frac{r}{b}.$$

Since

$$q'(b) = p - 2b - \frac{r}{b^2} = \frac{-(b-a)(b-c)}{b} \ge 0,$$

$q(b)$ is increasing on $[b_1, b_2]$, and hence $q(b)$ is maximal for $b = b_2$, when $b = c$, and is minimal for $b = b_1$, when $b = a$.

Remark 1. Substituting $1/a$, $1/b$, $1/c$ for a, b, c, respectively, we get the following similar statement.

- *Let $a \le b \le c$ be positive real numbers such that*

$$ab + bc + ca = q, \quad abc = r,$$

where q and r are fixed positive numbers satisfying $q^3 \ge 27r^2$. The sum

$$p = a + b + c$$

is minimal when $b = c$, and is maximal when $a = b$.

Remark 2. We can prove by the contradiction method (as in Remark of P 2.53) the following generalizations:

- *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that*

$$a_1 + a_2 + \cdots + a_n = np, \quad a_1 a_2 \cdots a_n = p_1,$$

where p and p_1 are fixed positive numbers satisfying $p^n \ge p_1$. Then, the sum

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

is maximal and minimal when $n - 1$ numbers of $a_1, a_2, \ldots, a_n$ are equal.

- *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that*

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}=\frac{n}{p}, \quad a_1a_2\cdots a_n=p_1,$$

where p and p_1 are fixed positive numbers satisfying $p^n \le p_1$. Then, the sum

$$a_1+a_2+\cdots+a_n$$

is maximal and minimal when $n-1$ numbers of $a_1, a_2, \ldots, a_n$ are equal.

□

P 3.84. *If a, b, c are positive real numbers, then*

$$(a+b+c-3)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-3\right)+abc+\frac{1}{abc}\ge 2.$$

(Vasile Cîrtoaje, 2004)

Solution. Since the inequality does not exchange by substituting a, b, c with $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$, respectively, we may consider only the case $abc \ge 1$. Using the notation $p = a+b+c$ and $r = abc$, $r \ge 1$, we can write the inequality as

$$(p-3)\left(\frac{ab+bc+ca}{r}-3\right)+r+\frac{1}{r}\ge 2.$$

For fixed p and r, by P 3.83, the expression $q = ab+bc+ca$ is minimal when two of a, b, c are equal. Since $p \ge 3\sqrt[3]{r} \ge 3$ (by the AM-GM inequality), it suffices to prove the desired inequality for $b = c$, when it becomes as follows

$$a\left(b^2+\frac{2}{b}-3\right)+\frac{1}{a}\left(\frac{1}{b^2}+2b-3\right)\ge 6\left(b+\frac{1}{b}-2\right),$$

$$(b-1)^2\left[\left(ab+\frac{1}{ab}-2\right)+2\left(a+\frac{1}{a}-2\right)\right]\ge 0.$$

Since $ab+\frac{1}{ab}\ge 2$ and $a+\frac{1}{a}\ge 2$, the conclusion follows. The equality holds for $a=b=1$, or $b=c=1$, or $c=a=1$.

□

P 3.85. *If a, b, c are positive real numbers such that $abc = 1$, then*

(a) $$\frac{3}{7}\left(ab+bc+ca-\frac{2}{3}\right) \ge \sqrt{\frac{2}{3}(a+b+c)-1};$$

(b) $$ab+bc+ca-3 \ge \frac{46}{27}(\sqrt{a+b+c-2}-1).$$

(Vasile Cîrtoaje, 2009)

Solution. Let us denote $p = a + b + c$.

(a) Write the inequality as

$$\frac{3}{7}\left(ab+bc+ca-\frac{2}{3}\right) \ge \sqrt{\frac{2p}{3}-1}.$$

For fixed p, the expression $q = ab + bc + ca$ is minimal when two of a, b, c are equal (see P 3.83). Thus, it suffices to consider the case $a = b$, when the inequality becomes

$$3a^3-2a+6 \ge 7\sqrt{\frac{4a^3-3a^2+2}{3}}.$$

By squaring, we get

$$(a-1)^2(3a-1)^2(3a^2+8a+10) \ge 0,$$

which is true. The equality holds for $a = b = c = 1$, and also for $(a, b, c) = \left(\frac{1}{3}, \frac{1}{3}, 9\right)$ or any cyclic permutation.

(b) Write the inequality as

$$ab+bc+ca-3 \ge \frac{46}{27}(\sqrt{p-2}-1).$$

For fixed p, the expression $q = ab + bc + ca$ is minimal when two of a, b, c are equal (see P 3.83). Thus, it suffices to consider the case $a = b$, when the inequality becomes

$$27a^3-35a+54 \ge 46\sqrt{2a^3-2a^2+1}.$$

By squaring, we get

$$(a-1)^2(9a-5)^2(9a^2+28a+32) \ge 0,$$

which is true. The equality holds for $a = b = c = 1$, and also for $(a, b, c) = \left(\frac{5}{9}, \frac{5}{9}, \frac{81}{25}\right)$ or any cyclic permutation.

□

P 3.86. *If a,b,c are positive real numbers such that $abc=1$, then*

$$ab+bc+ca+\frac{50}{a+b+c+5}\geq\frac{37}{4}.$$

(Michael Rozenberg, 2013)

Solution. Using the notation $p=a+b+c$, we can write the inequality as

$$ab+bc+ca+\frac{50}{p+5}\geq\frac{37}{4}.$$

For fixed p, the expression $q=ab+bc+ca$ is minimal when two of a,b,c are equal (see P 3.83). Thus,it suffices to prove the desired inequality for $a=b$; that is, to show that $a^2c=1$ involves

$$a^2+2ac+\frac{50}{2a+c+5}\geq\frac{37}{4}.$$

This is equivalent to

$$a^2+\frac{2}{a}+\frac{50a^2}{2a^3+5a^2+1}\geq\frac{37}{4},$$

which can be written in the obvious form

$$(a-1)^2(2a-1)^2(2a^2+11a+8)\geq 0.$$

The equality holds for $a=b=c=1$, and for $a=b=1/2$ and $c=4$ (or any cyclic permutation).

□

P 3.87. *Let a,b,c be positive real numbers.*

(a) If $abc=2$, then

$$(a+b+c-3)^2+1\geq\frac{a^2+b^2+c^2}{3};$$

(b) If $abc=\frac{1}{2}$, then

$$a^2+b^2+c^2+3(3-a-b-c)^2\geq 3.$$

(Vasile Cîrtoaje, 2007)

Solution. Let us denote $p = a + b + c$.

(a) Write the inequality as

$$(p-3)^2 + 1 \geq \frac{p^2 - 2(ab+bc+ca)}{3}.$$

For fixed p, the expression $q = ab + bc + ca$ is minimal when two of a, b, c are equal (see P 3.83). Thus, it suffices to consider the case $a = b$, when the inequality becomes in succession

$$\left(2a + \frac{2}{a^2} - 3\right)^2 + 1 \geq \frac{2a^2}{3} + \frac{4}{3a^4},$$

$$5a^6 - 18a^5 + 15a^4 + 12a^3 - 18a^2 + 4 \geq 0,$$

$$(a-1)^2(5a^4 - 8a^3 - 6a^2 + 8a + 4) \geq 0.$$

Since

$$5a^4 - 8a^3 - 6a^2 + 8a + 4 = 4(a-1)^4 + a(a^3 + 8a^3 - 30a + 24),$$

it suffices to prove that $a^3 + 8a^3 - 30a + 24 \geq 0$. Indeed, for $a \geq 1$, we have

$$\begin{aligned} a^3 + 8a^3 - 30a + 24 &= (a-1)^3 + 11a^2 - 33a + 25 \\ &= (a-1)^3 + 11(a - \frac{3}{2})^2 + \frac{1}{4} > 0, \end{aligned}$$

and for $a < 1$, we have

$$a^3 + 8a^3 - 30a + 24 = a(1-a)^2 + 2(2-a)(6-5a) > 0.$$

The equality holds for $(a, b, c) = (1, 1, 2)$ or any cyclic permutation.

(b) Write the inequality as

$$p^2 - 2(ab+bc+ca) + 3(3-p)^2 \geq 3.$$

For fixed p, the expression $q = ab + bc + ca$ is minimal when two of a, b, c are equal (see P 3.83). Thus, it suffices to consider the case $a = b$, when the inequality becomes in succession

$$2a^2 + \frac{1}{4a^2} + 3\left(3 - 2a - \frac{1}{2a^2}\right)^2 \geq 3,$$

$$14a^6 - 36a^5 + 24a^4 + 6a^3 - 9a^2 + 1 \geq 0,$$

$$(a-1)^2(14a^4 - 8a^3 - 6a^2 + 2a + 1) \geq 0.$$

Since

$$14a^4 - 8a^3 - 6a^2 + 2a + 1 = (a-1)^4 + a(13a^3 - 4a^2 - 12a + 6),$$

it suffices to prove that $13a^3-4a^2-12a+6\geq 0$. Indeed,

$$9(13a^3-4a^2-12a+6)=13(a+1)(3a-2)^2+2(a-1)^2+a^2>0.$$

The equality holds for $(a,b,c)=\left(1,1,\frac{1}{2}\right)$ or any cyclic permutation.

□

P 3.88. *If a,b,c are positive real numbers such that $a+b+c=3$, then*

$$4\left(\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\right)+9abc\geq 21.$$

Solution. Let $p=a+b+c$, $q=ab+bc+ca$, $r=abc$. We write the required inequality in the homogeneous form

$$\frac{4p^2q^2}{9r}+9r\geq\frac{5p^3}{3}.$$

For fixed p and r, it suffices to prove this inequality for the case when q is minimal; that is, when two of a,b,c are equal (see P 3.83). Due to symmetry and homogeneity, we can set $b=c=1$. Since $p=a+2$, $q=2a+1$, $r=a$, the inequality becomes

$$4(a+2)^2(2a+1)^2+81a^2\geq 15(a+2)^3,$$

which is equivalent to

$$(a-1)^2(a-4)^2\geq 0.$$

The equality holds for $a=b=c=1$, and for $a=2$ and $b=c=1/2$ (or any cyclic permutation).

□

P 3.89. *If a,b,c are nonnegative real numbers such that*

$$ab+bc+ca=abc+2,$$

then

$$a^2+b^2+c^2+abc\geq 4.$$

(Vasile Cîrtoaje, 2011)

First Solution. Among the numbers $1-a$, $1-b$ and $1-c$ there are always two with the same sign; let us say $(1-b)(1-c) \ge 0$. Thus, we have

$$a(1-b)(1-c) \ge 0,$$

$$a + abc \ge ab + ac,$$

$$a + (ab + bc + ca - 2) \ge ab + ac,$$

$$a + bc \ge 2,$$

and hence

$$\begin{aligned} a^2 + b^2 + c^2 + abc - 4 &\ge a^2 + 2bc + abc - 4 \\ &= (a+2)(a+bc-2) \ge 0. \end{aligned}$$

The equality holds for $a = b = c = 1$, and for $a = 0$ and $b = c = \sqrt{2}$ (or any cyclic permutation).

Second Solution For $a = 0$, we need to show that $bc = 2$ involves $b^2 + c^2 \ge 0$. This is true since

$$b^2 + c^2 \ge 2bc = 4.$$

Consider further that a, b, c are positive, and write the required inequality as

$$a^2 + b^2 + c^2 + abc \ge 2(ab + bc + ca - abc),$$

or

$$3abc \ge 2(ab + bc + ca) - a^2 - b^2 - c^2.$$

Let $p = a + b + c$, $q = ab + bc + ca$, $r = abc$. We need to show that $q = r + 2$ implies $3r \ge 4q - p^2$. For fixed q and r, the sum $p = a + b + c$ is minimal when two of a, b, c are equal (see Remark 1 from P 3.83). Thus, it suffices to consider the case $b = c$, when $p = a + 2b$, $q = 2ab + b^2$, $r = ab^2$. We need to prove that

$$2ab + b^2 = ab^2 + 2$$

implies

$$3ab^2 \ge 4(2ab + b^2) - (a + 2b)^2,$$

which reduces to

$$a(a + 3b^2 - 4b) \ge 0.$$

This is true since, for the nontrivial case $b < 4/3$, we have

$$a + 3b^2 - 4b = \frac{2 - b^2}{b(2-b)} + 3b^2 - 4b = \frac{(1-b)^2(2 + 4b - 3b^2)}{b(2-b)} \ge 0.$$

□

P 3.90. *If a,b,c are positive real numbers, then*

$$\left(\frac{b+c}{a}-2-\sqrt{2}\right)^2+\left(\frac{c+a}{b}-2-\sqrt{2}\right)^2+\left(\frac{a+b}{c}-2-\sqrt{2}\right)^2\ge 6.$$

(Vasile Cîrtoaje, 2012)

Solution. Let us denote $m=2+\sqrt{2}$. Without loss of generality, we can assume that $a=\max\{a,b,c\}$. We will prove first that

$$\left(\frac{c+a}{b}-m\right)^2+\left(\frac{a+b}{c}-m\right)^2\ge 2\left(\frac{2a}{b+c}-m+1\right)^2. \qquad (*)$$

