Vasile Cîrtoaje

MATHEMATICAL INEQUALITIES

Volume 4

EXTENSIONS AND REFINEMENTS OF JENSEN’S INEQUALITY

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MATHEMATICAL INEQUALITIES

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Chapter 1

Half Convex Function Method

1.1 Theoretical Basis

Let $\mathbb{I}$ be a real interval, $s$ an interior point of $\mathbb{I}$ and

$$\mathbb{I}_{\geq s} = \{ u | u \in \mathbb{I}, u \geq s \}, \quad \mathbb{I}_{\leq s} = \{ u | u \in \mathbb{I}, u \leq s \}.$$

The following statement is known as the Right Half Convex Function Theorem (RHCF-Theorem).

**Right Half Convex Function Theorem** (Vasile Cîrtoaje, 2004). Let $f$ be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\geq s}$, where $s \in \text{int}(\mathbb{I})$. The inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying $a_1 + a_2 + \cdots + a_n = ns$

if and only if

$$f(x) + (n-1)f(y) \geq nf(s)$$

for all $x, y \in \mathbb{I}$ so that $x \leq s \leq y$ and $x + (n-1)y = ns$.

**Proof.** For $a_1 = x, \quad a_2 = a_3 = \cdots = a_n = y$,

the inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s)$$

becomes

$$f(x) + (n-1)f(y) \geq nf(s);$$

therefore, the necessity is obvious. To prove the sufficiency, we assume that

$$a_1 \leq a_2 \leq \cdots \leq a_n.$$
If \( a_1 \geq s \), then the required inequality is just Jensen’s inequality for convex functions. Otherwise, if \( a_1 < s \), then there exists
\[
k \in \{1, 2, \ldots, n - 1\}
\]
so that
\[
a_1 \leq \cdots \leq a_k < s \leq a_{k+1} \leq \cdots \leq a_n.
\]
Since \( f \) is convex on \( \mathbb{I}_{\geq s} \), we may apply Jensen’s inequality to get
\[
f(a_{k+1}) + \cdots + f(a_n) \geq (n-k)f(z),
\]
where
\[
z = \frac{a_{k+1} + \cdots + a_n}{n-k}, \quad z \in \mathbb{I}.
\]
Thus, it suffices to show that
\[
f(a_1) + \cdots + f(a_k) + (n-k)f(z) \geq nf(s). \quad (*)
\]
Let \( b_1, \ldots, b_k \) be defined by
\[
a_i + (n-1)b_i = ns, \quad i = 1, \ldots, k.
\]
We claim that
\[
z \geq b_1 \geq \cdots \geq b_k > s,
\]
which involves
\[
b_1, \ldots, b_k \in \mathbb{I}_{\geq s}.
\]
Indeed, we have
\[
b_1 \geq \cdots \geq b_k,
\]
\[
b_k - s = \frac{s-a_k}{n-1} > 0,
\]
and
\[
z \geq b_1
\]
because
\[
(n-1)b_1 = ns - a_1 = (a_2 + \cdots + a_k) + a_{k+1} + \cdots + a_n
\]
\[
\leq (k-1)s + a_{k+1} + \cdots + a_n
\]
\[
= (k-1)s + (n-k)z \leq (n-1)z.
\]
Since \( b_1, \ldots, b_k \in \mathbb{I}_{\geq s} \), by hypothesis we have
\[
f(a_1) + (n-1)f(b_1) \geq nf(s),
\]
\[
\cdots
\]
\[
f(a_k) + (n-1)f(b_k) \geq nf(s),
\]
hence
\[ f(a_1) + \cdots + f(a_k) + (n-1)[f(b_1) + \cdots + f(b_k)] \geq knf(s), \]
\[ f(a_1) + \cdots + f(a_k) \geq knf(s) - (n-1)[f(b_1) + \cdots + f(b_k)]. \]

According to this result, the inequality (*) is true if
\[ knf(s) - (n-1)[f(b_1) + \cdots + f(b_k)] + (n-k)f(z) \geq nf(s), \]
which is equivalent to
\[ pf(z) + (k-p)f(s) \geq f(b_1) + \cdots + f(b_k), \quad p = \frac{n-k}{n-1} \leq 1. \]

By Jensen's inequality, we have
\[ pf(z) + (1-p)f(s) \geq f(w), \quad w = pz + (1-p)s \geq s. \]

Thus, we only need to show that
\[ f(w) + (k-1)f(s) \geq f(b_1) + \cdots + f(b_k). \]

Since the decreasingly ordered vector \( \vec{A}_k = (w, s, \ldots, s) \) majorizes the decreasingly ordered vector \( \vec{B}_k = (b_1, b_2, \ldots, b_k) \), this inequality follows from Karamata's inequality for convex functions.

Similarly, we can prove the Left Half Convex Function Theorem (LHCF-Theorem).

**Left Half Convex Function Theorem.** Let \( f \) be a real function defined on an interval \( \mathbb{I} \) and convex on \( \mathbb{I}_{\leq s} \), where \( s \in \text{int}(\mathbb{I}) \). The inequality
\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf \left( \frac{a_1 + a_2 + \cdots + a_n}{n} \right) \]
holds for all \( a_1, a_2, \ldots, a_n \in \mathbb{I} \) satisfying
\[ a_1 + a_2 + \cdots + a_n = ns \]
if and only if
\[ f(x) + (n-1)f(y) \geq nf(s) \]
for all \( x, y \in \mathbb{I} \) so that \( x \geq s \geq y \) and \( x + (n-1)y = ns \).

From the RHCF-Theorem and the LHCF-Theorem, we find the HCF-Theorem (Half Convex Function Theorem).

**Half Convex Function Theorem.** Let \( f \) be a real function defined on an interval \( \mathbb{I} \) and convex on \( \mathbb{I}_{\geq s} \) or \( \mathbb{I}_{\leq s} \), where \( s \in \text{int}(\mathbb{I}) \). The inequality
\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf \left( \frac{a_1 + a_2 + \cdots + a_n}{n} \right) \]
holds for all $a_1, a_2, \ldots, a_n \in I$ satisfying

\[ a_1 + a_2 + \cdots + a_n = ns \]

if and only if

\[ f(x) + (n-1)f(y) \geq nf(s) \]

for all $x, y \in I$ so that $x + (n-1)y = ns$.

The following LCRCF-Theorem is also useful to prove some symmetric inequalities.

**Left Convex-Right Concave Function Theorem (Vasile Cîrtoaje, 2004).** Let $a \leq c$ be real numbers, let $f$ be a continuous function defined on $I = [a, \infty)$, strictly convex on $[a, c]$ and strictly concave on $[c, \infty)$, and let

\[ E(a_1, a_2, \ldots, a_n) = f(a_1) + f(a_2) + \cdots + f(a_n). \]

If $a_1, a_2, \ldots, a_n \in I$ so that

\[ a_1 + a_2 + \cdots + a_n = S = \text{constant}, \]

then

(a) $E$ is minimum for $a_1 = a_2 = \cdots = a_{n-1} \leq a_n$;

(b) $E$ is maximum for either $a_i = a$ or $a < a_1 \leq a_2 = \cdots = a_n$.

**Proof.** Without loss of generality, assume that $a_1 \leq a_2 \leq \cdots \leq a_n$. Since the sum $E(a_1, a_2, \ldots, a_n)$ is a continuous function on the compact set

\[ \Lambda = \{(a_1, a_2, \ldots, a_n) : a_1 + a_2 + \cdots + a_n = S, a_1, a_2, \ldots, a_n \in I\}, \]

$E$ attains its minimum and maximum values.

(a) For the sake of contradiction, suppose that $E$ is minimum at $(b_1, b_2, \ldots, b_n)$ with

\[ b_1 \leq b_2 \leq \cdots \leq b_n, \quad b_1 < b_{n-1}. \]

For $b_{n-1} \leq c$, by Jensen's inequality for strictly convex functions we have

\[ f(b_1) + f(b_{n-1}) > 2f\left(\frac{b_1 + b_{n-1}}{2}\right), \]

while for $b_{n-1} > c$, by Karamata's inequality for strictly concave functions we have

\[ f(b_{n-1}) + f(b_n) > f(c) + f(b_{n-1} + b_n - c). \]

The both results contradict the assumption that $E$ is minimum at $(b_1, b_2, \ldots, b_n)$.

(b) For the sake of contradiction, suppose that $E$ is maximum at $(b_1, b_2, \ldots, b_n)$ with

\[ a < b_1 \leq b_2 \leq \cdots \leq b_n, \quad b_2 < b_n. \]
There are three cases to consider.

**Case 1:** \( b_2 \geq c \). By Jensen's inequality for strictly concave functions, we have

\[
f(b_2) + f(b_n) < 2f\left(\frac{b_2 + b_n}{2}\right).
\]

**Case 2:** \( b_2 < c \) and \( b_1 + b_2 - a \leq c \). By Karamata's inequality for strictly convex functions, we have

\[
f(b_1) + f(b_2) < f(a) + f(b_1 + b_2 - a).
\]

**Case 3:** \( b_2 < c \) and \( b_1 + b_2 - c \geq a \). By Karamata's inequality for strictly convex functions, we have

\[
f(b_1) + f(b_2) < f(b_1 + b_2 - c) + f(c).
\]

Clearly, all these results contradict the assumption that \( E \) is maximum at \((b_1, b_2, \ldots, b_n)\).

**Note 1.** Let us denote

\[
g(u) = \frac{f(u) - f(s)}{u - s}, \quad h(x, y) = \frac{g(x) - g(y)}{x - y}.
\]

In many applications, it is useful to replace the hypothesis

\[
f(x) + (n - 1)f(y) \geq nf(s)
\]

in the RHCF-Theorem, the LHCF-Theorem and the HCF-Theorem by the equivalent condition

\[
h(x, y) \geq 0 \text{ for all } x, y \in I \text{ so that } x + (n - 1)y = ns.
\]

This equivalence is true because

\[
f(x) + (n - 1)f(y) - nf(s) = [f(x) - f(s)] + (n - 1)[f(y) - f(s)]
\]

\[
= (x - s)g(x) + (n - 1)(y - s)g(y)
\]

\[
= \frac{n - 1}{n}(x - y)[g(x) - g(y)]
\]

\[
= \frac{n - 1}{n}(x - y)^2 h(x, y).
\]

**Note 2.** Assume that \( f \) is differentiable on \( I \), and let

\[
H(x, y) = \frac{f'(x) - f'(y)}{x - y}.
\]

The desired inequality of Jensen's type in the RHCF-Theorem, the LHCF-Theorem and the HCF-Theorem holds true by replacing the hypothesis

\[
f(x) + (n - 1)f(y) \geq nf(s)
\]
with the more restrictive condition
\[ H(x, y) \geq 0 \text{ for all } x, y \in \mathbb{I} \text{ so that } x + (n-1)y = ns. \]

To prove this, we will show that the new condition \( H(x, y) \geq 0 \) implies
\[ f(x) + (n-1)f(y) \geq nf(s) \]
for all \( x, y \in \mathbb{I} \) so that \( x + (n-1)y = ns \). Write this inequality as
\[ f_1(x) \geq nf(s), \]
where
\[ f_1(x) = f(x) + (n-1)f(y) = f(x) + (n-1)f\left(\frac{ns-x}{n-1}\right). \]

From
\[ f'_1(x) = f'(x) - f'\left(\frac{ns-x}{n-1}\right) \\
= f'(x) - f'(y) \\
= \frac{n}{n-1}(x-s)H(x, y), \]
it follows that \( f_1 \) is decreasing on \( \mathbb{I}_{\leq s} \) and increasing on \( \mathbb{I}_{\geq s} \); therefore,
\[ f_1(x) \geq f_1(s) = nf(s). \]

**Note 3.** From the proof of the RHCF-Theorem, it follows that the RHCF-Theorem, the LHCF-Theorem and the HCF-Theorem are also valid in the case when \( f \) is defined on \( \mathbb{I} \setminus \{u_0\} \), where \( u_0 \in \mathbb{I}_{<s} \) for the RHCF-Theorem, and \( u_0 \in \mathbb{I}_{>s} \) for the LHCF-Theorem.

**Note 4.** The desired inequalities in the RHCF-Theorem, the LHCF-Theorem and the HCF-Theorem become equalities for
\[ a_1 = a_2 = \cdots = a_n = s. \]

In addition, if there exist \( x, y \in \mathbb{I} \) so that
\[ x + (n-1)y = ns, \quad f(x) + (n-1)f(y) = nf(s), \quad x \neq y, \]
then the equality holds also for
\[ a_1 = x, \quad a_2 = \cdots = a_n = y \]
(or any cyclic permutation). Notice that these equality conditions are equivalent to
\[ x + (n-1)y = ns, \quad h(x, y) = 0 \]
(\(x < y\) for the RHCF-Theorem, and \(x > y\) for the LHCF-Theorem).

**Note 5.** The part (a) in LCRF-Theorem is also true in the case where \(I = (a, \infty)\) and \(f(a_+) = \infty\).

**Note 6.** Similarly, we can extend the *weighted* Jensen’s inequality to right and left half convex functions establishing the WRHCF-Theorem, the WLHCF-Theorem and the WHCF-Theorem (Vasile Cîrtoaje, 2008).

**WHCF-Theorem.** Let \(p_1, p_2, \ldots, p_n\) be positive real numbers so that

\[ p_1 + p_2 + \cdots + p_n = 1, \quad p = \min\{p_1, p_2, \ldots, p_n\}, \]

and let \(f\) be a real function defined on an interval \(I\) and convex on \(I_{\geq s}\) or \(I_{\leq s}\), where \(s \in \text{int}(I)\). The inequality

\[ p_1 f(a_1) + p_2 f(a_2) + \cdots + p_n f(a_n) \geq f(p_1 a_1 + p_2 a_2 + \cdots + p_n a_n) \]

holds for all \(a_1, a_2, \ldots, a_n \in I\) so that

\[ p_1 a_1 + p_2 a_2 + \cdots + p_n a_n = s, \]

if and only if

\[ p f(x) + (1 - p) f(y) \geq f(s) \]

for all \(x, y \in I\) satisfying

\[ p x + (1 - p) y = s. \]
1.2 Applications

1.1. If \( a, b, c \) are real numbers so that \( a + b + c = 3 \), then
\[
3(a^4 + b^4 + c^4) + a^2 + b^2 + c^2 + 6 \geq 6(a^3 + b^3 + c^3).
\]

1.2. If \( a_1, a_2, \ldots, a_n \geq \frac{1 - 2n}{n - 2} \) so that \( a_1 + a_2 + \cdots + a_n = n \), then
\[
a_1^3 + a_2^3 + \cdots + a_n^3 \geq n.
\]

1.3. If \( a_1, a_2, \ldots, a_n \geq \frac{-n}{n - 2} \) so that \( a_1 + a_2 + \cdots + a_n = n \), then
\[
a_1^3 + a_2^3 + \cdots + a_n^3 \geq a_1^2 + a_2^2 + \cdots + a_n^2.
\]

1.4. If \( a_1, a_2, \ldots, a_n \) are real numbers so that \( a_1 + a_2 + \cdots + a_n = n \), then
\[
(n^2 - 3n + 3)(a_1^4 + a_2^4 + \cdots + a_n^4 - n) \geq 2(n^2 - n + 1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n).
\]

1.5. If \( a_1, a_2, \ldots, a_n \) are nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n = n \), then
\[
(n^2 + n + 1)(a_1^3 + a_2^3 + \cdots + a_n^3 - n) \geq (n + 1)(a_1^4 + a_2^4 + \cdots + a_n^4 - n).
\]

1.6. If \( a, b, c \) are real numbers so that \( a + b + c = 3 \), then
\[
\begin{align*}
(a) & \quad a^4 + b^4 + c^4 - 3 + 2(7 + 3\sqrt{7})(a^3 + b^3 + c^3 - 3) \geq 0; \\
(b) & \quad a^4 + b^4 + c^4 - 3 + 2(7 - 3\sqrt{7})(a^3 + b^3 + c^3 - 3) \geq 0.
\end{align*}
\]

1.7. Let \( a_1, a_2, \ldots, a_n \) be nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n = n \). If \( k \) is a positive integer satisfying \( 3 \leq k \leq n + 1 \), then
\[
\frac{a_1^k + a_2^k + \cdots + a_n^k - n}{a_1^2 + a_2^2 + \cdots + a_n^2 - n} \geq (n - 1)\left[\left(\frac{n}{n-1}\right)^{k-1} - 1\right].
\]
1.8. Let $k \geq 3$ be an integer number. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then
\[
\frac{a_1^k + a_2^k + \cdots + a_n^k}{a_1^2 + a_2^2 + \cdots + a_n^2 - n} \leq \frac{n^{k-1} - 1}{n - 1}.
\]

1.9. If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then
\[
n^2 \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n \right) \geq 4(n - 1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n).
\]

1.10. If $a_1, a_2, \ldots, a_8$ are positive real numbers so that $a_1 + a_2 + \cdots + a_8 = 8$, then
\[
\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_8^2} \geq a_1^2 + a_2^2 + \cdots + a_8^2.
\]

1.11. If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = n$, then
\[
a_1^2 + a_2^2 + \cdots + a_n^2 - n \geq 2 \left( 1 + \frac{\sqrt{n-1}}{n} \right) (a_1 + a_2 + \cdots + a_n - n).
\]

1.12. If $a, b, c, d, e$ are positive real numbers so that $a^2 + b^2 + c^2 + d^2 + e^2 = 5$, then
\[
\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} - 5 + \frac{4(1 + \sqrt{5})}{5} (a + b + c + d + e - 5) \geq 0.
\]

1.13. If $a, b, c$ are nonnegative real numbers, no two of which are zero, then
\[
\frac{1}{3a + b + c} + \frac{1}{3b + c + a} + \frac{1}{3c + a + b} \leq \frac{2}{5} \left( \frac{1}{b + c} + \frac{1}{c + a} + \frac{1}{a + b} \right).
\]

1.14. If $a, b, c, d \geq 3 - \sqrt{7}$ so that $a + b + c + d = 4$, then
\[
\frac{1}{2 + a^2} + \frac{1}{2 + b^2} + \frac{1}{2 + c^2} + \frac{1}{2 + d^2} \geq \frac{4}{3}.
\]
1.15. If \( a_1, a_2, \ldots, a_n \in [-\sqrt{n}, n - 2] \) so that \( a_1 + a_2 + \cdots + a_n = n \), then
\[
\frac{1}{n + a_1^2} + \frac{1}{n + a_2^2} + \cdots + \frac{1}{n + a_n^2} \leq \frac{n}{n + 1}.
\]

1.16. If \( a, b, c \) are nonnegative real numbers so that \( a + b + c = 3 \), then
\[
\frac{3 - a}{9 + a^2} + \frac{3 - b}{9 + b^2} + \frac{3 - c}{9 + c^2} \geq \frac{3}{5}.
\]

1.17. If \( a, b, c \) are nonnegative real numbers so that \( a + b + c = 3 \), then
\[
\frac{1}{1 - a + 2a^2} + \frac{1}{1 - b + 2b^2} + \frac{1}{1 - c + 2c^2} \geq \frac{3}{2}.
\]

1.18. If \( a, b, c \) are nonnegative real numbers so that \( a + b + c = 3 \), then
\[
\frac{1}{5 + a + a^2} + \frac{1}{5 + b + b^2} + \frac{1}{5 + c + c^2} \geq \frac{3}{7}.
\]

1.19. If \( a, b, c, d \) are nonnegative real numbers so that \( a + b + c + d = 4 \), then
\[
\frac{1}{10 + a + a^2} + \frac{1}{10 + b + b^2} + \frac{1}{10 + c + c^2} + \frac{1}{10 + d + d^2} \leq \frac{1}{3}.
\]

1.20. Let \( a_1, a_2, \ldots, a_n \) be nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n = n \).
   If \( k \geq 1 - \frac{1}{n} \),
   then
\[
\frac{1}{1 + ka_1^2} + \frac{1}{1 + ka_2^2} + \cdots + \frac{1}{1 + ka_n^2} \geq \frac{n}{1 + k}.
\]

1.21. Let \( a_1, a_2, \ldots, a_n \) be real numbers so that \( a_1 + a_2 + \cdots + a_n = n \). If
   \[
   0 < k \leq \frac{n - 1}{n^2 - n + 1},
   \]
   then
\[
\frac{1}{1 + ka_1^2} + \frac{1}{1 + ka_2^2} + \cdots + \frac{1}{1 + ka_n^2} \leq \frac{n}{1 + k}.
\]
1.22. Let $a_1, a_2, \ldots, a_n$ be nonnegative numbers so that $a_1 + a_2 + \cdots + a_n = n$. If

$$k \geq \frac{n^2}{4(n-1)},$$

then

$$\frac{a_1(a_1 - 1)}{a_1^2 + k} + \frac{a_2(a_2 - 1)}{a_2^2 + k} + \cdots + \frac{a_n(a_n - 1)}{a_n^2 + k} \geq 0.$$  

1.23. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then

$$\frac{a_1 - 1}{(n - 2a_1)^2} + \frac{a_2 - 1}{(n - 2a_2)^2} + \cdots + \frac{a_n - 1}{(n - 2a_n)^2} \geq 0.$$  

1.24. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that

$$a_1 + a_2 + \cdots + a_n = n, \quad a_1, a_2, \ldots, a_n > -k, \quad k \geq 1 + \frac{n}{\sqrt{n - 1}},$$

then

$$\frac{a_1^2 - 1}{(a_1 + k)^2} + \frac{a_2^2 - 1}{(a_2 + k)^2} + \cdots + \frac{a_n^2 - 1}{(a_n + k)^2} \geq 0.$$  

1.25. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$.

If $0 < k \leq 1 + \sqrt{\frac{2n - 1}{n - 1}}$, then

$$\frac{a_1^2 - 1}{(a_1 + k)^2} + \frac{a_2^2 - 1}{(a_2 + k)^2} + \cdots + \frac{a_n^2 - 1}{(a_n + k)^2} \leq 0.$$  

1.26. If $a_1, a_2, \ldots, a_n \geq n - 1 - \sqrt{n^2 - n + 1}$ so that $a_1 + a_2 + \cdots + a_n = n$, then

$$\frac{a_1^2 - 1}{(a_1 + 2)^2} + \frac{a_2^2 - 1}{(a_2 + 2)^2} + \cdots + \frac{a_n^2 - 1}{(a_n + 2)^2} \leq 0.$$  

1.27. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$.

If $k \geq \frac{(n - 1)(2n - 1)}{n^2}$, then

$$\frac{1}{1 + ka_1^3} + \frac{1}{1 + ka_2^3} + \cdots + \frac{1}{1 + ka_n^3} \geq \frac{n}{1 + k}.$$
1.28. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If $0 < k \leq \frac{n-1}{n^2 - 2n + 2}$, then
\[
\frac{1}{1 + ka_1^3} + \frac{1}{1 + ka_2^3} + \cdots + \frac{1}{1 + ka_n^3} \leq \frac{n}{1 + k}.
\]

1.29. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If $k \geq \frac{n^2}{n-1}$, then
\[
\sqrt[3]{\frac{a_1}{k-a_1}} + \sqrt[3]{\frac{a_2}{k-a_2}} + \cdots + \sqrt[3]{\frac{a_n}{k-a_n}} \leq \frac{n}{\sqrt[3]{k-1}}.
\]

1.30. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then
\[
n^{-a_1^2} + n^{-a_2^2} + \cdots + n^{-a_n^2} \geq 1.
\]

1.31. If $a, b, c, d$ are nonnegative real numbers so that $a + b + c + d = 4$, then
\[
(3a^2 + 1)(3b^2 + 1)(3c^2 + 1)(3d^2 + 1) \leq 256.
\]

1.32. If $a, b, c, d, e \geq -1$ so that $a + b + c + d + e = 5$, then
\[
(a^2 + 1)(b^2 + 1)(c^2 + 1)(d^2 + 1)(e^2 + 1) \geq (a + 1)(b + 1)(c + 1)(d + 1)(e + 1).
\]

1.33. If $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ are positive numbers so that $a_1 + a_2 + \cdots + a_n = 1$, then
\[
\left( \frac{1}{\sqrt{a_1}} - \sqrt{a_1} \right) \left( \frac{1}{\sqrt{a_2}} - \sqrt{a_2} \right) \cdots \left( \frac{1}{\sqrt{a_n}} - \sqrt{a_n} \right) \geq \left( \sqrt{n} - \frac{1}{\sqrt{n}} \right)^n.
\]

1.34. Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If
\[
k \leq \left( 1 + \frac{2\sqrt{n-1}}{n} \right)^2,
\]
then
\[
(ka_1 + \frac{1}{a_1})(ka_2 + \frac{1}{a_2}) \cdots (ka_n + \frac{1}{a_n}) \geq (k + 1)^n.
\]
1.35. If \( a, b, c, d \) are nonzero real numbers so that
\[
a, b, c, d \geq -\frac{1}{2}, \quad a + b + c + d = 4,
\]
then
\[
3 \left( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \right) + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \geq 16.
\]

1.36. If \( a_1, a_2, \ldots, a_n \) are nonnegative real numbers so that \( a_1^2 + a_2^2 + \cdots + a_n^2 = n \), then
\[
a_1^3 + a_2^3 + \cdots + a_n^3 - n + \sqrt{\frac{n}{n-1}} (a_1 + a_2 + \cdots + a_n - n) \geq 0.
\]

1.37. If \( a, b, c, d, e \) are nonnegative real numbers so that \( a^2 + b^2 + c^2 + d^2 + e^2 = 5 \), then
\[
\frac{1}{7-2a} + \frac{1}{7-2b} + \frac{1}{7-2c} + \frac{1}{7-2d} + \frac{1}{7-2e} \leq 1.
\]

1.38. Let \( 0 \leq a_1, a_2, \ldots, a_n < k \) so that \( a_1^2 + a_2^2 + \cdots + a_n^2 = n \). If
\[
1 < k \leq 1 + \sqrt{\frac{n}{n-1}},
\]
then
\[
\frac{1}{k-a_1} + \frac{1}{k-a_2} + \cdots + \frac{1}{k-a_n} \geq \frac{n}{k-1}.
\]

1.39. If \( a, b, c \) are nonnegative real numbers, no two of which are zero, then
\[
\sqrt{1 + \frac{48a}{b+c}} + \sqrt{1 + \frac{48b}{c+a}} + \sqrt{1 + \frac{48c}{a+b}} \geq 15.
\]

1.40. If \( a, b, c \) are nonnegative real numbers, then
\[
\sqrt{\frac{3a^2}{7a^2 + 5(b+c)^2}} + \sqrt{\frac{3b^2}{7b^2 + 5(c+a)^2}} + \sqrt{\frac{3c^2}{7c^2 + 5(a+b)^2}} \leq 1.
\]
1.41. If \(a, b, c\) are nonnegative real numbers, then
\[
\sqrt{\frac{a^2}{a^2 + 2(b+c)^2}} + \sqrt{\frac{b^2}{b^2 + 2(c+a)^2}} + \sqrt{\frac{c^2}{c^2 + 2(a+b)^2}} \geq 1.
\]

1.42. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. If
\[
k \geq k_0, \quad k_0 = \frac{\ln 3}{\ln 2} - 1 \approx 0.585,
\]
then
\[
\left( \frac{2a}{b+c} \right)^k + \left( \frac{2b}{c+a} \right)^k + \left( \frac{2c}{a+b} \right)^k \geq 3.
\]

1.43. If \(a, b, c \in [1, 7 + 4\sqrt{3}]\), then
\[
\sqrt{\frac{2a}{b+c}} + \sqrt{\frac{2b}{c+a}} + \sqrt{\frac{2c}{a+b}} \geq 3.
\]

1.44. Let \(a, b, c\) be nonnegative real numbers so that \(a + b + c = 3\). If
\[
0 < k \leq k_0, \quad k_0 = \frac{\ln 2}{\ln 3 - \ln 2} \approx 1.71,
\]
then
\[
a^k(b+c) + b^k(c+a) + c^k(a+b) \leq 6.
\]

1.45. If \(a, b, c\) are nonnegative real numbers so that \(a + b + c = 3\), then
\[
\sqrt{a} + \sqrt{b} + \sqrt{c} - 3 \geq 13 \left( \sqrt{\frac{a+b}{2}} + \sqrt{\frac{b+c}{2}} + \sqrt{\frac{c+a}{2}} - 3 \right).
\]

1.46. If \(a_1, a_2, \ldots, a_n\) are nonnegative real numbers so that \(a_1 + a_2 + \cdots + a_n = n\), then
\[
\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n} + n(k-1) \leq k \left( \sqrt{\frac{n-a_1}{n-1}} + \sqrt{\frac{n-a_2}{n-1}} + \cdots + \sqrt{\frac{n-a_n}{n-1}} \right),
\]
where
\[
k = (\sqrt{n} - 1)(\sqrt{n} + \sqrt{n-1}).
\]
1.47. Let \( a, b, c \) be nonnegative real numbers so that \( a + b + c = 3 \). If \( k > 2 \), then
\[
a^k + b^k + c^k + 3 \geq 2 \left( \frac{a + b}{2} \right)^k + 2 \left( \frac{b + c}{2} \right)^k + 2 \left( \frac{c + a}{2} \right)^k.
\]

1.48. If \( a, b, c \) are the lengths of the sides of a triangle so that \( a + b + c = 3 \), then
\[
\frac{1}{a + b - c} + \frac{1}{b + c - a} + \frac{1}{c + a - b} - 3 \geq 4(2 + \sqrt{3}) \left( \frac{2}{a + b} + \frac{2}{b + c} + \frac{2}{c + a} - 3 \right).
\]

1.49. Let \( a_1, a_2, \ldots, a_5 \) be nonnegative numbers so that \( a_1 + a_2 + a_3 + a_4 + a_5 \leq 5 \). If
\[
k \geq k_0, \quad k_0 = \frac{29 + \sqrt{761}}{10} \approx 5.66,
\]
then
\[
\sum k a_i^2 + a_2 + a_3 + a_4 + a_5 \geq \frac{5}{k + 4}.
\]

1.50. Let \( a_1, a_2, \ldots, a_5 \) be nonnegative numbers so that \( a_1 + a_2 + a_3 + a_4 + a_5 \leq 5 \). If
\[
0 < k \leq k_0, \quad k_0 = \frac{11 - \sqrt{101}}{10} \approx 0.095,
\]
then
\[
\sum k a_i^2 + a_2 + a_3 + a_4 + a_5 \geq \frac{5}{k + 4}.
\]

1.51. Let \( a_1, a_2, \ldots, a_n \) be nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n \leq n \). If
\[
0 < k \leq \frac{1}{n + 1},
\]
then
\[
\frac{a_1}{k a_1^2 + a_2 + \cdots + a_n} + \frac{a_2}{a_1 + k a_2^2 + \cdots + a_n} + \cdots + \frac{a_n}{a_1 + a_2 + \cdots + k a_n^2} \geq \frac{n}{k + n - 1}.
\]

1.52. If \( a_1, a_2, a_3, a_4, a_5 \leq \frac{7}{2} \) so that \( a_1 + a_2 + a_3 + a_4 + a_5 = 5 \), then
\[
\frac{a_1}{a_1^2 - a_1 + 5} + \frac{a_2}{a_2^2 - a_2 + 5} + \frac{a_3}{a_3^2 - a_3 + 5} + \frac{a_4}{a_4^2 - a_4 + 5} + \frac{a_5}{a_5^2 - a_5 + 5} \leq 1.
\]
1.53. Let \( a_1, a_2, \ldots, a_n \) be nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n \geq n \). If
\[
0 < k \leq \frac{1}{1 + \frac{1}{4(n-1)^2}},
\]
then
\[
\frac{a_1^2}{ka_1^2 + a_2 + \cdots + a_n} + \frac{a_2^2}{a_1 + ka_2^2 + \cdots + a_n} + \cdots + \frac{a_n^2}{a_1 + a_2 + \cdots + ka_n^2} \geq \frac{n}{k + n - 1}.
\]

1.54. Let \( a_1, a_2, \ldots, a_n \) be nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n \leq n \). If \( k \geq n - 1 \), then
\[
\frac{a_1^2}{ka_1^2 + a_2 + \cdots + a_n} + \frac{a_2^2}{a_1 + ka_2^2 + \cdots + a_n} + \cdots + \frac{a_n^2}{a_1 + a_2 + \cdots + ka_n^2} \leq \frac{n}{k + n - 1}.
\]

1.55. Let \( a_1, a_2, \ldots, a_n \in [0, n] \) so that \( a_1 + a_2 + \cdots + a_n \geq n \). If \( 0 < k \leq \frac{1}{n} \), then
\[
\frac{a_1 - 1}{ka_1^2 + a_2 + \cdots + a_n} + \frac{a_2 - 1}{a_1 + ka_2^2 + \cdots + a_n} + \cdots + \frac{a_n - 1}{a_1 + a_2 + \cdots + ka_n^2} \geq 0.
\]

1.56. If \( a, b, c \) are positive real numbers so that \( abc = 1 \), then
\[
\sqrt{a^2 - a + 1} + \sqrt{b^2 - b + 1} + \sqrt{c^2 - c + 1} \geq a + b + c.
\]

1.57. If \( a, b, c, d \geq \frac{1}{1 + \sqrt{6}} \) so that \( abcd = 1 \), then
\[
\frac{1}{a + 2} + \frac{1}{b + 2} + \frac{1}{c + 2} + \frac{1}{d + 2} \leq \frac{4}{3}.
\]

1.58. If \( a, b, c \) are positive real numbers so that \( abc = 1 \), then
\[
a^2 + b^2 + c^2 - 3 \geq 2(ab + bc + ca - a - b - c).
\]

1.59. If \( a, b, c \) are positive real numbers so that \( abc = 1 \), then
\[
a^2 + b^2 + c^2 - 3 \geq 18(a + b + c - ab - bc - ca).
\]
1.60. If \( a_1, a_2, \ldots, a_n \) are positive real numbers so that \( a_1 a_2 \cdots a_n = 1 \), then
\[
a_1^2 + a_2^2 + \cdots + a_n^2 - n \geq 6\sqrt{3} \left( a_1 + a_2 + \cdots + a_n - \frac{1}{a_1} - \frac{1}{a_2} - \cdots - \frac{1}{a_n} \right).
\]

1.61. If \( a_1, a_2, \ldots, a_n \) (\( n \geq 4 \)) are positive real numbers so that \( a_1 a_2 \cdots a_n = 1 \), then
\[
(n - 1)(a_1^2 + a_2^2 + \cdots + a_n^2) + n(n + 3) \geq (2n + 2)(a_1 + a_2 + \cdots + a_n).
\]

1.62. Let \( a_1, a_2, \ldots, a_n \) (\( n \geq 3 \)) be positive real numbers so that \( a_1 a_2 \cdots a_n = 1 \). If \( p \\quad \text{and} \\quad q \) are nonnegative real numbers so that \( p + q \geq n - 1 \), then
\[
\frac{1}{1 + p a_1 + qa_1^2} + \frac{1}{1 + p a_2 + qa_2^2} + \cdots + \frac{1}{1 + p a_n + qa_n^2} \geq \frac{n}{1 + p + q}.
\]

1.63. Let \( a, b, c, d \) be positive real numbers so that \( abcd = 1 \). If \( p \\quad \text{and} \\quad q \) are nonnegative real numbers so that \( p + q = 3 \), then
\[
\frac{1}{1 + pa + qa^3} + \frac{1}{1 + pb + qb^3} + \frac{1}{1 + pc + qc^3} + \frac{1}{1 + pd + qd^3} \geq 1.
\]

1.64. If \( a_1, a_2, \ldots, a_n \) are positive real numbers so that \( a_1 a_2 \cdots a_n = 1 \), then
\[
\frac{1}{1 + a_1 + \cdots + a_n^{-1}} + \frac{1}{1 + a_2 + \cdots + a_n^{-1}} + \cdots + \frac{1}{1 + a_n + \cdots + a_n^{-1}} \geq 1.
\]

1.65. Let \( a_1, a_2, \ldots, a_n \) be positive real numbers so that \( a_1 a_2 \cdots a_n = 1 \). If
\[
k \geq n^2 - 1,
\]
then
\[
\frac{1}{\sqrt{1 + ka_1}} + \frac{1}{\sqrt{1 + ka_2}} + \cdots + \frac{1}{\sqrt{1 + ka_n}} \geq \frac{n}{\sqrt{1 + k}}.
\]

1.66. Let \( a_1, a_2, \ldots, a_n \) be positive real numbers so that \( a_1 a_2 \cdots a_n = 1 \). If \( p, q \geq 0 \) so that \( 0 < p + q \leq \frac{1}{n - 1} \), then
\[
\frac{1}{1 + pa_1 + qa_1^2} + \frac{1}{1 + pa_2 + qa_2^2} + \cdots + \frac{1}{1 + pa_n + qa_n^2} \leq \frac{n}{1 + p + q}.
\]
1.67. Let \(a_1, a_2, \ldots, a_n (n \geq 3)\) be positive real numbers so that \(a_1 a_2 \cdots a_n = 1\). If \(0 < k \leq \frac{2n-1}{(n-1)^2}\), then
\[
\frac{1}{\sqrt{1 + ka_1}} + \frac{1}{\sqrt{1 + ka_2}} + \cdots + \frac{1}{\sqrt{1 + ka_n}} \leq \frac{n}{\sqrt{1 + k}}.
\]

1.68. If \(a_1, a_2, \ldots, a_n\) are positive real numbers so that \(a_1 a_2 \cdots a_n = 1\), then
\[
\sqrt{a_1^4 + \frac{2n-1}{(n-1)^2}} + \sqrt{a_2^4 + \frac{2n-1}{(n-1)^2}} + \cdots + \sqrt{a_n^4 + \frac{2n-1}{(n-1)^2}} \geq \frac{1}{n-1}(a_1 + a_2 + \cdots + a_n)^2.
\]

1.69. If \(a_1, a_2, \ldots, a_n\) are positive real numbers so that \(a_1 a_2 \cdots a_n = 1\), then
\[
a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} + n(n-2) \geq (n-1)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).
\]

1.70. Let \(a_1, a_2, \ldots, a_n\) be positive real numbers so that \(a_1 a_2 \cdots a_n = 1\). If \(k \geq n\), then
\[
a_1^k + a_2^k + \cdots + a_n^k + kn \geq (k+1)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).
\]

1.71. If \(a_1, a_2, \ldots, a_n\) are positive real numbers so that \(a_1 a_2 \cdots a_n = 1\), then
\[
\left(1 - \frac{1}{n}\right)^{a_1} + \left(1 - \frac{1}{n}\right)^{a_2} + \cdots + \left(1 - \frac{1}{n}\right)^{a_n} \leq n - 1.
\]

1.72. If \(a, b, c\) are positive real numbers so that \(abc = 1\), then
\[
\frac{1}{1 + \sqrt{1+3a}} + \frac{1}{1 + \sqrt{1+3b}} + \frac{1}{1 + \sqrt{1+3c}} \leq 1.
\]

1.73. If \(a_1, a_2, \ldots, a_n\) are positive real numbers so that \(a_1 a_2 \cdots a_n = 1\), then
\[
\frac{1}{1 + \sqrt{1+4n(n-1)a_1}} + \frac{1}{1 + \sqrt{1+4n(n-1)a_2}} + \cdots + \frac{1}{1 + \sqrt{1+4n(n-1)a_n}} \geq \frac{1}{2}.
\]
1.74. If \(a, b, c\) are positive real numbers so that \(abc = 1\), then
\[
\frac{a^6}{1 + 2a^5} + \frac{b^6}{1 + 2b^5} + \frac{c^6}{1 + 2c^5} \geq 1.
\]

1.75. If \(a, b, c\) are positive real numbers so that \(abc = 1\), then
\[
\sqrt{25a^2 + 144} + \sqrt{25b^2 + 144} + \sqrt{25c^2 + 144} \leq 5(a + b + c) + 24.
\]