According to the identity $2p^2+2q^2=(p-q)^2+(p+q)^2$, we have

$$2\left(\frac{c+a}{b}-m\right)^2+2\left(\frac{a+b}{c}-m\right)^2=\left(\frac{c+a}{b}-\frac{a+b}{c}\right)^2+\left(\frac{c+a}{b}+\frac{a+b}{c}-2m\right)^2.$$

Thus, we can rewrite the inequality (*) as

$$\left(\frac{c+a}{b}-\frac{a+b}{c}\right)^2\ge 4\left(\frac{2a}{b+c}-m+1\right)^2-\left(\frac{c+a}{b}+\frac{a+b}{c}-2m\right)^2,$$

$$\frac{(a+b+c)^2(b-c)^2}{b^2c^2}+\left(\frac{4a}{b+c}+\frac{c+a}{b}+\frac{a+b}{c}-4m+2\right)\frac{(a+b+c)(b-c)^2}{bc(b+c)}\ge 0.$$

This is true if $f(a)\ge 0$, where

$$f(a)=\frac{(a+b+c)(b+c)}{bc}+\frac{4a}{b+c}+\frac{c+a}{b}+\frac{a+b}{c}-4m+2.$$

Since $f(a)$ is increasing and $a=\max\{a,b,c\}$, it suffices to show that $f\left(\frac{b+c}{2}\right)\ge 0$. Indeed,

$$f\left(\frac{b+c}{2}\right)=\frac{3(b-c)^2}{bc}+6-4\sqrt{2}\ge 6-4\sqrt{2}>0.$$

Using (*), we only need to show that

$$\left(\frac{b+c}{a}-m\right)^2+2\left(\frac{2a}{b+c}-m+1\right)^2\ge 6.$$

Setting $\frac{b+c}{a}=t$, this inequality becomes

$$(t-m)^2+2\left(\frac{2}{t}-m+1\right)^2\ge 6,$$

$$\frac{(t-2)^2(t-\sqrt{2})^2}{t^2} \ge 0.$$

The proof is completed. The equality holds for $a = b = c$, and for $\frac{a}{\sqrt{2}} = b = c$ (or any cyclic permutation).

□

P 3.91. *If a, b, c are positive real numbers, then*

$$2(a^3 + b^3 + c^3) + 9(ab + bc + ca) + 39 \ge 24(a + b + c).$$

(Vasile Cîrtoaje, 2010)

Solution. Let $p = a + b + c$ and $q = ab + bc + ca$. Since $a^3 + b^3 + c^3 = 3abc + p^3 - 3pq$, we can write the inequality as

$$6abc + 2p^3 + 3(3 - 2p)q + 39 \ge 24p.$$

By Schur's inequality of degree three, we have

$$9abc \ge 4pq - p^3.$$

Therefore, it suffices to show that

$$\frac{2}{3}(4pq - p^3) + 2p^3 + 3(3 - 2p)q + 39 \ge 24p,$$

which is equivalent to

$$4p^3 + 117 \ge 72p + (10p - 27)q.$$

Case 1: $10p - 27 \ge 0$. Since $3q \le p^2$, we have

$$4p^3 + 117 - 72p - (10p - 27)q \ge 4p^3 + 117 - 72p - \frac{(10p - 27)p^2}{3}$$
$$= \frac{1}{3}(p - 3)^2(2p + 39) \ge 0.$$

Case 2: $10p - 27 < 0$. From $(3p - 8)^2 \ge 0$, we get

$$9p^2 - 48p + 64 \ge 0,$$

$$18q \ge -9\sum a^2 + 48p - 64.$$

Using this inequality and

$$\sum(10a-9)=10p-27<0,$$

we get

$$\begin{aligned}
2\left[2\sum a^3+9q+39-24p\right] &\geq 4\sum a^3+\left(-9\sum a^2+48p-64\right)+78-48p\\
&=\sum\left(4a^3-9a^2+\frac{14}{3}\right)<\sum\left(4a^3-9a^2+\frac{14}{3}\right)+\frac{14}{27}\sum(10a-9)\\
&=\sum a\left(4a^2-9a+\frac{140}{27}\right)>\sum a\left(4a^2-9a+\frac{81}{16}\right)=\sum a\left(2a-\frac{9}{4}\right)^2\geq 0.
\end{aligned}$$

The equality holds for $a=b=c=1$.

□

P 3.92. *If a,b,c are positive real numbers such that $a^2+b^2+c^2=3$, then*

$$a^3+b^3+c^3-3\geq |(a-b)(b-c)(c-a)|.$$

Solution. Assume that $a\leq b\leq c$ and write the inequality in the homogeneous form

$$a^3+b^3+c^3-3\left(\frac{a^2+b^2+c^2}{3}\right)^{3/2}\geq |(a-b)(b-c)(c-a)|.$$

The left hand side is nonnegative because, by the Cauchy-Schwarz inequality, we have

$$3(a^3+b^3+c^3)^2=(1+1+1)(a^3+b^3+c^3)(a^3+b^3+c^3)\geq (a^2+b^2+c^2)^3.$$

Thus, it suffices to consider that $a<b<c$. Using the substitution $b=a+p$ and $c=a+q$, where $0<p<q$, we need to show that

$$f(a)\geq pq(q-p),$$

where

$$f(a)=a^3+(a+p)^3+(a+q)^3-3\left[\frac{a^2+(a+p)^2+(a+q)^2}{3}\right]^{3/2}.$$

From

$$\begin{aligned}
f'(a)&=3(a^2+b^2+c^2)-3(a+b+c)\sqrt{\frac{a^2+b^2+c^2}{3}}\\
&=9\sqrt{\frac{a^2+b^2+c^2}{3}}\left(\sqrt{\frac{a^2+b^2+c^2}{3}}-\frac{a+b+c}{3}\right)\geq 0,
\end{aligned}$$

it follows that f is increasing, hence $f(a) \geq f(0)$. Therefore, it suffices to show that $f(0) \geq pq(q-p)$; that is,

$$p^3 + q^3 - 3\left(\frac{p^2+q^2}{3}\right)^{3/2} \geq pq(q-p).$$

Due to homogeneity, we may assume that $p = 1$ and $q > 1$, when the inequality becomes as follows:

$$q^3 - q^2 + q + 1 \geq 3\left(\frac{1+q^2}{3}\right)^{3/2},$$

$$3(q^3 - q^2 + q + 1)^2 \geq (q^2+1)^3,$$

$$q^6 - 3q^5 + 3q^4 - 3q^2 + 3q + 1 \geq 0,$$

$$q^3(q-1)^3 + q^3 - 3q^2 + 3q + 1 \geq 0,$$

$$(q^3+1)(q-1)^3 + 2 \geq 0.$$

The last inequality is clearly true. The equality holds for $a = b = c = 1$.

□

P 3.93. *If a, b, c are nonnegative real numbers, then*

$$a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2 \geq 2|a^3b + b^3c + c^3a - ab^3 - bc^3 - ca^3|.$$

Solution. Assume that $a \leq b \leq c$ and write the inequality as

$$(a^2-b^2)^2 + (b^2-c^2)^2 + (c^2-a^2)^2 \geq 4(a+b+c)(a-b)(b-c)(c-a).$$

Using the substitution $b = a + p$ and $c = a + q$, where $q \geq p \geq 0$, the inequality can be restated as

$$4Aa^2 + 4Ba + C \geq 0,$$

where

$$A = p^2 - pq + q^2, \quad B = p^3 + q(p-q)^2,$$

$$C = p^4 + 2p^3q - p^2q^2 - 2pq^3 + q^4 = (p^2 + pq - q^2)^2.$$

Since $A \geq 0$, $B \geq 0$ and $C \geq 0$, the inequality is obviously true. The equality occurs for $a = b = c$, and also for $a = 0$ and $\frac{c}{b} = \frac{1+\sqrt{5}}{2}$ (or any cyclic permutation).

□

P 3.94. *If a,b,c are nonnegative real numbers, then*

$$a^4+b^4+c^4-abc(a+b+c)\geq 2\sqrt{2}\,|a^3b+b^3c+c^3a-ab^3-bc^3-ca^3|.$$

(Pham Kim Hung, 2006)

Solution. Assume that $a\leq b\leq c$ and write the inequality as

$$a^2(a^2-bc)+b^2(b^2-ca)+c^2(c^2-ab)\geq 2\sqrt{2}(a+b+c)(a-b)(b-c)(c-a).$$

Using the substitution $b=a+p$ and $c=a+q$, where $q\geq p\geq 0$, the inequality becomes

$$Aa^2+Ba+C\geq 0,$$

where

$$A=5(p^2-pq+q^2),\quad B=4p^3+(6\sqrt{2}-1)p^2q-(6\sqrt{2}+1)pq^2+4q^3,$$
$$C=p^4+q^4+2\sqrt{2}pq(p^2-q^2).$$

Since

$$A\geq 0,$$
$$B\geq \frac{25}{4}p^2q-10pq^2+4q^3=q\left(\frac{5p}{2}-2q\right)^2\geq 0$$

and

$$C=(p^2+\sqrt{2}pq-q^2)^2\geq 0,$$

the conclusion follows. The equality occurs for $a=b=c$, and also for $a=0$ and $\frac{c}{b}=\frac{\sqrt{2}+\sqrt{6}}{2}$ (or any cyclic permutation).

□

P 3.95. *If $a,b,c\geq -5$ such that $a+b+c=3$, then*

$$\frac{1-a}{1+a+a^2}+\frac{1-b}{1+b+b^2}+\frac{1-c}{1+c+c^2}\geq 0.$$

(Vasile Cîrtoaje, 2014)

First Solution. Using the substitution

$$a=x-5,\quad b=y-5,\quad c=z-5,$$

we need to prove that if $x,y,z\geq 0$ such that $x+y+z=18$, then

$$\frac{6-x}{x^2-9x+21}+\frac{6-y}{y^2-9y+21}+\frac{6-z}{z^2-9z+21}\geq 0.$$

Denoting
$$p=\frac{x+y+z}{18},$$
we can write this inequality as $f_5(x,y,z)\ge 0$, where
$$f_5(x,y,z)=\sum(6p-x)(y^2-9yp+21p^2)(z^2-9zp+21p^2)$$
is a symmetric homogeneous polynomial of degree 5. According to P 3.68, it suffices to prove this inequality for $y=z$ and for $x=0$. Therefore, we only need to prove the original inequality for $b=c$ and for $a=-5$.

Case 1: $b=c=\frac{3-a}{2}$. Since
$$\frac{1-b}{1+b+b^2}=\frac{1-c}{1+c+c^2}=\frac{2(a-1)}{a^2-8a+19},$$
we need to show that
$$\frac{1-a}{1+a+a^2}+\frac{4(a-1)}{a^2-8a+19}\ge 0,$$
which is equivalent to
$$(a-1)^2(a+5)\ge 0.$$

Case 2: $a=-5$, $b+c=8$. We can write the desired inequality as follows:
$$\left(\frac{1}{7}+\frac{1-b}{1+b+b^2}\right)+\left(\frac{1}{7}+\frac{1-c}{1+c+c^2}\right)\ge 0,$$
$$\frac{(b-4)(b-2)}{1+b+b^2}+\frac{(c-4)(c-2)}{1+c+c^2}\ge 0,$$
$$\frac{b-c}{2}\left(\frac{b-2}{1+b+b^2}-\frac{c-2}{1+c+c^2}\right)\ge 0,$$
$$\frac{(b-c)^2[3+2(b+c)-bc]}{2(1+b+b^2)(1+c+c^2)}\ge 0.$$
The last inequality is true since
$$2(b+c)-bc=\left(\frac{b+c}{2}\right)^2-bc=\left(\frac{b-c}{2}\right)^2\ge 0.$$
The proof is completed. The equality occurs for $a=b=c=1$, and also for $a=-5$ and $b=c=4$ (or any cyclic permutation).