1.76. If \(a, b, c\) are positive real numbers so that \(abc = 1\), then
\[
\sqrt{16a^2 + 9} + \sqrt{16b^2 + 9} + \sqrt{16c^2 + 9} \geq 4(a + b + c) + 3.
\]

1.77. If ABC is a triangle, then
\[
\sin A \left(2 \sin \frac{A}{2} - 1\right) + \sin B \left(2 \sin \frac{B}{2} - 1\right) + \sin C \left(2 \sin \frac{C}{2} - 1\right) \geq 0.
\]

1.78. If ABC is an acute or right triangle, then
\[
\sin 2A \left(1 - 2 \sin \frac{A}{2}\right) + \sin 2B \left(1 - 2 \sin \frac{B}{2}\right) + \sin 2C \left(1 - 2 \sin \frac{C}{2}\right) \geq 0.
\]

1.79. If \(a, b, c, d\) are real numbers so that \(a + b + c + d = 4\), then
\[
\frac{a}{a^2 - a + 4} + \frac{b}{b^2 - b + 4} + \frac{c}{c^2 - c + 4} + \frac{d}{d^2 - d + 4} \leq 1.
\]

1.80. Let \(a, b, c\) be nonnegative real numbers so that \(a + b + c = 2\). If
\[
k_0 \leq k \leq 3, \quad k_0 = \frac{\ln 2}{\ln 3 - \ln 2} \approx 1.71,
\]
then
\[
a^k(b + c) + b^k(c + a) + c^k(a + b) \leq 2.
\]

1.81. If \(a_1, a_2, \ldots, a_n\) are positive real numbers so that \(a_1 + a_2 + \cdots + a_n = n\), then
\[
(n + 1)^2 \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq 4(n + 2)(a_1^2 + a_2^2 + \cdots + a_n^2) + n(n^2 - 3n + 6).
\]
1.82. If $a, b, c$ are nonnegative real numbers so that $a + b + c = 12$, then

$$(a^2 + 10)(b^2 + 10)(c^2 + 10) \geq 13310.$$ 

1.83. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then

$$(a_1^2 + 1)(a_2^2 + 1) \cdots (a_n^2 + 1) \geq \frac{(n^2 - 2n + 2)^n}{(n - 1)^{2n-2}}.$$ 

1.84. If $a, b, c$ are nonnegative real numbers so that $a + b + c = 3$, then

$$(a^2 + 2)(b^2 + 2)(c^2 + 2) \leq 44.$$ 

1.85. If $a, b, c$ are nonnegative real numbers so that $a + b + c = 3$, then

$$(a^2 + 1)(b^2 + 1)(c^2 + 1) \leq \frac{169}{16}.$$ 

1.86. If $a, b, c$ are nonnegative real numbers so that $a + b + c = 3$, then

$$(2a^2 + 1)(2b^2 + 1)(2c^2 + 1) \leq \frac{121}{4}.$$ 

1.87. If $a, b, c, d$ are nonnegative real numbers so that $a + b + c + d = 4$, then

$$(a^2 + 3)(b^2 + 3)(c^2 + 3)(d^2 + 3) \leq 513.$$ 

1.88. If $a, b, c, d$ are nonnegative real numbers so that $a + b + c + d = 4$, then

$$(a^2 + 2)(b^2 + 2)(c^2 + 2)(d^2 + 2) \leq 144.$$
1.3 Solutions

P 1.1. If \( a, b, c \) are real numbers so that \( a + b + c = 3 \), then

\[
3(a^4 + b^4 + c^4) + a^2 + b^2 + c^2 + 6 \geq 6(a^3 + b^3 + c^3).
\]

(Vasile C., 2006)

Solution. Write the inequality as

\[
f(a) + f(b) + f(c) \geq 3f(s), \quad s = \frac{a + b + c}{3} = 1,
\]

where

\[
f(u) = 3u^4 - 6u^3 + u^2, \quad u \in \mathbb{R}.
\]

From

\[
f''(u) = 2(18u^2 - 18u + 1),
\]

it follows that \( f''(u) > 0 \) for \( u \geq 1 \), hence \( f \) is convex on \([s, \infty)\). By the RHCF-Theorem, it suffices to show that \( f(x) + 2f(y) \geq 3f(1) \) for all real \( x, y \) so that \( x + 2y = 3 \). Let

\[
E = f(x) + 2f(y) - 3f(1).
\]

We have

\[
E = [f(x) - f(1)] + 2[f(y) - f(1)]
\]

\[
= (3x^4 - 6x^3 + x^2 + 2) + 2(3y^4 - 6y^3 + y^2 + 2)
\]

\[
= (x - 1)(3x^3 - 3x^2 - 2x - 2) + 2(y - 1)(3y^3 - 3y^2 - 2y - 2)
\]

\[
= (x - 1)[(3x^3 - 3x^2 - 2x - 2) - (3y^3 - 3y^2 - 2y - 2)]
\]

\[
= (x - 1)[3(x^3 - y^3) - 3(x^2 - y^2) - 2(x - y)]
\]

\[
= (x - 1)(x - y)[3(x^2 + xy + y^2) - 3(x + y) - 2]
\]

\[
= (x - 1)^2[27(x^2 + xy + y^2) - 9(x + y)(x + 2y) - 2(x + 2y)^2]
\]

\[
= \frac{(x - 1)^2(4x - y)^2}{6} \geq 0.
\]

The equality holds for \( a = b = c = 1 \), and also for \( a = \frac{1}{3} \) and \( b = c = \frac{4}{3} \) (or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

- If \( a_1, a_2, \ldots, a_n \) are real numbers so that \( a_1 + a_2 + \cdots + a_n = n \), then

\[
(a_1^2 - a_1)^2 + (a_2^2 - a_2)^2 + \cdots + (a_n^2 - a_n)^2 \geq \frac{n - 1}{n^2 - 3n + 3}(a_1^2 + a_2^2 + \cdots + a_n^2 - n),
\]
with equality for \(a_1 = a_2 = \cdots = a_n = 1\), and also for
\[a_1 = \frac{1}{n^2 - 3n + 3}, \quad a_2 = a_3 = \cdots = a_n = 1 + \frac{n-2}{n^2 - 3n + 3}\]
(or any cyclic permutation).

\[\square\]

**P 1.2.** If \(a_1, a_2, \ldots, a_n \geq \frac{1-2n}{n-2}\) so that \(a_1 + a_2 + \cdots + a_n = n\), then
\[a_1^3 + a_2^3 + \cdots + a_n^3 \geq n.\]

(Vasile C., 2000)

**Solution.** Write the inequality as
\[f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,\]
where
\[f(u) = u^3, \quad u \geq \frac{1-2n}{n-2}.\]
From \(f''(u) = 6u\), it follows that \(f\) is convex on \([s, \infty)\). By the RHCF-Theorem and Note 1, it suffices to show that \(h(x, y) \geq 0\) for all \(x, y \geq \frac{1-2n}{n-2}\) so that \(x+(n-1)y = n\). We have
\[g(u) = \frac{f(u) - f(1)}{u-1} = u^2 + u + 1,\]
\[h(x, y) = \frac{g(x) - g(y)}{x-y} = x + y + 1 = \frac{(n-2)x + 2n-1}{n-1} \geq 0.\]
From \(x + (n-1)y = n\) and \(h(x, y) = 0\), we get
\[x = \frac{1-2n}{n-2}, \quad y = \frac{n+1}{n-2}.\]
Therefore, according to Note 4, the equality holds for \(a_1 = a_2 = \cdots = a_n = 1\), and also for
\[a_1 = \frac{1-2n}{n-2}, \quad a_2 = a_3 = \cdots = a_n = \frac{n+1}{n-2}\]
(or any cyclic permutation).

\[\square\]

**P 1.3.** If \(a_1, a_2, \ldots, a_n \geq \frac{-n}{n-2}\) so that \(a_1 + a_2 + \cdots + a_n = n\), then
\[a_1^3 + a_2^3 + \cdots + a_n^3 \geq a_1^2 + a_2^2 + \cdots + a_n^2.\]
**Solution.** Write the inequality as

\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1, \]

where

\[ f(u) = u^3 - u^2, \quad u \geq \frac{-n}{n-2}. \]

From \( f''(u) = 6u - 2 \), it follows that \( f \) is convex on \([s, \infty)\). According to the RHCF-Theorem and Note 1, it suffices to show that \( h(x, y) \geq 0 \) for \( x, y \geq \frac{-n}{n-2} \) so that \( x + (n-1)y = n \). We have

\[ g(u) = \frac{f(u) - f(1)}{u - 1} = u^2, \]

\[ h(x, y) = \frac{g(x) - g(y)}{x - y} = x + y = \frac{(n-2)x + n}{n-1} \geq 0. \]

From \( x + (n-1)y = n \) and \( h(x, y) = 0 \), we get

\[ x = \frac{-n}{n-2}, \quad y = \frac{n}{n-2}. \]

Therefore, in accordance with Note 4, the equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for

\[ a_1 = \frac{-n}{n-2}, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n-2} \]

(or any cyclic permutation).

\[ \square \]

**P 1.4.** If \( a_1, a_2, \ldots, a_n \) are real numbers so that \( a_1 + a_2 + \cdots + a_n = n \), then

\[ (n^2 - 3n + 3)(a_1^4 + a_2^4 + \cdots + a_n^4 - n) \geq 2(n^2 - n + 1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n). \]

*(Vasile C., 2009)*

**Solution.** Write the inequality as

\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1, \]

where

\[ f(u) = (n^2 - 3n + 3)u^4 - 2(n^2 - n + 1)u^2, \quad u \in \mathbb{I} = \mathbb{R}. \]

For \( u \geq s = 1 \), we have

\[ \frac{1}{4}f''(u) = 3(n^2 - 3n + 3)u^2 - (n^2 - n + 1) \]

\[ \geq 3(n^2 - 3n + 3) - (n^2 - n + 1) = 2(n-2)^2 \geq 0; \]
therefore, $f$ is convex on $I_{\geq s}$. By the RHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y \in \mathbb{R}$ so that $x + (n-1)y = n$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$ 

We have

$$g(u) = (n^2 - 3n + 3)(u^3 + u^2 + u + 1) - 2(n^2 - n + 1)(u + 1)$$

and

$$h(x, y) = (n^2 - 3n + 3)(x^2 + xy + y^2 + x + y + 1) - 2(n^2 - n + 1)
= [(n^2 - 3n + 3)y - n^2 + n + 1]^2 \geq 0.$$ 

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = -1 + \frac{2}{n^2 - 3n + 3}, \quad a_2 = a_3 = \cdots = a_n = 1 + \frac{2n - 4}{n^2 - 3n + 3}$$
(or any cyclic permutation). \qed

**P 1.5.** If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then

$$(n^2 + n + 1)(a_1^3 + a_2^3 + \cdots + a_n^3 - n) \geq (n + 1)(a_1^4 + a_2^4 + \cdots + a_n^4 - n).$$

*(Vasile C., 2009)*

**Solution.** Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = (n^2 + n + 1)u^3 - (n + 1)u, \quad u \in I = [0, n].$$

The function $f$ is convex on $I_{\leq s}$ because

$$f''(u) = 6u[n^2 + n + 1 - 2(n + 1)u] \geq 6u[n^2 + n + 1 - 2(n + 1)]$$
$$= 6(n^2 - n - 1)u \geq 0.$$ 

By the LHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y \geq 0$ so that $x + (n-1)y = n$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$
We have
\[
g(u) = (n^2 + n + 1)(u^2 + u + 1) - (n + 1)(u^3 + u^2 + u + 1)
  = -(n + 1)u^3 + n^2(u^2 + u + 1)
\]
and
\[
h(x, y) = -(n + 1)(x^2 + xy + y^2) + n^2(x + y + 1)
  = -(n + 1)(x^2 + xy + y^2) + n(x + y)[x + (n - 1)y] + [x + (n - 1)y]^2
  = (n^2 + n - 3)xy + 2n(n - 2)y^2 \geq 0.
\]
The equality holds for \(a_1 = a_2 = \cdots = a_n = 1\), and also for
\[
a_1 = n, \quad a_2 = a_3 = \cdots = a_n = 0
\]
(or any cyclic permutation).

\[\square\]

**P 1.6.** Let \(a, b, c\) be real numbers so that \(a + b + c = 3\). If
\[
-14 - 6\sqrt{7} \leq k \leq -14 + 6\sqrt{7},
\]
then
\[
a^4 + b^4 + c^4 - 3 \geq k(a^3 + b^3 + c^3 - 3).
\]

\textbf{(Vasile C., 2009)}

**Solution.** Write the desired inequalities as
\[
f(a) + f(b) + f(c) \geq 3f(s), \quad s = \frac{a + b + c}{3} = 1,
\]
where
\[
f(u) = u^4 - ku^3, \quad u \in \mathbb{R}.
\]
From
\[
f''(u) = 6u(2u^2 - k),
\]
it follows that \(f''(u) > 0\) for \(u \geq 1\), hence \(f\) is convex on \([s, \infty)\). By the RHCF-Theorem, it suffices to show that \(f(x) + 2f(y) \geq 3f(1)\) for all real \(x, y\) so that \(x + 2y = 3\). Using Note 1, we only need to show that \(h(x, y) \geq 0\), where
\[
h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.
\]
We have
\[
g(u) = u^3 + u^2 + u + 1 - k(u^2 + u + 1) + u + 1 = u^3 + (1 - k)(u^2 + u + 1),
\]
\[ h(x, y) = x^2 + xy + y^2 + (1 - k)(x + y + 1) = 3y^2 - (10 - k)y + 13 - 4k \]
\[ = 3 \left( y - \frac{10 - k}{6} \right)^2 + \frac{(6\sqrt{7} + 14 + k)(6\sqrt{7} - 14 - k)}{12} \geq 0. \]

The equality holds for \( a = b = c = 1 \). If \( k = -14 - 6\sqrt{7} \), then the equality holds also for
\[ a = -5 - 2\sqrt{7}, \quad b = c = 4 + \sqrt{7} \]
(or any cyclic permutation). If \( k = -14 + 6\sqrt{7} \), then the equality holds also for
\[ a = -5 + 2\sqrt{7}, \quad b = c = 4 - \sqrt{7} \]
(or any cyclic permutation).

**Remark.** Similarly, we can prove the following generalization:

- Let \( a_1, a_2, \ldots, a_n \) be real numbers so that \( a_1 + a_2 + \cdots + a_n = n \). If \( k_1 \leq k \leq k_2 \), where
  \[ k_1 = \frac{-2(n^2 - n + 1) - 2\sqrt{3}(n^2 - n + 1)(n^2 - 3n + 3)}{(n - 2)^2}, \]
  \[ k_2 = \frac{-2(n^2 - n + 1) + 2\sqrt{3}(n^2 - n + 1)(n^2 - 3n + 3)}{(n - 2)^2}, \]

then
\[ a_1^4 + a_2^4 + \cdots + a_n^4 - n \geq k(a_1^3 + a_2^3 + \cdots + a_n^3 - n). \]

The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \). If \( k \in \{k_1, k_2\} \), then the equality holds also for
\[ a_1 = \frac{-2(n^2 - 3n + 1) + (n - 1)(n - 2)k}{2(n^2 - 3n + 3)}, \]
\[ a_2 = a_3 = \cdots = a_n = \frac{2(n^2 - n - 1) - (n - 2)k}{2(n^2 - 3n + 3)} \]
(or any cyclic permutation).

**P 1.7.** Let \( a_1, a_2, \ldots, a_n \) be nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n = n \). If \( k \) is a positive integer satisfying \( 3 \leq k \leq n + 1 \), then
\[ \frac{a_1^k + a_2^k + \cdots + a_n^k - n}{a_1^2 + a_2^2 + \cdots + a_n^2 - n} \geq (n - 1) \left( \frac{n}{n - 1} \right)^{k-1} - 1. \]

*(Vasile C., 2012)*
Solution. Denote
\[ m = (n - 1) \left[ \left( \frac{n}{n-1} \right)^{k-1} - 1 \right] = \left( \frac{n}{n-1} \right)^{k-2} + \left( \frac{n}{n-1} \right)^{k-3} + \cdots + 1, \]
and write the inequality as
\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1, \]
where
\[ f(u) = u^k - mu^2, \quad u \in [0,n]. \]
We will show that \( f \) is convex on \([1,n]\). Since
\[ f''(u) = k(k-1)u^{k-2} - 2m \geq k(k-1) - 2m, \]
we need to show that
\[ \frac{k(k-1)}{2} \geq \left( \frac{n}{n-1} \right)^{k-2} + \left( \frac{n}{n-1} \right)^{k-3} + \cdots + 1. \]
Since \( n \geq k - 1 \), this inequality is true if
\[ \frac{k(k-1)}{2} \geq \left( \frac{k-1}{k-2} \right)^{k-2} + \left( \frac{k-1}{k-2} \right)^{k-3} + \cdots + 1. \]
By Bernoulli’s inequality, we have
\[ \left( \frac{k-1}{k-2} \right)^j = \frac{1}{(1 - \frac{1}{k-1})^j} \leq \frac{1}{1 - \frac{j}{k-1}} = \frac{k-1}{k-j-1}, \quad j = 0, 1, \ldots, k-2. \]
Therefore, it suffices to show that
\[ \frac{k(k-1)}{2} \geq (k-1) \left( 1 + \frac{1}{2} + \cdots + \frac{1}{k-1} \right). \]
This is true if
\[ \frac{k}{2} \geq 1 + \frac{1}{2} + \cdots + \frac{1}{k-1}, \]
which can be easily proved by induction. According to the RHCF-Theorem and Note 1, we only need to show that \( h(x,y) \geq 0 \) for \( x, y \geq 0 \) so that \( x + (n-1)y = n \), where
\[ h(x,y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}. \]
We have
\[ g(u) = \frac{(u^k - 1) - m(u^2 - 1)}{u - 1} = (u^{k-1} + u^{k-2} + \cdots + 1) - m(u+1), \]
As a consequence,

\[ h(x, y) = \left( \frac{x^{k-1} - y^{k-1}}{x - y} + \frac{x^{k-2} - y^{k-1}}{x - y} + \cdots + 1 \right) - m \]

\[ = \sum_{j=1}^{k-2} \left[ \frac{x^{j+1} - y^{j+1}}{x - y} - \left( \frac{n}{n-1} \right)^j \right]. \]

It suffices to show that \( f_j(y) \geq 0 \) for \( y \in \left[ 0, \frac{n}{n-1} \right] \) and \( j = 1, 2, \ldots, k-2 \), where

\[ f_j(y) = x^j + x^{j-1} y + \cdots + x y^{j-1} + y^j - \left( \frac{n}{n-1} \right)^j, \quad x = n - (n-1)y. \]

For \( j = 1 \), we have

\[ f_1(y) = x + y - \frac{n}{n-1} = \frac{(n-2)x}{n-1} \geq 0. \]

For \( j \geq 2 \), from \( x' = -(n-1) \) and \( n-1 \geq k-2 \geq j \), we get

\[ f_j'(y) = -(n-1)[j x^{j-1} + (j-1)x^{j-2} y + \cdots + y^{j-1}] + x^{j-1} + 2x^{j-2} y + \cdots + jy^{j-1} \]
\[ \leq -j[j x^{j-1} + (j-1)x^{j-2} y + \cdots + y^{j-1}] + x^{j-1} + 2x^{j-2} y + \cdots + jy^{j-1} \]
\[ = -(j - 1)x^{j-1} - (j \cdot (j-1) - 2)x^{j-2} y - \cdots - (j - 2 - j + 1)xy^{j-2} \leq 0. \]

As a consequence, \( f_j \) is decreasing, hence it is minimum for \( y = \frac{n}{n-1} \) (when \( x = 0 \)):

\[ f_j(y) \geq f_j \left( \frac{n}{n-1} \right) = 0. \]

From \( x + (n-1)y = n \) and \( h(x, y) = 0 \), we get

\[ x = 0, \quad y = \frac{n}{n-1}. \]

Therefore, the equality holds for

\[ a_1 = 0, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n-1} \]

(or any cyclic permutation).

**Remark.** For \( k = 3 \) and \( k = 4 \), we get the following statements (Vasile C., 2002):

- If \( a_1, a_2, \ldots, a_n \) are nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n = n \), then

\[ (n-1)(a_1^3 + a_2^3 + \cdots + a_n^3 - n) \geq (2n-1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n), \]

with equality for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for

\[ a_1 = 0, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n-1} \]
(or any cyclic permutation).

- If \(a_1, a_2, \ldots, a_n\) \((n \geq 3)\) are nonnegative real numbers so that
  \[ a_1 + a_2 + \cdots + a_n = n, \]
  then
  \[ (n - 1)^2(a_1^4 + a_2^4 + \cdots + a_n^4 - n) \geq (3n^2 - 3n + 1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n), \]
  with equality for \(a_1 = a_2 = \cdots = a_n = 1\), and also for
  \[ a_1 = 0, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n - 1} \]
  (or any cyclic permutation).

**P 1.8.** Let \(k \geq 3\) be an integer number. If \(a_1, a_2, \ldots, a_n\) are nonnegative real numbers so that \(a_1 + a_2 + \cdots + a_n = n\), then
  \[ \frac{a_1^k + a_2^k + \cdots + a_n^k - n}{a_1^2 + a_2^2 + \cdots + a_n^2 - n} \leq \frac{n^{k-1} - 1}{n - 1}. \]

*(Vasile C., 2012)*

**Solution.** Denote
  \[ m = \frac{n^{k-1} - 1}{n - 1} = n^{k-2} + n^{k-3} + \cdots + 1, \]
  and write the inequality as
  \[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1, \]
  where
  \[ f(u) = mu^2 - u^k, \quad u \in [0, n]. \]
  We will show that \(f\) is convex on \([0, 1]\). Since
  \[ f''(u) = 2m - k(k - 1)u^{k-2} \geq 2m - k(k - 1), \]
  we need to show that
  \[ n^{k-2} + n^{k-3} + \cdots + 1 \geq \frac{k(k - 1)}{2}. \]
  This is true if
  \[ 2^{k-2} + 2^{k-3} + \cdots + 1 \geq \frac{k(k - 1)}{2}, \]
which is equivalent to

\[ 2^{k-1} - 1 \geq \frac{k(k-1)}{2}, \]

\[ 2^k \geq k^2 - k + 2. \]

Since

\[ 2^k = (1 + 1)^k \geq 1 + \binom{k}{1} + \binom{k}{2} + \binom{k}{3} \]

\[ = 1 + k + \frac{k(k-1)}{2} + \frac{k(k-1)(k-2)}{6}, \]

it suffices to show that

\[ 1 + k + \frac{k(k-1)}{2} + \frac{k(k-1)(k-2)}{6} \geq k^2 - k + 2, \]

which reduces to

\[ (k - 1)(k - 2)(k - 3) \geq 0. \]

According to the LHCF-Theorem and Note 1, we only need to show that \( h(x, y) \geq 0 \) for \( x, y \geq 0 \) so that \( x + (n - 1)y = n \), where

\[ h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}. \]

We have

\[ g(u) = \frac{m(u^2 - 1) - (u^k - 1)}{u - 1} = m(u + 1) - (u^{k-1} + u^{k-2} + \cdots + 1) \]

and

\[ h(x, y) = m - \frac{x^{k-1} - y^{k-1}}{x - y} - \frac{x^{k-2} - y^{k-2}}{x - y} - \cdots - 1 \]

\[ = \left( n^{k-2} - \frac{x^{k-1} - y^{k-1}}{x - y} \right) + \left( n^{k-3} - \frac{x^{k-2} - y^{k-2}}{x - y} \right) + \cdots + \left( n - \frac{x^2 - y^2}{x - y} \right). \]

It suffices to show that

\[ n^j \geq \frac{x^{j+1} - y^{j+1}}{x - y}, \quad j = 1, 2, \ldots, k - 2. \]

We will show that

\[ n^j \geq (x + y)^j \geq \frac{x^{j+1} - y^{j+1}}{x - y}. \]

The left inequality is true since

\[ n - (x + y) = x + (n - 1)y - (x + y) = (n - 2)y \geq 0. \]
The right inequality is also true since
\[(x + y)^j = x^j + \binom{j}{1}x^{j-1}y + \cdots + \binom{j}{j-1}xy^{j-1} + y^j\]
and
\[
\frac{x^{j+1} - y^{j+1}}{x - y} = x^j + x^{j-1}y + \cdots + xy^{j-1} + y^j.
\]
The equality holds for \(a_1 = n\) and \(a_2 = a_3 = \cdots = a_n = 0\) (or any cyclic permutation).

Remark. For \(k = 3\) and \(k = 4\), we get the following statements (Vasile C., 2002):

- If \(a_1, a_2, \ldots, a_n\) are nonnegative real numbers so that \(a_1 + a_2 + \cdots + a_n = n\), then
  \[a_1^3 + a_2^3 + \cdots + a_n^3 - n \leq (n + 1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n),\]
  with equality for \(a_1 = a_2 = \cdots = a_n = 1\), and also for
  \[a_1 = n, \quad a_2 = a_3 = \cdots = a_n = 0\]
  (or any cyclic permutation).

- If \(a_1, a_2, \ldots, a_n\) are nonnegative real numbers so that \(a_1 + a_2 + \cdots + a_n = n\), then
  \[a_1^4 + a_2^4 + \cdots + a_n^4 - n \leq (n^2 + n + 1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n),\]
  with equality for \(a_1 = a_2 = \cdots = a_n = 1\), and also for
  \[a_1 = n, \quad a_2 = a_3 = \cdots = a_n = 0\]
  (or any cyclic permutation).

\[\square\]

**P 1.9.** If \(a_1, a_2, \ldots, a_n\) are positive real numbers so that \(a_1 + a_2 + \cdots + a_n = n\), then
\[
n^2\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n\right) \geq 4(n-1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n).
\]

(Vasile C., 2004)

**Solution.** Write the inequality as
\[f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,
\]
where
\[f(u) = \frac{n^2}{u} - 4(n-1)u^2, \quad u \in I = (0, n).
\]
For \( u \in (0, 1] \), we have
\[
f''(u) = \frac{2n^2}{u^3} - 8(n-1) \geq 2n^2 - 8(n-1) = 2(n-2)^2 \geq 0.
\]
Thus, \( f \) is convex on \( [1, s] \). By the LHCF-Theorem and Note 1, it suffices to show that
\[
h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.
\]
We have
\[
g(u) = \frac{-n^2}{u} - 4(n-1)(u+1)
\]
and
\[
h(x, y) = \frac{n^2}{xy} - 4(n-1) = \frac{[x + (n-1)y]^2}{xy} - 4(n-1) = \frac{[x - (n-1)y]^2}{xy}.
\]
In accordance with Note 4, the equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for
\[
a_1 = \frac{n}{2}, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{2n-2}
\]
(or any cyclic permutation).

\[\square\]

**P 1.10.** If \( a_1, a_2, \ldots, a_8 \) are positive real numbers so that \( a_1 + a_2 + \cdots + a_8 = 8 \), then
\[
\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_8^2} \geq a_1^2 + a_2^2 + \cdots + a_8^2.
\]

*(Vasile C., 2007)*

**Solution.** Write the inequality as
\[
f(a_1) + f(a_2) + \cdots + f(a_8) \geq 8f(s), \quad s = \frac{a_1 + a_2 + \cdots + a_8}{8} = 1,
\]
where
\[
f(u) = \frac{1}{u^2} - u^2, \quad u \in (0, 8).
\]
For \( u \in (0, 1] \), we have
\[
f''(u) = \frac{6}{u^4} - 2 \geq 6 - 2 > 0.
\]
Thus, $f$ is convex on $(0, s]$. By the LHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y > 0$ so that $x + 7y = 8$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$ 

We have

$$g(u) = -u - 1 - \frac{1}{u} - \frac{1}{u^2}$$

and

$$h(x, y) = -1 + \frac{1}{xy} + \frac{x + y}{x^2 y^2}.$$ 

From $8 = x + 7y \geq 2\sqrt{7xy}$, we get $xy \leq 16/7$. Therefore,

$$h(x, y) \geq -1 + \frac{1}{xy} + \frac{7(x + y)}{16xy} = \frac{112y^2 - 170y + 72}{16xy} > \frac{112y^2 - 176y + 72}{16xy} = \frac{14y^2 - 22y + 9}{2xy} > 0.$$ 

The equality holds for $a_1 = a_2 = \cdots = a_8 = 1$.

**Remark.** In the same manner, we can prove the following generalization:

- If $a_1, a_2, \ldots, a_n$ ($n \geq 4$) are positive real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then

$$\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2} + 8 - n \geq \frac{8}{n} (a_1^2 + a_2^2 + \cdots + a_n^2).$$

□

**P 1.11.** If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = n$, then

$$a_1^2 + a_2^2 + \cdots + a_n^2 - n \geq 2 \left(1 + \frac{\sqrt{n-1}}{n}\right) (a_1 + a_2 + \cdots + a_n - n).$$

*(Vasile C., 2006)*

**Solution.** Replacing each $a_i$ by $1/a_i$, we need to prove that

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{1}{u^2} - \frac{2k}{u}, \quad k = 1 + \frac{\sqrt{n-1}}{n}, \quad u \in (0, n).$$
For \( u \in (0, 1] \), we have

\[
 f''(u) = \frac{6 - 4ku}{u^4} \geq \frac{6 - 4k}{u^4} = \frac{2(\sqrt{n-1} - 1)^2}{nu^4} \geq 0.
\]

Thus, \( f \) is convex on \((0, s]\). By the LHCF-Theorem and Note 1, it suffices to show that \( h(x, y) \geq 0 \) for \( x, y > 0 \) so that \( x + (n-1)y = n \), where

\[
 h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.
\]

We have

\[
 g(u) = \frac{-1}{u^2} + \frac{2k-1}{u}
\]

and

\[
 h(x, y) = \frac{1}{xy} \left( \frac{1}{x} + \frac{1}{y} + 1 - 2k \right).
\]

We only need to show that

\[
 \frac{1}{x} + \frac{1}{y} \geq 2k - 1.
\]

Indeed, using the Cauchy-Schwarz inequality, we get

\[
 \frac{1}{x} + \frac{1}{y} \geq \frac{(1 + \sqrt{n-1})^2}{x + (n-1)y} = \frac{(1 + \sqrt{n-1})^2}{n} = 2k - 1,
\]

with equality for \( x = \sqrt{n-1}y \). From \( x + (n-1)y = n \) and \( h(x, y) = 0 \), we get

\[
 x = \frac{n}{1 + \sqrt{n-1}}, \quad y = \frac{n}{n-1 + \sqrt{n-1}}.
\]

In accordance with Note 4, the original equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for

\[
 a_1 = \frac{1 + \sqrt{n-1}}{n}, \quad a_2 = a_3 = \cdots = a_n = \frac{n-1 + \sqrt{n-1}}{n}
\]

(or any cyclic permutation).

\[
 \square
\]

**P 1.12.** If \( a, b, c, d, e \) are positive real numbers so that \( a^2 + b^2 + c^2 + d^2 + e^2 = 5 \), then

\[
 \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} - 5 + \frac{4(1 + \sqrt{5})}{5} (a + b + c + d + e - 5) \geq 0.
\]

(Vasile C., 2006)
Solution. Replacing \( a, b, c, d, e \) by \( \sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}, \sqrt{e} \), respectively, we need to prove that

\[
 f(a) + f(b) + f(c) + f(d) + f(e) \geq 5f(s), \quad s = \frac{a + b + c + d + e}{5} = 1,
\]

where

\[
 f(u) = \frac{1}{\sqrt{u}} + k\sqrt{u}, \quad k = \frac{4(1 + \sqrt{5})}{5} \approx 2.59, \quad u \in (0, 5).
\]

For \( u \in (0, 1] \), we have

\[
 f''(u) = \frac{3 - ku}{4u^2\sqrt{u}} > 0;
\]

therefore, \( f \) is convex on \((0, s]\). By the LHCF-Theorem and Note 1, it suffices to show that \( h(x, y) \geq 0 \) for \( x, y > 0 \) so that \( x + 4y = 5 \). We have

\[
 g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{k\sqrt{u} - 1}{u + \sqrt{u}}
\]

and

\[
 h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{\sqrt{x} + \sqrt{y} + 1 - k\sqrt{xy}}{\sqrt{xy}(\sqrt{x} + \sqrt{y})(\sqrt{x} + 1)(\sqrt{y} + 1)}.
\]

Thus, we only need to show that

\[
 \sqrt{x} + \sqrt{y} + 1 - k\sqrt{xy} \geq 0,
\]

which is true if

\[
 2\sqrt{xy} + 1 - k\sqrt{xy} \geq 0.
\]

Let

\[
 t = \sqrt{xy}.
\]

From

\[
 5 = x + 4y \geq 4\sqrt{xy} = 4t^2,
\]

we get

\[
 t \leq \frac{\sqrt{5}}{2}.
\]

Thus,

\[
 2\sqrt{xy} + 1 - k\sqrt{xy} = 2t + 1 - kt^2 = \left(1 - \frac{2}{\sqrt{5}}t\right)[1 + 2\left(1 + \frac{1}{\sqrt{5}}\right)t] \geq 0.
\]

The equality holds for \( a = b = c = d = e = 1 \).
P 1.13. If \(a, b, c\) are nonnegative real numbers, no two of which are zero, then
\[
\frac{1}{3a + b + c} + \frac{1}{3b + c + a} + \frac{1}{3c + a + b} \leq \frac{2}{5} \left( \frac{1}{b + c} + \frac{1}{c + a} + \frac{1}{a + b} \right).
\]

(Vasile C., 2006)

**Solution.** Due to homogeneity, we may assume that \(a + b + c = 3\). So, we need to show that
\[
f(a) + f(b) + f(c) \geq 3f(s), \quad s = \frac{a + b + c}{3} = 1,
\]
where
\[
f(u) = \frac{2}{3-u} - \frac{5}{2u+3}, \quad u \in [0, 3).
\]
For \(u \in [1, 3)\), we have
\[
f''(u) = \frac{4}{(3-u)^3} - \frac{40}{(2u+3)^3} = \frac{36[2u^3 + 3u^2 + 9(u-1)(3-u)]}{(3-u)^3(2u+3)^3} > 0;
\]
therefore, \(f\) is convex on \([s, 3)\). By the RHCF-Theorem and Note 1, it suffices to show that \(h(x, y) \geq 0\) for \(x, y \geq 0\) so that \(x + 2y = 3\), where
\[
h(x, y) = \frac{g(x) - g(y)}{x-y}, \quad g(u) = \frac{f(u) - f(1)}{u-1}.
\]
We have
\[
g(u) = \frac{1}{3-u} + \frac{2}{2u+3}
\]
and
\[
h(x, y) = \frac{1}{(3-x)(3-y)} - \frac{4}{(2x+3)(2y+3)}
\]
\[
= \frac{9(2x + 2y - 3)}{(3-x)(3-y)(2x+3)(2y+3)} = \frac{9x}{(3-x)(3-y)(2x+3)(2y+3)} \geq 0.
\]
The equality holds for \(a = b = c\), and also for \(a = 0\) and \(b = c\) (or any cyclic permutation).
\[\square\]

P 1.14. If \(a, b, c, d \geq 3 - \sqrt{7}\) so that \(a + b + c + d = 4\), then
\[
\frac{1}{2+a^2} + \frac{1}{2+b^2} + \frac{1}{2+c^2} + \frac{1}{2+d^2} \geq \frac{4}{3}.
\]

(Vasile C., 2008)
**Solution.** Write the inequality as
\[
f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a + b + c + d}{4} = 1,
\]
where
\[
f(u) = \frac{1}{2 + u^2}, \quad u \geq 3 - \sqrt{7}.
\]
For \(u \geq s = 1\), \(f(u)\) is convex because
\[
f''(u) = \frac{3(3u^2 - 2)}{(2 + u^2)^3} > 0.
\]
By the RHCF-Theorem and Note 1, it suffices to show that \(h(x, y) \geq 0\) for \(x, y \geq 3 - \sqrt{7}\) so that \(x + 3y = 4\). We have
\[
g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-1 - u}{3(2 + u^2)}
\]
and
\[
h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{xy + x + y - 2}{3(2 + x^2)(2 + y^2)},
\]
where
\[
xy + x + y - 2 = \frac{-x^2 + 6x - 2}{3} = \frac{(3 + \sqrt{7} - x)(x - 3 + \sqrt{7})}{3} = \frac{(-1 + \sqrt{7} + 3y)(x - 3 + \sqrt{7})}{3} \geq 0.
\]
In accordance with Note 4, the equality holds for \(a = b = c = d = 1\), and also for
\[
a = 3 - \sqrt{7}, \quad b = c = d = \frac{1 + \sqrt{7}}{3}
\]
(or any cyclic permutation).

**Remark.** Similarly, we can prove the following generalization:

- If \(a_1, a_2, \ldots, a_n \geq n - 1 - \sqrt{n^2 - 3n + 3}\) so that \(a_1 + a_2 + \cdots + a_n = n\), then
\[
\frac{1}{2 + a_1^2} + \frac{1}{2 + a_2^2} + \cdots + \frac{1}{2 + a_n^2} \geq \frac{n}{3},
\]
with equality for \(a_1 = a_2 = \cdots = a_n = 1\), and also for
\[
a_1 = n - 1 - \sqrt{n^2 - 3n + 3}, \quad a_2 = a_3 = \cdots = a_n = \frac{1 + \sqrt{n^2 - 3n + 3}}{n - 1}
\]
(or any cyclic permutation).
P 1.15. If $a_1, a_2, \ldots, a_n \in [-\sqrt{n}, n-2]$ so that $a_1 + a_2 + \cdots + a_n = n$, then

$$\frac{1}{n + a_1^2} + \frac{1}{n + a_2^2} + \cdots + \frac{1}{n + a_n^2} \leq \frac{n}{n + 1}.$$  

(Vasile C., 2008)

**Solution.** Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{-1}{n + u^2}, \quad n \geq 3, \quad u \in [-\sqrt{n}, n-2].$$

For $u \in [-\sqrt{n}, 1]$, we have

$$f''(u) = \frac{2(n-u^2)}{(n+u^2)^3} \geq 0,$$

hence $f$ is convex on $[-\sqrt{n}, s]$. By the LHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y \in [-\sqrt{n}, n-2]$ so that $x + (n-1)y = n$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{u + 1}{(n+1)(n+u^2)}$$

and

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{n - x - y - xy}{(n+1)(n+x^2)(n+y^2)}$$

$$= \frac{(n-x)(n-2-x)}{(n^2-1)(n+x^2)(n+y^2)} \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = n - 2, \quad a_2 = a_3 = \cdots = a_n = \frac{2}{n-1}$$

(or any cyclic permutation).

\[ \square \]

P 1.16. If $a, b, c$ are nonnegative real numbers so that $a + b + c = 3$, then

$$\frac{3-a}{9+a^2} + \frac{3-b}{9+b^2} + \frac{3-c}{9+c^2} \geq \frac{3}{5}.$$  

(Vasile C., 2013)
Solution. Write the inequality as
\[ f(a) + f(b) + f(c) \geq 3f(s), \quad s = \frac{a + b + c}{3} = 1, \]
where
\[ f(u) = \frac{3 - u}{9 + u^2}, \quad u \in [0, 3]. \]

For \( u \in [1, 3] \), we have
\[
\frac{1}{2}f''(u) = \frac{u^2(9-u) + 27(u-1)}{(9+u^2)^3} > 0.
\]
Thus, \( f \) is convex on \([s, 3]\). By the RHCF-Theorem and Note 1, it suffices to show that \( h(x, y) \geq 0 \) for \( x, y \geq 0 \) so that \( x + 2y = 3 \), where
\[
h(x, y) = \frac{g(x) - g(y)}{x-y}, \quad g(u) = \frac{f(u) - f(1)}{u-1}.
\]
We have
\[
g(u) = \frac{-(6+u)}{5(9+u^2)}
\]
and
\[
h(x, y) = \frac{xy + 6x + 6y - 9}{5(9+x^2)(9+y^2)} = \frac{x(9-x)}{10(9+x^2)(9+y^2)} \geq 0.
\]
The equality holds for \( a = b = c = 1 \), and also for \( a = 0 \) and \( b = c = \frac{3}{2} \) (or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

- If \( a_1, a_2, \ldots, a_n \) are nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n = n \), then
\[
\frac{n-a_1}{n^2 + (n^2 - 3n + 1)a_1^2} + \frac{n-a_2}{n^2 + (n^2 - 3n + 1)a_2^2} + \cdots + \frac{n-a_n}{n^2 + (n^2 - 3n + 1)a_n^2} \geq \frac{n}{2n-1},
\]
with equality for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for
\[
a_1 = 0, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n-1}
\]
(or any cyclic permutation).

\[ \square \]

P 1.17. If \( a, b, c \) are nonnegative real numbers so that \( a + b + c = 3 \), then
\[
\frac{1}{1-a+2a^2} + \frac{1}{1-b+2b^2} + \frac{1}{1-c+2c^2} \geq \frac{3}{2}.
\]

(Vasile C., 2012)
**Solution.** Write the inequality as

\[ f(a) + f(b) + f(c) \geq 3f(s), \quad s = \frac{a+b+c}{3} = 1, \]

where

\[ f(u) = \frac{1}{1-u+2u^2}, \quad u \in [0,3]. \]

For \( u \in [1,3] \), we have

\[ \frac{1}{2}f''(u) = \frac{12u^2 - 6u - 1}{(1-u+2u^2)^3} > 0. \]

Thus, \( f \) is convex on \([s,3]\). By the RHCF-Theorem and Note 1, it suffices to show that \( h(x, y) \geq 0 \) for \( x, y \geq 0 \) so that \( x + 2y = 3 \), where

\[ h(x, y) = \frac{g(x) - g(y)}{x-y}, \quad g(u) = \frac{f(u) - f(1)}{u-1}. \]

We have

\[ g(u) = \frac{-(1+2u)}{2(1-u+2u^2)} \]

and

\[ h(x, y) = \frac{4xy + 2x + 2y - 3}{2(1-x+2x^2)(1-y+2y^2)} = \frac{x(1+4y)}{2(1-x+2x^2)(1-y+2y^2)} \geq 0. \]

The equality holds for \( a = b = c = 1 \), and also for \( a = 0 \) and \( b = c = \frac{3}{2} \) (or any cyclic permutation).