Second Solution. Assume that $a\le b\le c$ and denote
$$t=\frac{b+c}{2},\qquad E(a,b,c)=\frac{1-a}{1+a+a^2}+\frac{1-b}{1+b+b^2}+\frac{1-c}{1+c+c^2}.$$

From

$$t \geq \frac{a+b+c}{3} = 1, \quad t = \frac{3-a}{2} \leq 4,$$

it follows that

$$t \in [1,4].$$

We will show that

$$E(a,b,c) \geq E(a,t,t) \geq 0.$$

Write the left inequality as follows:

$$\left(\frac{1-b}{1+b+b^2} - \frac{1-t}{1+t+t^2}\right) + \left(\frac{1-c}{1+c+c^2} - \frac{1-t}{1+t+t^2}\right) \geq 0,$$

$$(b-c)\left[\frac{(b-1)t-b-2}{1+b+b^2} - \frac{(c-1)t-c-2}{1+c+c^2}\right] \geq 0,$$

$$(b-c)^2[(2+b+c-bc)t+1+2(b+c)+bc] \geq 0,$$

$$(b-c)^2[2t^2+6t+1-bc(t-1)] \geq 0.$$

The last inequality is true since

$$2t^2+6t+1-bc(t-1) \geq 2t^2+6t+1-t^2(t-1)$$

$$= t(4-t)(1+t)+2t+1 > 0.$$

Also, we have

$$E(a,t,t) = \frac{1-a}{1+a+a^2} + \frac{2(1-t)}{1+t+t^2} = \frac{2(t-1)}{4t^2-14t+13} + \frac{2(1-t)}{1+t+t^2}$$

$$= \frac{6(1-t)^2(4-t)}{(4t^2-14t+13)(1+t+t^2)} \geq 0.$$

□

P 3.96. *Let $a,b,c \neq \frac{1}{k}$ be nonnegative real numbers such that $a+b+c=3$. If $k \geq \frac{4}{3}$, then*

$$\frac{1-a}{(1-ka)^2} + \frac{1-b}{(1-kb)^2} + \frac{1-c}{(1-kc)^2} \geq 0.$$

(Vasile Cîrtoaje, 2012)

First Solution. Denoting $p=(a+b+c)/3$, we may write the inequality as $f_5(x,y,z)\ge 0$, where

$$f_5(x,y,z)=\sum(p-a)(p-kb)^2(p-kc)^2$$

is a symmetric homogeneous polynomial of degree 5. According to P 3.68, it suffices to prove this inequality for $b=c$ and for $a=0$.

Case 1: $b=c$. Since $a=3-2b$, the original inequality is equivalent to the following sequence of inequalities:

$$\frac{1-a}{(1-ka)^2}+\frac{2(1-b)}{(1-kb)^2}\ge 0,$$

$$\frac{2(b-1)}{(1-3k+2kb)^2}+\frac{2(1-b)}{(1-kb)^2}\ge 0,$$

$$k(b-1)^2[k(3-b)-2]\ge 0.$$

The last inequality holds since

$$k(3-b)-2\ge\frac{4}{3}(3-b)-2=\frac{2(3-2b)}{3}=\frac{2a}{3}\ge 0.$$

Case 2: $a=0$. Since $b+c=3$, the original inequality becomes as follows:

$$1\ge\frac{b-1}{(1-kb)^2}+\frac{c-1}{(1-kc)^2},$$

$$(1-kb)^2(1-kc)^2\ge(b-1)(1-kc)^2+(c-1)(1-kb)^2,$$

$$(k^2bc-3k+1)^2\ge 1+6k-9k^2+(5k^2-4k)bc,$$

$$k^3b^2c^2+18k-12\ge(6k^2+3k-4)bc.$$

Since

$$k^3b^2c^2+18k-12\ge 2\sqrt{k^3(18k-12)}\;bc,$$

it suffices to show that

$$4k^3(18k-12)\ge(6k^2+3k-4)^2,$$

which is equivalent to

$$36k^4-84k^3+39k^2+24k-16\ge 0,$$

$$(3k-4)(12k^3-12k^2-3k+4)\ge 0.$$

The last inequality holds since

$$12k^3-12k^2-3k+4>12k^3-12k^2-4k+4=4(k-1)(3k^2-1)>0.$$

The proof is completed. The equality occurs for $a=b=c=1$. If $k=4/3$, then the equality holds also for $a=0$ and $b=c=3/2$ (or any cyclic permutation).

□

P 3.97. *Let a, b, c, d be nonnegative real numbers such that*

$$a^2 + b^2 + c^2 + d^2 = 1.$$

Prove that

$$(1-a)(1-b)(1-c)(1-d) \geq abcd.$$

(Vasile Cîrtoaje, 2001)

Solution. The desired inequality follows by multiplying the inequalities

$$(1-a)(1-b) \geq cd,$$

$$(1-c)(1-d) \geq ab.$$

With regard to the first inequality, we have

$$2cd \leq c^2 + d^2 = 1 - a^2 - b^2,$$

and hence

$$\begin{aligned}2(1-a)(1-b) - 2cd &\geq 2(1-a)(1-b) - 1 + a^2 + b^2 \\ &= (1-a-b)^2 \geq 0.\end{aligned}$$

The second inequality can be proved similarly. The equality holds for $a = b = c = d = 1/2$, and also for $a, b, c, d) = (1, 0, 0, 0)$ or any cyclic permutation.

□

P 3.98. *Let a, b, c, d and x be positive real numbers such that*

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} = \frac{4}{x^2}.$$

If $x \geq 2$, then

$$(a-1)(b-1)(c-1)(d-1) \geq (x-1)^4.$$

(Vasile Cîrtoaje, 2001)

Solution. The desired inequality follows by multiplying the inequalities

$$2(a-1)(b-1) \geq (x-1)\left(\frac{ab}{cd}x + x - 2\right),$$

$$2(c-1)(d-1) \geq (x-1)\left(\frac{cd}{ab}x + x - 2\right),$$

$$\left(\frac{ab}{cd}x+x-2\right)\left(\frac{cd}{ab}x+x-2\right)\geq 4(x-1)^2.$$

With regard to the first inequality, we write it as

$$2ab-2(a+b)+x(3-x)\geq x(x-1)\frac{ab}{cd}.$$

Since

$$\frac{2}{cd}\leq\frac{1}{c^2}+\frac{1}{d^2}=\frac{4}{x^2}-\frac{1}{a^2}-\frac{1}{b^2},$$

it suffices to show that

$$4ab-4(a+b)+2x(3-x)\geq x(x-1)ab\left(\frac{4}{x^2}-\frac{1}{a^2}-\frac{1}{b^2}\right).$$

This is equivalent to

$$4a^2b^2-4ab(a+b)x+2x^2(3-x)ab+x^2(x-1)(a^2+b^2)\geq 0,$$

which can be written in the obvious form

$$[2ab-(a+b)x]^2+x^2(x-2)(a-b)^2\geq 0.$$

The second inequality can be proved similarly. With regard to the third inequality, we have

$$\left(\frac{ab}{cd}x+x-2\right)\left(\frac{cd}{ab}x+x-2\right)=$$

$$=2x^2-4x+4+\left(\frac{ab}{cd}+\frac{cd}{ab}\right)x(x-2)$$

$$\geq 2x^2-4x+4+2x(x-2)=4(x-1)^2.$$

The equality holds for $a=b=c=d=x$.

Remark. Setting $x=2$ and substituting a,b,c,d by $1/a,1/b,1/c,1/d$, respectively, we get the inequality from P 3.97.

□

P 3.99. *If a,b,c,d are positive real numbers, then*

$$\frac{(1+a^3)(1+b^3)(1+c^3)(1+d^3)}{(1+a^2)(1+b^2)(1+c^2)(1+d^2)}\geq\frac{1+abcd}{2}.$$

(Vasile Cîrtoaje, 1992)

Solution. For $a = b = c = d$, the inequality can be written as

$$\left(\frac{1+a^3}{1+a^2}\right)^4 \ge \frac{1+a^4}{2}.$$

We will show that

$$\left(\frac{1+a^3}{1+a^2}\right)^4 \ge \left(\frac{1+a^3}{1+a}\right)^2 \ge \frac{1+a^4}{2}.$$

The left side inequality is equivalent to

$$(1+a^3)(1+a) \ge (1+a^2)^2,$$

which reduces to $a(1-a)^2 \ge 0$, while the right side inequality is equivalent to

$$2(1-a+a^2)^2 \ge 1+a^4,$$

which reduces to $(1-a)^4 \ge 0$. Multiplying the inequalities

$$\left(\frac{1+a^3}{1+a^2}\right)^4 \ge \frac{1+a^4}{2}, \quad \left(\frac{1+b^3}{1+b^2}\right)^4 \ge \frac{1+b^4}{2},$$

$$\left(\frac{1+c^3}{1+c^2}\right)^4 \ge \frac{1+c^4}{2}, \quad \left(\frac{1+d^3}{1+d^2}\right)^4 \ge \frac{1+d^4}{2},$$

yields

$$\frac{(1+a^3)(1+b^3)(1+c^3)(1+d^3)}{(1+a^2)(1+b^2)(1+c^2)(1+d^2)} \ge \frac{1}{2}\sqrt[4]{(1+a^4)(1+b^4)(1+c^4)(1+d^4)}.$$

Applying the Cauchy-Schwarz inequality produces

$$(1+a^4)(1+b^4)(1+c^4)(1+d^4) \ge (1+a^2b^2)^2(1+c^2d^2)^2 \ge (1+abcd)^4,$$

from which the desired inequality follows. The equality holds for $a = b = c = d = 1$.

□

P 3.100. *Let a,b,c,d be positive real numbers such that $a+b+c+d=4$. Prove that*

$$\left(a+\frac{1}{a}-1\right)\left(b+\frac{1}{b}-1\right)\left(c+\frac{1}{c}-1\right)\left(d+\frac{1}{d}-1\right)+3 \ge \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}.$$

Solution. Write the inequality as

$$\prod\left[1+\left(a+\frac{1}{a}-2\right)\right]\geq\sum\frac{1}{a}-3.$$

Since

$$a+\frac{1}{a}-2\geq 0,\quad b+\frac{1}{b}-2\geq 0,$$
$$c+\frac{1}{c}-2\geq 0,\quad d+\frac{1}{d}-2\geq 0,$$

applying Bernoulli's inequality, it suffices to show that

$$1+\sum\left(a+\frac{1}{a}-2\right)\geq\sum\frac{1}{a}-3.$$

This is an identity, and then the proof is completed. The equality holds for $a=b=c=d=1$.

□

P 3.101. *If a,b,c,d are nonnegative real numbers, then*

$$4(a^3+b^3+c^3+d^3)+15(abc+bcd+cda+dab)\geq(a+b+c+d)^3.$$

Solution. Let

$$E(a,b,c,d)=4(a^3+b^3+c^3+d^3)+15(abc+bcd+cda+dab)-(a+b+c+d)^3.$$

Without loss of generality, assume that $a\leq b\leq c\leq d$. We will show that

$$E(a,b,c,d)\geq E(0,a+b,c,d)\geq 0.$$

We have

$$\begin{aligned}E(a,b,c,d)-E(0,a+b,c,d)&=4[a^3+b^3-(a+b)^3]+15ab(c+d)\\&=3ab[5(c+d)-4(a+b)]\geq 0.\end{aligned}$$

Now, putting $x=a+b$, we need to show that $E(0,x,c,d)\geq 0$, where

$$E(0,x,c,d)=4(x^3+c^3+d^3)+15xcd-(x+c+x)^3.$$

This is equivalent to Schur's inequality

$$x^3+c^3+d^3+3xcd\geq xc(x+c)+cd(c+d)+dx(d+x).$$

The equality holds for $a=0$ and $b=c=d$ (or any cyclic permutation), and also for $a=b=0$ and $c=d$ (or any permutation thereof).

□

P 3.102. *Let a, b, c, d be positive real numbers such that*

$$a+b+c+d=4.$$

Prove that

$$1+2(abc+bcd+cda+dab) \geq 9\min\{a,b,c,d\}.$$

(Vasile Cîrtoaje, 2008)

Solution. Assume that $a=\min\{a,b,c,d\}$ and use the substitutions $b=a+x$, $c=a+y$, $d=a+z$ and $t=x+y+z$, where $x,y,z,t \geq 0$. First, write the inequality in the homogeneous forms

$$(a+b+c+d)^3+128bcd+128a(bc+cd+db) \geq 36a(a+b+c+d)^2,$$

$$(4a+t)^3+128bcd+128a(bc+cd+db) \geq 36a(4a+t)^2.$$

Since

$$bcd=(a+x)(a+y)(a+z) \geq a^3+a^2t$$

and

$$\begin{aligned} bc+cd+db &= (a+x)(a+y)+(a+y)(a+z)+(a+z)(a+x) \\ &= 3a^2+2at+xy+yz+zx \geq 3a^2+2at, \end{aligned}$$

it suffices to prove that

$$(4a+t)^3+128(a^3+a^2t)+128a(3a^2+2at) \geq 36a(4a+t)^2.$$

This inequality is equivalent to

$$t(t-12a)^2 \geq 0.$$

The equality holds for $a=b=c=d=1$, and also for $(a,b,c,d)=\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{13}{4}\right)$ or any cyclic permutation.

□

P 3.103. *Let a, b, c, d be nonnegative real numbers such that*

$$a+b+c+d=4.$$

Prove that

$$5(a^2+b^2+c^2+d^2) \geq a^3+b^3+c^3+d^3+16.$$

(Vasile Cîrtoaje, 2005)

Solution. Assume that $a \ge b \ge c \ge d$.

First Solution. We use the mixing variable method. Setting $x = (b+c+d)/3$, we have $a + 3x = 4$ and $x \le 1$. We will show that

$$E(a,b,c,d) \ge E(a,x,x,x) \ge 0,$$

where

$$E(a,b,c,d) = 5(a^2+b^2+c^2+d^2) - a^3 - b^3 - c^3 - d^3 - 16.$$

The left side inequality is equivalent to

$$5(b^2+c^2+d^2-3x^2) - (b^3+c^3+d^3-3x^3) \ge 0.$$

Since $b^2+c^2+d^2-3x^2 \ge 0$ and $x \le 1$, it suffices to prove the homogeneous inequality

$$5x(b^2+c^2+d^2-3x^2) - (b^3+c^3+d^3-3x^3) \ge 0,$$

which is equivalent to

$$2(b^3+c^3+d^3) + 3[b(c^2+d^2) + c(d^2+b^2) + d(b^2+c^2)] \ge 24bcd.$$

This is true, since $b^3+c^3+d^3 \ge 3bcd$ and

$$b(c^2+d^2) + c(d^2+b^2) + d(b^2+c^2 \ge 2bcd + 2cdb + 2dbc = 6bcd.$$

The right side inequality is also true, since

$$\begin{aligned} E(a,x,x,x) &= 5(a^2+3x^2) - a^3 - 3x^3 - 16 \\ &= 5(4-3x)^2 + 15x^2 - (4-3x)^3 - 3x^3 - 16 \\ &= 24x(x-1)^2 \ge 0. \end{aligned}$$

This completes the proof. The equality holds for $a = b = c = d = 1$, and also for $(a,b,c,d) = (0,0,0,4)$ or any cyclic permutation.

Second Solution. Write the inequality as

$$\sum (5a^2 - a^3 - 7a + 3) \ge 0,$$

$$\sum (1-a)^2(3-a) \ge 0.$$

For $a \le 3$, the inequality is clearly true. Otherwise, for $3 < a \le 4$, which involves $b \le 1$, $c \le 1$, $d \le 1$, we get the required inequality by summing the inequalities

$$5a^2 \ge a^3 + 16$$

and

$$5(b^2+c^2+d^2) \ge b^3+c^3+d^3.$$

We have

$$5a^2 - a^3 - 16 = (4-a)(a^2 - a - 4) = (4-a)[a(a-3) + 2(a-2)] \ge 0,$$

and

$$5(b^2 + c^2 + d^2) \ge b^2 + c^2 + d^2 \ge b^3 + c^3 + d^3.$$

Third Solution. Write the inequality as

$$\sum(a^3 - 5a^2 + 4a) \le 0,$$

or

$$\sum(1-a)f(a) \le 0,$$

where

$$f(a) = a^2 - 4a.$$

Since $a + b \le 4$, we have

$$f(a) - f(b) = (a-b)(a+b-4) \le 0,$$

and, similarly,

$$f(b) - f(c) \le 0, \quad f(c) - f(d) \le 0.$$

Since

$$a - 1 \ge b - 1 \ge c - 1 \ge d - 1$$

and

$$f(a) \le f(b) \le f(c) \le f(d),$$

by Chebyshev's inequality, we get

$$4\sum(a-1)f(a) \le \left[\sum(a-1)\right]\left[\sum f(a)\right] = 0.$$

Remark. Similarly, we can prove the following generalization.

- *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that*

$$a_1 + a_2 + \cdots + a_n = n,$$

then

$$(n+1)(a_1^2 + a_2^2 + \cdots + a_n^2) \ge a_1^3 + a_2^3 + \cdots + a_n^3 + n^2.$$

□

P 3.104. *Let a,b,c,d be nonnegative real numbers such that*

$$a+b+c+d=4.$$

Prove that

$$3(a^2+b^2+c^2+d^2)+4abcd \ge 16.$$

(Vasile Cîrtoaje, 2004)

Solution. We use the mixing variable method. Assume that $a=\min\{a,b,c,d\}$, $a \le 1$. Setting $x=(b+c+d)/3$, we have $a+3x=4$ and $x \le 4/3$. We will show that

$$E(a,b,c,d) \ge E(a,x,x,x) \ge 0,$$

where

$$E(a,b,c,d)=3(a^2+b^2+c^2+d^2)+4abcd-16.$$

The left side inequality is equivalent to

$$3(3x^2-bc-cd-db) \ge 2a(x^3-bcd).$$

By Schur's inequality

$$(b+c+d)^3+9bcd \ge 4(b+c+d)(bc+cd+db),$$

we get

$$x^3-bcd \le \frac{4x}{3}(3x^2-bc-cd-db).$$

Therefore, it suffices to prove that

$$3(3x^2-bc-cd-db) \ge \frac{8ax}{3}(3x^2-bc-cd-db);$$

that is,

$$(3x^2-bc-cd-db)(9-8ax) \ge 0.$$

This is true, since

$$6(3x^2-bc-cd-db)=(b-c)^2+(c-d)^2+(d-b)^2 \ge 0,$$

and

$$3(9-8ax)=27-8a(4-a)=8(1-a)^2+16(1-a)+3>0.$$

The right side inequality is also true, since

$$\begin{aligned}
E(a,x,x,x) &= 3a^2+9x^2+4ax^3-16\\
&= 3(4-3x)^2+9x^2+4(4-3x)x^3-16\\
&= 4(8-18x+9x^2+4x^3-3x^4)\\
&= 4(1-x)^2(2+x)(4-3x) \ge 0.
\end{aligned}$$

This completes the proof. The equality holds for $a = b = c = d = 1$, and also for $(a,b,c,d) = \left(0, \frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$ or any cyclic permutation.

Remark. The following generalization holds (*Vasile Cîrtoaje*, 2005).

- *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be nonnegative real numbers such that*

$$a_1 + a_2 + \cdots + a_n = n.$$

If k is a positive integer satisfying $2 \le k \le n+2$, and

$$r = \left(\frac{n}{n-1}\right)^{k-1} - 1,$$

then

$$a_1^k + a_2^k + \cdots + a_n^k - n \ge nr(1 - a_1 a_2 \cdots a_n).$$

□

P 3.105. *Let a, b, c, d be nonnegative real numbers such that*

$$a + b + c + d = 4.$$

Prove that

$$27(abc + cd + cda + dab) \le 44abcd + 64.$$

Solution. Use the mixing variable method. Without loss of generality, assume that $a \ge b \ge c \ge d$. Setting $x = (a+b+c)/3$, we have $3x + d = 4$, $d \le x \le 4/3$ and $x^3 \ge abc$. We will show that

$$E(a,b,c,d) \ge E(x,x,x,d) \ge 0.$$

The left inequality is equivalent to

$$27[(x^3 - abc) + d(3x^2 - ab - bc - ca)] \ge 44d(x^3 - abc).$$

By Schur's inequality

$$(a+b+c)^3 + 9abc \ge 4(a+b+c)(ab+bc+ca),$$

we get

$$9x^3 + 3abc \ge 4x(ab+bc+ca),$$

and hence

$$3x^2 - ab - bc - ca \ge \frac{3(x^3 - abc)}{4x} \ge 0.$$

Therefore, it suffices to prove that

$$27(1+\frac{3d}{4x}) \geq 44d.$$

Write this inequality in the homogeneous form

$$27(3x+d)(4x+3d) \geq 704xd,$$

or, equivalently,

$$81(4x^2+d^2) \geq 353xd.$$

This inequality is true, since

$$81(4x^2+d^2)-353xd \geq 81(4x^2+d^2-5ut) = 81(x-d)(4x-d) \geq 0.$$

The right inequality $E(x,x,x,t) \geq 0$ is also true, since

$$\begin{aligned} E(x,x,x,d) &= (44x^3-81x^2)d-27x^3+64 \\ &= 4(16-81x^2+98x^3-33x^4) \\ &= 4(1-x)^2(16+32x-33x^2) \\ &= 4(1-x)^2(4-3x)(4+11x) \geq 0. \end{aligned}$$

This completes the proof. The equality holds for $a=b=c=d=1$, and also for $(a,b,c,d) = \left(0,\frac{4}{3},\frac{4}{3},\frac{4}{3}\right)$ or any cyclic permutation.

□

P 3.106. *Let a,b,c,d be positive real numbers such that*

$$a+b+c+d = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}.$$

Prove that

$$(1-abcd)\left(a^2+b^2+c^2+d^2-\frac{1}{a^2}-\frac{1}{b^2}-\frac{1}{c^2}-\frac{1}{d^2}\right) \geq 0.$$

(Vasile Cîrtoaje, 2007)

Solution. From

$$(a+b+c+d)^2 = \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)^2,$$

we get

$$\sum a^2 - \sum \frac{1}{a^2} = 2\sum_{sym}\frac{1}{ab} - 2\sum_{sym} ab$$

$$= 2\sum_{sym}\left(\frac{1}{ab} - cd\right) = 2(1-abcd)\sum_{sym}\frac{1}{ab}.$$

Thus, the inequality can be restated as

$$2(1-abcd)^2\sum_{sym}\frac{1}{ab} \ge 0,$$

which is obviously true. The equality holds for $ab = cd = 1$, or $ac = bd = 1$, or $ad = bc = 1$.

Conjecture. *If a, b, c, d are positive real numbers such that*

$$a + b + c + d = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d},$$

then

$$(1-abcd)\left(a^n + b^n + c^n + d^n - \frac{1}{a^n} - \frac{1}{b^n} - \frac{1}{c^n} - \frac{1}{d^n}\right) \ge 0$$

for any integer $n \ge 2$.

□

P 3.107. *Let a, b, c, d be positive real numbers such that*

$$a + b + c + d = 1.$$

Prove that

$$(1-a)(1-b)(1-c)(1-d)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) \ge \frac{81}{16}.$$

(Keira, 2007)

Solution. Write the inequality as

$$E(a,b,c,d) \ge \frac{81}{16},$$

where

$$E(a,b,c,d) = (1-a)(1-b)(1-c)(1-d)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right).$$

Without loss of generality, assume that $a \le b \le c \le d$. First, we show that for $a \le b \le c \le d$ and $a + b + c + d = 1$, $F(a,b,c,d)$ is minimal when $a = c$. This is true if

$$E(a,b,c,d) \ge E(\frac{a+c}{2}, b, \frac{a+c}{2}, d).$$

Since

$$(1-a)(1-c) = 1 - a - c + ac = b + d + ac$$

and

$$(1-b)(1-d)=1-b-d+bd=a+c+bd,$$

we have

$$E(a,b,c,d)=(b+d+ac)(a+c+bd)\left(\frac{a+c}{ac}+\frac{b+d}{bd}\right),$$

and the inequality is equivalent to

$$(b+d+ac)\left(\frac{a+c}{ac}+\frac{b+d}{bd}\right)\ge\left[b+d+\left(\frac{a+c}{2}\right)^2\right]\left(\frac{4}{a+c}+\frac{b+d}{bd}\right),$$

or

$$(a-c)^2\left(\frac{4bd}{ac}-a-c\right)\ge 0.$$

Since

$$\frac{4bd}{ac}-a-c\ge 4-a-c=3+b+d>0,$$

the last inequality is clearly true. Since $E(a,b,c,d)$ is minimal when $a=c$, from $a\le b\le c\le d$ it follows that $E(a,b,c,d)$ is minimal when $a=b=c$. Therefore, it suffices to prove that $3a+d=4$ involves

$$E(a,a,a,d)\ge\frac{81}{16}.$$

This is equivalent to

$$21d^4+61d^3-57d^2-153d+128\ge 0,$$

$$(d-1)^2(21d^2+103d+128)\ge 0.$$

The equality holds for $a=b=c=d=1/4$.

□

P 3.108. *Let a,b,c,d be nonnegative real numbers such that*

$$a+b+c+d=a^3+b^3+c^3+d^3=2.$$

Prove that

$$a^2+b^2+c^2+d^2\ge\frac{7}{4}.$$

(Vasile Cîrtoaje, 2010)

Solution. Let us denote $x = a^2 + b^2 + c^2 + d^2$. From

$$2 = a^3 + b^3 + c^3 + d^3 \geq a^3 + \frac{1}{9}(b + c + d)^3 = a^3 + \frac{1}{9}(2 - a)^3,$$

it follows that $(4a - 5)(a + 1)^2 \leq 0$, and hence $a \leq \frac{5}{4}$. Similarly, we have $b, c, d \leq \frac{5}{4}$. On the other hand,

$$5x = 5\sum a^2 = 4\sum a^3 + \sum(5a^2 - 4a^3) = 8 + \sum a^2(5 - 4a)$$

and, by the Cauchy-Schwarz inequality,

$$\sum a^2(5 - 4a) \geq \frac{[\sum a(5 - 4a)]^2}{\sum(5 - 4a)} = \frac{(5\sum a - 4\sum a^2)^2}{20 - 4\sum a} = \frac{(5 - 2x)^2}{3}.$$

Therefore, we have

$$5x \geq 8 + \frac{(5 - 2x)^2}{3}.$$

This is equivalent to $(4x - 7)(x - 7) \leq 0$, and involves $x \geq \frac{7}{4}$. The equality holds for $(a, b, c, d) = \left(\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{7}{4}\right)$ or any cyclic permutation.