**Remark.** In the same manner, we can prove the following generalization:

- Let \( a_1, a_2, \ldots, a_n \) be nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n = n \). If

\[ k \geq k_1, \quad k_1 = \frac{3n-2 + \sqrt{5n^2-8n+4}}{2n}, \]

then

\[ \frac{1}{1-a_1 + ka_1^2} + \frac{1}{1-a_2 + ka_2^2} + \cdots + \frac{1}{1-a_n + ka_n^2} \geq \frac{n}{k}, \]

with equality for \( a_1 = a_2 = \cdots = a_n = 1 \). If \( k = k_1 \), then the equality holds also for

\[ a_1 = 0, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n-1} \]

(or any cyclic permutation).
P 1.18. If \( a, b, c \) are nonnegative real numbers so that \( a + b + c = 3 \), then
\[
\frac{1}{5 + a + a^2} + \frac{1}{5 + b + b^2} + \frac{1}{5 + c + c^2} \geq \frac{3}{7}.
\]
(Vasile C., 2008)

**Solution.** Write the inequality as
\[
f(a) + f(b) + f(c) \geq 3f(s), \quad s = \frac{a + b + c}{3} = 1,
\]
where
\[
f(u) = \frac{1}{5 + u + u^2}, \quad u \in [0, 3].
\]
For \( u \geq 1 \), from
\[
f''(u) = \frac{2(3u^2 + 3u - 4)}{(5 + u + u^2)^3} > 0,
\]
it follows that \( f \) is convex on \([s,3]\). By the RHCF-Theorem and Note 1, it suffices to show that \( h(x, y) \geq 0 \) for \( x, y \geq 0 \) so that \( x + 2y = 3 \). We have
\[
g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-2 - u}{7(5 + u + u^2)}
\]
and
\[
h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{xy + 2(x + y) - 3}{7(5 + x + x^2)(5 + y + y^2)}
\]
\[
= \frac{x(5 - x)}{14(5 + x + x^2)(5 + y + y^2)} \geq 0.
\]
According to Note 4, the equality holds for \( a = b = c = 1 \), and also for \( a = 0 \) and \( b = c = \frac{3}{2} \) (or any cyclic permutation).

**Remark.** Similarly, we can prove the following generalization:

- Let \( a_1, a_2, \ldots, a_n \) be nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n = n \). If
\[
0 < k \leq k_1, \quad k_1 = \frac{2(2n - 1)}{n - 1},
\]
then
\[
\frac{1}{k + a_1 + a_1^2} + \frac{1}{k + a_2 + a_2^2} + \cdots + \frac{1}{k + a_n + a_n^2} \geq \frac{n}{k + 2},
\]
with equality for \( a_1 = a_2 = \cdots = a_n = 1 \). If \( k = k_1 \), then the equality holds also for
\[
a_1 = 0, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n - 1}
\]
(or any cyclic permutation).
P 1.19. If \(a, b, c, d\) are nonnegative real numbers so that \(a + b + c + d = 4\), then 
\[
\frac{1}{10 + a + a^2} + \frac{1}{10 + b + b^2} + \frac{1}{10 + c + c^2} + \frac{1}{10 + d + d^2} \leq \frac{1}{3}.
\]
(Vasile C., 2008)

Solution. Write the inequality as
\[
f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a + b + c + d}{4} = 1,
\]
where
\[
f(u) = \frac{-1}{10 + u + u^2}, \quad u \in [0, 4].
\]
For \(u \in [0, 1]\), we have
\[
f''(u) = \frac{6(3-u-u^2)}{(10+u+u^2)^3} > 0.
\]
Thus, \(f\) is convex on \([0,s]\). By the LHCF-Theorem and Note 1, it suffices to show that \(h(x,y) \geq 0\) for \(x, y \geq 0\) so that \(x + 3y = 4\). We have
\[
g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{2 + u}{12(10 + u + u^2)}
\]
and
\[
h(x,y) = \frac{g(x) - g(y)}{x - y} = \frac{8 - 2(x + y) - xy}{12(10 + x + x^2)(10 + y + y^2)}
= \frac{3y^2}{12(10 + x + x^2)(10 + y + y^2)} \geq 0.
\]
The equality holds for \(a = b = c = d = 1\), and also for \(a = 4\) and \(b = c = d = 0\) (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

\[\quad \]
- Let \(a_1, a_2, \ldots, a_n (n \geq 4)\) be nonnegative real numbers so that 
  \[a_1 + a_2 + \cdots + a_n = n.\]
If \(k \geq 2n + 2\), then 
\[
\frac{1}{k + a_1 + a_1^2} + \frac{1}{k + a_2 + a_2^2} + \cdots + \frac{1}{k + a_n + a_n^2} \leq \frac{n}{k + 2},
\]
with equality for \(a_1 = a_2 = \cdots = a_n = 1\). If \(k = 2n + 2\), then the equality holds also for 
\[a_1 = n, \quad a_2 = a_3 = \cdots = a_n = 0\]
(or any cyclic permutation).
**P 1.20.** Let \(a_1, a_2, \ldots, a_n\) be nonnegative real numbers so that \(a_1 + a_2 + \cdots + a_n = n\). If
\[
k \geq 1 - \frac{1}{n},
\]
then
\[
\frac{1}{1 + ka_1^2} + \frac{1}{1 + ka_2^2} + \cdots + \frac{1}{1 + ka_n^2} \geq \frac{n}{1 + k}.
\]

*(Vasile C., 2005)*

**Solution.** Write the inequality as
\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,
\]
where
\[
f(u) = \frac{1}{1 + ku^2}, \quad u \in [0, n].
\]

For \(u \in [1, n]\), we have
\[
f''(u) = \frac{2k(3k^2 - 1)}{(1 + ku^2)^3} \geq \frac{2k(3k - 1)}{(1 + ku^2)^3} > 0.
\]

Thus, \(f\) is convex on \([s, n]\). By the RHCF-Theorem and Note 1, it suffices to show that \(h(x, y) \geq 0\) for \(x, y \geq 0\) so that \(x + (n-1)y = n\). We have
\[
g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-k(u + 1)}{(1 + k)(1 + ku^2)}
\]
and
\[
h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{k^2(x + y + xy) - k}{(1 + k)(1 + kx^2)(1 + ky^2)}.
\]

We need to show that
\[
k(x + y + xy) - 1 \geq 0.
\]

Indeed, we have
\[
k(x + y + xy) - 1 \geq \left(1 - \frac{1}{n}\right)(x + y + xy) - 1 = \frac{x(2n - 2 - x)}{n} \geq 0.
\]

The equality holds for \(a_1 = a_2 = \cdots = a_n = 1\). If \(k = 1 - \frac{1}{n}\), then the equality holds also for
\[
a_1 = 0, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n - 1}
\]
(or any cyclic permutation).
P 1.21. Let $a_1, a_2, \ldots, a_n$ be real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If

$$0 < k \leq \frac{n - 1}{n^2 - n + 1},$$

then

$$\frac{1}{1 + ka_1^2} + \frac{1}{1 + ka_2^2} + \cdots + \frac{1}{1 + ka_n^2} \leq \frac{n}{1 + k}.$$ 

(Vasile C., 2005)

Solution. Replacing all negative numbers $a_i$ by $-a_i$, we need to show the same inequality for

$$a_1, a_2, \ldots, a_n \geq 0, \quad a_1 + a_2 + \cdots + a_n \geq n.$$ 

Since the left side of the desired inequality is decreasing with respect to each $a_i$, it is sufficient to consider that $a_1 + a_2 + \cdots + a_n = n$. Write this inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{-1}{1 + ku^2}, \quad u \in [0, n].$$

For $u \in [0, 1]$, we have

$$f''(u) = \frac{2k(1 - 3ku^2)}{(1 + ku^2)^3} \geq 0,$$

since

$$1 - 3ku^2 \geq 1 - 3k \geq 1 - \frac{3(n - 1)}{n^2 - n + 1} = \frac{(n - 2)^2}{n^2 - n + 1} \geq 0.$$ 

Thus, $f$ is convex on $[0, s]$. By the LHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for $x, y \geq 0$ so that $x + (n - 1)y = n$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{k(u + 1)}{(1 + k)(1 + ku^2)}$$

and

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{k - k^2(x + y + xy)}{(1 + k)(1 + kx^2)(1 + ky^2)}.$$ 

It suffices to show that

$$1 - k(x + y + xy) \geq 0.$$ 

Indeed, we have

$$1 - k(x + y + xy) \geq 1 - \frac{n - 1}{n^2 - n + 1}(x + y + xy) = \frac{(x - n + 1)^2}{n^2 - n + 1} \geq 0.$$
The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \). If \( k = \frac{n - 1}{n^2 - n + 1} \), then the equality holds also for
\[
a_1 = n - 1, \quad a_2 = a_3 = \cdots = a_n = \frac{1}{n - 1}
\]
(or any cyclic permutation).

\[\]
\textbf{P 1.23.} If \(a_1, a_2, \ldots, a_n\) are nonnegative real numbers so that \(a_1 + a_2 + \cdots + a_n = n\), then

\[
\frac{a_1 - 1}{(n-2a_1)^2} + \frac{a_2 - 1}{(n-2a_2)^2} + \cdots + \frac{a_n - 1}{(n-2a_n)^2} \geq 0.
\]

(Vasile C., 2012)

\textbf{Solution.} For \(n = 2\), the inequality is an identity. Consider further \(n \geq 3\) and write the inequality as

\[f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,
\]

where

\[f(u) = \frac{u - 1}{(n-2u)^2}, \quad u \in \mathbb{I} = [0, n] \setminus \{n/2\}.
\]

From

\[f'(u) = \frac{2u + n - 4}{(n-2u)^3}, \quad f''(u) = \frac{8(u + n - 3)}{(n-2u)^4},
\]

it follows that \(f\) is convex on \(\mathbb{I}_{\leq s}\). By the LHCF-Theorem, Note 1 and Note 3, it suffices to show that \(h(x, y) \geq 0\) for \(x, y \in \mathbb{I}\) so that \(x + (n-1)y = n\). We have

\[g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{1}{(n-2u)^2}
\]

and

\[h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{4(n-x-y)}{(n-2x)^2(n-2y)^2} = \frac{4(n-2)y}{(n-2x)^2(n-2y)^2} \geq 0.
\]

In accordance with Note 4, the equality holds for \(a_1 = a_2 = \cdots = a_n = 1\), and also for

\[a_1 = n, \quad a_2 = a_3 = \cdots = a_n = 0
\]

(or any cyclic permutation).

\(\square\)

\textbf{P 1.24.} If \(a_1, a_2, \ldots, a_n\) are nonnegative real numbers so that

\[a_1 + a_2 + \cdots + a_n = n, \quad a_1, a_2, \ldots, a_n > -k, \quad k \geq 1 + \frac{n}{\sqrt{n-1}},
\]

then

\[
\frac{a_1^2 - 1}{(a_1 + k)^2} + \frac{a_2^2 - 1}{(a_2 + k)^2} + \cdots + \frac{a_n^2 - 1}{(a_n + k)^2} \geq 0.
\]

(Vasile C., 2008)
Solution. Write the inequality as

\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1, \]

where

\[ f(u) = \frac{u^2 - 1}{(u + k)^2}, \quad u > -k. \]

For \( u \in (-k, 1] \), we have

\[ f''(u) = \frac{2(k^2 - 3 - 2ku)}{(u + k)^4} \geq \frac{2(k^2 - 2k - 3)}{(u + k)^4} = \frac{2(k + 1)(k - 3)}{(u + k)^4} \geq 0. \]

Thus, \( f \) is convex on \((-k, s]\). By the LHCF-Theorem and Note 1, it suffices to show that \( h(x, y) \geq 0 \) for \( x, y > -k \) so that \( x + (n-1)y = n \). We have

\[ g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{u + 1}{(u + k)^2} \]

and

\[ h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{(k - 1)^2 - (1 + x)(1 + y)}{(x + k)^2(y + k)^2}. \]

Since

\[ (k - 1)^2 \geq \frac{n^2}{n - 1}, \]

we need to show that

\[ n^2 \geq (n - 1)(1 + x)(1 + y). \]

Indeed,

\[ n^2 - (n - 1)(1 + x)(1 + y) = n^2 - (1 + x)(2n - 1 - x) = (x - n + 1)^2 \geq 0. \]

The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \). If \( k = 1 + \frac{n}{\sqrt{n - 1}} \), then the equality holds also for

\[ a_1 = n - 1, \quad a_2 = a_3 = \cdots = a_n = \frac{1}{n - 1} \]

(or any cyclic permutation).

\[ \square \]

P 1.25. Let \( a_1, a_2, \ldots, a_n \) be nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n = n \). If \( 0 < k \leq 1 + \sqrt{\frac{2n - 1}{n - 1}} \), then

\[ \frac{a_1^2 - 1}{(a_1 + k)^2} + \frac{a_2^2 - 1}{(a_2 + k)^2} + \cdots + \frac{a_n^2 - 1}{(a_n + k)^2} \leq 0. \]

(Vasile C., 2008)
Solution. Write the inequality as
\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1, \]
where
\[ f(u) = \frac{1-u^2}{(u+k)^2}, \quad u \in [0, n]. \]

For \( u \geq 1 \), we have
\[ f''(u) = \frac{2(2k-2^2+3)}{(u+k)^4} \geq \frac{2(2k-2^2+3)}{(u+k)^4} = \frac{2(1+k)(3-k)}{(u+k)^4} > 0. \]

Thus, \( f \) is convex on \([s, n]\). By the RHCF-Theorem and Note 1, it suffices to show that \( h(x, y) \geq 0 \) for \( x, y \geq 0 \) so that \( x + (n-1)y = n \). We have
\[ g(u) = \frac{f(u) - f(1)}{u-1} = \frac{-u-1}{(u+k)^2} \]
and
\[ h(x, y) = \frac{g(x) - g(y)}{x-y} = \frac{2k-k^2 + x + y + xy}{(x+k)^2(y+k)^2} \geq \frac{2k-k^2 + x + y}{(x+k)^2(y+k)^2}. \]

Since
\[ x + y \geq \frac{x+(n-1)y}{n-1} = \frac{n}{n-1}, \]
we get
\[ 2k-k^2 + x + y \geq 2k-k^2 + \frac{n}{n-1} = -(k-1)^2 + \frac{2n-1}{n-1} \geq 0, \]
hence \( h(x, y) \geq 0 \).

The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \). If \( k = 1 + \sqrt{\frac{2n-1}{n-1}} \), then the equality holds also for
\[ a_1 = 0, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n-1} \]
(or any cyclic permutation).

\[ \square \]

P 1.26. If \( a_1, a_2, \ldots, a_n \geq n-1 - \sqrt{n^2-n+1} \) so that \( a_1 + a_2 + \cdots + a_n = n \), then
\[ \frac{a_1^2-1}{(a_1+2)^2} + \frac{a_2^2-1}{(a_2+2)^2} + \cdots + \frac{a_n^2-1}{(a_n+2)^2} \leq 0. \]

(Vasile C., 2008)
Solution. Write the inequality as

\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1, \]

where

\[ f(u) = \frac{1 - u^2}{(u + 2)^2}, \quad u \geq n - 1 - \sqrt{n^2 - n + 1}. \]

For \( u \geq 1 \), we have

\[ f''(u) = \frac{2(4u - 1)}{(u + 2)^4} > 0. \]

Thus, \( f(u) \) is convex for \( u \geq s \). By the RHCF-Theorem and Note 1, it suffices to show that \( h(x, y) \geq 0 \) for

\[ n - 1 - \sqrt{n^2 - n + 1} \leq x \leq 1 \leq y, \quad x + (n - 1)y = n. \]

Since

\[ g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-u - 1}{(u + 2)^2}, \]

\[ h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{x + y + xy}{(x + 2)^2(y + 2)^2} = \frac{-x^2 + 2(n - 1)x + n}{(n - 1)(x + 2)^2(y + 2)^2}, \]

we need to show that

\[ n - 1 - \sqrt{n^2 - n + 1} \leq x \leq n - 1 + \sqrt{n^2 - n + 1}. \]

This is true because

\[ n - 1 - \sqrt{n^2 - n + 1} \leq x \leq 1 < n - 1 + \sqrt{n^2 - n + 1}. \]

The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for

\[ a_1 = n - 1 - \sqrt{n^2 - n + 1}, \quad a_2 = a_3 = \cdots = a_n = \frac{1 + \sqrt{n^2 - n + 1}}{n - 1} \]

(or any cyclic permutation).

\[ \square \]

P 1.27. Let \( a_1, a_2, \ldots, a_n \) be nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n = n \). If \( k \geq \frac{(n - 1)(2n - 1)}{n^2} \), then

\[ \frac{1}{1 + ka_1^3} + \frac{1}{1 + ka_2^3} + \cdots + \frac{1}{1 + ka_n^3} \geq \frac{n}{1 + k}. \]

(Vasile C., 2008)
**Solution.** Write the inequality as

\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1, \]

where

\[ f(u) = \frac{1}{1+ku^3}, \quad u \in [0, n]. \]

For \( u \in [1, n] \), we have

\[ f''(u) = \frac{6ku(2ku^3 - 1)}{(1 + ku^3)^3} \geq \frac{6ku(2k - 1)}{(1 + ku^3)^3} > 0. \]

Thus, \( f \) is convex on \([s, n]\). By the RHCF-Theorem and Note 1, it suffices to show that \( h(x, y) \geq 0 \) for \( x, y \geq 0 \) so that \( x + (n - 1)y = n \), where

\[ h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}. \]

We have

\[ g(u) = \frac{-k(u^2 + u + 1)}{(1 + k)(1 + ku^3)} \]

and

\[ \frac{h(x, y)}{k^2} = \frac{x^2y^2 + xy(x + y - 1) + (x + y)^2 - (x + y + 1)/k}{(1 + k)(1 + kx^3)(1 + ky^3)}. \]

Since

\[ x + y \geq \frac{x + (n - 1)y}{n - 1} = \frac{n}{n - 1} > 1, \]

it suffices to show that

\[ (x + y)^2 \geq \frac{x + y + 1}{k}. \]

From \( x + y \geq \frac{n}{n - 1} \), we get

\[ k(x + y) \geq \frac{2n - 1}{n}, \]

hence

\[ k(x + y)^2 - x - y = (x + y)[k(x + y) - 1] \geq \frac{n}{n - 1} \left( \frac{2n - 1}{n} - 1 \right) = 1. \]

The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \). If \( k = \frac{(n - 1)(2n - 1)}{n^2} \), then the equality holds also for

\[ a_1 = 0, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n - 1} \]

(or any cyclic permutation). \( \square \)
**P 1.28.** Let \(a_1, a_2, \ldots, a_n\) be nonnegative real numbers so that \(a_1 + a_2 + \cdots + a_n = n\). If \(0 < k \leq \frac{n-1}{n^2-2n+2}\), then

\[
\frac{1}{1+ka_1^3} + \frac{1}{1+ka_2^3} + \cdots + \frac{1}{1+ka_n^3} \leq \frac{n}{1+k}.
\]

(Vasile C., 2008)

**Solution.** Write the inequality as

\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,
\]

where

\[f(u) = \frac{-1}{1+ku^3}, \quad u \in [0, n].\]

For \(u \in [0, 1]\), we have

\[f''(u) = \frac{6ku(1-2ku^3)}{(1+ku^3)^3} \geq \frac{6ku(1-2k)}{(1+ku^3)^3} \geq 0.
\]

Thus, \(f\) is convex on \([0,s]\). By the LHCF-Theorem and Note 1, it suffices to show that \(h(x,y) \geq 0\) for \(x, y \geq 0\) so that \(x + (n-1)y = n\), where

\[h(x,y) = \frac{g(x) - g(y)}{x-y}, \quad g(u) = \frac{f(u) - f(1)}{u-1}.
\]

We have

\[g(u) = \frac{k(u^2 + u + 1)}{(1+k)(1+ku^3)}
\]

and

\[
h(x,y) = \frac{(x+y+1)/k-x^2y^2-xy(x+y-1)-(x+y)^2}{(1+k)(1+kx^3)(1+ky^3)}.
\]

It suffices to show that

\[
\frac{(n^2-2n+2)(x+y+1)}{n-1} - x^2y^2 - xy(x+y-1) - (x+y)^2 \geq 0,
\]

which is equivalent to

\[
[2 + ny - (n-1)y^2][1 - (n-1)y]^2 \geq 0.
\]

This is true because

\[2 + ny - (n-1)y^2 = 2 + y[n - (n-1)y] = 2 + xy > 0.
\]

The equality holds for \(a_1 = a_2 = \cdots = a_n = 1\). If \(k = \frac{n-1}{n^2-2n+2}\), then the equality holds also for

\[a_1 = n-1, \quad a_2 = a_3 = \cdots = a_n = \frac{1}{n-1}
\]

(or any cyclic permutation).\[\Box\]
**P 1.29.** Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If $k \geq \frac{n^2}{n-1}$, then

$$\sqrt{\frac{a_1}{k-a_1}} + \sqrt{\frac{a_2}{k-a_2}} + \cdots + \sqrt{\frac{a_n}{k-a_n}} \leq \frac{n}{\sqrt{k-1}}.$$  

(Vasile C., 2008)

**Solution.** Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = -\sqrt{\frac{u}{k-u}}, \quad u \in [0, n].$$

For $u \in [0, 1]$, we have

$$f''(u) = \frac{k(4u)}{4u^{3/2}(k-u)^{5/2}} \geq \frac{k(4)}{4u^{3/2}(k-u)^{5/2}} \geq 0.$$

Thus, $f$ is convex on $[0, s]$. By the LHCF-Theorem, it suffices to prove that

$$f(x) + (n-1)f(y) \geq nf(1)$$

for $x \geq 1 \geq y \geq 0$ so that $x + (n-1)y = n$. We write the inequality as

$$\sqrt{\frac{(k-1)x}{k-x}} + (n-1)\sqrt{\frac{(k-1)y}{k-y}} \leq n,$$

$$\sqrt{1 + \frac{(n-1)(1-y)}{(n-1)y+k-n}} \leq 1 + (n-1)\left[1 - \sqrt{\frac{(k-1)y}{k-y}}\right].$$

Let

$$z = \sqrt{\frac{(k-1)y}{k-y}}, \quad z \leq 1,$$

which yields

$$y = \frac{kz^2}{z^2 + k-1},$$

$$1 - y = \frac{(k-1)(1-z^2)}{z^2 + k-1}, \quad (n-1)y + k - n = \frac{(k-1)(nz^2 + k-n)}{z^2 + k-1}.$$

Since

$$\frac{k(1-y)}{(n-1)y+k-n} = \frac{k(1-z^2)}{k-n(1-z^2)} = \frac{1-z^2}{1-n(1-z^2)/k} \leq \frac{1-z^2}{1-(1-z^2)(n-1)/n} = \frac{n(1-z^2)}{(n-1)z^2 + 1},$$
it suffices to show that
\[
\sqrt{1 + \frac{n(n-1)(1-z^2)}{(n-1)z^2+1}} \leq 1 + (n-1)(1-z).
\]

By squaring, we get the obvious inequality
\[
(z-1)^2[(n-1)z-1]^2 \geq 0.
\]

The equality holds for \(a_1 = a_2 = \cdots = a_n = 1\). If \(k = \frac{n^2}{n-1}\), then the equality holds also for
\[
a_1 = \frac{n(n-1)^2}{n^2-2n+2}, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{(n-1)(n^2-2n+2)}
\]
(or any cyclic permutation).

\(\square\)

**P 1.30.** If \(a_1, a_2, \ldots, a_n\) are nonnegative real numbers so that \(a_1 + a_2 + \cdots + a_n = n\), then
\[
n^{-a_1^2} + n^{-a_2^2} + \cdots + n^{-a_n^2} \geq 1.
\]

*(Vasile C., 2006)*

**Solution.** Let \(k = \ln n\). Write the inequality as
\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,
\]
where
\[
f(u) = n^{-u^2}, \quad u \in [0, n].
\]

For \(u \geq 1\), we have
\[
f''(u) = 2kn^{-u^2}(2ku^2 - 1) \geq 2kn^{-u^2}(2k-1) \geq 2kn^{-u^2}(2 \ln 2 - 1) > 0;
\]
therefore, \(f\) is convex on \([s, n]\). By the RHCF-Theorem, it suffices to show that
\[
f(x) + (n-1)f(y) \geq nf(1)
\]
for \(0 \leq x \leq 1 \leq y\) and \(x + (n-1)y = n\). The desired inequality is equivalent to \(g(x) \geq 0\), where
\[
g(x) = n^{-x^2} + (n-1)n^{-y^2} - 1, \quad y = \frac{n-x}{n-1}, \quad 0 \leq x \leq 1.
\]

Since \(y' = -1/(n-1)\), we get
\[
g'(x) = -2xkn^{-x^2} - 2(n-1)kyy' n^{-y^2} = 2k(y n^{-y^2} - x n^{-x^2}).
\]
The derivative $g'(x)$ has the same sign as $g_1(x)$, where
\[ g_1(x) = \ln(y n^{-x^2}) - \ln(x n^{-x^2}) = \ln y - \ln x + k(x^2 - y^2), \]
\[ g'_1(x) = \frac{y'}{y} - \frac{1}{x} + 2k(x - yy') = n \left[ \frac{-1}{x(n-x)} + \frac{2k(1 + nx - 2x)}{(n-1)^2} \right]. \]
For $0 < x \leq 1$, $g'_1(x)$ has the same sign as
\[ h(x) = \frac{-(n-1)^2}{2k} + x(n-x)(1+nx-2x). \]
Since
\[ h'(x) = n + 2(n^2 - 2n - 1)x - 3(n-2)x^2 \]
\[ \geq nx + 2(n^2 - 2n - 1)x - 3(n-2)x \]
\[ = 2(n-1)(n-2)x \geq 0, \]
h is strictly increasing on $[0, 1]$. From
\[ h(0) = \frac{-(n-1)^2}{2k} < 0, \quad h(1) = (n-1)^2 \left( 1 - \frac{1}{2k} \right) > 0, \]
it follows that there is $x_1 \in (0, 1)$ so that $h(x_1) = 0$, $h(x) < 0$ for $x \in [0, x_1)$ and $h(x) > 0$ for $x \in (x_1, 1]$. Therefore, $g_1$ is strictly decreasing on $(0, x_1]$ and strictly increasing on $[x_1, 1]$. Since $g_1(0+) = \infty$ and $g_1(1) = 0$, there is $x_2 \in (0, x_1)$ so that $g_1(x_2) = 0$, $g_1(x) > 0$ for $x \in (0, x_2)$ and $g_1(x) < 0$ for $x \in (x_2, 1)$. Consequently, $g$ is strictly increasing on $[0, x_2]$ and strictly decreasing on $[x_2, 1]$. Because $g(0) > 0$ and $g(1) = 0$, it follows that $g(x) \geq 0$ for $x \in [0, 1]$. The proof is completed.

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. 

\[ \square \]

**P 1.31.** If $a, b, c, d$ are nonnegative real numbers so that $a + b + c + d = 4$, then
\[ (3a^2 + 1)(3b^2 + 1)(3c^2 + 1)(3d^2 + 1) \leq 256. \]

*(Vasile C., 2006)*

**Solution.** Write the inequality as
\[ f(a) + f(b) + f(c) + f(d) \geq nf(s), \quad s = \frac{a + b + c + d}{4} = 1, \]
where
\[ f(u) = -\ln(3u^2 + 1), \quad u \in [0, 4]. \]
For \( u \in [1, 4] \), we have
\[
\frac{6(3u^2 - 1)}{(3u^2 + 1)^2} > 0.
\]
Therefore, \( f \) is convex on \([s, 4]\). By the RHCF-Theorem, we only need to show that
\[
f(x) + 3f(y) \geq 4f(1)
\]
for \( 0 \leq x \leq 1 \leq y \) so that \( x + 3y = 4 \); that is, to show that \( g(x) \geq 0 \) for \( x \in [0, 1] \), where
\[
g(x) = f(x) + 3f(y) - 4f(1), \quad y = \frac{4-x}{3}.
\]
Since \( y'(x) = -1/3 \), we have
\[
g'(x) = f'(x) + 3y'f'(y) = \frac{-6x}{3x^2 + 1} + \frac{6y}{3y^2 + 1}
\]
\[
= \frac{6(x-y)(xy-1)}{(3x^2 + 1)(3y^2 + 1)} = \frac{8(x-1)^2(3-x)}{3(3x^2 + 1)(3y^2 + 1)} \geq 0.
\]
Since \( g \) is strictly increasing on \([0, 1]\), it suffices to show that \( g(0) \geq 0 \); that is, to show that the original inequality holds for \( a = 0 \) and \( b = c = d = 4/3 \). This reduces to \( 19^3 \leq 27 \cdot 256 \), which is true because
\[
27 \cdot 256 - 19^3 = 53 > 0.
\]
The equality holds for \( a = b = c = d = 1 \).

\( \boxdot \)

**P 1.32.** If \( a, b, c, d, e \geq -1 \) so that \( a + b + c + d + e = 5 \), then
\[
(a^2 + 1)(b^2 + 1)(c^2 + 1)(d^2 + 1)(e^2 + 1) \geq (a + 1)(b + 1)(c + 1)(d + 1)(e + 1).
\]

*(Vasile C., 2007)*

**Solution.** Consider the nontrivial case \( a, b, c, d, e > -1 \), and write the inequality as
\[
f(a) + f(b) + f(c) + f(d) + f(e) \geq nf(s), \quad s = \frac{a + b + c + d + e}{5} = 1,
\]
where
\[
f(u) = \ln(u^2 + 1) - \ln(u + 1), \quad u > -1.
\]
For \( u \in (-1, 1) \), we have
\[
f''(u) = \frac{2(1-u^2)}{(u^2 + 1)^2} + \frac{1}{(u + 1)^2} > 0.
\]
Therefore, $f$ is convex on $(-1, s]$. By the LHCF-Theorem and Note 2, it suffices to show that $H(x, y) \geq 0$ for $x, y > -1$ so that $x + 4y = 5$, where

$$H(x, y) = \frac{f'(x) - f'(y)}{x - y} = \frac{2(1 - xy)}{(x^2 + 1)(y^2 + 1)} + \frac{1}{(x + 1)(y + 1)};$$

thus, we need to show that

$$2(1 - xy) + \frac{(x^2 + 1)(y^2 + 1)}{(x + 1)(y + 1)} \geq 0.$$

Since

$$\frac{x^2 + 1}{x + 1} \geq \frac{x + 1}{2}, \quad \frac{y^2 + 1}{y + 1} \geq \frac{y + 1}{2},$$

it suffices to prove that

$$2(1 - xy) + \frac{(x + 1)(y + 1)}{4} \geq 0,$$

which is equivalent to

$$x + y + 9 - 7xy \geq 0,$$

$$28x^2 - 38x + 14 \geq 0,$$

$$(28x - 19)^2 + 31 \geq 0.$$

The equality holds for $a = b = c = d = e = 1$.

\[\square\]

**P 1.33.** If $a_1, a_2, \ldots, a_n$ ($n \geq 3$) are positive numbers so that $a_1 + a_2 + \cdots + a_n = 1$, then

$$\left(\frac{1}{\sqrt{a_1}} - \sqrt{a_1}\right)\left(\frac{1}{\sqrt{a_2}} - \sqrt{a_2}\right)\cdots\left(\frac{1}{\sqrt{a_n}} - \sqrt{a_n}\right) \geq \left(\sqrt{n} - \frac{1}{\sqrt{n}}\right)^n.$$

*(Vasile C., 2006)*

**Solution.** Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{1}{n},$$

where

$$f(u) = \ln\left(\frac{1}{\sqrt{u}} - \sqrt{u}\right) = \ln(1 - u) - \frac{1}{2} \ln u, \quad u \in (0, 1).$$

From

$$f'(u) = \frac{-1}{1 - u} - \frac{1}{2u}, \quad f''(u) = \frac{1 - 2u - u^2}{2u^2(1 - u)^2},$$
it follows that \( f''(u) \geq 0 \) for \( u \in (0, \sqrt{2} - 1] \). Since
\[
s = \frac{1}{n} \leq \frac{1}{3} < \sqrt{2} - 1,
\]
f is convex on \((0, s]\). Thus, we can apply the LHCF-Theorem.

**First Solution.** By the LHCF-Theorem, it suffices to show that
\[
f(x) + (n - 1)f(y) \geq nf\left(\frac{1}{n}\right)
\]
for all \( x, y > 0 \) so that \( x + (n - 1)y = 1 \); that is, to show that
\[
\left(\frac{1}{\sqrt{x}} - \sqrt{x}\right)\left(\frac{1}{\sqrt{y}} - \sqrt{y}\right)^{n-1} \geq \left(\sqrt{n} - \frac{1}{\sqrt{n}}\right)^n.
\]
Write this inequality as
\[
n^{n/2}(1 - y)^{n-1} \geq (n - 1)^{n-1} x^{1/2} y^{(n-3)/2}.
\]
By squaring, this inequality becomes as follows:
\[
n^n(1 - y)^{2n-2} \geq (n - 1)^{2n-2} x y^{n-3},
\]
\[
(2 - 2y)^{2n-2} \geq \frac{(2n - 2)^{2n-2}}{n^n} x y^{n-3},
\]
\[
\left[n \cdot \frac{1}{n} + x + (n - 3)y\right]^{2n-2} \geq [n + 1 + (n - 3)]^{n+1+(n-3)} \cdot \frac{1}{n^n} \cdot x \cdot y^{n-3}.
\]
The last inequality follows from the AM-GM inequality. The proof is completed. The equality holds for \( a_1 = a_2 = \cdots = a_n = 1/n \).

**Second Solution.** By the LHCF-Theorem and Note 2, it suffices to prove that \( H(x, y) \geq 0 \) for \( x, y > 0 \) so that \( x + (n - 1)y = 1 \), where
\[
H(x, y) = \frac{f'(x) - f'(y)}{x - y}.
\]
We have
\[
H(x, y) = \frac{1 - x - y - xy}{2xy(1-x)(1-y)} = \frac{n(y + 1) - y - 3}{2x(1-x)(1-y)} \geq \frac{y}{x(1-x)(1-y)} > 0.
\]

**Remark 1.** We may write the inequality in P 1.33 in the form
\[
\prod_{i=1}^{n} \left(\frac{1}{\sqrt{a_i}} - 1\right) \cdot \prod_{i=1}^{n} (1 + \sqrt{a_i}) \geq \left(\sqrt{n} - \frac{1}{\sqrt{n}}\right)^n.
\]
On the other hand, by the AM-GM inequality and the Cauchy-Schwarz inequality, we have

\[
\prod_{i=1}^{n} (1 + \sqrt{a_i}) \leq \left(1 + \frac{1}{n} \sum_{i=1}^{n} \sqrt{a_i}\right)^n \leq \left(1 + \sqrt{\frac{1}{n} \sum_{i=1}^{n} a_i}\right)^n = \left(1 + \frac{1}{\sqrt{n}}\right)^n.
\]

Thus, the following statement follows:

- If \(a_1, a_2, \ldots, a_n\) (\(n \geq 3\)) are positive real numbers so that \(a_1 + a_2 + \cdots + a_n = 1\), then
  \[
  \left(\frac{1}{\sqrt{a_1}} - 1\right) \left(\frac{1}{\sqrt{a_2}} - 1\right) \cdots \left(\frac{1}{\sqrt{a_n}} - 1\right) \geq (\sqrt{n} - 1)^n,
  \]
  with equality for \(a_1 = a_2 = \cdots = a_n = 1/n\).