□

P 3.109. *Let $a, b, c, d \in (0, 4]$ such that $abcd = 1$. Prove that*

$$(1 + 2a)(1 + 2b)(1 + 2c)(1 + 2d) \geq (5 - 2a)(5 - 2b)(5 - 2c)(5 - 2d).$$

(Vasile Cîrtoaje, 2011)

Solution. Assume that $a \geq b \geq c \geq d$. For the nontrivial case where the right side of the inequality is positive, there are two cases to consider.

Case 1: $a < 5/2$. In virtue of the AM-GM inequality, we have

$$(1 + 2a)(1 + 2b)(1 + 2c)(1 + 2d) \geq (3\sqrt[3]{a^2})(3\sqrt[3]{b^2})(3\sqrt[3]{c^2})(3\sqrt[3]{d^2}) = 81,$$

$$(5 - 2a)(5 - 2b)(5 - 2c)(5 - 2d) \leq \left[\frac{(5 - 2a) + (5 - 2b) + (5 - 2c) + (5 - 2d)}{4}\right]^4$$

$$= \left[\frac{10 - (a + b + c + d)}{2}\right]^4 \leq \left(\frac{10 - 4\sqrt[4]{abcd}}{2}\right)^4 = 81,$$

from which the conclusion follows.

Case 2: $a \ge b > 5/2 > c \ge d$. Write the inequality as

$$\frac{(1+2a)(1+2b)}{(2a-5)(2b-5)} \ge \frac{(5-2c)(5-2d)}{(1+2c)(1+2d)},$$

$$\frac{1+4ab+2(a+b)}{25+4ab-10(a+b)} \ge \frac{25+4cd-10(c+d)}{1+4cd+2(c+d)}.$$

According to the AM-GM inequality, it suffices to prove that

$$\frac{1+4ab+4\sqrt{ab}}{25+4ab-20\sqrt{ab}} \ge \frac{25+4cd-20\sqrt{cd}}{1+4cd+4\sqrt{cd}}.$$

This is equivalent to

$$\frac{2\sqrt{ab}+1}{2\sqrt{ab}-5} \ge \frac{5-2\sqrt{cd}}{1+2\sqrt{cd}},$$

$$\frac{2\sqrt{ab}+1}{2\sqrt{ab}-5} \ge \frac{5\sqrt{ab}-2}{\sqrt{ab}+2},$$

$$(4\sqrt{ab}-1)(4-\sqrt{ab}) \ge 0.$$

The last inequality is true, since $a, b \in (0, 4]$ involves $4-\sqrt{ab} \ge 0$.
The equality holds for $a = b = c = d = 1$, and for $(a, b, c, d) = (4, 4, 1/4, 1/4)$ or any permutation thereof.

□

P 3.110. *Let a, b, c, d and k be positive real numbers such that*

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) = k.$$

If $16 \le k \le (1+\sqrt{10})^2$, then any three of a, b, c, d are the lengths of the sides of a triangle (non-degenerate or degenerate).

Solution. The condition $k \ge 16$ follows from the known inequality

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \ge 16.$$

Without loss of generality, assume that $a \ge b \ge c \ge d$. Clearly, any three of a, b, c, d are the lengths of the sides of a triangle if and only if $a \le c + d$. By virtue of the

Cauchy-Schwarz inequality, we have

$$(1+\sqrt{10})^2 \geq (b+a+c+d)\left(\frac{1}{b}+\frac{1}{a}+\frac{1}{c}+\frac{1}{d}\right)$$
$$\geq \left[1+\sqrt{(a+c+d)\left(\frac{1}{a}+\frac{1}{c}+\frac{1}{d}\right)}\right]^2$$
$$\geq \left[1+\sqrt{(a+c+d)\left(\frac{1}{a}+\frac{4}{c+d}\right)}\right]^2,$$

and hence

$$(a+c+d)\left(\frac{1}{a}+\frac{4}{c+d}\right) \leq (\sqrt{11+2\sqrt{10}}-1)^2 = 10.$$

Writing this inequality as

$$(a-c-d)(4a-c-d) \leq 0,$$

we get $a \leq c+d$. Thus, the proof is completed.

Remark 1. The interval $[16, 11+2\sqrt{10}]$ is the largest possible range of k such that any three of a, b, c, d are the lengths of the sides of a triangle. In order to prove this, for the sake of contradiction, assume that $k > 11+2\sqrt{10}$. The given relation

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) = k$$

is satisfied for $a = p+\sqrt{p^2-1}$, $b = \sqrt{\dfrac{a(a+2)}{2a+1}}$ and $c = d = 1$, where

$$p = \frac{(\sqrt{k}-1)^2-5}{4}.$$

If $k > 11+2\sqrt{10}$, then we get $p > 5/4$ and $a > 2$. Clearly, the numbers a, c and d are not the lengths of the sides of a triangle.

Remark 2. In the same manner, we can prove the following generalization.

- *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that*

$$(a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right) = k.$$

If $n \geq 3$ and $n^2 \leq k \leq (n+\sqrt{10}-3)^2$, then any three of $a_1, a_2, \ldots, a_n$ are the lengths of the sides of a triangle.

□

P 3.111. *Let a, b, c, d and k be positive real numbers such that*

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)=k.$$

If $16 \le k \le \frac{119}{6}$, then there exist three numbers of a, b, c, d which are the lengths of the sides of a triangle (non-degenerate or degenerate).

(Vasile Cîrtoaje, 2010)

Solution. The condition $k \ge 16$ follows from the known inequality

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \ge 16.$$

Without loss of generality, assume that $a \ge b \ge c \ge d$. We need to show that either $a \le b+c$ or $b \le c+d$. For the sake of contradiction, consider that $a > b+c$ and $b > c+d$. To complete the proof, it suffices to show that $k > 119/6$; that is,

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) > \frac{119}{6}.$$

Notice that for $a=3$, $b=2$ and $c=d=1$, we have $a=b+c$, $b=c+d$ and

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) = \frac{119}{6}.$$

Then, we apply the Cauchy-Schwarz inequality in the following manner

$$[a+(b+c+d)]\left(\frac{9}{a}+\frac{16}{b+c+d}\right) \ge (3+4)^2 = 49.$$

Thus, it suffices to show that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} > \frac{17}{42}\left(\frac{9}{a}+\frac{16}{b+c+d}\right).$$

Since

$$(b+c+d)\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)-10 \ge (b+c+d)\left(\frac{1}{b}+\frac{4}{c+d}\right)-10$$
$$= \frac{(b-c-d)(4b-c-d)}{b(c+d)} > 0,$$

we need to prove that

$$\frac{1}{a}+\frac{10}{b+c+d} \ge \frac{17}{42}\left(\frac{9}{a}+\frac{16}{b+c+d}\right).$$

This is equivalent to

$$4a \geq 3(b+c+d),$$

which is true, since

$$4a-3(b+c+d) = 4(a-b-c)+(b-c-d)+2(c-d) > 0.$$

Remark. The interval $[16, 119/6]$ is the largest domain of k such that among a, b, c, d there exist three numbers as lengths of the sides of a non-degenerate or degenerate triangle. In the case of the non-degenerate triangles, the largest domain of k is $[16, 119/6)$. To prove this, for the sake of contradiction, assume that $k \geq 119/6$. The hypothesis

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) = k$$

is satisfied for $a = (k-11+\sqrt{k^2-22k+81}\,)/5$, $b = 2$ and $c = d = 1$. Since $a \geq 3$ for $k \geq 119/6$, there exist not three numbers of a, b, c, d as lengths of the sides of a triangle.

□

P 3.112. *Let a, b, c, d and k be positive real numbers such that*

$$(a+b+c+d)^2 = k(a^2+b^2+c^2+d^2).$$

If $\frac{11}{3} \leq k \leq 4$, then any three of a, b, c, d are the lengths of the sides of a triangle (non-degenerate or degenerate).

(Vasile Cîrtoaje, 2010)

Solution. The condition $k \leq 4$ follows from the known inequality

$$(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2).$$

Without loss of generality, assume that $a \geq b \geq c \geq d$. Then, any three of a, b, c, d are the lengths of the sides of a triangle if and only if $a \leq c+d$. Let us denote $x = (c+d)/2$, $x \leq b$. Since $c^2+d^2 \geq 2x^2$, from the hypothesis, we have

$$(a+b+2x)^2 \geq k(a^2+b^2+2x^2),$$

which can be written as

$$(k-1)b^2-2(a+2x)b+k(a^2+2x^2)-(a+2x)^2 \leq 0,$$

$$(k-1)\left(b-\frac{a+2x}{k-1}\right)^2+k(a^2+2x^2)-\frac{k}{k-1}(a+2x)^2 \leq 0.$$

This involves

$$a^2+2x^2-\frac{1}{k-1}(a+2x)^2 \le 0.$$

Since $k \ge \frac{11}{3}$, we get

$$(a^2+2x^2)-\frac{3}{8}(a+2x)^2 \le 0,$$

$$(a-2x)(5a-2x) \le 0,$$

$$a-b-c \le 0.$$

Thus, the proof is completed.

Remark 1. The interval $[\frac{11}{3},4]$ is the largest possible. The obvious inequality

$$(a+b+c+d)^2 > a^2+b^2+c^2+d^2$$

involves $k > 1$. For the sake of contradiction, assume that $1 < k < 11/3$. It is easy to check that the hypothesis

$$(a+b+c+d)^2 = k(a^2+b^2+c^2+d^2)$$

is satisfied for $a = \frac{7+\sqrt{k(66-17k)}}{k-1} > 4$, $b = 3$ and $c = d = 2$. Since $a > 4$ for $1 < k < 11/3$, a, c and d are not the lengths of the sides of a triangle.

Remark 2. In the same manner, we can prove the following generalization.

- *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that*

$$(a_1+a_2+\cdots+a_n)^2 = k(a_1^2+a_2^2+\cdots+a_n^2).$$

If $n \ge 3$ and $n-\frac{1}{3} \le k \le n$, then any three of $a_1, a_2, \ldots, a_n$ are the lengths of the sides of a triangle.

Notice that the interval $[n-1/3, n]$ is the largest possible of k. From the known inequalities

$$(a_1+a_2+\cdots+a_n)^2 > a_1^2+a_2^2+\cdots+a_n^2$$

and

$$(a_1+a_2+\cdots+a_n)^2 \le n(a_1^2+a_2^2+\cdots+a_n^2),$$

it follows that $1 < k \le n$. For the sake of contradiction, assume that $1 < k < n-1/3$. The hypothesis $(a_1+a_2+\cdots+a_n)^2 = k(a_1^2+a_2^2+\cdots+a_n^2)$ is satisfied for

$$a_1 = \frac{3n-5+\sqrt{k[3(n-2)(3n-1)-k(9n-19)]}}{k-1},$$

$a_2 = \cdots = a_{n-2} = 3$ and $a_{n-1} = a_n = 2$. Since $a_1 > 4$, the numbers a_1, a_{n-1} and a_n are not the lengths of the sides of a triangle.

□

P 3.113. *Let a,b,c,d and k be positive real numbers such that*

$$(a+b+c+d)^2 = k(a^2+b^2+c^2+d^2).$$

If $49/15 \le k \le 4$, then there exist three numbers of a,b,c,d which are the lengths of the sides of a triangle (non-degenerate or degenerate).

(Vasile Cîrtoaje, 2010)

Solution. The condition $k \le 4$ follows from the known inequality

$$(a+b+c+d)^2 \le 4(a^2+b^2+c^2+d^2).$$

Without loss of generality, assume that $a \ge b \ge c \ge d$. We need to show that either $a \le b+c$ or $b \le c+d$. For the sake of contradiction, consider that $a > b+c$ and $b > c+d$. To complete the proof, it suffices to show that $k < 49/15$; that is,

$$\frac{49}{15}(a^2+b^2+c^2+d^2) > (a+b+c+d)^2.$$

Notice that for $a=3$, $b=2$ and $c=d=1$, we have $a=b+c$, $b=c+d$ and

$$\frac{49}{15}(a^2+b^2+c^2+d^2) = (a+b+c+d)^2.$$

Then, we apply the Cauchy-Schwarz inequality in the following manner

$$(3+4)\left[\frac{a^2}{3}+\frac{(b+c+d)^2}{4}\right] \ge (a+b+c+d)^2.$$

Thus, it suffices to show that

$$\frac{7}{15}(a^2+b^2+c^2+d^2) > \frac{a^2}{3}+\frac{(b+c+d)^2}{4}.$$

This is equivalent to

$$2a^2+7(b^2+c^2+d^2) > \frac{15}{4}(b+c+d)^2.$$

Since

$$\begin{aligned}
8(b^2+c^2+d^2)-3(b+c+d)^2 &= 5b^2-6b(c+d)+5(c^2+d^2)-6cd \\
&\ge 5b^2-6b(c+d)+(c+d)^2 \\
&= (b-c-d)(5b-c-d) > 0,
\end{aligned}$$

it is enough to prove that

$$2a^2 + \frac{21}{8}(b+c+d)^2 \ge \frac{15}{4}(b+c+d)^2.$$

This reduces to

$$4a \ge 3(b+c+d),$$

which is true, since

$$4a - 3(b+c+d) = 4(a-b-c) + (b-c-d) + 2(c-d) > 0.$$

Remark. The interval $[49/15, 4]$ is the largest possible domain of k. In the case of the non-degenerate triangles, the interval $(49/15, 4]$ of k is the largest possible domain. To prove this, we see that the known inequalities

$$(a+b+c+d)^2 > a^2 + b^2 + c^2 + d^2$$

and

$$(a+b+c+d)^2 \le 4(a^2 + b^2 + c^2 + d^2)$$

involves $1 < k \le 4$. For the sake of contradiction, assume that $1 < k \le 49/15$. The hypothesis

$$(a+b+c+d)^2 = k(a^2 + b^2 + c^2 + d^2)$$

is satisfied for $a = \dfrac{4 + \sqrt{2k(11-3k)}}{k-1}$, $b = 2$ and $c = d = 1$. Since $a \ge 3$ for $1 < k \le 49/15$, there exist not three numbers of a, b, c, d as lengths of the sides of a non-degenerate triangle.