**Remark 2.** By squaring, the inequality in P 1.33 becomes

\[
\prod_{i=1}^{n} \frac{(1 - a_i)^2}{a_i} \geq \frac{(n-1)^{2n}}{n^n}.
\]

On the other hand, since the function \(f(x) = \ln \frac{1+x}{1-x}\) is convex on \((0, 1)\), by Jensen’s inequality we have

\[
\prod_{i=1}^{n} \left(1 + \frac{a_i}{1-a_i}\right) \geq \left(1 + \frac{a_1 + a_2 + \cdots + a_n}{n} \right)^n = \left(\frac{n+1}{n-1}\right)^n.
\]

Multiplying these inequalities yields the following result (Kee-Wai Lau, 2000):

- If \(a_1, a_2, \ldots, a_n\) (\(n \geq 3\)) are positive real numbers so that \(a_1 + a_2 + \cdots + a_n = 1\), then
  \[
  \left(\frac{1}{a_1} - a_1\right) \left(\frac{1}{a_2} - a_2\right) \cdots \left(\frac{1}{a_n} - a_n\right) \geq \left(n - \frac{1}{n}\right)^n,
  \]
  with equality for \(a_1 = a_2 = \cdots = a_n = 1/n\).

\[\square\]

**P 1.34.** Let \(a_1, a_2, \ldots, a_n\) be positive real numbers so that \(a_1 + a_2 + \cdots + a_n = n\). If

\[
0 < k \leq \left(1 + \frac{2\sqrt{n-1}}{n}\right)^2,
\]

then

\[
\left(ka_1 + \frac{1}{a_1}\right) \left(ka_2 + \frac{1}{a_2}\right) \cdots \left(ka_n + \frac{1}{a_n}\right) \geq (k+1)^n.
\]

(Vasile C., 2006)
**Solution.** Write the inequality as

\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1, \]

where

\[ f(u) = \ln \left( ku + \frac{1}{u} \right), \quad u \in (0, n). \]

We have

\[ f'(u) = \frac{ku^2 - 1}{u(ku^2 + 1)}, \quad f''(u) = \frac{1 + 4ku^2 - k^2u^4}{u^2(ku^2 + 1)^2}. \]

For \( u \in (0, 1] \), we get \( f''(u) > 0 \) since

\[ 1 + 4ku^2 - k^2u^4 > ku^2(4 - ku^2) \geq ku^2(4 - k) \geq 0. \]

Therefore, \( f \) is convex on \((0, s] \). By the LHCF-Theorem and Note 2, it suffices to prove that \( H(x, y) \geq 0 \) for \( x, y > 0 \) so that \( x + (n - 1)y = n \), where

\[ H(x, y) = \frac{f'(x) - f'(y)}{x - y}. \]

Since

\[ H(x, y) = \frac{1 + k(x + y)^2 - k^2x^2y^2}{xy(kx^2 + 1)(ky^2 + 1)} > \frac{k[(x + y)^2 - kx^2y^2]}{xy(kx^2 + 1)(ky^2 + 1)}, \]

it suffices to show that

\[ x + y \geq \sqrt{k} xy. \]

Indeed, by the Cauchy-Schwarz inequality, we have

\[ (x + y)[(n - 1)y + x] \geq (\sqrt{n - 1} + 1)^2xy, \]

hence

\[ x + y \geq \frac{1}{n}(\sqrt{n - 1} + 1)^2xy = \left( 1 + \frac{2\sqrt{n - 1}}{n} \right)xy \geq \sqrt{k} xy. \]

The equality holds for \( a_1 = a_2 = \cdots = a_n = 1. \)


**P 1.35.** If \( a, b, c, d \) are nonzero real numbers so that

\[ a, b, c, d \geq -\frac{1}{2}, \quad a + b + c + d = 4, \]

then

\[ 3 \left( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \right) + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \geq 16. \]
Solution. Write the inequality as

\[ f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a + b + c + d}{4} = 1, \]

where

\[ f(u) = \frac{3}{u^2} + \frac{1}{u}, \quad u \in \mathbb{I} = \left[ -\frac{1}{2}, \frac{1}{2} \right] \setminus \{0\}, \]

is convex on \( \mathbb{I}_{\geq s} \) (because \( 3/u^2 \) and \( 1/u \) are convex). By the RHCF-Theorem, Note 1 and Note 3, it suffices to prove that \( h(x, y) \geq 0 \) for \( x, y \in \mathbb{I} \) so that

\[ x + 3y = 4, \]

where

\[ h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}. \]

Indeed, we have

\[ g(u) = -\frac{4}{u} - \frac{3}{u^2}, \]

\[ h(x, y) = \frac{4xy + 3x + 3y}{x^2y^2} = \frac{2(1 + 2x)(6 - x)}{3x^2y^2} \geq 0. \]

In accordance with Note 4, the equality holds for \( a = b = c = d = 1 \), and also for

\[ a = -\frac{1}{2}, \quad b = c = d = \frac{3}{2} \]

(or any cyclic permutation).

\[ \square \]

**P 1.36.** If \( a_1, a_2, \ldots, a_n \) are nonnegative real numbers so that \( a_1^2 + a_2^2 + \cdots + a_n^2 = n \), then

\[ a_1^3 + a_2^3 + \cdots + a_n^3 - n + \sqrt{\frac{n}{n-1}} (a_1 + a_2 + \cdots + a_n - n) \geq 0. \]

*(Vasile C., 2007)*

**Solution.** Replacing each \( a_i \) by \( \sqrt{a_i} \), we have to prove that

\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \]

where

\[ s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1 \]

and

\[ f(u) = u\sqrt{u} + k\sqrt{u}, \quad k = \sqrt{\frac{n}{n-1}}, \quad u \in [0, n]. \]
For \( u \geq 1 \), we have

\[
  f''(u) = \frac{3u-k}{4u\sqrt{u}} \geq \frac{3-k}{4u\sqrt{u}} > 0.
\]

Therefore, \( f \) is convex on \([s, n]\). According to the RHCF-Theorem and Note 1, it suffices to show that \( h(x, y) \geq 0 \) for \( x, y \geq 0 \) so that \( x + (n-1)y = n \). Since

\[
  g(u) = \frac{f(u) - f(1)}{u-1} = 1 + \frac{u+k}{\sqrt{u}+1}
\]

and

\[
  h(x, y) = \frac{g(x) - g(y)}{x-y} = \frac{\sqrt{x} + \sqrt{y} + \sqrt{xy} - k}{(\sqrt{x} + \sqrt{y})(\sqrt{x} + 1)(\sqrt{y} + 1)},
\]

we need to show that

\[
  \sqrt{x} + \sqrt{y} + \sqrt{xy} \geq k.
\]

Since

\[
  \sqrt{x} + \sqrt{y} + \sqrt{xy} \geq \sqrt{x} + \sqrt{y} \geq \sqrt{x+y},
\]

it suffices to show that

\[
  x + y \geq k^2.
\]

Indeed, we have

\[
  x + y \geq \frac{x}{n-1} + y = \frac{n}{n-1} = k^2.
\]

In accordance with Note 4, the equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for

\[
  a_1 = 0, \quad a_2 = \cdots = a_n = \sqrt{\frac{n}{n-1}}
\]

(or any cyclic permutation).

---

**P 1.37.** If \( a, b, c, d, e \) are nonnegative real numbers so that \( a^2 + b^2 + c^2 + d^2 + e^2 = 5 \), then

\[
  \frac{1}{7-2a} + \frac{1}{7-2b} + \frac{1}{7-2c} + \frac{1}{7-2d} + \frac{1}{7-2e} \leq 1.
\]

*(Vasile C., 2010)*

**Solution.** Replacing \( a, b, c, d, e \) by \( \sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}, \sqrt{e} \), we have to prove that

\[
  f(a) + f(b) + f(c) + f(d) + f(e) \geq 5f(s),
\]

where

\[
  s = \frac{a + b + c + d + e}{5} = 1
\]

and

\[
  f(u) = \frac{1}{2u - 7}, \quad u \in [0, 5].
\]
For \( u \in [0, 1] \), we have

\[
f''(u) = \frac{7 - 6\sqrt{u}}{2u\sqrt{u}(7 - 2\sqrt{u})^3} > 0.
\]

Therefore, \( f \) is convex on \([0, s]\). According to the LHCF-Theorem and Note 1, it suffices to show that \( h(x, y) \geq 0 \) for \( x, y \geq 0 \) so that \( x + 4y = 5 \). Since

\[
g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-2}{5(7 - 2\sqrt{u})(1 + \sqrt{u})}
\]
and

\[
h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{2(5 - 2\sqrt{x} - 2\sqrt{y})}{(\sqrt{x} + \sqrt{y})(1 + \sqrt{x})(1 + \sqrt{y})(7 - 2\sqrt{x})(7 - 2\sqrt{y})},
\]
we need to show that

\[
\sqrt{x} + \sqrt{y} \leq \frac{5}{2}.
\]

Indeed, by the Cauchy-Schwarz inequality, we have

\[
(\sqrt{x} + \sqrt{y})^2 \leq \left(1 + \frac{1}{4}\right)(x + 4y) = \frac{25}{4}.
\]

The proof is completed. The equality holds for \( a = b = c = d = e = 1 \), and also for

\[
a = 2, \quad b = c = d = e = \frac{1}{2}
\]
(or any cyclic permutation).

**Remark** In the same manner, we can prove the following generalization:

- Let \( a_1, a_2, \ldots, a_n \) be nonnegative real numbers so that \( a_1^2 + a_2^2 + \cdots + a_n^2 = n \). If \( k \geq 1 + \frac{n}{\sqrt{n-1}} \), then

\[
\frac{1}{k - a_1} + \frac{1}{k - a_2} + \cdots + \frac{1}{k - a_n} \leq \frac{n}{k - 1},
\]

with equality for \( a_1 = a_2 = \cdots = a_n = 1 \). If \( k = 1 + \frac{n}{\sqrt{n-1}} \), then the equality holds also for

\[
a_1 = \sqrt{n-1}, \quad a_2 = \cdots = a_n = \frac{1}{\sqrt{n-1}}
\]
(or any cyclic permutation).
**P 1.38.** Let $0 \leq a_1, a_2, \ldots, a_n < k$ so that $a_1^2 + a_2^2 + \cdots + a_n^2 = n$. If

$$1 < k \leq 1 + \sqrt{\frac{n}{n-1}},$$

then

$$1 \frac{1}{k-a_1} + \frac{1}{k-a_2} + \cdots + \frac{1}{k-a_n} \geq \frac{n}{k-1}.$$  

(Vasile C., 2010)

**Solution.** Replacing $a_1, a_2, \ldots, a_n$ by $\sqrt{a_1}, \sqrt{a_2}, \ldots, \sqrt{a_n}$, we have to prove that

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s),$$

where

$$s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1$$

and

$$f(u) = \frac{1}{k-\sqrt{u}}, \quad u \in [0,k^2).$$

From

$$f''(u) = \frac{3\sqrt{u} - k}{4u\sqrt{u}(k - \sqrt{u})^3},$$

it follows that $f$ is convex on $[s,k^2)$. According to the RHCF-Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for all $x, y \in [0,k^2)$ so that $x + (n-1)y = n$. Since

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{1}{(k-1)(k-\sqrt{u})(1+\sqrt{u}}$$

and

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{\sqrt{x} + \sqrt{y} + 1 - k}{(k-1)(\sqrt{x} + \sqrt{y})(1+\sqrt{x})(1+\sqrt{y})(k - \sqrt{x})(k - \sqrt{y})},$$

we need to show that

$$\sqrt{x} + \sqrt{y} \geq k - 1.$$ 

Indeed,

$$\sqrt{x} + \sqrt{y} \geq \sqrt{x + y} \geq \sqrt{\frac{x}{n-1} + y} = \sqrt{\frac{n}{n-1}} \geq k - 1.$$ 

The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = 0, \quad a_2 = \cdots = a_n = \sqrt{\frac{n}{n-1}}$$

(or any cyclic permutation). \qed
P 1.39. If \(a, b, c\) are nonnegative real numbers, no two of which are zero, then

\[
\sqrt{1 + \frac{48a}{b+c}} + \sqrt{1 + \frac{48b}{c+a}} + \sqrt{1 + \frac{48c}{a+b}} \geq 15.
\]

(Vasile C., 2005)

Solution. Due to homogeneity, we may assume that \(a + b + c = 1\). Thus, we need to show that

\[
f(a) + f(b) + f(c) \geq 3f\left(\frac{1}{3}\right),
\]

where

\[
s = \frac{a + b + c}{3} = \frac{1}{3}
\]

and

\[
f(u) = \sqrt{\frac{1+47u}{1-u}}, \quad u \in [0, 1).
\]

From

\[
f''(u) = \frac{48(47u - 11)}{\sqrt{(1-u)^3(1+47u)^3}},
\]

it follows that \(f\) is convex on \([s, 1]\). By the RHCF-Theorem, it suffices to show that

\[
f(x) + 2f(y) \geq 3f\left(\frac{1}{3}\right)
\]

for \(x, y \geq 0\) so that \(x + 2y = 1\); that is,

\[
\sqrt{\frac{1+47x}{1-x}} + 2\sqrt{\frac{49-47x}{1+x}} \geq 15.
\]

Setting

\[
t = \sqrt{\frac{49-47x}{1+x}}, \quad 1 < t \leq 7,
\]

the inequality turns into

\[
\sqrt{\frac{1175-23t^2}{t^2-1}} \geq 15 - 2t.
\]

By squaring, this inequality becomes

\[
350 - 15t - 61t^2 + 15t^3 - t^4 \geq 0,
\]

\[
(5-t)^2(2+t)(7-t) \geq 0.
\]

The original inequality is an equality for \(a = b = c\), and also for \(a = 0\) and \(b = c\) (or any cyclic permutation). \(\square\)
**P 1.40.** If \(a, b, c\) are nonnegative real numbers, then

\[
\sqrt{\frac{3a^2}{7a^2 + 5(b + c)^2}} + \sqrt{\frac{3b^2}{7b^2 + 5(c + a)^2}} + \sqrt{\frac{3c^2}{7c^2 + 5(a + b)^2}} \leq 1.
\]

(Vasile C., 2008)

**Solution.** Due to homogeneity, we may assume that \(a + b + c = 3\). Thus, we need to show that

\[
f(a) + f(b) + f(c) \geq 3f(s),
\]

where

\[
s = \frac{a + b + c}{3} = 1
\]

and

\[
f(u) = -\sqrt{\frac{3u^2}{7u^2 + 5(3 - u)^2}} = \frac{-u}{\sqrt{4u^2 - 10u + 15}}, \quad u \in [0, 3].
\]

From

\[
f''(u) = \frac{5(-8u^2 + 41u - 30)}{(4u^2 - 10u + 15)^{5/2}} \geq \frac{5(-8u^2 + 38u - 30)}{(4u^2 - 10u + 15)^{5/2}} = \frac{10(u - 1)(15 - 4u)}{(4u^2 - 10u + 15)^{5/2}},
\]

it follows that \(f\) is convex on \([s, 3]\). By the RHCF-Theorem, it suffices to prove the original homogeneous inequality for \(b = c = 0\) and \(b = c = 1\). For the nontrivial case \(b = c = 1\), we need to show that

\[
\sqrt{\frac{3a^2}{7a^2 + 20}} + 2\sqrt{\frac{3}{5a^2 + 10a + 12}} \leq 1.
\]

By squaring two times, the inequality becomes

\[
a(5a^3 + 10a^2 + 16a + 50) \geq 3a \sqrt{(7a^2 + 20)(5a^2 + 10a + 12)},
\]

\[
a^2(5a^6 + 20a^5 - 11a^4 + 38a^3 - 80a^2 - 40a + 68) \geq 0,
\]

\[
a^2(a - 1)^2(5a^4 + 30a^3 + 44a^2 + 96a + 68) \geq 0.
\]

The last inequality is clearly true.

The equality holds for \(a = b = c\), and also for \(a = 0\) and \(b = c\) (or any cyclic permutation).

\(\square\)

**P 1.41.** If \(a, b, c\) are nonnegative real numbers, then

\[
\sqrt{\frac{a^2}{a^2 + 2(b + c)^2}} + \sqrt{\frac{b^2}{b^2 + 2(c + a)^2}} + \sqrt{\frac{c^2}{c^2 + 2(a + b)^2}} \geq 1.
\]

(Vasile C., 2008)
Solution. Due to homogeneity, we may assume that \(a + b + c = 3\). Thus, we need to show that
\[
 f(a) + f(b) + f(c) \geq 3f(s),
\]
where
\[
 s = \frac{a + b + c}{3} = 1
\]
and
\[
 f(u) = \sqrt[3]{\frac{3u^2}{u^2 + 2(3-u)^2}} = \frac{u}{\sqrt{u^2 - 4u + 6}}, \quad u \in [0,3].
\]
From
\[
 f''(u) = \frac{2(2u^2 - 11u + 12)}{(u^2 - 4u + 6)^{5/2}} \geq \frac{2(-11u + 12)}{(u^2 - 4u + 6)^{5/2}},
\]
it follows that \(f\) is convex on \([0,s]\). By the LHCF-Theorem, it suffices to prove the original homogeneous inequality for \(b = c = 0\) and \(b = c = 1\). For the nontrivial case \(b = c = 1\), the inequality has the form
\[
 \frac{a}{\sqrt{a^2 + 8}} + \frac{2}{\sqrt{2a^2 + 4a + 3}} \geq 1.
\]
By squaring, the inequality becomes
\[
 a\sqrt{(a^2 + 8)(2a^2 + 4a + 3)} \geq 3a^2 + 8a - 2.
\]
For the nontrivial case \(3a^2 + 8a - 2 > 0\), by squaring both sides we get
\[
 a^6 + 2a^5 + 5a^4 - 8a^3 - 14a^2 + 16a - 2 \geq 0,
\]
\[
 (a - 1)^2[a^4 + 4a^3 + 9a^2 + 4a + (3a^2 + 8a - 2)] \geq 0.
\]
The equality holds for \(a = b = c\), and also for \(b = c = 0\) (or any cyclic permutation).
\[\square\]

P 1.42. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. If
\[
 k \geq k_0, \quad k_0 = \frac{\ln 3}{\ln 2} - 1 \approx 0.585,
\]
then
\[
 \left(\frac{2a}{b+c}\right)^k + \left(\frac{2b}{c+a}\right)^k + \left(\frac{2c}{a+b}\right)^k \geq 3.
\]
\[\text{(Vasile C., 2005)}\]
**Solution.** For \( k = 1 \), the inequality is just the well known Nesbitt’s inequality

\[
\frac{2a}{b+c} + \frac{2b}{c+a} + \frac{2c}{a+b} \geq 3.
\]

For \( k \geq 1 \), the inequality follows from Nesbitt’s inequality and Jensens’s inequality applied to the convex function \( f(u) = u^k \):

\[
\left( \frac{2a}{b+c} \right)^k + \left( \frac{2b}{c+a} \right)^k + \left( \frac{2c}{a+b} \right)^k \geq 3 \left( \frac{\frac{2a}{b+c} + \frac{2b}{c+a} + \frac{2c}{a+b}}{3} \right)^k \geq 3.
\]

Consider now that

\( k_0 \leq k < 1 \).

Due to homogeneity, we may assume that \( a + b + c = 1 \). Thus, we need to show that

\[
f(a) + f(b) + f(c) \geq 3f(s),
\]

where

\[
s = \frac{a + b + c}{3} = \frac{1}{3}
\]

and

\[
f(u) = \left( \frac{2u}{1-u} \right)^k, \quad u \in [0, 1).
\]

From

\[
f''(u) = \frac{4k}{(1-u)^4} \left( \frac{2u}{1-u} \right)^{k-2} (2u + k - 1),
\]

it follows that \( f \) is convex on \([s, 1)\) (because \( u \geq s = 1/3 \) involves \( 2u + k - 1 \geq 2/3 + k - 1 = k - 1/3 > 0 \)). By the RHCF-Theorem, it suffices to prove the original homogeneous inequality for \( b = c = 1 \) and \( a \in [0, 1] \); that is, to show that \( h(a) \geq 3 \), where

\[
h(a) = a^k + 2 \left( \frac{2}{a+1} \right)^k, \quad a \in [0, 1].
\]

For \( a \in (0, 1] \), the derivative

\[
h'(a) = ka^{k-1} - k \left( \frac{2}{a+1} \right)^{k+1}
\]

has the same sign as

\[
g(a) = (k-1) \ln a - (k+1) \ln \frac{2}{a+1}.
\]

From

\[
g'(a) = \frac{2ka + k - 1}{a(a+1)},
\]
it follows that \( g'(a_0) = 0 \) for \( a_0 = (1 - k)/(2k) < 1 \), \( g'(a) < 0 \) for \( a \in (0, a_0) \) and \( g'(a) > 0 \) for \( a \in (a_0, 1] \). Consequently, \( g \) is strictly decreasing on \( (0, a_0] \) and strictly increasing on \( (a_0, 1] \). Since \( g(0) = \infty \) and \( g(1) = 0 \), there exists \( a_1 \in (0, a_0) \) so that \( g(a_1) = 0 \), \( g(a) > 0 \) for \( a \in (0, a_1) \) and \( g(a) < 0 \) for \( a \in (a_1, 1) \); therefore, \( h(a) \) is strictly increasing on \( [0, a_1] \) and strictly decreasing on \( [a_1, 1] \). As a result,

\[
h(a) \geq \min\{h(0), h(1)\}.
\]

Since \( h(0) = 2^{k+1} \geq 3 \) and \( h(1) = 3 \), we get \( h(a) \geq 3 \). The proof is completed. The equality holds for \( a = b = c \). If \( k = k_0 \), then the equality holds also for \( a = 0 \) and \( b = c \) (or any cyclic permutation).

**Remark.** For \( k = 2/3 \), we can give the following solution (based on the AM-GM inequality):

\[
\sum \left( \frac{2a}{b+c} \right)^{2/3} = \sum \frac{2a}{\sqrt[3]{2a \cdot (b+c) \cdot (b+c)}} \geq \sum \frac{6a}{2a + (b+c) + (b+c)} = 3.
\]

\[\Box\]

**P 1.43.** If \( a, b, c \in [1, 7 + 4\sqrt{3}] \), then

\[
\sqrt{\frac{2a}{b+c}} + \sqrt{\frac{2b}{c+a}} + \sqrt{\frac{2c}{a+b}} \geq 3.
\]

*(Vasile C., 2007)*

**Solution.** Denoting

\[
s = \frac{a + b + c}{3}, \quad 1 \leq s \leq 7 + 4\sqrt{3},
\]

we need to show that

\[
f(a) + f(b) + f(c) \geq 3f(s),
\]

where

\[
f(u) = \sqrt{\frac{2u}{3s-u}}, \quad 1 \leq u < 3s.
\]

For \( u \geq s \), we have

\[
f''(u) = 3s \left( \frac{3s-u}{2u} \right)^{3/2} \frac{4u-3s}{(3s-u)^3} > 0.
\]
Therefore, $f(u)$ is convex for $u \geq s$. By the RHCF-Theorem, it suffices to prove the original inequality for $b = c$; that is,

$$\sqrt{\frac{a}{b}} + 2\sqrt{\frac{2b}{a+b}} \geq 3.$$ 

Putting $t = \sqrt{\frac{b}{a}}$, the condition $a, b \in [1, 7 + 4\sqrt{3}]$ involves

$$2 - \sqrt{3} \leq t \leq 2 + \sqrt{3}.$$ 

We need to show that

$$2\sqrt{\frac{2t^2}{t^2+1}} \geq 3 - \frac{1}{t}.$$ 

This is true if

$$\frac{8t^2}{t^2+1} \geq \left(3 - \frac{1}{t}\right)^2,$$

which is equivalent to the obvious inequality

$$(t - 1)^2(t - 2 + \sqrt{3})(t - 2 - \sqrt{3}) \leq 0.$$ 

The equality holds for $a = b = c$, and also for $a = 1$, and $b = c = 7 + 4\sqrt{3}$ (or any cyclic permutation). 

\[\square\]

**P 1.44.** Let $a, b, c$ be nonnegative real numbers so that $a + b + c = 3$. If

$$0 < k \leq k_0, \quad k_0 = \frac{\ln 2}{\ln 3 - \ln 2} \approx 1.71,$$

then

$$a^k(b+c) + b^k(c+a) + c^k(a+b) \leq 6.$$ 

**Solution.** For $0 < k \leq 1$, the inequality follows from Jensens’s inequality applied to the convex function $f(u) = -u^k$:

$$(b+c)a^k + (c+a)b^k + (a+b)c^k \leq 2(a + b + c)\left[\frac{(b+c)a+(c+a)b+(a+b)c}{2(a+b+c)}\right]^k$$

$$= 6\left(\frac{ab + bc + ca}{3}\right)^k \leq 6\left(\frac{a+b+c}{3}\right)^{2k} = 6.$$ 

Consider now that

$$1 < k \leq k_0,$$
and write the inequality as 
\[ f(a) + f(b) + f(c) \geq 3f(s), \]
where 
\[ s = \frac{a + b + c}{3} = 1 \]
and 
\[ f(u) = u^k(u - 3), \quad u \in [0, 3]. \]
For \( u \geq 1 \), we have 
\[ f''(u) = ku^{k-2}[(k + 1)u - 3k + 3] \geq ku^{k-2}[(k + 1) - 3k + 3] = 2k(2 - k)u^{k-2} > 0; \]
therefore, \( f \) is convex on \([1, s]\). By the RHCF-Theorem, it suffices to consider the case \( a \leq b = c \). So, we only need to prove the homogeneous inequality
\[ a^k(b + c) + b^k(c + a) + c^k(a + b) \leq 6\left(\frac{a + b + c}{3}\right)^{k+1} \]
for \( b = c = 1 \) and \( a \in [0, 1] \); that is, to show that \( g(a) \geq 0 \) for \( a \geq 0 \), where
\[ g(a) = 3\left(\frac{a + 2}{3}\right)^{k+1} - a^k - a - 1. \]
We have
\[ g'(a) = (k + 1)\left(\frac{a + 2}{3}\right)^k - ka^{k-1} - 1, \quad \frac{1}{k}g''(a) = \frac{k + 1}{3} \left(\frac{a + 2}{3}\right)^{k-1} - \frac{k - 1}{a^{2-k}}. \]
Since \( g'' \) is strictly increasing, \( g''(0) = -\infty \) and \( g''(1) = 2k(2 - k)/3 > 0 \), there exists \( a_1 \in (0, 1) \) so that \( g''(a_1) = 0 \), \( g''(a) < 0 \) for \( a \in (0, a_1) \), \( g''(a) > 0 \) for \( a \in (a_1, 1) \). Therefore, \( g' \) is strictly decreasing on \([0, a_1]\) and strictly increasing on \([a_1, 1]\). Since
\[ g'(0) = (k + 1)(2/3)^k - 1 \geq (k + 1)(2/3)^{k_0} - 1 = \frac{k + 1}{2} - 1 = \frac{k - 1}{2} > 0, \]
\[ g'(1) = 0, \]
there exists \( a_2 \in (0, a_1) \) so that \( g'(a_2) = 0 \), \( g'(a) > 0 \) for \( a \in [0, a_2] \), \( g'(a) < 0 \) for \( a \in (a_2, 1] \). Thus, \( g \) is strictly increasing on \([0, a_2]\) and strictly decreasing on \([a_2, 1]\); consequently,
\[ g(a) \geq \min\{g(0), g(1)\}. \]
From
\[ g(0) = 3(2/3)^{k+1} - 1 \geq 3(2/3)^{k_0+1} - 1 = 1 - 1 = 0, \quad g(1) = 0, \]
we get \( g(a) \geq 0 \). This completes the proof. The equality holds for \( a = b = c = 1 \). If \( k = k_0 \), then the equality holds also for \( a = 0 \) and \( b = c = 3/2 \) (or any cyclic permutation).

**Remark 1.** Using the Cauchy-Schwarz inequality and the inequality in P 1.44, we get
\[
\sum \frac{a}{b^k + c^k} \geq \frac{(a + b + c)^2}{\sum a(b^k + c^k)} = \frac{9}{\sum a^k(b + c)} \geq \frac{3}{2}.
\]
Thus, the following statement holds:

- Let \( a, b, c \) be nonnegative real numbers so that \( a + b + c = 3 \). If
  \[0 < k \leq k_0, \quad k_0 = \frac{\ln 2}{\ln 3 - \ln 2} \approx 1.71,\]
  then
  \[
  \frac{a}{b^k + c^k} + \frac{b}{c^k + a^k} + \frac{c}{a^k + b^k} \geq \frac{3}{2},
  \]
with equality for \( a = b = c = 1 \). If \( k = k_0 \), then the equality holds also for \( a = 0 \) and \( b = c = 3/2 \) (or any cyclic permutation).

**Remark 2.** Also, the following statement holds:

- Let \( a, b, c \) be nonnegative real numbers so that \( a + b + c = 3 \). If
  \[k \geq k_1, \quad k_1 = \frac{\ln 9 - \ln 8}{\ln 3 - \ln 2} \approx 0.2905,\]
  then
  \[
  \frac{a^k}{b + c} + \frac{b^k}{c + a} + \frac{c^k}{a + b} \geq \frac{3}{2},
  \]
with equality for \( a = b = c = 1 \). If \( k = k_1 \), then the equality holds also for \( a = 0 \) and \( b = c = 3/2 \) (or any cyclic permutation).

For \( k_1 \leq k \leq 2 \), the inequality can be proved using the Cauchy-Schwarz inequality and the inequality in P 1.44, as follows:
\[
\sum \frac{a^k}{b + c} \geq \frac{(a + b + c)^2}{\sum a^{2-k}(b + c)} = \frac{9}{\sum a^{2-k}(b + c)} \geq \frac{3}{2}.
\]

For \( k \geq 2 \), the inequality can be deduced from the Cauchy-Schwarz inequality and Bernoulli’s inequality, as follows:
\[
\sum \frac{a^k}{b + c} \geq \frac{(\sum a^{k/2})^2}{\sum (b + c)} = \frac{(\sum a^{k/2})^2}{6},
\]
\[
\sum a^{k/2} \geq \sum \left[1 + \frac{k}{2}(a - 1)\right] = 3.
\]
\[\square\]
\textbf{P 1.45.} If \(a, b, c\) are nonnegative real numbers so that \(a + b + c = 3\), then

\[
\sqrt{a} + \sqrt{b} + \sqrt{c} - 3 \geq 13 \left( \sqrt{\frac{a + b}{2}} + \sqrt{\frac{b + c}{2}} + \sqrt{\frac{c + a}{2}} - 3 \right).
\]

(Vasile C., 2008)

\textbf{Solution.} Write the inequality as

\[
f(a) + f(b) + f(c) \geq 3f(s), \quad s = \frac{a + b + c}{3} = 1,
\]

where

\[
f(u) = \sqrt{u} - 13 \sqrt{\frac{3-u}{2}}, \quad u \in [0, 3].
\]

For \(u \in [1, 3]\), we have

\[
4f''(u) = -u^{-3/2} + \frac{13}{4} \left( \frac{3-u}{2} \right)^{-3/2} \geq -1 + \frac{13}{4} > 0.
\]

Therefore, \(f\) is convex on \([s, 3]\). By the RHCF-Theorem, it suffices to consider only the case \(a \leq b = c\). Write the original inequality in the homogeneous form

\[
\sqrt{a} + \sqrt{b} + \sqrt{c} - 3 \sqrt{\frac{a + b + c}{3}} \geq 13 \left( \sqrt{\frac{a + b}{2}} + \sqrt{\frac{b + c}{2}} + \sqrt{\frac{c + a}{2}} - 3 \sqrt{\frac{a + b + c}{3}} \right).
\]

Due to homogeneity, we may assume that \(b = c = 1\). Moreover, it is convenient to use the notation \(\sqrt{a} = x\). Thus, we need to show that \(g(x) \geq 0\) for \(x \in [0, 1]\), where

\[
g(x) = x - 11 + 36 \sqrt{\frac{x^2 + 2}{3}} - 26 \sqrt{\frac{x^2 + 1}{2}}.
\]

We have

\[
g'(x) = 1 + 12x \sqrt{\frac{3}{x^2 + 2} - 13x} \sqrt{\frac{2}{x^2 + 1}},
\]

\[
g''(x) = \frac{13}{2} \left( \frac{2}{x^2 + 1} \right)^{3/2} \left[ \left( m \cdot \frac{x^2 + 1}{x^2 + 2} \right)^{3/2} - 1 \right],
\]

where

\[
m = \frac{6 \sqrt{52}}{13} \approx 1.72.
\]

Clearly, \(g''(x)\) has the same sign as \(h(x)\), where

\[
h(x) = m \cdot \frac{x^2 + 1}{x^2 + 2} - 1.
\]
Since $h$ is strictly increasing,

$$h(0) = \frac{m}{2} - 1 < 0, \quad h(1) = \frac{2m}{3} - 1 > 0,$$

there is $x_1 \in (0, 1)$ so that $h(x_1) = 0$, $h(x) < 0$ for $x \in [0, x_1)$ and $h(x) > 0$ for $x \in (x_1, 1]$. Therefore, $g'$ is strictly decreasing on $[0, x_1]$ and strictly increasing on $[x_1, 1]$. Since $g'(0) = 1$ and $g'(1) = 0$, there is $x_2 \in (0, x_1)$ so that $g'(x_2) = 0$, $g'(x) > 0$ for $x \in (0, x_2)$ and $g'(x) < 0$ for $x \in (x_2, 1)$. Thus, $g(x)$ is strictly increasing on $[0, x_2]$ and strictly decreasing on $[x_2, 1]$. Therefore, $g$ is strictly decreasing on $[0, x_1]$ and strictly increasing on $[x_1, 1]$. From

$$g(0) = -11 + 12\sqrt{6} - 13\sqrt{2} > 0$$

and $g(1) = 0$, it follows that $g(x) \geq 0$ for $x \in [0, 1]$. This completes the proof. The equality holds for $a = b = c = 1$.

**Remark.** Similarly, we can prove the following generalizations:

- Let $a, b, c$ be nonnegative real numbers so that $a + b + c = 3$. If $k \geq k_0$, where

$$k_0 = \frac{\sqrt{6} - 2}{\sqrt{6} - \sqrt{2} - 1} = (2 + \sqrt{2})(2 + \sqrt{3}) \approx 12.74,$$

then

$$\sqrt{a} + \sqrt{b} + \sqrt{c} - 3 \geq k \left( \sqrt{\frac{a + b}{2}} + \sqrt{\frac{b + c}{2}} + \sqrt{\frac{c + a}{2}} - 3 \right),$$

with equality for $a = b = c = 1$. If $k = k_0$, then the equality holds also for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

- Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If $k \geq k_0$, where

$$k_0 = \frac{\sqrt{n} - \sqrt{n - 1}}{\sqrt{n} - \sqrt{n - 2} - \frac{1}{\sqrt{n - 1}}},$$

then

$$\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n} - n \geq k \left( \sqrt{\frac{n - a_1}{n - 1}} + \sqrt{\frac{n - a_2}{n - 1}} + \cdots + \sqrt{\frac{n - a_n}{n - 1}} - n \right),$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$. If $k = k_0$, then the equality holds also for $a_1 = 0$ and $a_2 = a_3 = \cdots = a_n = \frac{n}{n - 1}$ (or any cyclic permutation).
If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then
\[
\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n} + n(k-1) \leq k \left( \sqrt{\frac{n-a_1}{n-1}} + \sqrt{\frac{n-a_2}{n-1}} + \cdots + \sqrt{\frac{n-a_n}{n-1}} \right),
\]
where
\[
k = (\sqrt{n} - 1)(\sqrt{n} + \sqrt{n-1}).
\]

**Solution.** For $n = 2$, the inequality is an identity. Consider further that $n \geq 3$. We will show first that
\[
n - 1 < k < 2(n - 1).
\]
The left inequality reduces to
\[
(\sqrt{n} - 1)(\sqrt{n} - 1 - 1) > 0,
\]
while the right inequality is equivalent to
\[
(\sqrt{n} - 1)(\sqrt{n} - \sqrt{n-1} + 2) > 0.
\]
Write the inequality as
\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,
\]
where
\[
f(u) = -u + k \sqrt{\frac{n-u}{n-1}}, \quad u \in [0,n].
\]
For $u \leq 1$, we have
\[
4f''(u) = u^{-3/2} - \frac{k}{\sqrt{n-1}}(n-u)^{-3/2} \geq 1 - \frac{k}{\sqrt{n-1}}(n-1)^{-3/2}
\]
\[
= 1 - \frac{k}{(n-1)^2} \geq 1 - \frac{k}{2(n-1)} > 0.
\]
Therefore, $f$ is convex on $[0,s]$. By the LHCF-Theorem, it suffices to consider the case
\[
a_1 \geq a_2 = \cdots = a_n.
\]
Write the original inequality in the homogeneous form
\[
\sum \sqrt{a_1} + n(k-1) \sqrt[2]{\frac{a_1 + a_2 + \cdots + a_n}{n}} \leq k \sum \sqrt[2]{\frac{a_2 + \cdots + a_n}{n-1}}.
\]
Do to homogeneity, we need to prove this inequality for $a_2 = \cdots = a_n = 1$ and $\sqrt{a_1} = x \geq 1$; that is, to show that $g(x) \leq 0$ for $x \geq 1$, where
\[
g(x) = x + n - 1 - k + (k-1)\sqrt{n(x^2 + n-1)} - k\sqrt{(n-1)(x^2 + n-2)}.
\]
We have
\[ g'(x) = 1 + (k-1)\sqrt{\frac{nx^2}{x^2+n-1}} - k\sqrt{\frac{(n-1)x^2}{x^2+n-2}}, \]
\[ g''(x) = \frac{k(n-2)\sqrt{n-1}}{(x^2+n-2)^{3/2}} \left[ \left( \frac{m \cdot x^2 + n - 2}{x^2 + n - 1} \right)^{3/2} - 1 \right], \]
where
\[ m = \sqrt{\frac{(k-1)^2n(n-1)}{k^2(n-2)^2}}. \]

Clearly, \( g''(x) \) has the same sign as \( h(x) \), where
\[ h(x) = \frac{m(x^2 + n - 2)}{x^2 + n - 1} - 1 = m \left( 1 - \frac{1}{x^2 + n - 1} \right) - 1. \]

We have
\[ h(1) = \frac{m(n-1)}{n} - 1, \quad \lim_{x \to \infty} h(x) = m - 1. \]

We will show that \( h(1) < 0 \) and \( \lim_{x \to \infty} h(x) > 0 \); that is, to show that
\[ 1 < m < \frac{n}{n - 1}. \]

The inequality \( m > 1 \) is equivalent to
\[ 1 - \frac{1}{k} > \frac{n-2}{\sqrt{n(n-1)}}, \]
which is true since
\[ 1 - \frac{1}{k} > 1 - \frac{1}{n-1} = \frac{n-2}{n-1} > \frac{n-2}{\sqrt{n(n-1)}}. \]

The inequality \( m < \frac{n}{n - 1} \) is equivalent to
\[ 1 - \frac{1}{k} < \frac{n(n-2)}{(n-1)^2}, \]
which is also true because
\[ 1 - \frac{1}{k} < 1 - \frac{1}{2(n-1)} = \frac{2n-3}{2(n-1)} \leq \frac{n(n-2)}{(n-1)^2}. \]

Since \( h \) is strictly increasing on \([1, \infty)\), \( h(1) < 0 \) and \( \lim_{x \to \infty} h(x) > 0 \), there is \( x_1 \in (1, \infty) \) so that \( h(x_1) = 0 \); \( h(x) < 0 \) for \( x \in [1, x_1] \) and \( h(x) > 0 \) for \( x \in (x_1, \infty) \). Therefore, \( g' \) is strictly decreasing on \([1, x_1]\) and strictly increasing on \([x_1, \infty)\). Since \( g'(1) = 0 \) and \( \lim_{x \to \infty} g'(x) = 0 \), it follows that \( g'(x) < 0 \) for \( x \in (1, \infty) \). Thus, \( g(x) \) is strictly decreasing on \([1, \infty)\), hence \( g(x) \leq g(1) = 0 \).
The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for
\[
a_1 = n, \quad a_2 = a_3 = \cdots = a_n = 0
\]
(or any cyclic permutation).

**Remark.** Since \( k > n - 1 \) for \( n \geq 3 \), the inequality in P 1.46 is sharper than Popoviciu's inequality applied to the convex function \( f(x) = -\sqrt{x}, \quad x \geq 0 \):
\[
\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n} + n(n - 2) \leq (n - 1) \left( \sqrt{\frac{n - a_1}{n - 1}} + \sqrt{\frac{n - a_2}{n - 1}} + \cdots + \sqrt{\frac{n - a_n}{n - 1}} \right).
\]

**P 1.47.** Let \( a, b, c \) be nonnegative real numbers so that \( a + b + c = 3 \). If \( k > 2 \), then
\[
a^k + b^k + c^k + 3 \geq 2 \left( \frac{a + b}{2} \right)^k + 2 \left( \frac{b + c}{2} \right)^k + 2 \left( \frac{c + a}{2} \right)^k.
\]

**Solution.** Write the inequality as
\[
f(a) + f(b) + f(c) \geq 3f(s), \quad s = \frac{a + b + c}{3} = 1,
\]
where
\[
f(u) = u^k - 2 \left( \frac{3 - u}{2} \right)^k, \quad u \in [0, 3].
\]
For \( u \geq 1 \), we have
\[
\frac{f''(u)}{k(k - 1)} = u^{k-2} - \frac{1}{2} \left( \frac{3 - u}{2} \right)^{k-2} \geq 1 - \frac{1}{2} > 0.
\]
Therefore, \( f \) is convex on \([s, 3]\). By the RHCF-Theorem, it suffices to consider only the case \( a \leq b = c \). Write the original inequality in the homogeneous form
\[
a^k + b^k + c^k + 3 \left( \frac{a + b + c}{3} \right)^k \geq 2 \left( \frac{a + b}{2} \right)^k + 2 \left( \frac{b + c}{2} \right)^k + 2 \left( \frac{c + a}{2} \right)^k.
\]
Due to homogeneity, we may assume that \( b = c = 1 \). Thus, we need to prove that
\[
a^k + 3 \left( \frac{a + 1}{3} \right)^k \geq 4 \left( \frac{a + 1}{2} \right)^k
\]
for \( a \in [0, 1] \). Substituting
\[
a^k = t, \quad t \in [0, 1],
\]
we need to show that \( g(t) \geq 0 \), where
\[
g(t) = t + 3 \left( \frac{t^{1/k} + 2}{3} \right)^k - 4 \left( \frac{t^{1/k} + 1}{2} \right)^k.
\]

We have
\[
g'(t) = 1 + t^{1/k - 1} \left( \frac{t^{1/k} + 2}{3} \right)^{k-1} - 2 t^{1/k - 1} \left( \frac{t^{1/k} + 1}{2} \right)^{k-1},
\]
\[
\frac{kt^{2-1/k}}{k-1} g''(t) = \left( \frac{t^{1/k} + 1}{2} \right)^{k-2} - 2 \left( \frac{t^{1/k} + 2}{3} \right)^{k-2}.
\]

Setting
\[
m = \left( \frac{2}{3} \right)^{\frac{1}{k-2}}, \quad 0 < m < 1,
\]
we see that \( g''(t) \) has the same sign as \( h(t) \), where
\[
h(t) = 6 \left( \frac{t^{1/k} + 1}{2} - m \frac{t^{1/k} + 2}{3} \right) = (3 - 2m) t^{1/k} + 3 - 4m
\]
is strictly increasing. There are two cases to consider: \( 0 < m \leq 3/4 \) and \( 3/4 < m < 1 \).

**Case 1**: \( 0 < m \leq 3/4 \). Since \( h(0) = 3 - 4m \geq 0 \), we have \( h(t) > 0 \) for \( t \in (0,1) \), hence \( g' \) is strictly increasing on \( (0,1) \). From \( g'(1) = 0 \), it follows that \( g'(t) < 0 \) for \( t \in (0,1) \), hence \( g \) is strictly decreasing on \( [0,1] \). Since \( g(1) = 0 \), we get \( g(t) > 0 \) for \( t \in [0,1] \).

**Case 2**: \( 3/4 < m < 1 \). From \( m > 3/4 \), we get
\[
2^{2k-3} > 3^{k-1}.
\]
Since \( h(0) = 3 - 4m < 0 \) and \( h(1) = 3(1 - m) > 0 \), there is \( t_1 \in (0,1) \) so that \( h(t_1) = 0 \), \( h(t) < 0 \) for \( t \in [0,t_1) \) and \( h(t) > 0 \) for \( t \in (t_1,1] \). Thus, \( g'(t) \) is strictly decreasing on \( (0,t_1] \) and strictly increasing on \( [t_1,1] \). Since \( g'(0) = +\infty \) and \( g'(1) = 0 \), there exists \( t_2 \in (0,t_1) \) so that \( g'(t_2) = 0 \), \( g'(t) > 0 \) for \( t \in (0,t_2) \) and \( g'(t) < 0 \) for \( t \in (t_2,1) \). Therefore, \( g(t) \) is strictly increasing on \( [0,t_2] \) and strictly decreasing on \( [t_2,1] \). Since
\[
g(0) = \frac{2^{2k-2} - 3^{k-1}}{2k3^{k-1}} > 0
\]
and \( g(1) = 0 \), we have \( g(t) \geq 0 \) for \( t \in [0,1] \). The equality holds for \( a = b = c = 1 \).
Remark 1. The inequality in P 1.47 is Popoviciu’s inequality
\[ f(a) + f(b) + f(c) + 3f\left(\frac{a + b + c}{3}\right) \geq 2f\left(\frac{a + b}{2}\right) + 2f\left(\frac{b + c}{2}\right) + 2f\left(\frac{c + a}{2}\right) \]

applied to the convex function \( f(x) = x^k \) defined on \([0, \infty)\).

Remark 2. In the same manner, we can prove the following refinements (Vasile C., 2008):

- Let \( a, b, c \) be nonnegative real numbers so that \( a + b + c = 3 \). If \( k > 2 \) and \( m \leq m_0 \), where
  \[
  m_0 = \frac{2^k(3^{k-1} - 2^{k-1})}{6^{k-1} + 3^{k-1} - 2^{2k-1}} > 2,
  \]
  then
  \[
  a^k + b^k + c^k - 3 \geq m \left( \left(\frac{a + b}{2}\right)^k + \left(\frac{b + c}{2}\right)^k + \left(\frac{c + a}{2}\right)^k - 3 \right),
  \]
with equality for \( a = b = c = 1 \). If \( m = m_0 \), then the equality holds also for \( a = 0 \) and \( b = c = 3/2 \) (or any cyclic permutation).

- Let \( a_1, a_2, \ldots, a_n \) be nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n = n \). If \( k > 2 \) and \( m \leq m_1 \), where
  \[
  m_1 = \frac{1}{\left(\frac{1}{n-1}\right)^{k-1} - \frac{1}{n^{k-1}}} \frac{1}{\left(\frac{n-2}{n-1}\right)^{k-1} - \frac{1}{n^{k-1}}} > n - 1,
  \]
  then
  \[
  a_1^k + a_2^k + \cdots + a_n^k - n \geq m \left[ \left(\frac{n-a_1}{n-1}\right)^k + \left(\frac{n-a_2}{n-1}\right)^k + \cdots + \left(\frac{n-a_n}{n-1}\right)^k - n \right],
  \]
with equality for \( a_1 = a_2 = \cdots = a_n = 1 \). If \( m = m_1 \), then the equality holds also for \( a_1 = 0 \) and \( a_2 = a_3 = \cdots = a_n = \frac{n}{n-1} \) (or any cyclic permutation).

\( \square \)

P 1.48. If \( a, b, c \) are the lengths of the sides of a triangle so that \( a + b + c = 3 \), then
\[
\frac{1}{a + b - c} + \frac{1}{b + c - a} + \frac{1}{c + a - b} - 3 \geq 4(2 + \sqrt{3}) \left( \frac{2}{a + b} + \frac{2}{b + c} + \frac{2}{c + a} - 3 \right).
\]

(Vasile C., 2008)
Solution. Write the inequality as
\[ f(a) + f(b) + f(c) \geq 3f(s), \quad s = \frac{a + b + c}{3} = 1, \]
where
\[ f(u) = \frac{1}{3-2u} - \frac{4k}{3-u}, \quad k = 2(2 + \sqrt{3}) \approx 7.464, \quad u \in [0, 3/2). \]
For \( u \geq 1 \), we have
\[ f''(u) = \frac{8}{(3-2u)^3} - \frac{8k}{(3-u)^3} > 8 \left[ \left( \frac{1}{3-2u} \right)^3 - \left( \frac{2}{3-u} \right)^3 \right]. \]
Since \( \frac{1}{3-2u} \geq \frac{2}{3-u} \), \( u \in [1, 3/2) \), it follows that \( f \) is convex on \([s, 3/2)\). By the RHCF-Theorem and Note 1, it suffices to show that \( h(x, y) \geq 0 \) for \( x, y \in [0, 3/2) \) so that \( x + 2y = 3 \). We have
\[ g(u) = \frac{f(u) - f(1)}{u-1} = \frac{2}{3-2u} - \frac{2k}{3-u}, \]
and
\[ h(x, y) = \frac{g(x) - g(y)}{x-y} = \frac{2}{(3-2x)(3-2y)} - \frac{k}{(3-x)(3-y)} = \frac{2}{(2y-x)x} - \frac{2y(x+y)}{2xy(x+y)(2y-x)} \]
\[ = \frac{kx^2 - 2(k-2)xy + 4y^2}{2xy(x+y)(2y-x)} \geq 0. \]
According to Note 4, the equality holds for \( a = b = c = 1 \), and also for
\[ a = 3(2 - \sqrt{3}), \quad b = c = \frac{3(\sqrt{3} - 1)}{2} \]
(or any cyclic permutation).

\[ \square \]

P 1.49. Let \( a_1, a_2, \ldots, a_5 \) be nonnegative numbers so that \( a_1 + a_2 + a_3 + a_4 + a_5 \leq 5 \). If
\[ k \geq k_0, \quad k_0 = \frac{29 + \sqrt{761}}{10} \approx 5.66, \]
then
\[ \sum \frac{1}{ka_i^2 + a_2 + a_3 + a_4 + a_5} \geq \frac{5}{k + 4}. \]

(Vasile C., 2006)
Solution. Since each term of the left hand side of the inequality decreases by increasing any number \(a_i\), it suffices to consider the case
\[
a_1 + a_2 + a_3 + a_4 + a_5 = 5,
\]
when the desired inequality can be written as
\[
f(a_1) + f(a_2) + f(a_3) + f(a_4) + f(a_5) \geq 5f(s),
\]
where
\[
s = \frac{a_1 + a_2 + a_3 + a_4 + a_5}{5} = 1
\]
and
\[
f(u) = \frac{1}{ku^2-u+5}, \quad u \in [0,5].
\]
For \(u \geq 1\), we have
\[
f''(u) = \frac{2[3ku(ku-1)-5k+1]}{(ku^2-u+5)^3} \\
> \frac{2[3k(k-1)-5k+1]}{(ku^2-u+5)^3} \\
= \frac{2[k(3k-8)+1]}{(ku^2-u+5)^3} > 0;
\]
therefore, \(f\) is convex on \([s,5]\). By the RHCF-Theorem, it suffices to show that
\[
\frac{1}{kx^2-x+5} + \frac{4}{ky^2-y+5} \geq \frac{5}{k+4}
\]
for
\[
0 \leq x \leq 1 \leq y, \quad x + 4y = 5.
\]
Write this inequality as follows:
\[
\frac{1}{kx^2-x+5} - \frac{1}{k+4} + 4\left(\frac{1}{ky^2-y+5} - \frac{1}{k+4}\right) \geq 0,
\]
\[
\frac{(x-1)(1-k-kx)}{kx^2-x+5} + \frac{4(y-1)(1-k-ky)}{ky^2-y+5} \geq 0.
\]
Since
\[
4(y-1) = 1 - x,
\]
the inequality is equivalent to
\[
(x-1)\left(\frac{1-k-kx}{kx^2-x+5} - \frac{1-k-ky}{ky^2-y+5}\right) \geq 0,
\]
\[
\frac{5(x-1)^2g(x,y,k)}{4(kx^2-x+5)(ky^2-y+5)} \geq 0,
\]
where 
\[ g(x, y, k) = k^2xy + k(k - 1)(x + y) - 6k + 1. \]
For fixed \( x \) and \( y \), let \( h(k) = g(x, y, k) \). Since
\[
\frac{h'(k)}{2} = 2kxy + (2k - 1)(x + y) - 6 \geq (2k - 1)(x + y) - 6
\]
\[ \geq (2k - 1) \left( x + \frac{y}{4} \right) - 6 = \frac{10k - 29}{4} > 0, \]
it suffices to show that \( g(x, y, k_0) \geq 0 \). We have
\[ g(x, y, k_0) = k_0^2xy + k_0(k_0 - 1)(x + y) - 6k_0 + 1 \]
\[ = (-4k_0^2y^2 + k_0(2k_0 + 3)y + 5k_0^2 - 11k_0 + 1). \]
Since
\[ 5k_0^2 - 29k_0 + 4 = 0, \]
we get
\[ g(x, y, k_0) = (5 - 4y) \left( k_0^2y + k_0^2 - \frac{11k_0 - 1}{5} \right) = x \left( k_0^2y + k_0^2 - \frac{11k_0 - 1}{5} \right). \]
It suffices to show that
\[ k_0^2 - \frac{11k_0 - 1}{5} \geq 0. \]
Indeed,
\[ k_0^2 - \frac{11k_0 - 1}{5} = \frac{k_0(5k_0 - 11) + 1}{5} > 0. \]
The equality holds for \( a_1 = a_2 = a_3 = a_4 = a_5 = 1 \). If \( k = k_0 \), then the equality holds also for
\[ a_1 = 0, \quad a_2 = a_3 = a_4 = a_5 = \frac{5}{4} \]
(or any cyclic permutation).

**Remark.** In the same manner, we can prove the following statement:
- Let \( a_1, a_2, \ldots, a_n \) be nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n \leq n \). If
\[ k \geq k_0, \quad k_0 = \frac{n^2 + n - 1 + \sqrt{n^4 + 2n^3 - 5n^2 + 2n + 1}}{2n}, \]
then
\[ \sum \frac{1}{ka_1^2 + a_2 + \cdots + a_n} \geq \frac{n}{k + n - 1}, \]
with equality for \( a_1 = a_2 = \cdots = a_n = 1 \). If \( k = k_0 \), then the equality holds also for
\[ a_1 = 0, \quad a_2 = \cdots = a_n = \frac{n}{n - 1} \]
(or any cyclic permutation).
**P 1.50.** Let \( a_1, a_2, \ldots, a_5 \) be nonnegative numbers so that \( a_1 + a_2 + a_3 + a_4 + a_5 \leq 5 \). If
\[
0 < k \leq k_0, \quad k_0 = \frac{11 - \sqrt{101}}{10} \approx 0.095,
\]
then
\[
\sum \frac{1}{ka_1^2 + a_2 + a_3 + a_4 + a_5} \geq \frac{5}{k+4}.
\]

*(Vasile C., 2006)*

**Solution.** As shown at the preceding P 1.49, it suffices to consider the case
\[
a_1 + a_2 + a_3 + a_4 + a_5 = 5,
\]
when the desired inequality can be written as
\[
f(a_1) + f(a_2) + f(a_3) + f(a_4) + f(a_5) \geq 5f(s),
\]
where
\[
s = \frac{a_1 + a_2 + a_3 + a_4 + a_5}{5} = 1,
\]
and
\[
f(u) = \frac{1}{ku^2 - u + 5}, \quad u \in [0, 5].
\]
For \( u \in [0, 1] \), we have
\[
u(ku - 1) - (k - 1) = (1 - u)(1 - ku) \geq 0,
\]
hence
\[
f''(u) = \frac{2[3ku(ku - 1) - 5k + 1]}{(ku^2 - u + 5)^3}
\geq \frac{2[3k(k - 1) - 5k + 1]}{(ku^2 - u + 5)^3}
= \frac{2[(1 - 8k) + 3k^2]}{(ku^2 - u + 5)^3} > 0;
\]
therefore, \( f \) is convex on \([0, s]\). By the LHCF-Theorem, it suffices to show that
\[
\frac{1}{kx^2 - x + 5} + \frac{4}{ky^2 - y + 5} \geq \frac{5}{k+4}
\]
for
\[
x \geq 1 \geq y \geq 0, \quad x + 4y = 5.
\]
Write this inequality as follows:
\[
\frac{1}{kx^2 - x + 5} - \frac{1}{k+4} + 4 \left[ \frac{1}{ky^2 - y + 5} - \frac{1}{k+4} \right] \geq 0,
\]
\[
\frac{(x-1)(1-k-kx)}{kx^2-x+5} + \frac{4(y-1)(1-k-ky)}{ky^2-y+5} \geq 0.
\]

Since \(4(y-1) = 1 - x\), the inequality is equivalent to

\[
(x-1) \left( \frac{1-k-kx}{kx^2-x+5} - \frac{1-k-ky}{ky^2-y+5} \right) \geq 0,
\]

\[
\frac{5(x-1)^2 g(x, y, k)}{4(kx^2-x+5)(ky^2-y+5)} \geq 0,
\]

where

\[
g(x, y, k) = k^2 xy - k(1-k)(x+y) - 6k + 1.
\]

For fixed \(x\) and \(y\), let \(h(k) = g(x, y, k)\). Since

\[
h'(k) = 2kxy - (1-2k)(x+y) - 6 \leq 2kxy - 6 \leq \frac{k(x+4y)^2}{8} - 6 = \frac{25k}{8} - 6 < 0,
\]

it suffices to show that \(g(x, y, k_0) \geq 0\). We have

\[
g(x, y, k_0) = k_0^2 xy + k_0(1-k_0)(x+y) - 6k_0 + 1
= -4k_0^2y^2 + k_0(2k_0 + 3)y + 5k_0^2 - 11k_0 + 1.
\]

Since

\[
5k_0^2 - 11k_0 + 1 = 0,
\]

we get

\[
g(x, y, k_0) = k_0y(-4k_0y + 2k_0 + 3) \geq k_0y(-4k_0 + 2k_0 + 3) = k_0(3-2k_0)y \geq 0.
\]

The equality holds for \(a_1 = a_2 = a_3 = a_4 = a_5 = 1\). If \(k = k_0\), then the equality holds also for

\[
a_1 = 5, \quad a_2 = a_3 = a_4 = a_5 = 0
\]

(or any cyclic permutation).

**Remark.** Similarly, we can prove the following statement:

- Let \(a_1, a_2, \ldots, a_n\) be nonnegative real numbers so that \(a_1 + a_2 + \cdots + a_n \leq n\). If

\[
0 \leq k \leq k_0, \quad k_0 = \frac{2n + 1 - \sqrt{4n^2 + 1}}{2n},
\]

then

\[
\sum \frac{1}{ka_1^2 + a_2 + \cdots + a_n} \geq \frac{n}{k + n - 1}.
\]
with equality for \( a_1 = a_2 = \cdots = a_n = 1 \). If \( k = k_0 \), then the equality holds also for

\[
a_1 = n, \quad a_2 = \cdots = a_n = 0
\]

(or any cyclic permutation).

\[\square\]

\textbf{P 1.51.} Let \( a_1, a_2, \ldots, a_n \) be nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n \leq n \). If

\[
0 < k \leq \frac{1}{n+1},
\]

then

\[
\frac{a_1}{ka_1^2 + a_2 + \cdots + a_n} + \frac{a_2}{a_1 + ka_2^2 + \cdots + a_n} + \cdots + \frac{a_n}{a_1 + a_2 + \cdots + ka_n^2} \geq \frac{n}{k + n - 1}.
\]

(Vasile C., 2006)

\textbf{Solution.} Using the notation

\[
x_1 = \frac{a_1}{s}, \quad x_2 = \frac{a_2}{s}, \ldots, \quad x_n = \frac{a_n}{s},
\]

where

\[
s = \frac{a_1 + a_2 + \cdots + a_n}{n} \leq 1,
\]

we need to show that \( x_1 + x_2 + \cdots + x_n = n \) involves

\[
\frac{x_1}{ksx_1^2 + x_2 + \cdots + x_n} + \cdots + \frac{x_n}{x_1 + x_2 + \cdots + ksx_n^2} \geq \frac{n}{k + n - 1}.
\]

Since \( s \leq 1 \), it suffices to prove the inequality for \( s = 1 \); that is, to show that

\[
\frac{a_1}{ka_1^2 - a_1 + n} + \frac{a_2}{ka_2^2 - a_2 + n} + \cdots + \frac{a_n}{ka_n^2 - a_n + n} \geq \frac{n}{k + n - 1}
\]

for

\[
a_1 + a_2 + \cdots + a_n = n.
\]

Write the desired inequality as

\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s),
\]

where

\[
s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1
\]

and

\[
f(u) = \frac{u}{u^2 - u + n}, \quad u \in [0, n].
\]
We have
\[ f'(u) = \frac{n - ku^2}{(ku^2 - u + n)^2}, \quad f''(u) = \frac{f_1(u)}{(u^2 - u + n)^3}, \]
where
\[ f_1(u) = k^2u^3 - 3knu + n. \]
For \( u \in [0, 1] \), we have
\[ f_1(u) \geq -3knu + n \geq -3kn + n \geq -\frac{3n}{n+1} + n = \frac{n(n-2)}{n+1} \geq 0. \]
Since \( f''(u) > 0 \), it follows that \( f \) is convex on \([0, s]\). By the LHCF-Theorem, we only need to show that
\[ \frac{x}{kx^2 - x + n} + \frac{(n-1)y}{ky^2 - y + n} \geq \frac{n}{k + n - 1} \]
for all nonnegative \( x, y \) which satisfy \( x + (n-1)y = n \). Write this inequality as follows:
\[ \frac{x}{kx^2 - x + n} - \frac{1}{k + n - 1} + (n-1)\left[ \frac{y}{ky^2 - y + n} - \frac{1}{k + n - 1} \right] \geq 0, \]
\[ (x - 1)\left( \frac{n-kx}{kx^2 - x + n} - \frac{n-ky}{ky^2 - y + n} \right) \geq 0, \]
\[ (x - 1)^2h(x, y) \geq 0, \]
where
\[ h(x, y) = k^2xy - kn(x + y) + n - nk. \]
We need to show that \( h(x, y) \geq 0 \). Indeed,
\[ h(x, y) = ky[n(k + n - 2) - k(n-1)y] + n[1 - k(n+1)] \]
\[ = ky[n(n-2) + kx] + n[1 - k(n+1)] \geq 0. \]
The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \). If \( k = \frac{1}{n+1} \), then the equality holds also for
\[ a_1 = n, \quad a_2 = a_3 = \cdots = a_n = 0 \]
(or any cyclic permutation).
P 1.52. If \( a_1, a_2, a_3, a_4, a_5 \leq \frac{7}{2} \) so that \( a_1 + a_2 + a_3 + a_4 + a_5 = 5 \), then

\[
\frac{a_1}{a_1^2 - a_1 + 5} + \frac{a_2}{a_2^2 - a_2 + 5} + \frac{a_3}{a_3^2 - a_3 + 5} + \frac{a_4}{a_4^2 - a_4 + 5} + \frac{a_5}{a_5^2 - a_5 + 5} \leq 1.
\]

(Vasile C., 2006)

Solution. Write the desired inequality as

\[
f(a_1) + f(a_2) + f(a_3) + f(a_4) + f(a_5) \geq 5f(s),
\]

where

\[
s = \frac{a_1 + a_2 + a_3 + a_4 + a_5}{5} = 1
\]

and

\[
f(u) = \frac{-u}{u^2 - u + 5}, \quad u \leq \frac{7}{2}.
\]

For \( u \in \left[ 1, \frac{7}{2} \right] \), we have

\[
f''(u) = \frac{-u^3 + 15u - 5}{(u^2 - u + 5)^3} = \frac{(2u + 9)(u - 1)(7 - 2u) + 43 - 7u}{4(u^2 - u + 5)^3} > 0.
\]

Thus, \( f \) is convex on \( \left[ s, \frac{7}{2} \right] \). By the RHCF-Theorem, it suffices to show that

\[
\frac{x}{x^2 - x + 5} + \frac{4y}{y^2 - y + 5} \leq 1
\]

for all nonnegative \( x, y \leq \frac{7}{2} \) which satisfy \( x + 4y = 5 \). Write this inequality as follows:

\[
\frac{x}{x^2 - x + 5} - \frac{1}{5} + 4\left(\frac{y}{y^2 - y + 5} - \frac{1}{5}\right) \leq 0,
\]

\[
(x - 1)\left(\frac{5 - x}{x^2 - x + 5} - \frac{5 - y}{y^2 - y + 5}\right) \leq 0,
\]

\[
(x - 1)^2[5(x + y) - xy] \geq 0,
\]

\[
(x - 1)^2[(x + 4y)(x + y) - xy] \geq 0,
\]

\[
\frac{(x - 1)^2(x + 2y)^2}{(x^2 - x + 5)(y^2 - y + 5)} \geq 0.
\]
The equality holds for \( a_1 = a_2 = a_3 = a_4 = a_5 = 1 \), and also for
\[
a_1 = -5, \quad a_2 = a_3 = a_4 = a_5 = \frac{5}{2}
\]
(or any cyclic permutation).

**Remark.** In the same manner, we can prove the following generalization:

- Let \( a_1, a_2, \ldots, a_n \leq \sqrt{3} \) so that \( a_1 + a_2 + \cdots + a_n \leq n \). If
\[
k = \frac{n^2 + 2n - 2 - 2\sqrt{(n-1)(2n^2 - 1)}}{n},
\]
then
\[
\frac{a_1}{ka_1^2 - a_1 + n} + \frac{a_2}{ka_2^2 - a_2 + n} + \cdots + \frac{a_n}{ka_n^2 - a_n + n} \leq \frac{n}{k - 1 + n},
\]
with equality for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for
\[
a_1 = \frac{n(k - n + 2)}{2k}, \quad a_2 = \cdots = a_n = \frac{n(k + n - 2)}{2k(n-1)}
\]
(or any cyclic permutation).

\[\square\]

**P 1.53.** Let \( a_1, a_2, \ldots, a_n \) be nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n \geq n \).

If
\[
0 < k \leq \frac{1}{1 + \frac{1}{4(n-1)^2}},
\]
then
\[
\frac{a_1^2}{ka_1^2 + a_2 + \cdots + a_n} + \frac{a_2^2}{a_1 + ka_2^2 + \cdots + a_n} + \cdots + \frac{a_n^2}{a_1 + a_2 + \cdots + ka_n^2} \geq \frac{n}{k + n - 1}.
\]

*(Vasile C., 2006)*

**Solution.** Using the substitution
\[
x_1 = \frac{a_1}{s}, \quad x_2 = \frac{a_2}{s}, \quad \ldots, \quad x_n = \frac{a_n}{s},
\]
where
\[
s = \frac{a_1 + a_2 + \cdots + a_n}{n} \geq 1,
\]
we need to show that \( x_1 + x_2 + \cdots + x_n = n \) involves
\[
\frac{x_1^2}{kx_1^2 + (x_2 + \cdots + x_n)/s} + \cdots + \frac{x_n^2}{(x_1 + \cdots + x_{n-1})/s + kx_n^2} \geq \frac{n}{k + n - 1}.
\]
Since $s \geq 1$, it suffices to prove the inequality for $s = 1$; that is, to show that
\[
\frac{a_1^2}{ka_1^2 - a_1 + n} + \frac{a_2^2}{ka_2^2 - a_2 + n} + \cdots + \frac{a_n^2}{ka_n^2 - a_n + n} \geq \frac{n}{k + n - 1}
\]
for
\[a_1 + a_2 + \cdots + a_n = n.
\]
Write the desired inequality as
\[f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s),
\]
where
\[s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1
\]
and
\[f(u) = \frac{u^2}{u^2 - u + n}, \quad u \in [0, n].
\]
We have
\[f'(u) = \frac{u(2n-u)}{(ku^2 - u + n)^2}, \quad f''(u) = \frac{2f_1(u)}{(u^2 - u + n)^3},
\]
where
\[f_1(u) = ku^3 - 3kn^2 + n^2.
\]
For $u \in [0, 1]$ and $n \geq 3$, we have
\[f_1(u) \geq -3kn^2 + n^2 \geq -3kn + n^2 > -3n + n^2 \geq 0.
\]
Also, for $u \in [0, 1]$ and $n = 2$, we have
\[f_1(u) = 4 - ku^2(6-u) \geq 4 - \frac{4}{5}u^2(6-u)
\]
\[\geq 4 - \frac{4}{5}u(6-u) = \frac{4(1-u)(5-u)}{5} \geq 0.
\]
Since $f''(u) \geq 0$ for $u \in [0, 1]$, it follows that $f$ is convex on $[0, s]$. By the LHCF-Theorem, we need to show that
\[\frac{x^2}{kx^2 - x + n} + \frac{(n-1)y^2}{ky^2 - y + n} \geq \frac{n}{k + n - 1}
\]
for all nonnegative $x, y$ which satisfy $x + (n-1)y = n$. Write this inequality as follows:
\[
\frac{x^2}{kx^2 - x + n} - \frac{1}{k + n - 1} + (n-1)\left[\frac{y^2}{ky^2 - y + n} - \frac{1}{k + n - 1}\right] \geq 0,
\]
\[
\frac{(x-1)(nx-x+n)}{kx^2 - x + 5} + \frac{4(y-1)(ny-y+n)}{ky^2 - y + 5} \geq 0,
\]
\[ (x - 1) \left( \frac{nx - x + n}{kx^2 - x + n} - \frac{ny - y + n}{ky^2 - y + n} \right) \geq 0, \]

\[ \frac{(x - 1)^2 h(x, y)}{(kx^2 - x + n)(ky^2 - y + n)} \geq 0, \]

where

\[ h(x, y) = n^2 - kn(x + y) - k(n - 1)xy. \]

Since

\[ 0 < k \leq k_0, \quad k_0 = \frac{1}{1 + \frac{1}{4(n-1)^2}}, \]

we have

\[ h(x, y) \geq n^2 - k_0 n(x + y) - k_0(n - 1)xy \]

\[ = (n - 1)^2 k_0 y^2 - nk_0 y + n^2(1 - k_0) \]

\[ = k_0 \left[ (n - 1)y - \frac{n}{2(n-1)} \right]^2 \geq 0. \]

The equality holds for \( a_1 = a_2 = \cdots = a_n = 1. \) If \( k = k_0, \) then the equality holds also for

\[ a_1 = \frac{n(2n-3)}{2(n-1)}, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{2(n-1)^2} \]

(or any cyclic permutation).

\[ \square \]

**P 1.54.** Let \( a_1, a_2, \ldots, a_n \) be nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n \leq n. \) If \( k \geq n - 1, \) then

\[ \frac{a_1^2}{ka_1^2 + a_2 + \cdots + a_n} + \frac{a_2^2}{a_1 + ka_2^2 + \cdots + a_n} + \cdots + \frac{a_n^2}{a_1 + a_2 + \cdots + ka_n^2} \leq \frac{n}{k + n - 1}. \]

(Vasile C., 2006)

**Solution.** Using the notation

\[ x_1 = \frac{a_1}{s}, \quad x_2 = \frac{a_2}{s}, \ldots, \quad x_n = \frac{a_n}{s}, \]

where

\[ s = \frac{a_1 + a_2 + \cdots + a_n}{n} \leq 1, \]

we need to show that \( x_1 + x_2 + \cdots + x_n = n \) involves

\[ \frac{x_1^2}{kx_1^2 + (x_2 + \cdots + x_n)/s} + \cdots + \frac{x_n^2}{(x_1 + \cdots + x_{n-1})/s + kx_n^2} \leq \frac{n}{k + n - 1}. \]
Since \( s \leq 1 \), it suffices to prove the inequality for \( s = 1 \); that is, to show that
\[
\frac{a_1^2}{ka_1^2 - a_1 + n} + \frac{a_2^2}{ka_2^2 - a_2 + n} + \cdots + \frac{a_n^2}{ka_n^2 - a_n + n} \leq \frac{n}{k + n - 1}
\]
for
\[
a_1 + a_2 + \cdots + a_n = n.
\]
Write the desired inequality as
\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,
\]
where
\[
f(u) = \frac{-u^2}{u^2 - u + n}, \quad u \in [0, n].
\]
We have
\[
f'(u) = \frac{u(u - 2n)}{(ku^2 - u + n)^2}, \quad f''(u) = \frac{2f_1(u)}{(u^2 - u + n)^3},
\]
where
\[
f_1(u) = -ku^3 + 3knu^2 - n^2.
\]
For \( u \in [1, n] \), we have
\[
f_1(u) \geq -knu^2 + 3knu^2 - n^2 = 2knu^2 - n^2
\]
\[
\geq 2kn - n^2 \geq 2(n - 1)n - n^2 = n(n - 2) \geq 0.
\]
Since \( f''(u) \geq 0 \) for \( u \in [1, n] \), it follows that \( f \) is convex on \( [s, n] \). By the RHCF-Theorem, it suffices to show that
\[
\frac{x^2}{kx^2 - x + n} + \frac{(n - 1)y^2}{ky^2 - y + n} \leq \frac{n}{k + n - 1}
\]
for all nonnegative \( x, y \) which satisfy \( x + (n - 1)y = n \). As shown in the proof of the preceding P 1.53, we only need to show that \( h(x, y) \geq 0 \), where
\[
h(x, y) = kn(x + y) + k(n - 1)xy - n^2.
\]
Since \( k \geq n - 1 \), we have
\[
h(x, y) \geq n(n - 1)(x + y) + (n - 1)^2xy - n^2
\]
\[
= -(n - 1)^3y^2 + n(n - 1)y + n^2(n - 2)
\]
\[
= [n - (n - 1)y][n(n - 2) + (n - 1)^2y]
\]
\[
= x[n(n - 2) + (n - 1)^2y] \geq 0.
\]
The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \). If \( k = n - 1 \), then the equality holds also for
\[
a_1 = 0, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n - 1}
\]
(or any cyclic permutation). \( \square \)
P 1.55. Let \( a_1, a_2, \ldots, a_n \in [0, n] \) so that \( a_1 + a_2 + \cdots + a_n \geq n \). If \( 0 < k \leq \frac{1}{n} \), then

\[
\frac{a_1 - 1}{ka_1^2 + a_2 + \cdots + a_n} + \frac{a_2 - 1}{a_1 + ka_2^2 + \cdots + a_n} + \cdots + \frac{a_n - 1}{a_1 + a_2 + \cdots + ka_n^2} \geq 0.
\]

(Vasile C., 2006)

Solution. Let

\[
s = \frac{a_1 + a_2 + \cdots + a_n}{n}, \quad s \geq 1.
\]

Case 1: \( s > 1 \) Without loss of generality, assume that

\[
a_1 \geq \cdots \geq a_j > 1 \geq a_{j+1} \cdots \geq a_n, \quad j \in \{1, 2, \ldots, n\}.
\]

Clearly, there are \( b_1, b_2, \ldots, b_n \) so that \( b_1 + b_2 + \cdots + b_n = n \) and

\[
a_1 \geq b_1 \geq 1, \quad \ldots, \quad a_j \geq b_j \geq 1, \quad b_{j+1} = a_{j+1}, \quad \ldots, \quad b_n = a_n.
\]

Write the desired inequality as

\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq 0,
\]

where

\[
f(u) = \frac{u - 1}{ku^2 - u + ns}, \quad u \in [0, n],
\]

\[
f'(u) = \frac{f_1(u)}{(ku^2 - u + ns)^2}, \quad f_1(u) = k(-u^2 + 2u) + ns - 1.
\]

For \( u \in [1, n] \), we have

\[
f_1(u) \geq k(-nu + 2u) + ns - 1 = -k(n-2)u + ns - 1
\]

\[
\geq -k(n-2)n + ns - 1 \geq -(n-2) + ns - 1 = n(s-1) + 1 > 0.
\]

Consequently, \( f \) is strictly increasing on \([1, n]\) and

\[
f(b_1) \leq f(a_1), \quad \ldots, \quad f(b_j) \leq f(a_j), \quad f(b_{j+1}) = f(a_{j+1}), \quad \ldots, \quad f(b_n) = f(a_n).
\]

Since

\[
f(b_1) + f(b_2) + \cdots + f(b_n) \leq f(a_1) + f(a_2) + \cdots + f(a_n),
\]

it suffices to show that \( f(b_1) + f(b_2) + \cdots + f(b_n) \geq 0 \) for \( b_1 + b_2 + \cdots + b_n = n \).

This inequality is proved at Case 2.

Case 2: \( s = 1 \). Write the inequality as

\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,
\]
where
\[ f(u) = \frac{u - 1}{ku^2 - u + n}, \quad u \in [0, n], \]
\[ f''(u) = \frac{2g(u)}{(ku^2 - u + n)^3}, \quad g(u) = k^2u^3 - 3k^2u^2 - 3k(n - 1)u + kn + n - 1. \]

We will show that \( f''(u) \geq 0 \) for \( u \in [0, 1] \). From
\[ g'(u) = 3k^2u(u - 2) - 3k(n - 1), \]
it follows that \( g'(u) < 0 \), \( g \) is decreasing, hence
\[ g(u) \geq g(1) = -2k^2 - (2n - 3)k + n - 1 \]
\[ \geq \frac{-2}{n^2} - \frac{2n - 3}{n} + n - 1 \]
\[ = \frac{(n - 1)^3 - 1}{n^2} \geq 0. \]

Thus, \( f \) is convex on \([0,s]\). By the LHCF-Theorem, it suffices to show that
\[ \frac{x - 1}{kx^2 - x + n} + \frac{(n - 1)(y - 1)}{ky^2 - y + n} \geq 0 \]
for all nonnegative real \( x, y \) so that \( x + (n - 1)y = n \). Since \((n - 1)(y - 1) = 1 - x\), we have
\[ \frac{x - 1}{kx^2 - x + n} + \frac{(n - 1)(y - 1)}{ky^2 - y + n} = (x - 1)\left( \frac{\frac{1}{kx^2 - x + n} - \frac{1}{ky^2 - y + n}}{} \right) \]
\[ = \frac{(x - 1)(x - y)(1 - kx - ky)}{(kx^2 - x + n)(ky^2 - y + n)} \]
\[ = \frac{n(x - 1)^2(1 - kx - ky)}{(n - 1)(kx^2 - x + n)(ky^2 - y + n)} \]
\[ \geq \frac{(n - 2)y(x - 1)^2}{(n - 1)(kx^2 - x + n)(ky^2 - y + n)} \geq 0. \]

The proof is completed. The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \). If \( k = \frac{1}{n} \), then the equality holds also for
\[ a_1 = n, \quad a_2 = a_3 = \cdots = a_n = 0. \]
**P 1.56.** If \( a, b, c \) are positive real numbers so that \( abc = 1 \), then

\[
\sqrt{a^2 - a + 1} + \sqrt{b^2 - b + 1} + \sqrt{c^2 - c + 1} \geq a + b + c.
\]

**Solution.** Using the substitution

\[ a = e^x, \quad b = e^y, \quad c = e^z, \]

we need to show that

\[ f(x) + f(y) + f(z) \geq 3f(s), \quad s = \frac{x + y + z}{3} = 0, \]

where

\[ f(u) = \sqrt{e^{2u} - e^u + 1} - e^u, \quad u \in \mathbb{I} = \mathbb{R}. \]

We claim that \( f \) is convex on \( \mathbb{I}_{\geq 1} \). Since

\[ e^{-u}f''(u) = \frac{4e^{3u} - 6e^{2u} + 9e^u - 2}{4(e^{2u} - e^u + 1)^{3/2}} - 1, \]

we need to show that \( 4x^3 - 6x^2 + 9x - 2 > 0 \) and

\[ (4x^3 - 6x^2 + 9x - 2)^2 \geq 16(x^2 - x + 1)^3, \]

where \( x = e^u \geq 1 \). Indeed,

\[ 4x^3 - 6x^2 + 9x - 2 = x(x - 3)^2 + (3x^3 - 2) > 0 \]

and

\[ (4x^3 - 6x^2 + 9x - 2)^2 - 16(x^2 - x + 1)^3 = 12x^3(x - 1) + 9x^2 + 12(x - 1) > 0. \]

By the RHCF-Theorem, it suffices to prove the original inequality for

\[ b = c := t, \quad a = 1/t^2, \quad t > 0; \]

that is,

\[
\frac{\sqrt{t^4 - t^2 + 1}}{t^2} + 2\sqrt{t^2 - t + 1} \geq \frac{1}{t^2} + 2t,
\]

\[
\frac{t^2 - 1}{\sqrt{t^4 - t^2 + 1}} + \frac{2(1-t)}{\sqrt{t^2 - t + 1 + t}} \geq 0.
\]

Since

\[
\frac{t^2 - 1}{\sqrt{t^4 - t^2 + 1}} \geq \frac{t^2 - 1}{t^2 + 1},
\]

it suffices to show that

\[
\frac{t^2 - 1}{t^2 + 1} + \frac{2(1-t)}{\sqrt{t^2 - t + 1 + t}} \geq 0,
\]
which is equivalent to
\[
(t-1) \left[ \frac{t+1}{t^2+1} - \frac{2}{\sqrt{t^2-t+1+t}} \right] \geq 0,
\]
\[
(t-1) \left[ (t+1)\sqrt{t^2-t+1-t^2+t-2} \right] \geq 0,
\]
\[
\frac{(t-1)^2(3t^2-2t+3)}{(t+1)\sqrt{t^2-t+1+t^2-t+2}} \geq 0.
\]
The equality holds for \(a = b = c = 1\).

\[\square\]

**P 1.57.** If \(a, b, c, d \geq \frac{1}{1+\sqrt{6}}\) so that \(abcd = 1\), then
\[
\frac{1}{a+2} + \frac{1}{b+2} + \frac{1}{c+2} + \frac{1}{d+2} \leq \frac{4}{3},
\]

(Vasile C., 2005)

**Solution.** Using the notation
\[a = e^x, \quad b = e^y, \quad c = e^z, \quad d = e^w,\]
we need to show that
\[f(x) + f(y) + f(z) + f(w) \geq 4f(s), \quad s = \frac{x+y+z+w}{4} = 0,\]
where
\[f(u) = \frac{-1}{e^u + 2}, \quad u \in \mathbb{I} = \mathbb{R}.\]

For \(u \leq 0\), we have
\[f''(u) = \frac{e^u(2-e^u)}{(e^u+2)^3} > 0,\]
hence \(f\) is convex on \(\mathbb{I}_{\leq 0}\). By the LHCF-Theorem, it suffices to prove the original inequality for
\[b = c = d := t, \quad a = 1/t^3, \quad t \geq \frac{1}{1+\sqrt{6}};\]
that is,
\[\frac{t^3}{2t^3+1} + \frac{3}{t+2} \leq \frac{4}{3},\]
which is equivalent to the obvious inequality
\[(t-1)^2(5t^2+2t-1) \geq 0.\]
According to Note 4, the equality holds for \( a = b = c = d = 1 \), and also for 
\[
a = 19 + 9\sqrt{6}, \quad b = c = d = \frac{1}{1 + \sqrt{6}}
\]
(or any cyclic permutation).