□

P 3.114. *Let a, b, c, d, e be nonnegative real numbers.*

(a) If $a + b + c = 3(d + e)$, then

$$4(a^4 + b^4 + c^4 + d^4 + e^4) \ge (a^2 + b^2 + c^2 + d^2 + e^2)^2;$$

(b) If $a + b + c = d + e$, then

$$12(a^4 + b^4 + c^4 + d^4 + e^4) \le 7(a^2 + b^2 + c^2 + d^2 + e^2)^2.$$

(Vasile Cîrtoaje, 2010)

Solution. (a) Let

$$E(a,b,c,d,e) = 4(a^4+b^4+c^4+d^4+e^4) - (a^2+b^2+c^2+d^2+e^2)^2.$$

We will show that

$$E(a,b,c,d,e) \ge E(a,b,c,d+e,0) \ge 0.$$

The left side inequality is equivalent to

$$de(a^2+b^2+c^2-3d^2-3e^2-5de) \ge 0.$$

This is true, since

$$\begin{aligned} a^2+b^2+c^2-3d^2-3e^2-5de &\ge \frac{1}{3}(a+b+c)^2-3d^2-3e^2-5de \\ &= 3(d+e)^2-3d^2-3e^2-5de = de \ge 0. \end{aligned}$$

Also, in virtue of the Cauchy-Schwarz inequality, we have

$$E(a,b,c,d+e,0) = 4[a^4+b^4+c^4+(d+e)^4] - [a^2+b^2+c^2+(d+e)^2]^2 \ge 0.$$

The equality holds for $a=b=c=d$ and $e=0$, or for $a=b=c=e$ and $d=0$.

(b) Let

$$E(a,b,c,d,e) = 7(a^2+b^2+c^2+d^2+e^2)^2 - 12(a^4+b^4+c^4+d^4+e^4).$$

We will show that

$$E(a,b,c,d,e) \ge E(a,b,c,d+e,0) \ge 0.$$

The left side inequality is equivalent to

$$de[12(d^2+e^2)+11de-7(a^2+b^2+c^2)] \ge 0.$$

This is true, since

$$\begin{aligned} 12(d^2+e^2)+11de-7(a^2+b^2+c^2) &\ge 12(d^2+e^2)+11de-7(a+b+c)^2 \\ &= 12(d^2+e^2)+11de-7(d+e)^2 \\ &= 5(d^2+e^2)-3de \ge 0. \end{aligned}$$

Also, we have

$$\begin{aligned} \frac{1}{4}E(a,b,c,d+e,0) &= \frac{1}{4}E(a,b,c,a+b+c,0) \\ &= \sum a^4 + 2\sum ab(a^2+b^2) + 3\sum a^2b^2 - 8abc\sum a \\ &\ge \sum a^2b^2 + 4\sum a^2b^2 + 3\sum a^2b^2 - 8abc\sum a \\ &= 8(\sum a^2b^2 - abc\sum a) = 4\sum a^2(b-c)^2 \ge 0. \end{aligned}$$

The equality holds for $a=b=c=d/3$ and $e=0$, or for $a=b=c=e/3$ and $d=0$.

□

P 3.115. *Let a,b,c,d,e be nonnegative real numbers such that*

$$a+b+c+d+e=5.$$

Prove that

$$a^4+b^4+c^4+d^4+e^4+150 \le 31(a^2+b^2+c^2+d^2+e^2).$$

(Vasile Cîrtoaje, 2007)

Solution. Write the inequality as

$$\sum(a^4-31a^2+30a) \le 0,$$

or

$$\sum(1-a)f(a) \le 0,$$

where

$$f(a)=a^3+a^2-30a.$$

Without loss of generality, assume that $a \ge b \ge c \ge d \ge e$. Since $a+b \le 5$, we have

$$\begin{aligned} f(a)-f(b) &= (a-b)(a^2+ab+b^2+a+b-30) \\ &\le (a-b)[(a+b)^2+a+b-30] \\ &= (a-b)(a+b-5)(a+b+6) \le 0. \end{aligned}$$

Similarly,

$$f(b)-f(c) \le 0, \quad f(c)-f(d) \le 0, \quad f(d)-f(e) \le 0.$$

Since

$$a-1 \ge b-1 \ge c-1 \ge d-1 \ge e-1$$

and

$$f(a) \le f(b) \le f(c) \le f(d) \le f(e),$$

by Chebyshev's inequality, we get

$$5\sum(a-1)f(a) \le \left[\sum(a-1)\right]\left[\sum f(a)\right]=0.$$

The equality holds for $a=b=c=d=e=1$, and for $(a,b,c,d,e)=(5,0,0,0,0)$ or any cyclic permutation.

Remark. Similarly, we can prove the following generalization.

- *If $a_1,a_2,\ldots,a_n$ are nonnegative real numbers such that*

$$a_1+a_2+\cdots+a_n=n,$$

then

$$a_1^4+a_2^4+\cdots+a_n^4+n^2(n+1) \le (n^2+n+1)(a_1^2+a_2^2+\cdots+a_n^2).$$

□

P 3.116. *Let a, b, c, d, e be positive real numbers such that*

$$a^2 + b^2 + c^2 + d^2 + e^2 = 5.$$

Prove that

$$abcde(a^4 + b^4 + c^4 + d^4 + e^4) \le 5.$$

(Vasile Cîrtoaje, 2006)

First Solution. Without loss of generality, assume that $a \le b \le c \le d \le e$. First, we prove that the expression

$$E(a, b, c, d, e) = abcde(a^4 + b^4 + c^4 + d^4 + e^4)$$

is maximal for $a = d$. We need to show that

$$E(a, b, c, d, e) \le E\left(\sqrt{\frac{a^2 + d^2}{2}}, b, c, \sqrt{\frac{a^2 + d^2}{2}}, e\right).$$

This inequality is true if

$$4ad(a^4 + b^4 + c^4 + d^4 + e^4) \le (a^2 + d^2)[(a^2 + d^2)^2 + 2b^4 + 2c^4 + 2e^4],$$

which is equivalent to

$$(a^2 + d^2)(a - d)^4 - 4a^2d^2(a - d)^2 + 2(b^4 + c^4 + e^4)(a - d)^2 \ge 0.$$

To prove this inequality, it suffices to show that

$$b^4 + c^4 + e^4 \ge 2a^2d^2.$$

Indeed, we have

$$b^4 + c^4 + e^4 - 2a^2d^2 \ge b^4 + a^4 + d^4 - 2a^2d^2 = b^4 + (a^2 - d^2)^2 > 0.$$

Since $E(a, b, c, d, e)$ is maximal for $a = d$, from $a \le b \le c \le d$ it follows that $E(a, b, c, d, e)$ is maximal for $a = b = c = d$. Then, it suffices to show that the desired homogeneous inequality

$$\left(\frac{a^2 + b^2 + c^2 + d^2 + e^2}{5}\right)^9 \ge (abcde)^2\left(\frac{a^4 + b^4 + c^4 + d^4 + e^4}{5}\right)^2$$

holds for $a = b = c = d = 1$. Setting $x = e^2$, we need to show that $f(x) \ge 0$ for $x > 0$, where

$$f(x) = 9\ln\frac{4 + x}{5} - \ln x - 2\ln\frac{4 + x^2}{5}.$$

From
$$f'(x) = \frac{9}{4+x} - \frac{1}{x} - \frac{4x}{4+x^2} = \frac{4(x-1)(x-2)^2}{x(4+x)(4+x^2)},$$
it follows that $f(x)$ is decreasing for $0 < x \le 1$ and increasing for $x \ge 1$. Therefore, $f(x) \ge f(1) = 0$. This completes the proof. The equality holds if and only if $a = b = c = d = e = 1$.

Second Solution. Replacing a, b, c, d, e by $\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}, \sqrt{e}$, we need to show the homogeneous inequality
$$\sqrt{abcde}(a^2 + b^2 + c^2 + d^2 + e^2) \le 5\left(\frac{a+b+c+d+e}{5}\right)^4,$$
where a, b, c, d, e are positive real numbers. According to Remark from P 3.57, it suffices to prove this inequality for $b = c = d = e = 1$; that is, to show that
$$\sqrt{a}(a^2+4) \le 5\left(\frac{a+4}{5}\right)^{9/2}.$$
Taking logarithms of both sides, we need to prove that $f(a) \ge 0$, where
$$f(a) = 9\ln(a+4) - 7\ln 5 - \ln a - 2\ln(a^2+4).$$
From
$$f'(a) = \frac{9}{a+4} - \frac{1}{a} - \frac{4a}{a^2+4} = \frac{4(a-1)(a-2)^2}{a(a+4)(a^2+4)},$$
it follows that $f(a)$ is decreasing for $0 < a \le 1$ and increasing for $a \ge 1$; therefore, $f(a) \ge f(1) = 0$.

Remark. The following more general statement holds (*Vasile Cirtoaje*, 2006):

- *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that*
$$a_1 + a_2 + \cdots + a_n = n,$$
then
$$(a_1a_2\cdots a_n)^{\frac{1}{\sqrt{n-1}}}(a_1^2 + a_2^2 + \cdots + a_n^2) \le n.$$

□

P 3.117. *Let a, b, c, d, e be positive real numbers such that*
$$a + b + c + d + e = 5.$$
Prove that
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} + \frac{20}{a^2+b^2+c^2+d^2+e^2} \ge 9.$$
(Vasile Cîrtoaje, 2006)

Solution. Without loss of generality, assume that $a \le b \le c \le d \le e$. First, we prove that the expression

$$E(a,b,c,d,e) = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{e} + \frac{20}{a^2+b^2+c^2+d^2+e^2}$$

is minimal when $a = d$. If this is true, then $E(a,b,c,d,e)$ is minimal when $a = b = c = d$, and it suffices to prove the desired inequality for $a = b = c = d$, when it is equivalent to the obvious inequality

$$(a-1)^2(6a-5)^2 \ge 0.$$

Therefore, it remains to show that

$$E(a,b,c,d,d) \ge E\left(\frac{a+d}{2}, b, c, \frac{a+d}{2}, e\right).$$

This inequality is equivalent to

$$\frac{(a-d)^2}{ad(a+d)} \ge \frac{20(a-d)^2}{(a^2+b^2+c^2+d^2+e^2)[(a+d)^2+2b^2+2c^2+2e^2]}.$$

Since $(a-d)^2 \ge 0$ and

$$a^2+b^2+c^2+d^2+e^2 \ge \frac{1}{5}(a+b+c+d+e)^2 = a+b+c+d+e,$$

it suffices to show that

$$(a+b+c+d+e)[(a+d)^2+2b^2+2c^2+2e^2] \ge 20ad(a+d).$$

Since

$$a+b+c+d+e \ge a+a+a+d+d = 3a+2d$$

and

$$(a+d)^2+2b^2+2c^2+2e^2 \ge (a+d)^2+2a^2+2a^2+2d^2 = 5a^2+2ad+3d^2,$$

it is enough to prove that

$$(3a+2d)(5a^2+2ad+3d^2) \ge 20ad(a+d).$$

This is true, since

$$\begin{aligned}(3a+2d)(5a^2+2ad+3d^2) - 20ad(a+d) &= 15a^3 - 4a^2d - 7ad^2 + 6d^3 \\ &> 5a^3 - 4a^2d - 7ad^2 + 6d^3 \\ &= (a-d)^2(5a+6d) \ge 0.\end{aligned}$$

Thus, the proof is completed. The equality holds for $a = b = c = d = e = 1$, and again for $(a, b, c, d, e) = \left(\frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \frac{5}{3}\right)$ or any cyclic permutation.