**P 1.58.** If \( a, b, c \) are positive real numbers so that \( abc = 1 \), then
\[
a^2 + b^2 + c^2 - 3 \geq 2(ab + bc + ca - a - b - c).
\]

**Solution.** Using the substitution
\[
a = e^x, \quad b = e^y, \quad c = e^z,
\]
we need to show that
\[
f(x) + f(y) + f(z) \geq 3f(s), \quad s = \frac{x + y + z}{3} = 0,
\]
where
\[
f(u) = e^{2u} - 1 + 2(e^u - e^{-u}), \quad u \in \mathbb{R} = \mathbb{R}.
\]
For \( u \geq 0 \), we have
\[
f''(u) = 4e^{2u} + 2(e^u - e^{-u}) > 0,
\]
hence \( f \) is convex on \( \mathbb{R}_+ \). By the RHCF-Theorem, it suffices to prove the original inequality for \( b = c := t \) and \( a = 1/t^2 \), where \( t > 0 \); that is, to show that
\[
4t^5 - 3t^4 - 4t^3 + 2t^2 + 1 \geq 0,
\]
which is equivalent to
\[
(t - 1)^2(4t^3 + 5t^2 + 2t + 1) \geq 0.
\]
The equality holds for \( a = b = c = 1 \).

**P 1.59.** If \( a, b, c \) are positive real numbers so that \( abc = 1 \), then
\[
a^2 + b^2 + c^2 - 3 \geq 18(a + b + c - ab - bc - ca).
\]
Solution. Using the substitution
\[ a = e^x, \quad b = e^y, \quad c = e^z, \]
we need to show that
\[ f(x) + f(y) + f(z) \geq 3f(s), \quad s = \frac{x + y + z}{3} = 0, \]
where
\[ f(u) = e^{2u} - 1 - 18(e^u - e^{-u}), \quad u \in \mathbb{R}. \]
For \( u \leq 0 \), we have
\[ f''(u) = 4e^{2u} + 18(e^{-u} - e^u) > 0, \]
hence \( f \) is convex on \( I \leq s \). By the LHCF-Theorem, it suffices to prove the original inequality for \( b = c := t \) and \( a = 1/t^2 \), where \( t > 0 \). Since
\[ a^2 + b^2 + c^2 - 3 = \frac{1}{t^4} + 2t^2 - 3 = \frac{(t^2 - 1)^2(2t^2 + 1)}{t^4} \]
and
\[ a + b + c - ab - bc - ca = \frac{-(t^4 - 2t^3 + 2t - 1)}{t^2} = \frac{-(t - 1)^3(t + 1)}{t^2}, \]
we get
\[ a^2 + b^2 + c^2 - 3 - 18(a + b + c - ab - bc - ca) = \frac{(t - 1)^2(2t - 1)^2(t + 1)(5t + 1)}{t^4} \geq 0. \]
The equality holds for \( a = b = c = 1 \), and also for \( a = 4 \) and \( b = c = 1/2 \) (or any cyclic permutation).

\[ \Box \]

P 1.60. If \( a_1, a_2, \ldots, a_n \) are positive real numbers so that \( a_1a_2\cdots a_n = 1 \), then
\[ a_1^2 + a_2^2 + \cdots + a_n^2 - n \geq 6\sqrt{3} \left( a_1 + a_2 + \cdots + a_n - \frac{1}{a_1} - \frac{1}{a_2} - \cdots - \frac{1}{a_n} \right). \]

Solution. Using the notation \( a_i = e^{x_i} \) for \( i = 1, 2, \ldots, n \), we need to show that
\[ f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s), \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0, \]
where
\[ f(u) = e^{2u} - 1 - 6\sqrt{3} (e^u - e^{-u}), \quad u \in \mathbb{I} = \mathbb{R}. \]
For $u \leq 0$, we have
\[ f''(u) = 4e^{2u} + 6\sqrt{3}(e^{-u} - e^u) > 0, \]
hence $f$ is convex on $I \subseteq \mathbb{R}$. By the LHCF-Theorem and Note 2, it suffices to show that $H(x, y) \geq 0$ for $x, y \in \mathbb{R}$ so that $x + (n-1)y = 0$, where
\[ H(x, y) = \frac{f'(x) - f'(y)}{x - y}. \]

From
\[ f'(u) = 2e^{2u} - 6\sqrt{3}\ (e^{u} + e^{-u}), \]
we get
\[ H(x, y) = \frac{2(e^x - e^y)}{x - y} \left( e^x + e^y - 3\sqrt{3} + 3\sqrt{3} e^{-x-y} \right). \]

Since $(e^x - e^y)/(x - y) > 0$, we need to prove that
\[ e^x + e^y + 3\sqrt{3} e^{-x-y} \geq 3\sqrt{3}. \]

Indeed, by the AM-GM inequality, we have
\[ e^x + e^y + 3\sqrt{3} e^{-x-y} \geq 3\sqrt[3]{e^x \cdot e^y \cdot 3\sqrt{3} e^{-x-y}} = 3\sqrt{3}. \]

The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

\[ \square \]

**P 1.61.** If $a_1, a_2, \ldots, a_n$ ($n \geq 4$) are positive real numbers so that $a_1 a_2 \cdots a_n = 1$, then
\[ (n-1)(a_1^2 + a_2^2 + \cdots + a_n^2) + n(n+3) \geq (2n+2)(a_1 + a_2 + \cdots + a_n). \]

**Solution.** Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \ldots, n$, we need to show that
\[ f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s), \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0, \]
where
\[ f(u) = (n-1)e^{2u} - (2n+2)e^u, \quad u \in \mathbb{R}. \]

For $u \geq 0$, we have
\[ f''(u) = 4(n-1)e^{2u} - (2n+2)e^u \]
\[ = 2e^u[2(n-1)e^{u} - n-1] \]
\[ \geq 2e^u[2(n-1) - n-1] = 2(n-3)e^u \geq 0. \]
Therefore, \( f \) is convex on \( I \geq s \). By the RHCF-Theorem and Note 2, it suffices to show that \( H(x, y) \geq 0 \) for \( x, y \in \mathbb{R} \) so that \( x + (n - 1)y = 0 \), where

\[
H(x, y) = \frac{f'(x) - f'(y)}{x - y}.
\]

From

\[
f'(u) = 2(n - 1)e^{2u} - (2n + 2)e^u,
\]

we get

\[
H(x, y) = \frac{2(e^x - e^y)}{x - y} [(n - 1)(e^x + e^y) - (n + 1)].
\]

Since \((e^x - e^y)/(x - y) > 0\), we need to prove that \((n - 1)(e^x + e^y) \geq n + 1\). Using the AM-GM inequality, we have

\[
(n - 1)(e^x + e^y) = (n - 1)e^x + e^y + e^y + \cdots + e^y
\]

\[
\geq n\sqrt{(n - 1)e^x \cdot e^y \cdot e^y \cdots e^y}
\]

\[
= n\sqrt{(n - 1)e^{x+(n-1)y}} = n\sqrt{n-1}.
\]

Thus, it suffices to show that

\[
n\sqrt{n-1} \geq n + 1,
\]

which is equivalent to

\[
n - 1 \geq \left(1 + \frac{1}{n}\right)^n.
\]

This is true for \( n \geq 4 \), since

\[
n - 1 \geq 3 > \left(1 + \frac{1}{n}\right)^n.
\]

The proof is completed. The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \).

**Remark.** From the proof above, the following sharper inequality follows (Gabriel Dospinescu and Calin Popa):

- If \( a_1, a_2, \ldots, a_n \) are positive real numbers so that \( a_1a_2 \cdots a_n = 1 \), then

\[
a_1^2 + a_2^2 + \cdots + a_n^2 - n \geq \frac{2n\sqrt{n-1}}{n-1}(a_1 + a_2 + \cdots + a_n - n).
\]

**P 1.62.** Let \( a_1, a_2, \ldots, a_n \) \((n \geq 3)\) be positive real numbers so that \( a_1a_2 \cdots a_n = 1 \). If \( p, q \geq 0 \) so that \( p + q \geq n - 1 \), then

\[
\frac{1}{1 + pa_1 + qa_1^2} + \frac{1}{1 + pa_2 + qa_2^2} + \cdots + \frac{1}{1 + pa_n + qa_n^2} \geq \frac{n}{1 + p + q}.
\]

(Vasile C., 2007)
Solution. Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \ldots, n$, we need to show that
\[ f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s), \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0, \]
where
\[ f(u) = \frac{1}{1 + pe^u + qe^{2u}}, \quad u \in \mathbb{I} = \mathbb{R}. \]
For $u \geq 0$, we have
\[
\begin{align*}
    f''(u) &= \frac{e^u[4q^2e^{2u} + 3pe^{2u} + (p^2 - 4q)e^u - p]}{(1 + pe^u + qe^{2u})^3} \\
    &\geq \frac{e^{2u}[4q^2 + 3pq + (p^2 - 4q) - p]}{(1 + pe^u + qe^{2u})^3} \\
    &= \frac{e^{2u}[(p + 2q)(p + q - 2) + 2q^2 + p]}{(1 + pe^u + qe^{2u})^3} > 0,
\end{align*}
\]
therefore $f$ is convex on $\mathbb{I}_{\geq s}$. By the RHCF-Theorem, it suffices to prove the original inequality for
\[ a_1 = 1/t^{n-1}, \quad a_2 = \cdots = a_n = t, \quad t > 0. \]
Write this inequality as
\[ \frac{t^{2n-2}}{t^{2n-2} + pt^{n-1} + q} + \frac{n-1}{1 + pt + qt^2} \geq \frac{n}{1 + p + q}. \]
Applying the Cauchy-Schwarz inequality, it suffices to prove that
\[ \frac{(t^{n-1} + n - 1)^2}{(t^{2n-2} + pt^{n-1} + q) + (n-1)(1 + pt + qt^2)} \geq \frac{n}{1 + p + q}, \]
which is equivalent to
\[ pB + qC \geq A, \]
where
\[ A = (n-1)(t^{n-1} - 1)^2 \geq 0, \]
\[ B = (t^{n-1} - 1)^2 + nE = \frac{A}{n-1} + nE, \quad E = t^{n-1} + n - 2 - (n-1)t, \]
\[ C = (t^{n-1} - 1)^2 + nF = \frac{A}{n-1} + nF, \quad F = 2t^{n-1} + n - 3 - (n-1)t^2. \]
By the AM-GM inequality applied to $n-1$ positive numbers, we have $E \geq 0$ and $F \geq 0$ for $n \geq 3$. Since $A \geq 0$ and $p + q \geq n - 1$, we have
\[ pB + qC - A \geq pB + qC - \frac{(p + q)A}{n-1} = n(pE + qF) \geq 0. \]
The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \).

**Remark 1.** For \( p = 2k \) and \( q = k^2 \), we get the following result:

- Let \( a_1, a_2, \ldots, a_n \) (\( n \geq 3 \)) be positive real numbers so that \( a_1 a_2 \cdots a_n = 1 \). If \( k \geq \sqrt{n} - 1 \), then

\[
\frac{1}{(1 + ka_1)^2} + \frac{1}{(1 + ka_2)^2} + \cdots + \frac{1}{(1 + ka_n)^2} \geq \frac{n}{(1 + k)^2},
\]

with equality for \( a_1 = a_2 = \cdots = a_n = 1 \).

In addition, for \( n = 4 \) and \( k = 1 \), we get the known inequality (Vasile C., 1999):

\[
\frac{1}{(1 + a)^2} + \frac{1}{(1 + b)^2} + \frac{1}{(1 + c)^2} + \frac{1}{(1 + d)^2} \geq 1,
\]

where \( a, b, c, d > 0 \) so that \( abcd = 1 \).

**Remark 2.** For \( p + q = n - 1 \) (\( n \geq 3 \)), we get the beautiful inequality

\[
\frac{1}{1 + pa_1 + qa_1^2} + \frac{1}{1 + pa_2 + qa_2^2} + \cdots + \frac{1}{1 + pa_n + qa_n^2} \geq 1,
\]

which is a generalization of the following inequalities:

\[
\frac{1}{1 + (n - 1)a_1} + \frac{1}{1 + (n - 1)a_2} + \cdots + \frac{1}{1 + (n - 1)a_n} \geq 1,
\]

\[
\frac{1}{[1 + (\sqrt{n} - 1)a_1]^2} + \frac{1}{[1 + (\sqrt{n} - 1)a_1]^2} + \cdots + \frac{1}{[1 + (\sqrt{n} - 1)a_1]^2} \geq 1,
\]

\[
\frac{1}{2 + (n - 1)(a_1 + a_1^2)} + \frac{1}{2 + (n - 1)(a_2 + a_2^2)} + \cdots + \frac{1}{2 + (n - 1)(a_n + a_n^2)} \geq \frac{1}{2}.
\]

\( \square \)

**P 1.63.** Let \( a, b, c, d \) be positive real numbers so that \( abcd = 1 \). If \( p \) and \( q \) are nonnegative real numbers so that \( p + q = 3 \), then

\[
\frac{1}{1 + pa + qa^3} + \frac{1}{1 + pb + qb^3} + \frac{1}{1 + pc + qc^3} + \frac{1}{1 + pd + qd^3} \geq 1.
\]

(Vasile C., 2007)
Solution. Using the notation

\[ a = e^x, \quad b = e^y, \quad c = e^z, \quad d = e^w, \]

we need to show that

\[ f(x) + f(y) + f(z) + f(w) \geq 4f(s), \quad s = \frac{x + y + z + w}{4} = 0, \]

where

\[ f(u) = \frac{1}{1 + pe^u + qe^{3u}}, \quad u \in \mathbb{I} = \mathbb{R}. \]

We will show that \( f''(u) > 0 \) for \( u \geq 0 \), hence \( f \) is convex on \( \mathbb{I}_{\geq 0} \). Since

\[ f''(u) = \frac{th(t)}{(1 + pt + qt^3)^3}, \]

where

\[ h(t) = 9q^2t^5 + 2pt^3 - 9qt^2 + p^2t - p, \quad t = e^u, \]

we need to show that \( h(t) \geq 0 \) for \( t \geq 1 \). Indeed, we have

\[ h(t) \geq 9q^2t^3 + 2pt^3 - 9qt^2 + p^2t - pt = tg(t), \]

where

\[ g(t) = (9q^2 + 2pq)t^2 - 9qt + p^2 - p \]
\[ \geq (9q^2 + 2pq)(2t - 1) - 9qt + p^2 - p \]
\[ = q(18q + 4p - 9)t - 9q^2 - 2pq + p^2 - p \]
\[ \geq q(18q + 4p - 9) - 9q^2 - 2pq + p^2 - p \]
\[ = p^2 + 2pq + 9q^2 - p - 9q \]
\[ = p^2 + 2pq + 9q^2 - \frac{(p + 9q)(p + q)}{3} \]
\[ = \frac{2(p-q)^2 + 16q^2}{3} \geq 0. \]

By the RHCF-Theorem, it suffices to prove the original inequality for

\[ b = c = d = t, \quad a = 1/t^3, \quad t > 0; \]

that is,

\[ \frac{t^9}{t^9 + pt^6 + q} + \frac{3}{1 + pt + qt^3} \geq 1, \]

\[ \frac{3}{1 + pt + qt^3} \geq \frac{pt^6 + q}{t^9 + pt^6 + q}, \]

\[ (3 - pq)t^9 - p^2t^7 + 2pt^6 - q^2t^3 - pqt + 2q \geq 0, \]
\[(p + q)^2 - 3pq \geq t^9 - 3p^2t^7 + 2p(p + q)t^6 - 3q^2t^3 - 3pqt + 2q(p + q) \geq 0,\]

\[Ap^2 + Bq^2 \geq Cpq,\]

where

\[A = t^9 - 3t^7 + 2t^6 = t^6(t - 1)^2(t + 2) \geq 0,\]

\[B = t^9 - 3t^3 + 2 = (t^3 - 1)^2(t^3 + 2) \geq 0,\]

\[C = t^9 - 2t^6 + 3t - 2.\]

Since \(A \geq 0\) and \(B \geq 0\), it suffices to consider the case \(C \geq 0\). Since

\[Ap^2 + Bq^2 \geq 2\sqrt{ABpq},\]

we only need to show that \(4AB \geq C^2\). From

\[t^3 - 3t + 2 = (t - 1)^2(t + 2) \geq 0,\]

we get \(3t - 2 \leq t^3\). Therefore

\[C \leq t^9 - 2t^6 + 3t = t^3(t^3 - 1)^2,\]

hence

\[4AB - C^2 \geq 4AB - t^6(t^3 - 1)^4\]

\[= t^6(t - 1)^2(t^3 - 1)^2[4(t + 2)(t^3 + 2) - (t^2 + t + 1)^2]\]

\[= t^6(t - 1)^2(t^3 - 1)^2(3t^4 + 6t^3 - 3t^2 + 6t + 15) \geq 0.\]

The proof is completed. The inequality holds for \(a = b = c = d = 1\).

**Remark 1.** For \(p = 1\) and \(p = 2\), we get the following nice inequalities:

\[\frac{1}{1 + a + 2a^3} + \frac{1}{1 + b + 2b^3} + \frac{1}{1 + c + 2c^3} + \frac{1}{1 + d + 2d^3} \geq 1,\]

\[\frac{1}{1 + 2a + a^3} + \frac{1}{1 + 2b + b^3} + \frac{1}{1 + 2c + c^3} + \frac{1}{1 + 2d + d^3} \geq 1.\]

**Remark 2.** Similarly, we can prove the following generalizations:

- Let \(a, b, c, d\) be positive real numbers so that \(abcd = 1\). If \(p \) and \(q \) are nonnegative real numbers so that \(p + q \geq 3\), then

\[\frac{1}{1 + pa + qa^3} + \frac{1}{1 + pb + qb^3} + \frac{1}{1 + pc + qc^3} + \frac{1}{1 + pd + qd} \geq \frac{4}{1 + p + q}.\]
Let \( a_1, a_2, \ldots, a_n \) \((n \geq 4)\) be positive real numbers so that \(a_1 a_2 \cdots a_n = 1\). If \(p, q, r \geq 0\) so that \(p + q + r \geq n - 1\), then
\[
\sum_{i=1}^{n} \frac{1}{1 + pa_i + qa_i^2 + ra_i^3} \geq \frac{n}{1 + p + q + r}.
\]

For \(n = 4\) and \(p + q + r = 3\), we get the beautiful inequality
\[
\sum_{i=1}^{4} \frac{1}{1 + pa_i + qa_i^2 + ra_i^3} \geq 1.
\]

Since
\[
a_i^2 \leq \frac{a_i + a_i^3}{2},
\]
the best inequality with respect to \(q\) if for \(q = 0\):
\[
\sum_{i=1}^{4} \frac{1}{1 + pa_i + ra_i^3} \geq 1, \quad p + r = 3.
\]

\[\Box\]

**P 1.64.** If \(a_1, a_2, \ldots, a_n\) are positive real numbers so that \(a_1 a_2 \cdots a_n = 1\), then
\[
\frac{1}{1 + a_1 + \cdots + a_1^{n-1}} + \frac{1}{1 + a_2 + \cdots + a_2^{n-1}} + \cdots + \frac{1}{1 + a_n + \cdots + a_n^{n-1}} \geq 1.
\]

(Vasile C., 2007)

**Solution.** Using the substitutions \(a_i = e^{x_i}\) for \(i = 1, 2, \ldots, n\), we need to show that
\[
f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s), \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,
\]
where
\[
f(u) = \frac{1}{1 + e^u + \cdots + e^{(n-1)u}}, \quad u \in \mathbb{I} = \mathbb{R}.
\]

We will show by induction on \(n\) that \(f\) is convex on \(\mathbb{I}_{\geq 0}\). Setting \(t = e^u\), the condition \(f''(u) \geq 0\) for \(u \geq 0\) \((t \geq 1)\) is equivalent to
\[
2A^2 \geq B(1 + C),
\]
where
\[
A = t + 2t^2 + \cdots + (n - 1)t^{n-1},
\]
\[
B = t + 4t^2 + \cdots + (n - 1)^2t^{n-1},
\]
\[
C = t + t^2 + \cdots + t^{n-1}.
\]
For \( n = 2 \), the inequality becomes \( t(t-1) \geq 0 \). Assume now that the inequality is true for \( n \) and prove it for \( n+1, n \geq 2 \). So, we need to show that \( 2A^2 \geq B(1+C) \) involves
\[
2(A + nt^n)^2 \geq (B + n^2t^n)(1 + C + t^n),
\]
which is equivalent to
\[
2A^2 - B(1+C) + t^n[n^2(t^n - 1) + D] \geq 0,
\]
where
\[
D = 4nA - B - n^2C = \sum_{i=1}^{n-1} b_i t^i, \quad b_i = 3n^2 - (2n-i)^2.
\]
Since \( 2A^2 - B(1+C) \geq 0 \) (by the induction hypothesis), it suffices to show that \( D \geq 0 \). Since
\[
b_1 < b_2 < \cdots < b_{n-1}, \quad t \leq t^2 \leq \cdots \leq t^{n-1},
\]
we may apply Chebyshev's inequality to get
\[
D \geq \frac{1}{n} (b_1 + b_2 + \cdots + b_{n-1})(t + t^2 + \cdots + t^{n-1}).
\]
Thus, it suffices to show that \( b_1 + b_2 + \cdots + b_{n-1} \geq 0 \). Indeed,
\[
b_1 + b_2 + \cdots + b_{n-1} = \sum_{i=1}^{n-1} \left[ 3n^2 - (2n-i)^2 \right] = \frac{n(n-1)(4n+1)}{6} > 0.
\]
By the RHCF-Theorem, it suffices to prove the original inequality for
\[
a_1 = 1/t^{n-1}, \quad a_2 = \cdots = a_n = t, \quad t \geq 1,
\]
Setting \( k = n-1 \) (\( k \geq 1 \)), we need to show that
\[
\frac{t^k}{1 + t + \cdots + t^k} + \frac{k}{1 + t + \cdots + t^k} \geq 1.
\]
For the nontrivial case \( t > 1 \), this inequality is equivalent to each of the following inequalities:
\[
\frac{k}{1 + t + \cdots + t^k} \geq \frac{1 + t^k + \cdots + t^{(k-1)k}}{1 + t + \cdots + t^k},
\]
\[
\frac{k(t-1)}{t^{k+1} - 1} \geq \frac{t^{k^2} - 1}{t^k - 1}, \quad \frac{t^k - 1}{t^{(k+1)k} - 1},
\]
\[
\frac{k(t-1)}{t^{k+1} - 1} \geq \frac{t^{k^2} - 1}{t(t^{(k+1)k} - 1)},
\]
\[
\frac{k}{t^{k+1} - 1} \geq \frac{t^k - 1}{t-1}.
\]
\[ k \left[ 1 + t^{k+1} + t^{2(k+1)} + \cdots + t^{(k-1)(k+1)} \right] \geq 1 + t^2 + \cdots + t^{(k-1)(k+1)}, \]
\[ k \left[ 1 \cdot 1 + t \cdot t^k + \cdots + t^{k(k-1)} \right] \geq (1 + t + \cdots + t^{k(k-1)}) \left[ 1 + t^k + \cdots + t^{(k-1)k} \right]. \]

Since \( 1 < t < \cdots < t^{k-1} \) and \( 1 < t^k < \cdots < t^{(k-1)k} \), the last inequality follows from Chebyshev's inequality.

The equality holds for \( a_1 = a_2 = \cdots = a_n = 1. \)

**Remark.** Actually, the following generalization holds:

\( \bullet \) Let \( a_1, a_2, \ldots, a_n \) be positive numbers so that \( a_1 a_2 \cdots a_n = 1 \), and let \( k_1, k_2, \ldots, k_m \geq 0 \) so that \( k_1 + k_2 + \cdots + k_m \geq n - 1 \). If \( m \leq n - 1 \), then
\[
\sum_{i=1}^{n} \frac{1}{1 + k_1 a_i + k_2 a_i^2 + \cdots + k_m a_i^m} \geq \frac{n}{1 + k_1 + k_2 + \cdots + k_m}.
\]

In addition, since
\[
a_i^k \leq \frac{(m-k)a_i + (k-1)a_i^m}{m-1}, \quad k = 2, 3, \ldots, m-1
\]
(by the AM-GM inequality applied to \( m-1 \) positive numbers), the best inequality with respect to \( k_2, \ldots, k_{m-1} \) is for \( k_2 = 0, \ldots, k_{m-1} = 0 \); that is,
\[
\sum_{i=1}^{n} \frac{1}{1 + k_1 a_i + k_m a_i^m} \geq \frac{n}{1 + k_1 + k_m}, \quad k_1 + k_m \geq n - 1, \quad 1 \leq m \leq n - 1.
\]

If \( k_1 + k_m = n - 1 \), then
\[
\sum_{i=1}^{n} \frac{1}{1 + k_1 a_i + k_m a_i^m} \geq 1, \quad 1 \leq m \leq n - 1,
\]
therefore
\[
\sum_{i=1}^{n} \frac{1}{1 + k_1 a_i + k_{n-1} a_i^{n-1}} \geq 1, \quad k_1 + k_{n-1} = n - 1.
\]

For \( k_1 = 1 \) and \( k_1 = n-2 \), we get the following strong inequalities:
\[
\sum_{i=1}^{n} \frac{1}{1 + a_i + (n-2)a_i^{n-1}} \geq 1,
\]
\[
\sum_{i=1}^{n} \frac{1}{1 + (n-2)a_i + a_i^{n-1}} \geq 1.
\]
**P 1.65.** Let \( a_1, a_2, \ldots, a_n \) be positive real numbers so that \( a_1 a_2 \cdots a_n = 1 \). If 
\[
k \geq n^2 - 1,
\]
then
\[
\frac{1}{\sqrt{1 + ka_1}} + \frac{1}{\sqrt{1 + ka_2}} + \cdots + \frac{1}{\sqrt{1 + ka_n}} \geq \frac{n}{\sqrt{1 + k}}.
\]

**Solution.** Using the substitutions \( a_i = e^{x_i} \) for \( i = 1, 2, \ldots, n \), we need to show that
\[
f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s), \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,
\]
where
\[
f(u) = \frac{1}{\sqrt{1 + ke^u}}, \quad u \in \mathbb{I} = \mathbb{R}.
\]
For \( u \geq 0 \), we have
\[
f''(u) = \frac{ke^u(k - 2)}{4(1 + ke^u)^{5/2}} \geq \frac{ke^u(k - 2)}{4(1 + ke^u)^{5/2}} > 0.
\]
Therefore, \( f \) is convex on \( \mathbb{I}_{\geq 0} \). By the RHCF-Theorem, it suffices to prove the original inequality for 
\[
a_1 = 1/t^{n-1}, \quad a_2 = \cdots = a_n = t, \quad t \geq 1.
\]
Write this inequality as \( h(t) \geq 0 \), where
\[
h(t) = \sqrt{\frac{t^{n-1}}{t^n + k}} + \frac{n-1}{\sqrt{1 + kt}} - \frac{n}{\sqrt{1 + k}}.
\]
The derivative
\[
h'(t) = \frac{(n-1)kt^{(n-3)/2}}{2(t^{n-1} + k)^{3/2}} - \frac{(n-1)k}{2(kt + 1)^{3/2}}
\]
has the same sign as
\[
h_1(t) = t^{n-1}(kt + 1) - t^{n-1} - k.
\]
Denoting \( m = n/3 \) (\( m \geq 2/3 \)), we see that
\[
h_1(t) = kt^m + t^{m-1} - t^{3m-1} - k = k(t^m - 1) - t^{m-1}(t^{3m-1} - 1) = (t^m - 1)h_2(t),
\]
where
\[
h_2(t) = k - t^{m-1} - t^{2m-1}.
\]
For \( t > 1 \), we have
\[
h_2'(t) = t^{m-2}[-m + 1 - (2m - 1)t^m] < t^{m-2}[-m + 1 - (2m - 1)]
\]
\[
= -(3m - 2)t^{m-2} \leq 0,
\]
hence \( h_2(t) \) is strictly decreasing for \( t \geq 1 \). Since
\[
h_2(1) = k - 2 > 0, \quad \lim_{t \to \infty} h_2(t) = -\infty,
\]
there exists \( t_1 > 1 \) so that \( h_2(t_1) = 0, h_2(t) > 0 \) for \( t \in [1, t_1), h_2(t) < 0 \) for \( t \in (t_1, \infty) \). Since \( h_2(t) \), \( h_1(t) \) and \( h'(t) \) has the same sign for \( t > 1 \), \( h(t) \) is strictly increasing for \( t \in [1, t_1] \) and strictly decreasing for \( t \in (t_1, \infty) \); this yields
\[
h(t) \geq \min\{h(1), h(\infty)\}.
\]
From \( h(1) = 0 \) and \( h(\infty) = 1 - \frac{n}{\sqrt{1 + k}} \geq 0 \), it follows that \( h(t) \geq 0 \) for all \( t \geq 1 \).
The proof is completed. The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \).

**Remark.** The following generalization holds (Vasile C., 2005):
- Let \( a_1, a_2, \ldots, a_n \) be positive real numbers so that \( a_1 a_2 \cdots a_n = 1 \). If \( k \) and \( m \) are positive numbers so that \( m \leq n - 1, \quad k \geq n^{1/m} - 1 \),
then
\[
\frac{1}{(1 + ka_1)^m} + \frac{1}{(1 + ka_2)^m} + \cdots + \frac{1}{(1 + ka_n)^m} \geq \frac{n}{(1 + k)^m},
\]
with equality for \( a_1 = a_2 = \cdots = a_n = 1 \).

For \( 0 < m \leq n - 1 \) and \( k = n^{1/m} - 1 \), we get the beautiful inequality
\[
\frac{1}{(1 + ka_1)^m} + \frac{1}{(1 + ka_2)^m} + \cdots + \frac{1}{(1 + ka_n)^m} \geq 1.
\]

**P 1.66.** Let \( a_1, a_2, \ldots, a_n \) be positive real numbers so that \( a_1 a_2 \cdots a_n = 1 \). If \( p, q \geq 0 \) so that \( 0 < p + q \leq \frac{1}{n - 1} \), then
\[
\frac{1}{1 + pa_1 + qa_1^2} + \frac{1}{1 + pa_2 + qa_2^2} + \cdots + \frac{1}{1 + pa_n + qa_n^2} \leq \frac{n}{1 + p + q}.
\]
(Vasile C., 2007)

**Solution.** Using the notation \( a_i = e^{x_i} \) for \( i = 1, 2, \ldots, n \), we need to show that
\[
f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s), \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,
\]
where
\[
f(u) = \frac{-1}{1 + pe^u + qe^{2u}}, \quad u \in \mathbb{I} = \mathbb{R}.
\]
For \( u \leq 0 \), we have

\[
f''(u) = \frac{e^u[-4q^2e^{3u} - 3pqe^{2u} + (4q - p^2)e^u + p]}{(1 + pe^u + qe^{2u})^3}
\]

\[
= \frac{e^{2u}[-4q^2e^{2u} - 3pqe^u + (4q - p^2) + pe^{-u}]}{(1 + pe^u + qe^{2u})^3}
\]

\[
\geq \frac{e^{2u}[-4q^2 - 3pq + (4q - p^2) + p]}{(1 + pe^u + qe^{2u})^3}
\]

\[
= \frac{e^{2u}[(p + 4q)(1 - p - q) + 2pq]}{(1 + pe^u + qe^{2u})^3} \geq 0,
\]

therefore \( f \) is convex on \( \mathbb{I}_{\leq} \). By the LHCF-Theorem, it suffices to prove the original inequality for

\[
a_1 = 1/|t|^{n-1}, \quad a_2 = \cdots = a_n = t, \quad t > 0.
\]

Write this inequality as

\[
t^{2n-2} + pt^{n-1} + q + \frac{n-1}{1 + pt + qt^2} \leq \frac{n}{1 + p + q},
\]

\[
p^2A + q^2B + pqC \leq pD + qE,
\]

where

\[
A = t^{n-1}(t^n - nt + n - 1), \quad B = t^{2n} - nt^2 + n - 1,
\]

\[
C = t^{2n-1} + t^{2n} - nt^{n+1} + (n-1)t^{n-1} - nt + n - 1,
\]

\[
D = t^{n-1}[(n-1)t^{n} + 1 - nt^{n-1}], \quad E = (n-1)t^{2n} - nt^{2n-2}.
\]

Applying the AM-GM inequality to \( n \) positive numbers yields \( D \geq 0 \) and \( E \geq 0 \). Since \((n-1)(p+q) \leq 1\) involves \( pD + qE \geq (n-1)(p+q)(pD + qE) \), it suffices to show that

\[
p^2A + q^2B + pqC \leq (n-1)(p+q)(pD + qE).
\]

Write this inequality as

\[
p^2A_1 + q^2B_1 + pqC_1 \geq 0,
\]

where

\[
A_1 = (n-1)D - A = nt^n[(n-2)t^{n-1} + 1 - (n-1)tn^{-2}],
\]

\[
B_1 = (n-1)E - B = nt^2[(n-2)t^{2n-2} + 1 - (n-1)t^{2n-4}],
\]

\[
C_1 = (n-1)(D + E) - C = nt[(n-2)(t^{2n-1} + t^{2n-2}) - 2(n-1)t^{2n-3} + t^n + 1].
\]

Applying the AM-GM inequality to \( n - 1 \) nonnegative numbers yields \( A_1 \geq 0 \) and \( B_1 \geq 0 \). So, it suffices to show that \( C_1 \geq 0 \). Indeed, we have

\[
(n-2)(t^{2n-1} + t^{2n-2}) - 2(n-1)t^{2n-3} + t^n + 1 = A_2 + B_2 + C_2,
\]

where

\[
A_2 = (n-2)t^{2n-1} + t - (n-1)t^{2n-3} \geq 0,
\]
\[ B_2 = (n - 2)t^{2n-2} + t^{n-1} - (n - 1)t^{2n-3} \geq 0, \]
\[ C_2 = t^n - t^{n-1} - t + 1 = (t - 1)(t^{n-1} - 1) \geq 0. \]

The inequalities \[A_2 \geq 0\] and \[B_2 \geq 0\] follow by applying the AM-GM inequality to \(n - 1\) nonnegative numbers.

The equality holds for \(a_1 = a_2 = \cdots = a_n = 1\).

**Remark 1.** For \(p + q = \frac{1}{n-1}\), we get the inequality

\[
\frac{1}{1 + pa_1 + qa_1^2} + \frac{1}{1 + pa_2 + qa_2^2} + \cdots + \frac{1}{1 + pa_n + qa_n^2} \leq n - 1,
\]

which is a generalization of the following inequalities:

\[
\frac{1}{n-1 + a_1} + \frac{1}{n-1 + a_2} + \cdots + \frac{1}{n-1 + a_n} \leq 1,
\]
\[
\frac{1}{2n - 2 + a_1 + a_1^2} + \frac{1}{2n - 2 + a_2 + a_2^2} + \cdots + \frac{1}{2n - 2 + a_n + a_n^2} \leq \frac{1}{2}.
\]

**Remark 2.** For

\[ p = \frac{4n - 3}{2(n-1)(2n-1)}, \quad q = \frac{1}{2(n-1)(2n-1)}, \]

we get the inequality

\[
\frac{1}{(a_1 + 2n - 2)(a_1 + 2n - 1)} + \cdots + \frac{1}{(a_n + 2n - 2)(a_n + 2n - 1)} \leq \frac{1}{4n - 2},
\]

which is equivalent to

\[
\frac{1}{a_1 + 2n - 2} + \cdots + \frac{1}{a_n + 2n - 2} \leq \frac{1}{4n - 2} + \frac{1}{a_1 + 2n - 1} + \cdots + \frac{1}{a_n + 2n - 1}.
\]

**Remark 3.** For \(p = 2k\) and \(q = k^2\), we get the following statement:

- Let \(a_1, a_2, \ldots, a_n\) be positive real numbers so that \(a_1a_2 \cdots a_n = 1\). If

\[ 0 < k \leq \sqrt{\frac{n}{n-1}} - 1, \]

then

\[
\frac{1}{(1 + ka_1)^2} + \frac{1}{(1 + ka_2)^2} + \cdots + \frac{1}{(1 + ka_n)^2} \leq \frac{n}{(1 + k)^2},
\]

with equality for \(a_1 = a_2 = \cdots = a_n = 1\). \(\square\)
**P 1.67.** Let \( a_1, a_2, \ldots, a_n \) \((n \geq 3)\) be positive real numbers so that \( a_1 a_2 \cdots a_n = 1 \). If \( 0 < k \leq \frac{2n-1}{(n-1)^2} \), then

\[
\frac{1}{\sqrt{1 + k a_1}} + \frac{1}{\sqrt{1 + k a_2}} + \cdots + \frac{1}{\sqrt{1 + k a_n}} \leq \frac{n}{\sqrt{1 + k}}.
\]

**Solution.** Using the substitutions \( a_i = e^u_i \) for \( i = 1, 2, \ldots, n \), we need to show that

\[
f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s), \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,
\]

where

\[
f(u) = \frac{-1}{\sqrt{1 + ke^u}}, \quad u \in \mathbb{I} = \mathbb{R}.
\]

For \( u \leq 0 \), we have

\[
f''(u) = \frac{ke^u(2 - ke^u)}{4(1 + ke^u)^{5/2}} \geq \frac{ke^u(2 - k)}{4(1 + ke^u)^{5/2}} > 0.
\]

Therefore, \( f \) is convex on \( \mathbb{I} \leq s \). By the LHCF-Theorem, it suffices to prove the original inequality for

\[
a_1 = \frac{1}{t^{n-1}}, \quad a_2 = \cdots = a_n = t. \quad 0 < t \leq 1.
\]

Write this inequality as \( h(t) \leq 0 \), where

\[
h(t) = \sqrt{\frac{t^{n-1}}{t^{n-1} + k}} + \frac{n-1}{\sqrt{1 + kt}} - \frac{n}{\sqrt{1 + k}}.
\]

The derivative

\[
h'(t) = \frac{(n-1)kt^{(n-3)/2}}{2(t^{n-1} + k)^{3/2}} - \frac{(n-1)k}{2(kt + 1)^{3/2}}
\]

has the same sign as

\[
h_1(t) = t^{n/3-1}(kt + 1) - t^{n-1} - k.
\]

Denoting \( m = n/3, m \geq 1 \), we see that

\[
h_1(t) = kt^m + t^{m-1} - t^{3m-1} - k = -k(1 - t^m) + t^{m-1}(1 - t^{2m}) = (1 - t^m)h_2(t),
\]

where

\[
h_2(t) = t^{m-1} + t^{2m-1} - k
\]

is strictly increasing for \( t \in [0,1] \). There are two possible cases: \( h_2(0) \geq 0 \) and \( h_2(0) < 0 \).
Case 1: \( h_2(0) \geq 0 \). This case is possible only for \( m = 1 \) and \( k \leq 1 \), when \( h_2(t) = t + 1 - k > 0 \) for \( t \in (0, 1] \). Also, we have \( h_1(t) > 0 \) and \( h'(t) > 0 \) for \( t \in (0, 1) \). Therefore, \( h \) is strictly increasing on \([0, 1]\), hence \( h(t) \leq h(1) = 0 \).