Remark. The following more general statement holds (*Vasile Cirtoaje*, 2006):

- *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 + a_2 + \cdots + a_n = n$, then*

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} + \frac{2n\sqrt{n-1}}{a_1^2 + a_2^2 + \cdots + a_n^2} \geq n + 2\sqrt{n-1}.$$

□

P 3.118. *If $a, b, c, d, e \geq 1$, then*

$$\left(a + \frac{1}{a}\right)\left(b + \frac{1}{b}\right)\left(c + \frac{1}{c}\right)\left(d + \frac{1}{d}\right)\left(e + \frac{1}{e}\right) + 68 \geq$$

$$\geq 4(a + b + c + d + e)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}\right).$$

(Vo Quoc Ba Can and Vasile Cîrtoaje, 2011)

Solution. Write the inequality as $E(a, b, c, d, e) \geq 0$, and denote

$$A = \left(a + \frac{1}{a}\right)\left(b + \frac{1}{b}\right)\left(c + \frac{1}{c}\right)\left(d + \frac{1}{d}\right).$$

We claim that

$$E(a, b, c, d, e) \geq E(a, b, c, d, 1).$$

If this is true, then (by symmetry)

$$E(a, b, c, d, e) \geq E(a, b, c, d, 1) \geq \ldots \geq E(a, 1, 1, 1, 1) = 0,$$

and the proof is completed. Since

$$E(a, b, c, d, e) - E(a, b, c, d, 1) = (e - 1)\left(B - \frac{C}{e}\right),$$

we need to show that

$$B - \frac{C}{e} \geq 0,$$

where

$$B = A - 4\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right),$$

$$C = A - 4(a + b + c + d).$$

From $A \geq 16$ and

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} \leq 4,$$

it follows that $B \geq 0$. If $C \leq 0$, then $B - C/e \geq 0$. If $C \geq 0$, then

$$B-\frac{C}{e} \geq B-C = 4\left(a-\frac{1}{a}\right)+4\left(b-\frac{1}{b}\right)+4\left(c-\frac{1}{c}\right)+4\left(d-\frac{1}{d}\right) \geq 0.$$

The equality holds for $a = b = c = d = 1$ or any cyclic permutation.

□

P 3.119. *Let a, b, c and x, y, z be positive real numbers such that*

$$(a+b+c)(x+y+z) = (a^2+b^2+c^2)(x^2+y^2+z^2) = 4.$$

Prove that

$$abcxyz < \frac{1}{36}.$$

(Vasile Cîrtoaje and Mircea Lascu, 1997)

Solution. Using the given relations and the AM-GM inequality, we have

$$4(ab+bc+ca)(xy+yz+zx) =$$

$$= [(a+b+c)^2-(a^2+b^2+c^2)][(x+y+z)^2-(x^2+y^2+z^2)]$$

$$= 20-(a+b+c)^2(x^2+y^2+z^2)-(x+y+z)^2(a^2+b^2+c^2)$$

$$\leq 20-2(a+b+c)(x+y+z)\sqrt{(a^2+b^2+c^2)(x^2+y^2+z^2)} = 4,$$

and hence

$$(ab+bc+ca)(xy+yz+zx) \leq 1.$$

On the other hand, multiplying the well-known inequalities

$$(ab+bc+ca)^2 \geq 3abc(a+b+c)$$

and

$$(xy+yz+zx)^2 \geq 3xyz(x+y+z),$$

we get

$$(ab+bc+ca)^2(xy+yz+zx)^2 \geq 36abcxyz,$$

and hence

$$1 \geq (ab+bc+ca)^2(xy+yz+zx)^2 \geq 36abcxyz.$$

In order to have $36abcxyz = 1$, it is necessary that $(ab + bc + ca)^2 = 3abc(a + b + c)$ and $(xy+yz+zx)^2 = 3xyz(x+y+z)$. These relations imply $a = b = c$ and $x = y = z$, which contradict the hypothesis

$$(a + b + c)(x + y + z) = (a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = 4.$$

Consequently, we have $abcxyz < \dfrac{1}{36}$.

□

P 3.120. *Let $a_1, a_2, \cdots, a_n$ $(n \ge 3)$ be positive real numbers such that*

$$a_1 + a_2 + \cdots + a_n = a_1^2 + a_2^2 + \cdots + a_n^2 = n - 1.$$

Prove that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \ge \frac{n^2(2n-3)}{2(n-1)(n-2)}.$$

(Vasile Cîrtoaje, 2010)

Solution. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} n-1 = a_1^2 + a_2^2 + \cdots + a_n^2 &\ge \frac{(a_1 + a_2 + \cdots + a_{n-1})^2}{n-1} + a_n^2 \\ &= \frac{(n-1-a_n)^2}{n-1} + a_n^2, \end{aligned}$$

which provides $a_n \le 2(n-1)/n$. Similarly, $a_i \le 2(n-1)/n$ for all i. The hint for proving the given inequality is to apply the Cauchy-Schwarz inequality after we made the numerators nonnegative and as small as possible. So, since $2n - 2 - na_i \ge 0$, we have

$$\begin{aligned} \sum \frac{1}{a_1} &= \sum \left(\frac{1}{a_1} - \frac{n}{2n-2} \right) + \frac{n^2}{2n-2} \\ &= \frac{1}{2(n-1)} \sum \frac{2n-2-na_1}{a_1} + \frac{n^2}{2n-2} \\ &\ge \frac{1}{2(n-1)} \cdot \frac{[\sum(2n-2-na_1)]^2}{\sum a_1(2n-2-na_1)} + \frac{n^2}{2n-2} \\ &= \frac{1}{2(n-1)} \cdot \frac{[n(2n-2) - n\sum a_1]^2}{(2n-2)\sum a_1 - n\sum a_1^2} + \frac{n^2}{2n-2} \\ &= \frac{1}{2(n-1)} \cdot \frac{n^2(n-1)^2}{(n-1)(n-2)} + \frac{n^2}{2n-2} = \frac{n^2(2n-3)}{2(n-1)(n-2)}, \end{aligned}$$

from where the conclusion follows. The equality holds for $a_1 = a_2 = \cdots = a_{n-1} = 1 - 2/n$ and $a_n = 2 - 2/n$ (or any cyclic permutation).

□

P 3.121. *Let $a_1, a_2, \cdots, a_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = n$. Prove that*

$$n^2\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}-n\right) \ge 4(n-1)(a_1^2+a_2^2+\cdots+a_n^2-n).$$

(Vasile Cîrtoaje, 2004)

Solution. Since

$$n = \frac{1}{n}(a_1+a_2+\cdots+a_n)^2 \le a_1^2+a_2^2+\cdots+a_n^2 < (a_1+a_2+\cdots+a_n)^2 = n^2,$$

we can use the substitution

$$a_1^2+a_2^2+\cdots+a_n^2 = n+n(n-1)t^2,$$

where $0 \le t < 1$. On the other hand, from

$$a_1^2+a_2^2+\cdots+a_n^2 \ge a_1^2 + \frac{(a_2+\cdots+a_n)^2}{n-1} = a_1^2 + \frac{(n-a_1)^2}{n-1},$$

we get

$$n+n(n-1)t^2 \ge a_1^2 + \frac{(n-a_1)^2}{n-1},$$

which involves $a_1 \le 1+(n-1)t$; similarly, we get

$$a_i \le 1+(n-1)t$$

for any i. The hint for proving the given inequality is to apply the Cauchy-Schwarz inequality after we made the numerators nonnegative and as small as possible. Since

$$\sum \frac{1}{a_1} = \sum \frac{1}{1+(n-1)t} + \sum\left[\frac{1}{a_1} - \frac{1}{1+(n-1)t}\right]$$
$$= \frac{n}{1+(n-1)t} + \frac{1}{1+(n-1)t}\sum \frac{1+(n-1)t-a_1}{a_1},$$

we can write the desired inequality as

$$\sum \frac{1+(n-1)t-a_1}{a_1} \ge n(n-1)t + \frac{4(n-1)^2t^2(1+(n-1)t)}{n}.$$

By virtue of the Cauchy-Schwarz inequality, we have

$$\sum \frac{1+(n-1)t-a_1}{a_1} \ge \frac{[\sum(1+(n-1)t-a_1)]^2}{\sum a_1(1+(n-1)t-a_1)}$$
$$= \frac{[n+n(n-1)t-\sum a_1]^2}{(1+(n-1)t)\sum a_1 - \sum a_1^2} = \frac{n(n-1)t}{1-t},$$

Therefore, it suffices to prove that

$$\frac{n(n-1)t}{1-t} \geq n(n-1)t + \frac{4(n-1)^2t^2[(1+(n-1)t])}{n}.$$

This inequality is true if

$$4(n-1)(1-t)[1+(n-1)t]) \leq n^2.$$

Indeed,

$$4(n-1)(1-t)[1+(n-1)t] \leq [(n-1)(1-t)+1+(n-1)t]^2 = n^2.$$

The equality holds when $a_1 = a_2 = \cdots = a_n = 1$, as well as when one of $a_1, a_2, \ldots, a_n$ is $n/2$ and the others are $n/(2n-2)$.

□

P 3.122. *Let $a_1, a_2, \cdots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$. Prove that*

$$(n+1)(a_1^2 + a_2^2 + \cdots + a_n^2) \geq n^2 + a_1^3 + a_2^3 + \cdots + a_n^3.$$

(Vasile Cîrtoaje, 2002)

First Solution. If $a_1 = a_2 = \cdots = a_n$, then the equality holds. Otherwise, as in the preceding proof, we will use the substitution

$$a_1^2 + a_2^2 + \cdots + a_n^2 = n + n(n-1)t^2,$$

where $0 < t \leq 1$; in addition, for any i, we have

$$a_i \leq 1 + (n-1)t.$$

By the Cauchy-Schwarz inequality, we have

$$\sum [1+(n-1)t-a_1]a_1^2 \geq \frac{[\sum(1+(n-1)t-a_1)a_1]^2}{\sum[1+(n-1)t-a_1]} = n(n-1)t(1-t)^2,$$

hence

$$\sum a_1^3 \leq [1+(n-1)t]\sum a_1^2 - n(n-1)t(1-t)^2$$
$$= n[(n-1)(n-2)t^3 + 3(n-1)t^2 + 1].$$

Therefore, it suffices to show that

$$(n+1)[n+n(n-1)t^2] \geq n^2 + n[(n-1)(n-2)t^3 + 3(n-1)t^2 + 1],$$

which is equivalent to the obvious inequality

$$n(n-1)(n-2)t^2(1-t) \geq 0.$$

For $n = 2$, the original inequality is an identity. For $n \geq 3$, the equality holds when $a_1 = a_2 = \cdots = a_n = 1$, as well as when $n-1$ of $a_1, a_2, \ldots, a_n$ are zero.

Second Solution. Assume that $a_1 \geq a_2 \geq \cdots \geq a_n$. Replacing n^2 by $n(a_1 + a_2 + \cdots + a_n)$, the desired inequality becomes as follows

$$\sum[(n+1)a_1^2 - na_1 - a_1^3] \geq 0,$$

$$\sum(a_1 - 1)(na_1 - a_1^2) \geq 0.$$

Since

$$a_1 - 1 \geq a_2 - 1 \geq \cdots \geq a_n - 1$$

and

$$na_1 - a_1^2 \geq na_2 - a_2^2 \geq \cdots \geq na_n - a_n^2,$$

we apply Chebyshev's inequality to get

$$n\sum(a_1 - 1)(na_1 - a_1^2) \geq \left[\sum(a_1 - 1)\right]\left[\sum(na_1 - a_1^2)\right] = 0.$$

□

P 3.123. *Let $a_1, a_2, \cdots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$. Prove that*

$$(n-1)(a_1^3 + a_2^3 + \cdots + a_n^3) + n^2 \geq (2n-1)(a_1^2 + a_2^2 + \cdots + a_n^2).$$

(Vasile Cîrtoaje, 2002)

Solution. If $a_1 = a_2 = \cdots = a_n$, then the equality holds. Otherwise, as in the proof of problem P 3.121, we will use the substitution

$$a_1^2 + a_2^2 + \cdots + a_n^2 = n + n(n-1)t^2,$$

where $0 < t \leq 1$; in addition, for any i, we have

$$a_i \geq 1 - (n-1)t.$$

By the Cauchy-Schwarz inequality, we have

$$\sum[a_1 - 1 + (n-1)t]a_1^2 \geq \frac{[\sum(a_1 - 1 + (n-1)t)a_1]^2}{\sum[a_1 - 1 + (n-1)t]} = n(n-1)t(t+1)^2,$$

which yields

$$\sum a_1^3 \ge n(n-1)t(t+1)^2 + [1-(n-1)t]\sum a_1^2$$
$$= n[1+3(n-1)t^2-(n-1)(n-2)t^3].$$

Therefore, it suffices to show that

$$(n-1)n[1+3(n-1)t^2-(n-1)(n-2)t^3]+n^2 \ge (2n-1)[n+n(n-1)t^2],$$

which is equivalent to the obvious inequality

$$n(n-1)(n-2)t^2[1-(n-1)t] \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = 0$ and $a_2 = a_3 = \cdots = a_n = \frac{n}{n-1}$ (or any cyclic permutation).