Case 2: \( h_2(0) < 0 \). This case is possible for either \( m = 1 \) \((n = 3)\) and \( 1 < k \leq 5/4 \), or \( m > 1 \) \((n \geq 4)\). Since \( h_2(1) = 2 - k > 0 \), there exists \( t_1 \in (0, 1) \) so that \( h_2(t_1) = 0 \), \( h_2(t) < 0 \) for \( t \in (0, t_1) \), and \( h_2(t) > 0 \) for \( t \in (t_1, 1) \). Since \( h' \) has the same sign as \( h_2 \) on \((0, 1)\), it follows that \( h \) is strictly decreasing on \([0, t_1]\) and strictly increasing on \([t_1, 1]\). Therefore, \( h(t) \leq \max\{h(0), h(1)\} \). Since \( h(0) = n - 1 - \frac{n}{\sqrt{1 + k}} \leq 0 \) and \( h(1) = 0 \), we have \( h(t) \leq 0 \) for all \( t \in (0, 1] \).

The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \).

Remark. The following generalization holds \((Vasile C., 2005)\):

- Let \( a_1, a_2, \ldots, a_n \) \((n \geq 3)\) be positive real numbers so that \( a_1 a_2 \cdots a_n = 1 \). If \( k \) and \( m \) are positive numbers so that

\[
m \geq \frac{1}{n - 1}, \quad k \leq \left( \frac{n}{n - 1} \right)^{1/m} - 1,
\]

then

\[
\frac{1}{(1 + ka_1)^m} + \frac{1}{(1 + ka_2)^m} + \cdots + \frac{1}{(1 + ka_n)^m} \leq \frac{n}{(1 + k)^m},
\]

with equality for \( a_1 = a_2 = \cdots = a_n = 1 \).

For \( n \geq 3 \), \( m \geq \frac{1}{n - 1} \) and \( k = \left( \frac{n}{n - 1} \right)^{1/m} - 1 \), we get the beautiful inequality

\[
\frac{1}{(1 + ka_1)^m} + \frac{1}{(1 + ka_2)^m} + \cdots + \frac{1}{(1 + ka_n)^m} \leq n - 1.
\]

\[\square\]

P 1.68. If \( a_1, a_2, \ldots, a_n \) are positive real numbers so that \( a_1 a_2 \cdots a_n = 1 \), then

\[
\sqrt{a_1^4 + \frac{2n - 1}{(n - 1)^2}} + \sqrt{a_2^4 + \frac{2n - 1}{(n - 1)^2}} + \cdots + \sqrt{a_n^4 + \frac{2n - 1}{(n - 1)^2}} \geq \frac{1}{n - 1}(a_1 + a_2 + \cdots + a_n)^2.
\]

\((Vasile C., 2006)\)

Solution. According to the preceding P 1.67, the following inequality holds

\[
\sum \frac{1}{\sqrt{1 + \frac{2n - 1}{(n - 1)^2} a_i^4}} \leq n - 1.
\]
On the other hand, by the Cauchy-Schwarz inequality
\[
\left( \sum \frac{1}{\sqrt{1 + \frac{2n-1}{(n-1)^2} a_i^2}} \right) \left( \sum a_i^2 \sqrt{1 + \frac{2n-1}{(n-1)^2} a_i^{-4}} \right) \geq \left( \sum a_i \right)^2.
\]
From these inequalities, we get
\[
(n-1) \left( \sum a_i^2 \sqrt{1 + \frac{2n-1}{(n-1)^2} a_i^{-4}} \right) \geq \left( \sum a_i \right)^2,
\]
which is the desired inequality.

The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \). \( \square \)

**P 1.69.** If \( a_1, a_2, \ldots, a_n \) are positive real numbers so that \( a_1 a_2 \cdots a_n = 1 \), then
\[
a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} + n(n-2) \geq (n-1) \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right).
\]

**Solution.** Using the notation \( a_i = e^{x_i} \) for \( i = 1, 2, \ldots, n \), we need to show that
\[
f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s), \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,
\]
where
\[
f(u) = e^{(n-1)u} - (n-1)e^{-u}, \quad u \in \mathbb{I} = \mathbb{R}.
\]
For \( u \geq 0 \), we have
\[
f''(u) = (n-1)^2 e^{(n-1)u} - (n-1)e^{-u} = (n-1)e^{-u}[(n-1)e^{nu} - 1] \geq 0;
\]
therefore, \( f \) is convex on \( \mathbb{I} \). By the RHCF-Theorem and Note 2, it suffices to show that \( H(x, y) \geq 0 \) for \( x, y \in \mathbb{R} \) so that \( x + (n-1)y = 0 \), where
\[
H(x, y) = \frac{f'(x) - f'(y)}{x - y}.
\]
From
\[
f'(u) = (n-1)[e^{(n-1)u} + e^{-u}],
\]
we get
\[
H(x, y) = \frac{(n-1)(e^x - e^y)}{x - y} \left[ e^{(n-2)x} + e^{(n-3)x+y} + \cdots + e^{x+(n-3)y} + e^{(n-2)y} - e^{-x-y} \right]
= \frac{(n-1)(e^x - e^y)}{x - y} \left[ e^{(n-2)x} + e^{(n-3)x+y} + \cdots + e^{x+(n-3)y} \right].
\]
Since \( (e^x - e^y)/(x - y) > 0 \), we have \( H(x, y) > 0 \).

The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \). \( \square \)
**P 1.70.** Let \( a_1, a_2, \ldots, a_n \) be positive real numbers so that \( a_1a_2\cdots a_n = 1 \). If \( k \geq n \), then
\[
a_1^k + a_2^k + \cdots + a_n^k + kn \geq (k + 1) \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right). \tag{Vasile C., 2006}
\]

**Solution.** Using the notations \( a_i = e^{x_i} \) for \( i = 1, 2, \ldots, n \), we need to show that
\[
f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s), \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,
\]
where
\[
f(u) = e^{ku} - (k + 1)e^{-u}, \quad u \in \mathbb{I} = \mathbb{R}.
\]
For \( u \geq 0 \), we have
\[
f''(u) = k^2 e^{ku} - (k + 1)e^{-u} = e^{-u}[k^2 e^{(k+1)u} - k - 1] \geq e^{-u}(k^2 - k - 1) > 0;
\]
therefore, \( f \) is convex on \( \mathbb{I}_{\geq} \). By the RHCF-Theorem, it suffices to prove the original inequality for \( a_1 \leq 1 \leq a_2 = \cdots = a_n \); that is, to show that
\[
a_k + (n-1)b^k - \frac{k+1}{a} - \frac{(k+1)(n-1)}{b} + kn \geq 0
\]
for
\[
ab^{n-1} = 1, \quad 0 < a \leq 1 \leq b.
\]
By the weighted AM-GM inequality, we have
\[
a_k + (kn - k - 1) \geq \left[ 1 + (kn - k - 1) \right] a^{\frac{k}{(kn - k - 1)}} = \frac{k(n-1)}{b}.
\]
Thus, we still have to show that
\[
(n-1)\left( b^k - \frac{1}{b} \right) - (k+1)\left( \frac{1}{a} - 1 \right) \geq 0,
\]
which is equivalent to \( h(b) \geq 0 \) for \( b \geq 1 \), where
\[
h(b) = (n-1)(b^{k+1} - 1) - (k+1)(b^n - b).
\]
Since
\[
\frac{h'(b)}{k+1} = (n-1)b^k - nb^{n-1} + 1 \geq (n-1)b^n - nb^{n-1} + 1
\]
\[
= nb^{n-1}(b-1) - (b^n - 1)
\]
\[
= (b-1)\left[ (b^{n-1} - b^{n-2}) + (b^{n-1} - b^{n-3}) + \cdots + (b^{n-1} - 1) \right] \geq 0,
\]
h is increasing on \([1, \infty)\), hence \( h(b) \geq h(1) = 0 \). The proof is completed. The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \).
P 1.71. If \( a_1, a_2, \ldots, a_n \) are positive real numbers so that \( a_1 a_2 \cdots a_n = 1 \), then
\[
\left( 1 - \frac{1}{n} \right)^{a_1} + \left( 1 - \frac{1}{n} \right)^{a_2} + \cdots + \left( 1 - \frac{1}{n} \right)^{a_n} \leq n - 1.
\] (Vasile C., 2006)

Solution. Let
\[
k = \frac{n}{n - 1}, \quad k > 1,
\]
and
\[
m = \ln k, \quad 0 < m \leq \ln 2 < 1.
\]
Using the substitutions \( a_i = e^{x_i} \) for \( i = 1, 2, \ldots, n \), we need to show that
\[
f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s), \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,
\]
where
\[
f(u) = -k^{-e^u}, \quad u \in \mathbb{I} = \mathbb{R}.
\]
From
\[
f''(u) = me^u k^{-e^u} (1 - me^u),
\]
it follows that \( f''(u) > 0 \) for \( u \leq 0 \), since
\[
1 - me^u \geq 1 - m \geq 1 - \ln 2 > 0.
\]
Therefore, \( f \) is convex on \( \mathbb{I} \leq s \). By the LHCF-Theorem and Note 5, it suffices to prove the original inequality for
\[
a_2 = \cdots = a_n := t, \quad a_1 = t^{-n+1}, \quad 0 < t \leq 1.
\]
Write this inequality as
\[
h(t) \leq n - 1,
\]
where
\[
h(t) = k^{-t^{-n+1}} + (n - 1)k^{-t}, \quad t \in (0, 1].
\]
We have
\[
h'(t) = (n - 1)mt^{-n}k^{-t^{-n+1}}h_1(t), \quad h_1(t) = 1 - t^n k^{-t^{-n+1} - t},
\]
\[
h'_1(t) = k^{t^{-n+1} - t} h_2(t), \quad h_2(t) = m(n - 1 + t^n) - nt^{n-1}.
\]
Since
\[
h'_2(t) = nt^{n-2}(mt - n + 1) \leq nt^{n-2}(m - n + 1) \leq nt^{n-2}(m - 1) < 0,
\]
h_2 is strictly decreasing on \([0, 1]\). From
\[
h_2(0) = (n - 1)m > 0, \quad h_2(1) = n(m - 1) < 0,
\]
it follows that there is \( t_1 \in (0, 1) \) so that \( h_2(t_1) = 0 \), \( h_2(t) > 0 \) for \( t \in [0, t_1) \) and \( h_2(t) < 0 \) for \( t \in (t_1, 1] \). Therefore, \( h_1 \) is strictly increasing on \((0, t_1)\) and strictly decreasing on \([t_1, 1]\). Since \( h_1(0_+) = -\infty \) and \( h_1(1) = 0 \), there is \( t_2 \in (0, t_1) \) so that \( h_1(t_2) = 0 \), \( h_1(t) < 0 \) for \( t \in (0, t_2) \), \( h_1(t) > 0 \) for \( t \in (t_2, 1) \). Thus, \( h \) is strictly decreasing on \((0, t_2)\) and strictly increasing on \([t_2, 1]\). Since \( h(0_+) = n - 1 \) and \( h(1) = n - 1 \), we have \( h(t) \leq n - 1 \) for all \( t \in (0, 1] \). This completes the proof. The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \).

\[ \square \]

**P 1.72.** If \( a, b, c \) are positive real numbers so that \( abc = 1 \), then

\[
\frac{1}{1 + \sqrt{1+3a}} + \frac{1}{1 + \sqrt{1+3b}} + \frac{1}{1 + \sqrt{1+3c}} \leq 1.
\]

*(Vasile C., 2008)*

**Solution.** Write the inequality as

\[
\frac{\sqrt{1+3a} - 1}{3a} + \frac{\sqrt{1+3b} - 1}{3b} + \frac{\sqrt{1+3c} - 1}{3c} \leq 1,
\]

\[
\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 3 \geq \sqrt{\frac{1}{a^2} + \frac{3}{a}} + \sqrt{\frac{1}{b^2} + \frac{3}{b}} + \sqrt{\frac{1}{c^2} + \frac{3}{c}}.
\]

Replacing \( a, b, c \) by \( 1/a, 1/b, 1/c \), respectively, we need to prove that \( abc = 1 \) involves

\[
a + b + c + 3 \geq \sqrt{a^2 + 3a} + \sqrt{b^2 + 3b} + \sqrt{c^2 + 3c}.
\]

Using the notation

\[
a = e^x, \quad b = e^y, \quad c = e^z,
\]

we need to show that

\[
f(x) + f(y) + f(z) \geq 3f(s), \quad s = \frac{x + y + z}{3} = 0,
\]

where

\[
f(u) = e^u - \sqrt{e^{2u} + 3e^u}, \quad u \in \mathbb{R}.
\]

We have

\[
f''(u) = t \left[ 1 - \frac{4t^2 + 18t + 9}{4(t+3)\sqrt{t(t+3)}} \right], \quad t = e^u \geq 1.
\]

For \( u \geq 0 \), which involves \( t \geq 1 \), from

\[16t(t+3)^3 - (4t^2 + 18t + 9)^2 = 9(4t^2 + 12t - 9) > 0,
\]
it follows that \( f'' > 0 \), hence \( f \) is convex on \( \mathbb{I} \geq s \). By the RHCF-Theorem, it suffices to prove the inequality (*) for \( b = c \). Thus, we need to show that

\[ a - \sqrt{a^2 + 3a} + 2(b - \sqrt{b^2 + 3b}) + 3 \geq 0 \]

for \( ab^2 = 1 \). Write this inequality as

\[ 2b^3 + 3b^2 + 1 \geq \sqrt{3b^2 + 1} \]

Squaring and dividing by \( b^2 \), the inequality becomes

\[ 9b^2 + 4b + 3 \geq 4(b^2 + 3b)(3b^2 + 1) \]

Since

\[ 2\sqrt{(b^2 + 3b)(3b^2 + 1)} \leq (b^2 + 3b) + (3b^2 + 1) = 4b^2 + 3b + 1, \]

it suffices to show that

\[ 9b^2 + 4b + 3 \geq 2(4b^2 + 3b + 1), \]

which is equivalent to \((b - 1)^2 \geq 0\). The equality holds for \( a = b = c = 1 \).

**Remark.** In the same manner, we can prove the following generalization:

- Let \( a_1, a_2, \ldots, a_n \) be positive real numbers so that \( a_1a_2\cdots a_n = 1 \). If

\[ 0 < k \leq \frac{4n}{(n-1)^2}, \]

then

\[ \frac{1}{1 + \sqrt{1 + ka_1}} + \frac{1}{1 + \sqrt{1 + ka_2}} + \cdots + \frac{1}{1 + \sqrt{1 + ka_n}} \leq \frac{n}{1 + \sqrt{1 + k}}. \]
Solution. Denote

\[ k = 4n(n-1), \quad k \geq 8, \]

and write the inequality as follows:

\[
\frac{\sqrt{1+ka_1} - 1}{ka_1} + \frac{\sqrt{1+ka_2} - 1}{ka_2} + \cdots + \frac{\sqrt{1+ka_n} - 1}{ka_n} \geq \frac{1}{2},
\]

\[
\sqrt{\frac{1}{a_1^2} + \frac{k}{a_1}} + \sqrt{\frac{1}{a_2^2} + \frac{k}{a_2}} + \cdots + \sqrt{\frac{1}{a_n^2} + \frac{k}{a_n}} \geq \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} + \frac{k}{2}.
\]

Replacing \(a_1, a_2, \ldots, a_n\) by \(1/a_1, 1/a_2, \ldots, 1/a_n\), we need to prove that \(a_1a_2 \cdots a_n = 1\) implies

\[
\sqrt{a_1^2 + ka_1} + \sqrt{a_2^2 + ka_2} + \cdots + \sqrt{a_n^2 + ka_n} \geq a_1 + a_2 + \cdots + a_n + \frac{k}{2}. \quad (\ast)
\]

Using the substitutions \(a_i = e^{x_i}\) for \(i = 1, 2, \ldots, n\), we need to show that

\[ f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s), \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0, \]

where

\[ f(u) = e^{2u} + ke^u - e^u, \quad u \in \mathbb{I} = \mathbb{R}. \]

We will show that \(f''(u) > 0\) for \(u \leq 0\). Indeed, denoting \(t = e^u, t \in (0, 1]\), we have

\[ f''(u) = t \left[ \frac{4t^2 + 6kt + k^2}{4(t+k)\sqrt{t(t+k)}} - 1 \right] > 0 \]

because

\[
(4t^2 + 6kt + k^2)^2 - 16t(t+k)^3 = k^2(k^2 - 4kt - 4t^2) \geq k^2(k^2 - 4k - 4) > 0.
\]

Thus, \(f\) is convex on \(\mathbb{I}_{\leq 0}\). By the LHCF-Theorem, it suffices to prove the inequality (\(\ast\)) for \(a_2 = a_3 = \cdots = a_n\); that is, to show that

\[
\sqrt{a^2 + ka - a + (n-1)}(\sqrt{b^2 + kb - b}) \geq n(\sqrt{1+k} - 1),
\]

for all positive \(a, b\) satisfying \(ab^{n-1} = 1\). Write this inequality as

\[
\sqrt{kb^{n-1} + 1 + (n-1)\sqrt{kb^{2n-1} + b^{2n}}} \geq (n-1)b^n + 2n(n-1)b^{n-1} + 1.
\]

By Minkowski’s inequality, we have

\[
\sqrt{kb^{n-1} + 1 + (n-1)\sqrt{kb^{2n-1} + b^{2n}}} \geq \sqrt{kb^{n-1}[1 + (n-1)b^{n/2}]^2 + [1 + (n-1)b^n]^2}.
\]
Thus, it suffices to show that
\[ kb^{n-1}[(n-1)b^{n/2}]^2 + [1+(n-1)b^n]^2 \geq [(n-1)b^n + 2(n-1)b^{n-1} + 1]^2, \]
which is equivalent to
\[ 4n(n-1)^2b^{3n-2}\left[2 + (n-2)b^{n} - nb^{n-2}\right] \geq 0. \]
This inequality follows immediately by the AM-GM inequality applied to \( n \) positive numbers.

The equality holds for \( a_1 = a_2 = \cdots = a_n = 1. \)

\[ \square \]

**P 1.74.** If \( a, b, c \) are positive real numbers so that \( abc = 1 \), then
\[ \frac{a^6}{1+2a^5} + \frac{b^6}{1+2b^5} + \frac{c^6}{1+2c^5} \geq 1. \]

*(Vasile C., 2008)*

**Solution.** Using the substitution
\[ a = e^x, \quad b = e^y, \quad c = e^z, \]
we need to show that
\[ f(x) + f(y) + f(z) \geq 3f(s), \quad s = \frac{x+y+z}{3} = 0, \]
where
\[ f(u) = \frac{e^{6u}}{1+2e^{5u}}, \quad u \in \mathbb{I} = \mathbb{R}. \]
For \( u \leq 0 \), which involves \( w = e^u \in (0, 1) \), we have
\[ f''(u) = \frac{2w^6(2-w^5)(9-2w^5)}{(1+2w^5)^3} > 0. \]
Therefore, \( f \) is convex on \( \mathbb{I}_{\leq 0} \). By the LHCF-Theorem, it suffices to prove the original inequality for \( b = c \) and \( ab^2 = 1 \); that is,
\[ \frac{1}{b^2(b^{10}+2)} + \frac{2b^6}{1+2b^5} \geq 1. \]
Since
\[ 1 + 2b^5 \leq 1 + b^4 + b^6, \]
it suffices to show that
\[
\frac{1}{x(x^5 + 2)} + \frac{2x^2}{1 + x^2 + x^3} \geq 1, \quad x = \sqrt{b}.
\]
This inequality can be written as follows:
\[
x^3(x^6 - x^5 - x^3 + 2x - 1) + (x - 1)^2 \geq 0,
\]
\[
x^3(x - 1)^2(x^4 + x^3 + x^2 - 1) + (x - 1)^2 \geq 0,
\]
\[
(x - 1)^2[x^7 + x^5 + (x^6 - x^3 + 1)] \geq 0.
\]
The equality holds for \(a = b = c = 1\).

\(\Box\)

**P 1.75.** If \(a, b, c\) are positive real numbers so that \(abc = 1\), then
\[
\sqrt{25a^2 + 144} + \sqrt{25b^2 + 144} + \sqrt{25c^2 + 144} \leq 5(a + b + c) + 24.
\]

\((\text{Vasile C., 2008})\)

**Solution.** Using the notation
\[
a = e^x, \quad b = e^y, \quad c = e^z,
\]
we need to show that
\[
f(x) + f(y) + f(z) \geq 3f(s), \quad s = \frac{x + y + z}{3} = 0,
\]
where
\[
f(u) = 5e^u - \sqrt{25e^{2u} + 144}, \quad u \in \mathbb{R}.
\]
We will show that \(f(u)\) is convex for \(u \leq 0\). From
\[
f''(u) = 5w\left[1 - \frac{5w(25w^2 + 288)}{(25w^2 + 144)^{3/2}}\right], \quad w = e^u \in (0, 1],
\]
we need to show that
\[
(25w^2 + 144)^3 \geq 25w^2(25w^2 + 288)^2.
\]
Setting \(25w^2 = 144z\), we have \(z \in \left(0, \frac{25}{144}\right]\) and
\[
(25w^2 + 144)^3 - 25w^2(25w^2 + 288)^2 = 144^3(z + 1)^3 - 144^3z(z + 2)^2
\]
\[
= 144^3(1 - z - z^2) > 0.
\]
By the LHCF-Theorem, it suffices to prove the original inequality for 
\[ a = t^2, \quad b = c = 1/t, \quad t > 0; \]
that is,
\[ 5t^3 + 24t + 10 \geq \sqrt{25t^6 + 144t^2} + 2\sqrt{25 + 144t^2}. \]
Squaring and dividing by 4t give
\[ 60t^3 + 25t^2 - 36t + 120 \geq \sqrt{(25t^4 + 144)(144t^2 + 25)}. \]
Squaring again and dividing by 120, the inequality becomes
\[ 25t^5 - 36t^4 + 105t^3 - 112t^2 - 72t + 90 \geq 0, \]
\[ (t - 1)^2(25t^2 + 14t^2 + 108t + 90) \geq 0. \]
The equality holds for \( a = b = c = 1. \)

\[ \square \]

**P 1.76.** If \( a, b, c \) are positive real numbers so that \( abc = 1 \), then
\[ \sqrt{16a^2 + 9} + \sqrt{16b^2 + 9} + \sqrt{16c^2 + 9} \geq 4(a + b + c) + 3. \]

*(Vasile C., 2008)*

**Solution.** Using the substitution
\[ a = e^x, \quad b = e^y, \quad c = e^z, \]
we need to show that
\[ f(x) + f(y) + f(z) \geq 3f(s), \quad s = \frac{x + y + z}{3} = 0, \]
where
\[ f(u) = \sqrt{16e^{2u} + 9} - 4e^u, \quad u \in \mathbb{R}. \]
We will show that \( f(u) \) is convex for \( u \geq 0 \). From
\[ f''(u) = 4w \left[ \frac{4w(16w^2 + 18)}{(16w^2 + 9)^{3/2}} - 1 \right], \quad w = e^u \geq 1, \]
we need to show that
\[ 16w^2(16w^2 + 18)^2 \geq (16w^2 + 9)^3. \]
Setting $16w^2 = 9z$, we have $z \geq \frac{16}{9}$ and

\[
16w^2(16w^2 + 18)^2 - (16w^2 + 9)^3 = 729z(z + 2)^2 - 729(z + 1)^3
\]
\[
= 729(z^2 + z - 1) > 0.
\]

By the RHCF-Theorem, it suffices to prove the original inequality for

\[
a = t^2, \quad b = c = 1/t, \quad t > 0;
\]

that is,

\[
\sqrt{16t^6 + 9t^2 + 2\sqrt{16 + 9t^2}} \geq 4t^3 + 3t + 8.
\]

Squaring and dividing by $4t$ give

\[
\sqrt{(16t^4 + 9)(9t^2 + 16)} \geq 6t^3 + 16t^2 - 9t + 12.
\]

Squaring again and dividing by $12t$, the inequality becomes

\[
9t^5 - 16t^4 + 9t^3 + 12t^2 - 32t + 18 \geq 0,
\]

\[
(t - 1)^2(9t^3 + 2t^2 + 4t + 18) \geq 0.
\]

The equality holds for $a = b = c = 1$.

\[\square\]

**P 1.77.** If $ABC$ is a triangle, then

\[
\sin A \left(2 \sin \frac{A}{2} - 1\right) + \sin B \left(2 \sin \frac{B}{2} - 1\right) + \sin C \left(2 \sin \frac{C}{2} - 1\right) \geq 0.
\]

*(Lorian Saceanu, 2015)*

**Solution.** Write the inequality as

\[
f(A) + f(B) + f(C) \geq 3f(s), \quad s = \frac{A+B+C}{3} = \frac{\pi}{3},
\]

where

\[
f(u) = \sin u \left(2 \sin \frac{u}{2} - 1\right) = \cos \frac{u}{2} - \cos \frac{3u}{2} - \sin u, \quad u \in \mathbb{I} = [0, \pi].
\]

We will show that $f$ is convex on $\mathbb{I}$. Indeed, for $u \in [0, \pi/3]$, we have

\[
f''(u) = \cos \frac{u}{2} \left(2 + 2 \sin \frac{u}{2} - 9 \sin^2 \frac{u}{2}\right) \geq \cos \frac{u}{2} \left(2 + 2 \sin \frac{u}{2} - 12 \sin^2 \frac{u}{2}\right)
\]
\[
= 2 \cos \frac{u}{2} \left(1 + 3 \sin \frac{u}{2}\right) \left(1 - 2 \sin \frac{u}{2}\right) \geq 0.
\]
By the LHCF-Theorem, it suffices to prove the original inequality for $B = C$, when it transforms into

$$\sin 2B(2 \cos B - 1) + 2 \sin B \left(2 \sin \frac{B}{2} - 1\right) \geq 0,$$

$$\sin B \sin \frac{B}{2} \left(\sin \frac{B}{2} + 1\right) \left(2 \sin \frac{B}{2} - 1\right)^2 \geq 0.$$

The equality occurs for an equilateral triangle, and for a degenerate triangle with $A = \pi$ and $B = C = 0$ (or any cyclic permutation).

**Remark.** Based on this inequality, we can prove the following statement:

- If $ABC$ is a triangle, then

$$\sin 2A(2 \cos A - 1) + \sin 2B(2 \cos B - 1) + \sin 2C(2 \cos C - 1) \geq 0,$$

*with equality for an equilateral triangle, for a degenerate triangle with $A = 0$ and $B = C = \pi/2$ (or any cyclic permutation), and for a degenerate triangle with $A = \pi$ and $B = C = 0$ (or any cyclic permutation).*

If $ABC$ is an acute or right triangle, then this inequality follows by replacing $A, B$ and $C$ with $\pi - 2A, \pi - 2B$ and $\pi - 2C$ in the inequality from P 1.77. Consider now that

$$A > \frac{\pi}{2} > B \geq C \geq 0.$$

The inequality is true for $B \leq \pi/3$, because

$$\sin 2A(2 \cos A - 1) \geq 0, \quad \sin 2B(2 \cos B - 1) \geq 0, \quad \sin 2C(2 \cos C - 1) \geq 0.$$

Consider further that

$$\frac{2\pi}{3} > A > \frac{\pi}{2} > B > \frac{\pi}{3} \geq C \geq 0.$$

From

$$1 - 2 \cos A > 1 - 2 \cos B,$$

it follows that

$$(-\sin 2A)(1 - 2 \cos A) > (-\sin 2A)(1 - 2 \cos B).$$

Therefore it suffices to

$$(-\sin 2A)(1 - 2 \cos B) + \sin 2B(2 \cos B - 1) + \sin 2C(2 \cos C - 1) \geq 0,$$

which is equivalent to

$$(\sin 2A + \sin 2B)(2 \cos B - 1) + \sin 2C(2 \cos C - 1) \geq 0,$$
Half Convex Function Method

\[2 \sin C \cos(A - B)(2 \cos B - 1) + 2 \sin C \cos(2 \cos C - 1) \geq 0.\]

This inequality is true if

\[\cos(A - B)(2 \cos B - 1) + \cos(2 \cos C - 1) \geq 0,\]

which can be written as

\[\cos(2 \cos C - 1) \geq \cos(A - B)(1 - 2 \cos B).\]

Since

\[C < A - B < \frac{2\pi}{3} - \frac{\pi}{3} = \frac{\pi}{3},\]

we have \(\cos C > \cos(A - B)\). Therefore, it suffices to show that

\[2 \cos C - 1 \geq 1 - 2 \cos B,\]

which is equivalent to

\[\cos B + \cos C \geq 1.\]

From \(B + C < \pi/2\), we get \(\cos B > \cos(\pi/2 - C) = \sin C\), hence

\[\cos B + \cos C > \sin C + \cos C = \sqrt{1 + \sin 2C} \geq 1.\]

\[\square\]

**P 1.78.** If \(ABC\) is an acute or right triangle, then

\[\sin 2A \left(1 - 2 \sin \frac{A}{2}\right) + \sin 2B \left(1 - 2 \sin \frac{B}{2}\right) + \sin 2C \left(1 - 2 \sin \frac{C}{2}\right) \geq 0.\]

*(Vasile C., 2015)*

**Solution.** Write the inequality as

\[f(A) + f(B) + f(C) \geq 3f(s), \quad s = \frac{A + B + C}{3} = \frac{\pi}{3},\]

where

\[f(u) = \sin 2u \left(1 - 2 \sin \frac{u}{2}\right) = \sin 2u - \cos \frac{3u}{2} + \cos \frac{5u}{2}, \quad u \in \mathbb{I} = [0, \pi/2].\]

We will show that \(f\) is convex on \([s, \pi/2]\). From

\[f''(u) = -4 \sin 2u + \frac{9}{4} \cos \frac{3u}{2} - \frac{25}{4} \cos \frac{5u}{2}\]

and

\[\cos \frac{3u}{2} - \cos \frac{5u}{2} = 2 \sin \frac{u}{2} \sin 2u \geq 0,\]
we get
\[
f''(u) \geq -4 \sin 2u + \frac{9}{4} \cos \frac{5u}{2} - \frac{25}{4} \cos \frac{5u}{2}
\]
\[
= -4 \left[ \sin 2u + \sin \frac{\pi - 5u}{2} \right] = 8 \sin \frac{\pi - u}{4} \cos \frac{5\pi - 9u}{4}.
\]
For \(\pi/3 \leq u \leq \pi/2\), we have
\[
\frac{\pi}{8} \leq \frac{5\pi - 9u}{4} \leq \frac{\pi}{2},
\]
hence \(f''(u) \geq 0\). By the RHCF-Theorem, it suffices to prove the original inequality for \(B = C\), \(0 \leq B \leq \pi/2\), when it becomes
\[
-\sin 4B(1 - 2\cos B) + 2\sin 2B \left( 1 - 2\sin \frac{B}{2} \right) \geq 0,
\]
\[
2\sin 2B \left[ \cos 2B(2\cos B - 1) + 1 - \sin \frac{B}{2} \right] \geq 0.
\]
We need to show that
\[
\cos 2B(2\cos B - 1) + 1 - \sin \frac{B}{2} \geq 0,
\]
which is equivalent to \(g(t) \geq 0\), where
\[
g(t) = (1 - 8t^2 + 8t^4)(1 - 4t^2) + 1 - 2t, \quad t = \sin \frac{B}{2}, \quad 0 \leq t \leq \frac{1}{\sqrt{2}}.
\]
Indeed, we have
\[
g(t) = 2(1 - t)^2(1 + 3t + 2t^2 - 4t^3 - 4t^4) \geq 0
\]
because
\[
1 + 3t + 2t^2 - 4t^3 - 4t^4 \geq 1 + 3t + 2t^2 - 2t - 2t^2 = 1 + t > 0.
\]
The equality occurs for an equilateral triangle, for a degenerate triangle with \(A = 0\) and and \(B = C = \pi/2\) (or any cyclic permutation), and for a degenerate triangle with \(A = \pi\) and \(B = C = 0\) (or any cyclic permutation).

**Remark 1.** Actually, the inequality holds also for an obtuse triangle ABC. To prove this, consider that
\[
A > \frac{\pi}{2} > B \geq C > 0.
\]
The inequality is true for \(B \leq \pi/3\), because
\[
\sin 2A \left( 1 - 2\sin \frac{A}{2} \right) \geq 0, \quad \sin 2B \left( 1 - 2\sin \frac{B}{2} \right) \geq 0, \quad \sin 2C \left( 1 - 2\sin \frac{C}{2} \right) \geq 0.
\]
Consider further that 
\[ \frac{2\pi}{3} > A > \frac{\pi}{2} > B > \frac{\pi}{3} > C \geq 0. \]

From 
\[ 2 \sin \frac{A}{2} - 1 > 2 \sin \frac{B}{2} - 1, \]

it follows that 
\[ (-\sin 2A) \left( 2 \sin \frac{A}{2} - 1 \right) > (-\sin 2A) \left( 2 \sin \frac{B}{2} - 1 \right). \]

Therefore it suffices to 
\[ (-\sin 2A) \left( 2 \sin \frac{B}{2} - 1 \right) + \sin 2B \left( 1 - 2 \sin \frac{B}{2} \right) + \sin 2C \left( 1 - 2 \sin \frac{C}{2} \right) \geq 0, \]

which is equivalent to 
\[ (\sin 2A + \sin 2B) \left( 1 - 2 \sin \frac{B}{2} \right) + \sin 2C \left( 1 - 2 \sin \frac{C}{2} \right) \geq 0, \]
\[ 2 \sin C \cos (A - B) \left( 1 - 2 \sin \frac{B}{2} \right) + 2 \sin C \cos (1 - 2 \sin \frac{C}{2}) \geq 0. \]

This inequality is true if 
\[ \cos (A - B) \left( 1 - 2 \sin \frac{B}{2} \right) + \cos C \left( 1 - 2 \sin \frac{C}{2} \right) \geq 0, \]

which can be written as 
\[ \cos C \left( 1 - 2 \sin \frac{C}{2} \right) \geq \cos (A - B) \left( 2 \sin \frac{B}{2} - 1 \right). \]

Since 
\[ C < A - B < \frac{2\pi}{3} - \frac{\pi}{3} = \frac{\pi}{3}, \]
we have \( \cos C > \cos (A - B) \). Therefore, it suffices to show that 
\[ 1 - 2 \sin \frac{C}{2} \geq 2 \sin \frac{B}{2} - 1, \]

which is equivalent to 
\[ \sin \frac{B}{2} + \sin \frac{C}{2} \leq 1, \]
\[ 2 \sin \frac{B + C}{4} \cos \frac{B - C}{4} \leq 1. \]

This is true since 
\[ 2 \sin \frac{B + C}{4} < 2 \sin \frac{\pi}{8} < 1, \quad \cos \frac{B - C}{4} < 1. \]
Remark 2. Replacing $A$, $B$ and $C$ in P 1.78 by $\pi - 2A$, $\pi - 2B$ and $\pi - 2C$, respectively, we get the following inequality for an acute or right triangle $ABC$:

$$\sin 4A(2\cos A - 1) + \sin 4B(2\cos B - 1) + \sin 4C(2\cos C - 1) \geq 0,$$

with equality for an equilateral triangle, for a triangle with $A = \pi/2$ and $B = C = \pi/4$ (or any cyclic permutation), and for a degenerate triangle with $A = 0$ and and $B = C = \pi/2$ (or any cyclic permutation).

P 1.79. If $a, b, c, d$ are real numbers so that $a + b + c + d = 4$, then

$$\frac{a}{a^2 - a + 4} + \frac{b}{b^2 - b + 4} + \frac{c}{c^2 - c + 4} + \frac{d}{d^2 - d + 4} \leq 1.$$

(Sqing, 2015)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a + b + c + d}{4} = 1,$$

where

$$f(u) = \frac{-u}{u^2 - u + 4}, \quad u \in \mathbb{R}.$$

We see that

$$f(u) - f(2) = \frac{(u - 2)^2}{3(u^2 - u + 4)} \geq 0.$$

From

$$f''(u) = \frac{2(-u^3 + 12u - 4)}{(u^2 - u + 4)^3},$$

it follows that $f$ is convex on $[1, 2]$. Define the function

$$f_0(u) = \begin{cases} f(u), & u \leq 2 \\ f(2), & u > 2 \end{cases}.$$

Since $f_0(u) \leq f(u)$ for $u \in \mathbb{R}$ and $f_0(1) = f(1)$, it suffices to show that

$$f_0(a) + f_0(b) + f_0(c) + f_0(d) \geq 4f_0(s).$$

The function $f_0$ is convex on $[1, \infty)$ because it is differentiable on $[1, \infty)$ and its derivative

$$f'_0(u) = \begin{cases} f'(u), & u \leq 2 \\ 0, & u > 2 \end{cases}.$$
is continuous and increasing on $[1, \infty)$. Therefore, by the RHCF-Theorem, we only need to show that $f_0(x) + 3f_0(y) \geq 4f_0(1)$ for all $x, y \in \mathbb{R}$ so that $x \leq 1 \leq y$ and $x + 3y = 4$. There are two cases to consider: $y \leq 2$ and $y > 2$.

**Case 1:** $y \leq 2$. The inequality $f_0(x) + 3f_0(y) \geq 4f_0(1)$ is equivalent to $f(x) + 3f(y) \geq 4f(1)$. According to Note 1, this is true if $h(x, y) \geq 0$ for $x + 3y = 4$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{u - 4}{4(u^2 - u + 4)},$$

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{4(x + y) - xy}{4(x^2 - x + 4)(y^2 - y + 4)} - \frac{3(y - 2)^2 + 4}{4(x^2 - x + 4)(y^2 - y + 4)} > 0.$$

**Case 2:** $y > 2$. From $y > 2$ and $x + 3y = 4$, we get $x < -2$ and

$$f_0(x) + 3f_0(y) - 4f_0(1) = f(x) + 3f(2) - 4f(1) = \frac{-x}{x^2 - x + 4} > 0.$$

The equality holds for $a = b = c = d = 1$.