□

P 3.124. *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be positive real numbers such that*

$$a_1 + a_2 + \cdots + a_n = 1.$$

Prove that

$$\left(a_1+\frac{1}{a_1}-2\right)\left(a_2+\frac{1}{a_2}-2\right)\left(a_n+\frac{1}{a_n}-2\right) \ge \left(n+\frac{1}{n}-2\right)^n.$$

Solution. Applying Popoviciu's inequality for the convex function $f(x) = -\ln x$, $x > 0$, gives

$$(b_1b_2\cdots b_n)^{n-1} \ge (a_1a_2\cdots a_n)\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)^{n(n-2)},$$

where

$$b_i = \frac{1}{n-1}\sum_{j\ne i} a_j, \quad i = 1,2,\cdots,n.$$

Under the hypothesis $a_1 + a_2 + \cdots + a_n = 1$, this inequality becomes

$$(1-a_1)^{n-1}(1-a_2)^{n-1}\cdots(1-a_n)^{n-1} \ge n^n\left(1-\frac{1}{n}\right)^{n^2-n} a_1a_2\cdots a_n. \tag{*}$$

On the other hand, by the AM-GM inequality, we get in succession:

$$(1-a_1)+(1-a_2)+\cdots+(1-a_n) \ge n\sqrt[n]{(1-a_1)(1-a_2)\cdots(1-a_n)},$$

$$\left(1-\frac{1}{n}\right)^n \ge (1-a_1)(1-a_2)\cdots(1-a_n).$$

$$\left(1-\frac{1}{n}\right)^{n(n-3)} \ge (1-a_1)^{n-3}(1-a_2)^{n-3}\cdots(1-a_n)^{n-3},$$

$$\left(1-\frac{1}{n}\right)^{n(n-3)}(1-a_1)^2(1-a_2)^2\cdots(1-a_n)^2 \ge (1-a_1)^{n-1}(1-a_2)^{n-1}\cdots(1-a_n)^{n-1}.$$

Multiplying this inequality and (*) yields the desired inequality. The equality holds for $a_1 = a_2 = \cdots = a_n = 1/n$.

□

P 3.125. *Let $a_1, a_2, \ldots, a_n$ ($n \ge 3$) be positive real numbers such that*

$$a_1 + a_2 + \cdots + a_n = n.$$

Prove that

$$a_1^2 + a_2^2 + \cdots + a_n^2 - n \ge \frac{n}{n-1}(1 - a_1a_2\cdots a_n).$$

Solution. For fixed $a_1^2 + a_2^2 + \cdots + a_n^2$, according to Remark from P 3.57, the product $a_1a_2\cdots a_n$ is minimal when one of $a_1, a_2, \ldots, a_n$ is zero or $n-1$ numbers of $a_1, a_2, \ldots, a_n$ are equal. Therefore, it suffices to consider these cases.

Case 1: $a_1 = 0$. We need to show that $a_2 + a_3 + \cdots + a_n = n$ involves

$$a_2^2 + a_3^2 + \cdots + a_n^2 \ge \frac{n^2}{n-1}.$$

Indeed, by the Cauchy-Schwarz inequality, we have

$$a_2^2 + a_3^2 + \cdots + a_n^2 \ge \frac{1}{n-1}(a_2 + a_3 + \cdots + a_n)^2 = \frac{n^2}{n-1}.$$

Case 2: $a_2 = a_3 = \cdots = a_n$. Setting $a_1 = x$ and $a_2 = y$, we need to show that

$$x + (n-1)y = n$$

involves

$$x^2 + (n-1)y^2 - n + \frac{n}{n-1}\left(xy^{n-1} - 1\right) \ge 0.$$

By Bernoulli's inequality, we have

$$y^{n-1} = \left(1 + \frac{1-x}{n-1}\right)^{n-1} \ge 1 + (1-x) = 2 - x.$$

Therefore, it suffices to prove that

$$x^2+(n-1)y^2-n+\frac{n}{n-1}[x(2-x)-1]\ge 0,$$

which is an identity.
The equality holds for $a_1=a_2=\cdots=a_n=1/n$, and also for $a_1=0$ and $a_2=\cdots=a_n=\dfrac{n}{n-1}$ (or any cyclic permutation).

□

P 3.126. *Let $a_1, a_2, \dots, a_n$ ($n\ge 3$) be positive real numbers such that*

$$a_1\le a_2\le\cdots\le a_n,$$

$$(a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)=k.$$

(a) If $n^2\le k\le n^2+\dfrac{i(n-i)}{2}$, $i\in\{2,3,\cdots,n-1\}$, then a_{i-1}, a_i and a_{i+1} are the lengths of the sides of a non-degenerate or degenerate triangle;

(b) If $n^2\le k\le\alpha_n$, where $\alpha_n=\dfrac{9n^2}{8}$ for even n, and $\alpha_n=\dfrac{9n^2-1}{8}$ for odd n, then there exist three numbers a_i which are the lengths of the sides of a non-degenerate or degenerate triangle.

(Vasile Cîrtoaje, 2010)

Solution. The condition $k\ge n^2$ follows from the AM-HM inequality

$$(a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)\ge n^2.$$

(a) For the sake of contradiction, assume that a_{i-1}, a_i and a_{i+1} are not the lengths of the sides of a triangle; that is, $a_{i+1}>a_{i-1}+a_i$. Let x and y be positive numbers such that

$$(i-1)x=a_1+\cdots+a_{i-1},$$

$$(n-i)y=a_{i+1}+\cdots+a_n,$$

$$x\le a_{i-1}\le a_i<a_{i+1}\le y,\quad x<y.$$

Let us denote

$$A(x,y)=(i-1)x+a_i+(n-i)y,$$

$$B(x,y)=\frac{i-1}{x}+\frac{1}{a_i}+\frac{n-i}{y},$$

$$f(x,y)=A(x,y)B(x,y).$$

By the Cauchy-Schwarz inequality, we have

$$B(x,y)\le \frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}.$$

Since $A(x,y)=a_1+a_2+\cdots+a_n$, we obtain

$$f(x,y)\le (a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right),$$

and hence

$$f(x,y)\le k.$$

On the other hand, we claim that

$$f(x,y)>f(a_{i-1},a_{i-1}+a_i).$$

This inequality is equivalent to

$$[a_{i-1}y-(a_{i-1}+a_i)x][(a_{i-1}+a_i)y-a_{i-1}x]\ge 0,$$

and is true since $y\ge a_{i+1}>a_{i-1}+a_i$ and $x\le a_{i-1}$ imply

$$a_{i-1}y-(a_{i-1}+a_i)x>a_{i-1}(a_{i-1}+a_i)-(a_{i-1}+a_i)a_{i-1}=0.$$

Then, we have

$$\begin{aligned}k\ge f(x,y)&>f(a_{i-1},a_{i-1}+a_i)\\&=[(n-1)a_{i-1}+(n-i+1)a_i]\left(\frac{i-1}{a_{i-1}}+\frac{1}{a_i}+\frac{n-i}{a_{i-1}+a_i}\right).\end{aligned}$$

Setting $a_{i-1}=1$ and $a_i=t$, $t\ge 1$, we get

$$k>\frac{[n-1+(n-i+1)t][1+nt+(i-1)t^2]}{t(1+t)}\ge n^2+\frac{i(n-i)}{2},$$

which contradicts the hypothesis $k\le n^2+i(n-i)/2$. The last inequality is equivalent to $(t-1)(Ct^2+Dt+E)\ge 0$, where

$$C=2(i-1)(n-i+1),\quad D=(i-2)(n-i),\quad E=-2(n-1).$$

This is true, since

$$Ct^2+Dt+E\ge C+D+E=3(i-2)(n-i)\ge 0.$$

(b) We apply the result of (a). If n is even, then $n^2+\dfrac{i(n-i)}{2}$ attains its maximum value $\dfrac{9n^2}{8}$ for $i=\dfrac{n}{2}$. If n is odd, then $n^2+\dfrac{i(n-i)}{2}$ attains its maximum value $\dfrac{9n^2-1}{8}$ for $i=\dfrac{n\pm 1}{2}$.

□

P 3.127. *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be positive real numbers such that*

$$a_1 \le a_2 \le \cdots \le a_n,$$

$$(a_1 + a_2 + \cdots + a_n)^2 = k(a_1^2 + a_2^2 + \cdots + a_n^2).$$

(a) If $\frac{(2n-i)^2}{4n-3i} \le k \le n$, $i \in \{2, 3, \cdots, n-1\}$, then a_{i-1}, a_i and a_{i+1} are the lengths of the sides of a non-degenerate or degenerate triangle;

(b) If $\frac{8n+1}{9} \le k \le n$, then there exist three numbers a_i which are the lengths of the sides of a non-degenerate or degenerate triangle.

(Vasile Cîrtoaje, 2010)

Solution. The condition $k \le n$ follows from the Cauchy-Schwarz inequality

$$(a_1 + a_2 + \cdots + a_n)^2 \le n(a_1^2 + a_2^2 + \cdots + a_n^2).$$

(a) For the sake of contradiction, assume that a_{i-1}, a_i and a_{i+1} are not the lengths of the sides of a triangle; that is, $a_{i+1} > a_{i-1} + a_i$. Let x and y be positive numbers such that

$$(i-1)x = a_1 + \cdots + a_{i-1},$$

$$(n-i)y = a_{i+1} + \cdots + a_n,$$

$$x \le a_{i-1} \le a_i < a_{i+1} \le y, \quad x < y.$$

Let us denote

$$A(x, y) = (i-1)x + a_i + (n-i)y,$$

$$B(x, y) = (i-1)x^2 + a_i^2 + (n-i)y^2,$$

$$f(x, y) = \frac{A^2(x, y)}{B(x, y)}.$$

By the Cauchy-Schwarz inequality, we have

$$(i-1)x^2 \le a_1^2 + \cdots + a_{i-1}^2,$$

$$(n-i)y^2 \le a_{i+1}^2 + \cdots + a_n^2,$$

and hence

$$B(x, y) \le a_1^2 + a_2^2 + \cdots + a_n^2.$$

Since

$$A(x, y) = a_1 + a_2 + \cdots + a_n,$$

we obtain

$$f(x,y) \geq \frac{(a_1+a_2+\cdots+a_n)^2}{a_1^2+a_2^2+\cdots+a_n^2},$$

and hence

$$f(x,y) \geq k.$$

On the other hand, from

$$\frac{\partial f(x,y)}{\partial x} = \frac{2(i-1)AC}{B^2} > 0$$

and

$$\frac{\partial f(x,y)}{\partial y} = \frac{2(n-i)AD}{B^2} < 0,$$

where

$$C = a_i(a_i-x)+(n-i)y(y-x) > 0,$$

$$D = a_i(a_i-y)+(i-1)x(x-y) < 0,$$

it follows that $f(x,y)$ is strictly increasing with respect to $x > 0$ and strictly decreasing with respect to $y > 0$. Then, since $x \leq a_{i-1}$ and $y \geq a_{i+1} > a_{i-1}+a_i$, we have

$$f(x,y) < f(a_{i-1}, a_{i-1}+a_i).$$

This involves

$$k < f(a_{i-1}, a_{i-1}+a_i),$$

and hence

$$\begin{aligned} k &< \frac{[(i-1)a_{i-1}+a_i+(n-i)(a_{i-1}+a_i)]^2}{(i-1)a_{i-1}^2+a_i^2+(n-i)(a_{i-1}+a_i)^2} \\ &= \frac{[(n-1)a_{i-1}+(n-i+1)a_i]^2}{(n-1)a_{i-1}^2+2(n-i)a_{i-1}a_i+(n-i+1)a_i^2} \\ &\leq \frac{(2n-i)^2}{4n-3i}, \end{aligned}$$

which is false. Setting $a_{i-1}=1$ and $a_i=t$, $t \geq 1$, the last inequality

$$\frac{[n-1+(n-i+1)t]^2}{n-1+2(n-i)t+(n-i+1)t^2} \leq \frac{(2n-i)^2}{4n-3i}$$

is equivalent to

$$(t-1)(Et-F) \geq 0,$$

where

$$E = (n-i+1)[(3i-4)n-2i^2+3i],$$

$$F = (n-1)[(4-i)n+i^2-3i].$$

Since

$$E=(n-i+1)[(3i-4)(n-i+1)+i^2+2i-4]\geq 0,$$

we get

$$Et-F\geq E-F=2(i-2)(n-i)(2n-i)\geq 0.$$

(b) According to (a), it suffices to show that there exists $i\in\{2,3,\cdots,n-1\}$ such that

$$\frac{(2n-i)^2}{4n-3i}\leq\frac{8n+1}{9};$$

that is,

$$4n-3i\geq(2n-3i)^2.$$

Since one of the numbers $\frac{2n-1}{3}$, $\frac{2n}{3}$ and $\frac{2n+1}{3}$ is integer, it suffices to prove this inequality for all $i\in\{\frac{2n-1}{3},\frac{2n}{3},\frac{2n+1}{3}\}$. Indeed, for these cases, the inequality reduces to $2n\geq 0$, $2n\geq 0$ and $2n-2\geq 0$, respectively.

□

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Chapter 4

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