**P 1.80.** Let $a, b, c$ be nonnegative real numbers so that $a + b + c = 2$. If

$$k_0 \leq k \leq 3, \quad k_0 = \frac{\ln 2}{\ln 3 - \ln 2} \approx 1.71,$$

then

$$a^k(b + c) + b^k(c + a) + c^k(a + b) \leq 2.$$

**Solution.** Write the inequality as

$$f(a) + f(b) + f(c) \leq 2,$$

where

$$f(u) = u^k(2 - u), \quad u \in [0, \infty).$$

From

$$f''(u) = ku^{k-2}[2k - 2 - (k + 1)u],$$

it follows that $f$ is convex on $\left[0, \frac{2k - 2}{k + 1}\right]$ and concave on $\left[\frac{2k - 2}{k + 1}, 2\right]$. According to LCRF-Theorem, the sum $f(a) + f(b) + f(c)$ is maximum when either $a = 0$ or $0 < a \leq b = c$. 

\[\square\]
Case 1: \( a = 0 \). We need to show that
\[
bc(b^{k-1} + c^{k-1}) \leq 2
\]
for \( b + c = 2 \). Since \( 0 < (k-1)/2 \leq 1 \), Bernoulli’s inequality gives
\[
b^{k-1} + c^{k-1} = (b^2)^{(k-1)/2} + (c^2)^{(k-1)/2} \leq 1 + \frac{k-1}{2}(b^2 - 1) + 1 + \frac{k-1}{2}(c^2 - 1) = 3 - k + \frac{k-1}{2}(b^2 + c^2).
\]
Thus, it suffices to show that
\[
(3-k)bc + \frac{k-1}{2}bc(b^2 + c^2) \leq 2.
\]
Since
\[
bc \leq \left(\frac{b+c}{2}\right)^2 = 1,
\]
we only need to show that
\[
3-k + \frac{k-1}{2}bc(b^2 + c^2) \leq 2,
\]
which is equivalent to
\[
bc(b^2 + c^2) \leq 2.
\]
Indeed, we have
\[
8[2 - bc(b^2 + c^2)] = (b + c)^4 - 8bc(b^2 + c^2) = (b - c)^4 \geq 0.
\]
Case 2: \( 0 < a \leq b = c \). We only need to prove the homogeneous inequality
\[
a^k(b + c) + b^k(c + a) + c^k(a + b) \leq 2 \left(\frac{a + b + c}{2}\right)^{k+1}
\]
for \( b = c = 1 \) and \( 0 < a \leq 1 \); that is,
\[
\left(1 + \frac{a}{2}\right)^{k+1} - a^k - a - 1 \geq 0.
\]
Since \( \left(1 + \frac{a}{2}\right)^{k+1} \) is increasing and \( a^k \) is decreasing with respect to \( k \), it suffices to consider the case \( k = k_0 \); that is, to prove that \( g(a) \geq 0 \), where
\[
g(a) = \left(1 + \frac{a}{2}\right)^{k_0+1} - a^{k_0} - a - 1, \quad 0 < a \leq 1.
\]
We have
\[
g'(a) = \frac{k_0 + 1}{2}\left(1 + \frac{a}{2}\right)^{k_0} - k_0a^{k_0-1} - 1,
\]
\[
\frac{1}{k_0} g''(a) = \frac{k_0 + 1}{4} (1 + \frac{a}{2})^{k_0-1} - \frac{k_0 - 1}{a^{2-k_0}}.
\]

Since \( g'' \) is increasing on \((0, 1] \), \( g''(0^+) = -\infty \) and
\[
\frac{1}{k_0} g''(1) = \frac{k_0 + 1}{4} (3/2)^{k_0-1} - k_0 + 1 = \frac{k_0 + 1}{3} - k_0 + 1 = \frac{2(2-k_0)}{3} > 0,
\]
there exists \( a_1 \in (0, 1) \) so that \( g''(a_1) = 0 \), \( g''(a) < 0 \) for \( a \in (0, a_1) \), \( g''(a) > 0 \) for \( a \in (a_1, 1] \). Therefore, \( g' \) is strictly decreasing on \([0, a_1]\) and strictly increasing on \([a_1, 1]\). Since
\[
g'(0) = \frac{k_0 - 1}{2} > 0, \quad g'(1) = \frac{k_0 + 1}{2} [(3/2)^{k_0} - 2] = 0,
\]
there exists \( a_2 \in (0, a_1) \) so that \( g'(a_2) = 0 \), \( g'(a) > 0 \) for \( a \in [0, a_2) \), \( g'(a) < 0 \) for \( a \in (a_2, 1) \). Thus, \( g \) is strictly increasing on \([0, a_2]\) and strictly decreasing on \([a_2, 1]\). Consequently,
\[
g(a) \geq \min\{g(0), g(1)\},
\]
and from
\[
g(0) = 0, \quad g(1) = (3/2)^{k_0+1} - 3 = 0,
\]
we get \( g(a) \geq 0 \).

The equality holds for \( a = 0 \) and \( b = c \) (or any cyclic permutation). If \( k = k_0 \), then the equality holds also for \( a = b = c \).

\[\square\]

**P 1.81.** If \( a_1, a_2, \ldots, a_n \) are positive real numbers so that \( a_1 + a_2 + \cdots + a_n = n \), then
\[
(n+1)^2 \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right) \geq 4(n+1)(a_1^2 + a_2^2 + \cdots + a_n^2) + n(n^2 - 3n - 6).
\]

*(Vasile C., 2006)*

**Solution.** Write the inequality as
\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq n(n^2 - 3n - 6),
\]
where
\[
f(u) = \frac{(n+1)^2}{u} - 4(n+2)u^2, \quad u \in (0, \infty).
\]

From
\[
f''(u) = \frac{2(n+1)^2}{u^3} - 8(n+2),
\]
it follows that \( f \) is strictly convex on \((0, c]\) and strictly concave on \([c, \infty)\), where
\[
c = \sqrt[4]{\frac{(n+1)^2}{4(n+2)}}.
\]
According to LCRCF-Theorem and Note 5, it suffices to consider the case
\[ a_1 = a_2 = \cdots = a_{n-1} = x, \quad a_n = n - (n-1)x, \quad 0 < x \leq 1, \]
when the inequality becomes as follows:
\[
(n+1)^2 \left( \frac{n-1}{x} + \frac{1}{a_n} \right) \geq 4(n+2)[(n-1)x^2 + a_n^2] + n(n^2 - 3n - 6),
\]
\[
n(n-1)(2x-1)^2[(n+2)(n-1)x^2 - (n+2)(2n-1)x + (n+1)^2] \geq 0.
\]
The last inequality is true since
\[
(n-1)x^2 - (2n-1)x + \frac{(n+1)^2}{n+2} = (n-1)\left(x - \frac{2n-1}{2n-2}\right)^2 + \frac{3(n-2)}{4(n-1)(n+2)} \geq 0.
\]
The equality holds for
\[
a_1 = a_2 = \cdots = a_{n-1} = \frac{1}{2}, \quad a_n = \frac{n+1}{2}
\]
(or any cyclic permutation).

\[\square\]

**P 1.82.** If \( a, b, c \) are nonnegative real numbers so that \( a + b + c = 12 \), then
\[
(a^2 + 10)(b^2 + 10)(c^2 + 10) \geq 13310.
\]

(\textit{Vasile C., 2006})

**Solution.** Write the inequality as
\[
f(a) + f(b) + f(c) \geq 2 \ln 11 + \ln 110,
\]
where
\[
f(u) = \ln(u^2 + 10), \quad u \in [0, 12].
\]
From
\[
f''(u) = \frac{2(10-u^2)}{(u^2+10)^2},
\]
it follows that \( f \) is convex on \([0, \sqrt{10}]\) and concave on \([\sqrt{10}, 12]\). According to LCRCF-Theorem, the sum \( f(a) + f(b) + f(c) \) is minimum when \( a = b \leq c \). Therefore, it suffices to prove that \( g(a) \geq 0 \), where
\[
g(a) = 2f(a) + f(c) - 2 \ln 11 - \ln 110, \quad c = 12 - 2a, \quad a \in [0, 4].
\]
Since \( c'(a) = -2 \), we have
\[
g'(a) = 2f'(a) - 2f'(c) = 4 \left( \frac{a}{a^2 + 10} - \frac{c}{c^2 + 10} \right) = \frac{4(a - c)(10 - ac)}{(a^2 + 10)(c^2 + 10)} = \frac{24(4 - a)(5 - a)(a - 1)}{(a^2 + 10)(c^2 + 10)}.
\]

Therefore, \( g'(a) < 0 \) for \( a \in [0, 1] \) and \( g'(a) > 0 \) for \( a \in (1, 4) \), hence \( g \) is strictly decreasing on \([0, 1]\) and strictly increasing on \([1, 4]\). Thus, we have
\[
g(a) \geq g(1) = 0.
\]
The equality holds for \( a = b = 1 \) and \( c = 10 \) (or any cyclic permutation).

**Remark.** Similarly, we can prove the following generalization:

- Let \( a_1, a_2, \ldots, a_n \) be nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n = 2n(n-1) \). If \( k = (n-1)(2n-1) \), then
\[
(a^2 + k)(a_2^2 + k) \cdots (a_n^2 + k) \geq k(k+1)^n,
\]
with equality for \( a_1 = k \) and \( a_2 = \cdots = a_n = 1 \) (or any cyclic permutation).

\[\Box\]

**P 1.83.** If \( a_1, a_2, \ldots, a_n \) are nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n = n \), then
\[
(a_1^2 + 1)(a_2^2 + 1) \cdots (a_n^2 + 1) \geq \frac{(n^2 - 2n + 2)^n}{(n-1)^{2n-2}}.
\]

(Vasile C., 2006)

**Solution.** Write the inequality as
\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq \ln k, \quad k = \frac{(n^2 - 2n + 2)^n}{(n-1)^{2n-2}},
\]
where
\[
f(u) = \ln(u^2 + 1), \quad u \in [0, n].
\]
From
\[
f''(u) = \frac{2(1-u^2)}{(u^2 + 1)^2},
\]
it follows that \( f \) is strictly convex on \([0, 1]\) and strictly concave on \([1, n]\). According to LRCF-Theorem, it suffices to consider the case \( a_1 = a_2 = \cdots = a_{n-1} \leq a_n \); that is, to show that \( g(x) \geq 0 \), where
\[
g(x) = (n-1)f(x) + f(y) - \ln k, \quad y = n - (n-1)x, \quad x \in [0, 1].
\]
Since \(y'(x) = -(n-1)\), we get
\[
g'(x) = (n-1)f'(x) - (n-1)f'(y) = (n-1)[f'(x) - f'(y)]
= 2(n-1)\left(\frac{x}{x^2 + 1} - \frac{y}{y^2 + 1}\right) = \frac{2(n-1)(x-y)(1-xy)}{(x^2 + 1)(y^2 + 1)}
= \frac{2n(n-1)(x-y)^2}{(x^2 + 1)(y^2 + 1)}.
\]

Therefore, \(g'(x) \leq 0\) for \(x \in \left[0, \frac{1}{n-1}\right]\) and \(g'(x) \geq 0\) for \(x \in \left[\frac{1}{n-1}, n\right]\), hence \(g\) is decreasing on \(\left[0, \frac{1}{n-1}\right]\) and increasing on \(\left[\frac{1}{n-1}, 1\right]\). Since \(g\left(\frac{1}{n-1}\right) = 0\), the conclusion follows.

The equality holds for \(a_1 = a_2 = \cdots = a_{n-1} = \frac{1}{n-1}\) and \(a_n = n-1\) (or any cyclic permutation). 

\[\square\]

**P 1.84.** If \(a, b, c\) are nonnegative real numbers so that \(a + b + c = 3\), then
\[
(a^2 + 2)(b^2 + 2)(c^2 + 2) \leq 44.
\]

*(Vasile C., 2006)*

**Solution.** Write the inequality as
\[
f(a) + f(b) + f(c) \leq \ln 44,
\]
where
\[
f(u) = \ln(u^2 + 2), \quad u \in [0, 3].
\]
From
\[
f''(u) = \frac{2(2-u^2)}{(u^2 + 2)^2},
\]
it follows that \(f\) is strictly convex on \([0, \sqrt{2}]\) and strictly concave on \([\sqrt{2}, 3]\). According to LCRCF-Theorem, the sum \(f(a) + f(b) + f(c)\) is maximum for either \(a = 0\) or \(0 < a \leq b = c\).

**Case 1:** \(a = 0\). We need to show that \(b + c = 3\) involves
\[
(b^2 + 2)(c^2 + 2) \leq 22,
\]
which is equivalent to
\[
bc(bc - 4) \leq 0.
\]
This is true because
\[
bc \leq \left(\frac{b + c}{2}\right)^2 = \frac{9}{4} < 4.
\]
Case 2: $0 < a \leq b = c$. We need to show that $a + 2b = 3$ $(0 < a \leq 1)$ involves

$$(a^2 + 2)(b^2 + 2) \leq 44,$$

which is equivalent to $g(a) \leq 0$, where

$$g(a) = \ln(a^2 + 2) + 2\ln(b^2 + 2) - \ln 44, \quad b = \frac{3-a}{2}, \quad a \in (0, 1].$$

Since $b'(a) = -1/2$, we have

$$g'(a) = \frac{2a}{a^2 + 2} - \frac{2b}{b^2 + 2} = \frac{2(a-b)(2-ab)}{(a^2 + 2)(b^2 + 2)}$$

$$= \frac{3(a-1)(a^2 - 3a + 4)}{2(a^2 + 2)(b^2 + 2)}.$$ 

Because

$$a^2 - 3a + 4 = (a-2)^2 + a > 0,$$

we have $g'(a) < 0$ for $a \in (0, 1)$, $g$ is strictly decreasing on $[0, 1]$, hence it suffices to show that $g(0) \leq 0$. This reduces to $16 \cdot 22 \geq 17^2$, which is true because

$$16 \cdot 22 - 17^2 = 63 > 0.$$ 

The equality holds for $a = b = 0$ and $c = 3$ (or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

- Let $a, b, c$ be nonnegative real numbers so that $a + b + c = 3$. If $k \geq \frac{9}{8}$, then

$$(a^2 + k)(b^2 + k)(c^2 + k) \leq k^2(k + 9),$$

with equality for $a = b = 0$ and $c = 3$ (or any cyclic permutation). If $k = 9/8$, then the equality holds also for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

\[\Box\]

P 1.85. If $a, b, c$ are nonnegative real numbers so that $a + b + c = 3$, then

$$(a^2 + 1)(b^2 + 1)(c^2 + 1) \leq \frac{169}{16}.$$ 

(Vasile C., 2006)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) \leq \ln 169 - \ln 16,$$
where
\[ f(u) = \ln(u^2 + 1), \quad u \in [0, 3]. \]

From
\[ f''(u) = \frac{2(1-u^2)}{(u^2 + 1)^2}, \]

it follows that \( f \) is strictly convex on \([0, 1]\) and strictly concave on \([1, 3]\). According to LCRCF-Theorem, it suffices to consider the cases \( a = 0 \) and \( 0 < a \leq b = c \).

**Case 1**: \( a = 0 \). We need to show that \( b + c = 3 \) involves
\[ (b^2 + 1)(c^2 + 1) \leq \frac{169}{16}, \]

which is equivalent to
\[ (4bc + 1)(4bc - 9) \leq 0. \]

This is true because
\[ 4bc \leq (b + c)^2 = 9. \]

**Case 2**: \( 0 < a \leq b = c \). We need to show that \( a + 2b = 3 \) \((0 < a \leq 1)\) involves
\[ (a^2 + 1)(b^2 + 1)^2 \leq \frac{169}{16}, \]

which is equivalent to \( g(a) \leq 0 \), where
\[ g(a) = \ln(a^2 + 1) + 2 \ln(b^2 + 1) - \ln 169 + \ln 16, \quad b = \frac{3-a}{2}, \quad a \in (0, 1]. \]

Since \( b'(a) = -1/2 \), we have
\[ g'(a) = \frac{2a}{a^2 + 1} - \frac{2b}{b^2 + 1} = \frac{2(a - b)(1 - ab)}{(a^2 + 1)(b^2 + 1)} \]
\[ = \frac{3(a - 1)^2(a - 2)}{2(a^2 + 1)(b^2 + 1)} \leq 0, \]

hence \( g \) is strictly decreasing. Consequently, we have
\[ g(a) < g(0) = 0. \]

The equality holds for \( a = 0 \) and \( b = c = 3/2 \) (or any cyclic permutation).

\( \Box \)

**P 1.86.** If \( a, b, c \) are nonnegative real numbers so that \( a + b + c = 3 \), then
\[ (2a^2 + 1)(2b^2 + 1)(2c^2 + 1) \leq \frac{121}{4}. \]

(Vasile C., 2006)
**Solution.** Write the inequality as

\[ f(a) + f(b) + f(c) \leq \ln 121 - \ln 4, \]

where

\[ f(u) = \ln(2u^2 + 1), \quad u \in [0, 3]. \]

From

\[ f''(u) = \frac{4(1 - 2u^2)}{(2u^2 + 1)^2}, \]

it follows that \( f \) is strictly convex on \([0, 1/\sqrt{2}]\) and strictly concave on \([1/\sqrt{2}, 3]\).

By LCRCF-Theorem, it suffices to consider the cases \( a = 0 \) and \( 0 < a \leq b = c \).

**Case 1:** \( a = 0 \). We need to show that \( b + c = 3 \) involves

\[ (2b^2 + 1)(2c^2 + 1) \leq \frac{121}{4}, \]

which is equivalent to

\[ (4bc + 5)(4bc - 9) \leq 0. \]

This is true because

\[ 4bc \leq (b + c)^2 = 9. \]

**Case 2:** \( 0 < a \leq b = c \). We need to show that \( a + 2b = 3 \) \((0 < a \leq 1)\) involves

\[ (2a^2 + 1)(2b^2 + 1)^2 \leq \frac{121}{4}, \]

which is equivalent to \( g(a) \leq 0 \), where

\[ g(a) = \ln(2a^2 + 1) + 2\ln(2b^2 + 1) - \ln 121 + \ln 4, \quad b = \frac{3 - a}{2}, \quad a \in (0, 1]. \]

Since \( b'(a) = -1/2 \), we have

\[
g'(a) = \frac{4a}{2a^2 + 1} - \frac{4b}{2b^2 + 1} = \frac{4(a - b)(1 - 2ab)}{(2a^2 + 1)(2b^2 + 1)}
= \frac{6(a - 1)(a^2 - 3a + 1)}{(2a^2 + 1)(2b^2 + 1)}
= \frac{3(1 - a)(3 + \sqrt{5} - 2a)(2a - 3 + \sqrt{5})}{2(2a^2 + 1)(2b^2 + 1)},
\]

hence \( g'(\frac{3 - \sqrt{5}}{2}) = 0 \), \( g'(a) < 0 \) for \( a \in \left[ 0, \frac{3 - \sqrt{5}}{2} \right) \), \( g'(a) > 0 \) for \( a \in \left( \frac{3 - \sqrt{5}}{2}, 1 \right) \).

Therefore, \( g \) is strictly decreasing on \( \left[ 0, \frac{3 - \sqrt{5}}{2} \right] \) and strictly increasing on \( \left[ \frac{3 - \sqrt{5}}{2}, 1 \right] \).

Since \( g(0) = 0 \), it suffices to show that \( g(1) \leq 0 \), which reduces to \( 27 \cdot 4 \leq 121 \).

The equality holds for \( a = 0 \) and \( b = c = 3/2 \) (or any cyclic permutation). \( \square \)
P 1.87. If \(a, b, c, d\) are nonnegative real numbers so that \(a + b + c + d = 4\), then
\[
(a^2 + 3)(b^2 + 3)(c^2 + 3)(d^2 + 3) \leq 513.
\]
(Vasile C., 2006)

Solution. Write the inequality as
\[
f(a) + f(b) + f(c) + f(d) \leq \ln 513,
\]
where
\[
f(u) = \ln(u^2 + 3), \quad u \in [0, 4].
\]
From
\[
f''(u) = \frac{2(3 - u^2)}{(u^2 + 3)^2},
\]
it follows that \(f\) is strictly convex on \([0, \sqrt{3}]\) and strictly concave on \([\sqrt{3}, 4]\). By LCRF-Theorem, it suffices to consider the cases \(a = 0\) and \(0 < a \leq b = c\).

Case 1: \(a = 0\). We need to show that \(b + c + d = 4\) involves
\[
(b^2 + 3)(c^2 + 3)(d^2 + 3) \leq 171.
\]
Substituting \(b, c, d\) by \(\frac{4b}{3}, \frac{4c}{3}, \frac{4d}{3}\), respectively, we need to show that \(b + c + d = 3\) involves
\[
(b^2 + k)(c^2 + k)(d^2 + k) \leq k^2(k + 9),
\]
where \(k = \frac{27}{16}\). According to Remark from the proof of P 1.84, this inequality holds for all \(k \geq \frac{9}{8}\).

Case 2: \(0 < a \leq b = c\). We need to show that \(a + 3b = 4\) (\(0 < a \leq 1\)) involves
\[
(a^2 + 3)(b^2 + 3)^3 \leq 513,
\]
which is equivalent to \(g(a) \leq 0\), where
\[
g(a) = \ln(a^2 + 3) + 3 \ln(b^2 + 3) - \ln 513, \quad b = \frac{4-a}{3}, \quad a \in (0, 1].
\]
Since \(b'(a) = -1/3\), we have
\[
g'(a) = \frac{2a}{a^2 + 3} - \frac{2b}{b^2 + 3} = \frac{2(a - b)(3 - ab)}{(a^2 + 3)(b^2 + 3)}
\]
\[
= \frac{8(a - 1)(a^2 - 4a + 9)}{9(a^2 + 3)(b^2 + 3)}.
\]
Because
\[
a^2 - 4a + 9 = (a - 2)^2 + 5 > 0,
\]
we have \(g'(a) > 0\) for \(a \in [0, 1]\), \(g\) is strictly decreasing on \([0, 1]\), hence it suffices to show that \(g(0) \leq 0\). This reduces to show that the original inequality holds for \(a = 0\) and \(b = c = d = 4/3\), which follows immediately from the case 1.

The equality holds for \(a = b = c = 0\) and \(d = 4\) (or any cyclic permutation). \(\Box\)
**P 1.88.** If \( a, b, c, d \) are nonnegative real numbers so that \( a + b + c + d = 4 \), then
\[
(a^2 + 2)(b^2 + 2)(c^2 + 2)(d^2 + 2) \leq 144.
\]

(Vasile C., 2006)

**Solution.** Write the inequality as
\[
f(a) + f(b) + f(c) + f(d) \leq \ln 144,
\]
where
\[
f(u) = \ln(u^2 + 2), \quad u \in [0, 4].
\]
From
\[
f''(u) = \frac{2(2 - u^2)}{(u^2 + 2)^2},
\]
it follows that \( f \) is strictly convex on \([0, \sqrt{2}]\) and strictly concave on \([\sqrt{2}, 4]\). By LCRF-Theorem, it suffices to consider the cases \( a = 0 \) and \( 0 < a \leq b = c \).

**Case 1:** \( a = 0 \). We need to show that \( b + c + d = 4 \) involves
\[
(b^2 + 2)(c^2 + 2)(d^2 + 2) \leq 72.
\]
Substituting \( b, c, d \) by \( 4b/3, 4c/3, 4d/3 \), respectively, we need to show that \( b + c + d = 3 \) involves
\[
(8b^2 + 9)(8c^2 + 9)(8d^2 + 9) \leq 9^4.
\]
(see Remark from the proof of P 1.84).

**Case 2:** \( 0 < a \leq b = c = d \). We need to show that \( a + 3b = 4 \) \((0 < a \leq 1)\) involves
\[
(a^2 + 2)(b^2 + 2)^3 \leq 144,
\]
which is equivalent to \( g(a) \leq 0 \), where
\[
g(a) = \ln(a^2 + 2) + 3 \ln(b^2 + 2) - \ln 144, \quad b = \frac{4 - a}{3}, \quad a \in (0, 1].
\]
Since \( b'(a) = -1/3 \), we have
\[
g'(a) = \frac{2a}{a^2 + 2} - \frac{2b}{b^2 + 2} = \frac{2(a - b)(2 - ab)}{(a^2 + 2)(b^2 + 2)}
= \frac{8(a - 1)(a^2 - 4a + 6)}{9(a^2 + 2)(b^2 + 2)}.
\]
Because
\[
a^2 - 4a + 6 = (a - 2)^2 + 2 > 0,
\]
we have \( g'(a) > 0 \) for \( a \in [0, 1) \), \( g \) is strictly decreasing on \([0, 1]\), hence it suffices to show that \( g(0) \leq 0 \). This reduces to show that the original inequality holds for \( a = 0 \) and \( b = c = d = 4/3 \), which follows immediately from the case 1.

The equality holds for \( a = b = c = 0 \) and \( d = 4 \) (or any cyclic permutation), and also for \( a = b = 0 \) and \( c = d = 2 \) (or any permutation). \( \Box \)
Chapter 2

Half Convex Function Method for Ordered Variables

2.1 Theoretical Basis

The following statement is known as the Right Half Convex Function Theorem for Ordered Variables (RHCF-OV Theorem).

**RHCF-OV Theorem** (Vasile Cîrtoaje, 2008). Let \( f \) be a real function defined on an interval \( \mathbb{I} \) and convex on \( \mathbb{I}_{\geq s} \), where \( s \in \text{int}(\mathbb{I}) \). The inequality

\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)
\]

holds for all \( a_1, a_2, \ldots, a_n \in \mathbb{I} \) satisfying

\[
a_1 + a_2 + \cdots + a_n = ns
\]

and

\[
a_1 \leq a_2 \leq \cdots \leq a_m \leq s, \quad m \in \{1, 2, \ldots, n-1\},
\]

if and only if

\[
f(x) + (n-m)f(y) \geq (1 + n-m)f(s)
\]

for all \( x, y \in \mathbb{I} \) so that

\[
x \leq s \leq y, \quad x + (n-m)y = (1 + n-m)s.
\]

**Proof.** For

\[
a_1 = x, \quad a_2 = \cdots = a_m = s, \quad a_{m+1} = \cdots = a_n = y,
\]

the inequality

\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s)
\]

becomes

\[
f(x) + (n-m)f(y) \geq (1 + n-m)f(s);
\]
thus, the necessity is proved. To prove the sufficiency, we assume that
\[ a_1 \leq a_2 \leq \cdots \leq a_n. \]
From \( a_1 \leq a_2 \leq \cdots \leq a_m \leq s \), it follows that there is an integer
\[ k \in \{m, m + 1, \ldots, n - 1\} \]
so that
\[ a_1 \leq \cdots \leq a_k \leq s \leq a_{k+1} \leq \cdots \leq a_n. \]
Since \( f \) is convex on \( I_{\geq s} \), we may apply Jensen's inequality to get
\[ f(a_{k+1}) + \cdots + f(a_n) \geq (n-k)f(z), \]
where
\[ z = \frac{a_{k+1} + \cdots + a_n}{n-k}, \quad z \in I. \]
Therefore, to prove the desired inequality
\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq f(s), \]
it suffices to show that
\[ f(a_1) + \cdots + f(a_k) + (n-k)f(z) \geq nf(s). \quad (*) \]
Let \( b_1, \ldots, b_k \) be defined by
\[ a_i + (n-m)b_i = (1+n-m)s, \quad i = 1, \ldots, k. \]
We claim that
\[ z \geq b_1 \geq \cdots \geq b_k \geq s, \quad b_1, \ldots, b_k \in I. \]
Indeed, we have
\[ b_1 \geq \cdots \geq b_k, \]
\[ b_k - s = \frac{s - a_k}{n-m} \geq 0, \]
and
\[ z \geq b_1 \]
because
\[
(n-m)b_1 = (1+n-m)s - a_1 \\
= -(m-1)s + (a_2 + \cdots + a_k) + (a_{k+1} + \cdots + a_n) \\
\leq -(m-1)s + (k-1)s + (a_{k+1} + \cdots + a_n) = \\
= (k-m)s + (n-k)s \leq (n-m)s.
\]
Since $b_1, \ldots, b_k \in \mathbb{I}_{\geq s}$, by hypothesis we have

\[
f(a_1) + (n-m)f(b_1) \geq (1+n-m)f(s),
\]

\[
\ldots
\]

\[
f(a_k) + (n-m)f(b_k) \geq (1+n-m)f(s),
\]

hence

\[
f(a_1) + \cdots + f(a_k) + (n-m)[f(b_1) + \cdots + f(b_k)] \geq k(1+n-m)f(s),
\]

\[
f(a_1) + \cdots + f(a_k) \geq k(1+n-m)f(s) - (n-m)[f(b_1) + \cdots + f(b_k)].
\]

According to this result, the inequality \( (*) \) is true if

\[
k(1+n-m)f(s) - (n-m)[f(b_1) + \cdots + f(b_k)] + (n-k)f(z) \geq nf(s),
\]

which is equivalent to

\[
 pf(z) + (k-p)f(s) \geq f(b_1) + \cdots + f(b_k), \quad p = \frac{n-k}{n-m} \leq 1.
\]

By Jensen's inequality, we have

\[
 pf(z) + (1-p)f(s) \geq f(w), \quad w = pz + (1-p)s \geq s.
\]

Thus, we only need to show that

\[
f(w) + (k-1)f(s) \geq f(b_1) + \cdots + f(b_k).
\]

Since the decreasingly ordered vector \( \vec{A}_k = (w,s, \ldots, s) \) majorizes the decreasingly ordered vector \( \vec{B}_k = (b_1, b_2, \ldots, b_k) \), this inequality follows from Karamata's inequality for convex functions.

Similarly, we can prove the Left Half Convex Function Theorem for Ordered Variables \( \text{(LHCF-OV Theorem)} \).

**LHCF-OV Theorem.** Let \( f \) be a real function defined on an interval \( \mathbb{I} \) and convex on \( \mathbb{I}_{\leq s} \), where \( s \in \text{int}(\mathbb{I}) \). The inequality

\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)
\]

holds for all \( a_1, a_2, \ldots, a_n \in \mathbb{I} \) satisfying

\[
a_1 + a_2 + \cdots + a_n = ns
\]

and

\[
a_1 \geq a_2 \geq \cdots \geq a_m \geq s, \quad m \in \{1, 2, \ldots, n-1\},
\]
if and only if
\[ f(x) + (n - m)f(y) \geq (1 + n - m)f(s) \]
for all \( x, y \in I \) so that
\[ x \geq s \geq y, \quad x + (n - m)y = (1 + n - m)s. \]

From the RHCF-OV Theorem and the LHCF-OV Theorem, we find the HCF-OV Theorem (Half Convex Function Theorem for Ordered Variables).

**HCF-OV Theorem.** Let \( f \) be a real function defined on an interval \( I \) and convex on \( I_{\geq s} \) (or \( I_{\leq s} \)), where \( s \in \text{int}(I) \). The inequality
\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \]
holds for all \( a_1, a_2, \ldots, a_n \in I \) so that \( a_1 + a_2 + \cdots + a_n = ns \)
and at least \( m \) of \( a_1, a_2, \ldots, a_n \) are smaller (greater) than \( s \), where \( m \in \{1, 2, \ldots, n-1\} \), if and only if
\[ f(x) + (n - m)f(y) \geq (1 + n - m)f(s) \]
for all \( x, y \in I \) satisfying \( x + (n - m)y = (1 + n - m)s \).

The RHCF-OV Theorem, the LHCF-OV Theorem and the HCF-OV Theorem are respectively generalizations of the RHCF-Theorem, the LHCF Theorem and the HCF-Theorem, because the last theorems can be obtained from the first theorems for \( m = 1 \).

**Note 1.** Let us denote
\[ g(u) = \frac{f(u) - f(s)}{u - s}, \quad h(x, y) = \frac{g(x) - g(y)}{x - y}. \]
In many applications, it is useful to replace the hypothesis
\[ f(x) + (n - m)f(y) \geq (1 + n - m)f(s) \]
in the RHCF-OV Theorem and the LHCF-OV Theorem by the equivalent condition
\[ h(x, y) \geq 0 \quad \text{for all} \quad x, y \in I \quad \text{so that} \quad x + (n - m)y = (1 + n - m)s. \]
This equivalence is true since
\[
\begin{align*}
 f(x) + (n - m)f(y) - (1 + n - m)f(s) &= [f(x) - f(s)] + (n - m)[f(y) - f(s)] \\
 &= (x - s)g(x) + (n - m)(y - s)g(y) \\
 &= \frac{n - m}{1 + n - m}(x - y)[g(x) - g(y)] \\
 &= \frac{n - m}{1 + n - m}(x - y)^2h(x, y).
\end{align*}
\]
**Note 2.** Assume that \( f \) is differentiable on \( I \), and let

\[
H(x, y) = \frac{f'(x) - f'(y)}{x - y}.
\]

The desired inequality of Jensen’s type in the RHCF-OV Theorem and the LHCF-OV Theorem holds true by replacing the hypothesis

\[
f(x) + (n - m)f(y) \geq (1 + n - m)f(s)
\]

with the more restrictive condition

\[
H(x, y) \geq 0 \text{ for all } x, y \in I \text{ so that } x + (n - m)y = (1 + n - m)s.
\]

To prove this, we will show that the new condition implies

\[
f(x) + (n - m)f(y) \geq (1 + n - m)f(s)
\]

for all \( x, y \in I \) so that \( x + (n - m)y = (1 + n - m)s \). Write this inequality as

\[
f_1(x) \geq (1 + n - m)f(s),
\]

where

\[
f_1(x) = f(x) + (n - m)f\left(\frac{(1 + n - m)s - x}{n - m}\right).
\]

From

\[
f'_1(x) = f'(x) - f'\left(\frac{(1 + n - m)s - x}{n - m}\right)
= f'(x) - f'(y)
= \frac{1 + n - m}{n - m}(x - s)H(x, y),
\]

it follows that \( f_1 \) is decreasing on \( I_{\leq s} \) and increasing on \( I_{\geq s} \); therefore,

\[
f_1(x) \geq f_1(s) = (1 + n - m)f(s).
\]

**Note 3.** The RHCF-OV Theorem and the LHCF-OV Theorem are also valid in the case when \( f \) is defined on \( I \setminus \{u_0\} \), where \( u_0 \in I_{\leq s} \) for the RHCF-OV Theorem, and \( u_0 \in I_{> s} \) for the LHCF-OV Theorem.

**Note 4.** The desired inequalities in the RHCF-OV Theorem and the LHCF-OV Theorem become equalities for

\[
a_1 = a_2 = \cdots = a_n = s.
\]

In addition, if there exist \( x, y \in I \) so that

\[
x + (n - m)y = (1 + n - m)s, \quad f(x) + (n - m)f(y) = (1 + n - m)f(s), \quad x \neq y,
\]
then the equality holds also for
\[ a_1 = x, \quad a_2 = \cdots = a_m = s, \quad a_{m+1} = \cdots = a_n = y. \]

Notice that these equality conditions are equivalent to
\[ x + (n - m)y = (1 + n - m)s, \quad h(x, y) = 0 \]
\((x < y)\) for the RHCF-OV Theorem, and \((x > y)\) for the LHCF-OV Theorem.

**Note 5.** The WRHCF-OV Theorem and the WLHCF-OV Theorem are extensions of the *weighted* Jensen’s inequality to right and left half convex functions with ordered variables (Vasile Cirtoaje, 2008).

**WRHCF-OV Theorem.** Let \( p_1, p_2, \ldots, p_n \) be positive real numbers so that
\[ p_1 + p_2 + \cdots + p_n = 1, \]
and let \( f \) be a real function defined on an interval \( \mathbb{I} \) and convex on \( \mathbb{I}_{\geq s} \), where \( s \in \text{int}(\mathbb{I}) \). The inequality
\[ p_1 f(x_1) + p_2 f(x_2) + \cdots + p_n f(x_n) \geq f(p_1 x_1 + p_2 x_2 + \cdots + p_n x_n) \]
holds for all \( x_1, x_2, \ldots, x_n \in \mathbb{I} \) so that \( p_1 x_1 + p_2 x_2 + \cdots + p_n x_n = s \) and
\[ x_1 \leq x_2 \leq \cdots \leq x_n, \quad x_m \leq s, \quad m \in \{1, 2, \ldots, n - 1\}, \]
if and only if
\[ f(x) + k f(y) \geq (1 + k) f(s) \]
for all \( x, y \in \mathbb{I} \) satisfying
\[ x \leq s \leq y, \quad x + ky = (1 + k)s, \]
where
\[ k = \frac{p_{m+1} + p_{m+2} + \cdots + p_n}{p_1}. \]

**WLHCF-OV Theorem.** Let \( p_1, p_2, \ldots, p_n \) be positive real numbers so that
\[ p_1 + p_2 + \cdots + p_n = 1, \]
and let \( f \) be a real function defined on an interval \( \mathbb{I} \) and convex on \( \mathbb{I}_{\leq s} \), where \( s \in \text{int}(\mathbb{I}) \). The inequality
\[ p_1 f(x_1) + p_2 f(x_2) + \cdots + p_n f(x_n) \geq f(p_1 x_1 + p_2 x_2 + \cdots + p_n x_n) \]
holds for all \( x_1, x_2, \ldots, x_n \in \mathbb{I} \) so that \( p_1 x_1 + p_2 x_2 + \cdots + p_n x_n = s \) and
\[ x_1 \geq x_2 \geq \cdots \geq x_n, \quad x_m \geq s, \quad m \in \{1, 2, \ldots, n - 1\}, \]
if and only if

\[ f(x) + kf(y) \geq (1 + k)f(s) \]

for all \( x, y \in \mathbb{I} \) satisfying

\[ x \geq s \geq y, \quad x + ky = (1 + k)s, \]

where

\[ k = \frac{p_{m+1} + p_{m+2} + \cdots + p_n}{p_1}. \]
2.2 Applications

2.1. If \(a, b, c, d\) are real numbers so that

\[
a \leq b \leq 1 \leq c \leq d, \quad a + b + c + d = 4,
\]

then

\[
(3a^2 - 2)(a - 1)^2 + (3b^2 - 2)(b - 1)^2 + (3c^2 - 2)(c - 1)^2 + (3d^2 - 2)(d - 1)^2 \geq 0.
\]

2.2. If \(a, b, c, d\) are nonnegative real numbers so that

\[
a \geq b \geq 1 \geq c \geq d, \quad a + b + c + d = 4,
\]

then

\[
\frac{1}{2a^3 + 5} + \frac{1}{2b^3 + 5} + \frac{1}{2c^3 + 5} + \frac{1}{2d^3 + 5} \leq \frac{4}{7}.
\]

2.3. If

\[
-\frac{2n - 1}{n - 1} \leq a_1 \leq \cdots \leq a_n \leq 1 \leq a_{n+1} \leq \cdots \leq a_{2n}, \quad a_1 + a_2 + \cdots + a_{2n} = 2n,
\]

then

\[
a_1^3 + a_2^3 + \cdots + a_{2n}^3 \geq 2n.
\]

2.4. Let \(a_1, a_2, \ldots, a_n \ (n \geq 3)\) be real numbers so that \(a_1 + a_2 + \cdots + a_n = n\). Prove that

(a) if \(-3 \leq a_1 \leq \cdots \leq a_{n-2} \leq 1 \leq a_{n-1} \leq a_n,\) then

\[
a_1^3 + a_2^3 + \cdots + a_n^3 \geq a_1^2 + a_2^2 + \cdots + a_n^2,
\]

(b) if \(-\frac{n - 1}{n - 3} \leq a_1 \leq a_2 \leq 1 \leq \cdots \leq a_n,\) then

\[
a_1^3 + a_2^3 + \cdots + a_n^3 + n \geq 2(a_1^2 + a_2^2 + \cdots + a_n^2).
\]

2.5. Let \(a_1, a_2, \ldots, a_n\) be nonnegative real numbers so that \(a_1 + a_2 + \cdots + a_n = n\) and let \(m \in \{1, 2, \ldots, n - 1\}\). Prove that

(a) if \(a_1 \leq a_2 \leq \cdots \leq a_m \leq 1,\) then

\[(n - m)(a_1^3 + a_2^3 + \cdots + a_n^3 - n) \geq (2n - 2m + 1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n);\]

(b) if \(a_1 \geq a_2 \geq \cdots \geq a_m \geq 1,\) then

\[
a_1^3 + a_2^3 + \cdots + a_n^3 - n \leq (n - m + 2)(a_1^2 + a_2^2 + \cdots + a_n^2 - n).
\]
2.6. Let $a_1, a_2, \ldots, a_n \ (n \geq 3)$ be real numbers so that $a_1 + a_2 + \cdots + a_n = n$. Prove that

(a) if $a_1 \leq \cdots \leq a_{n-1} \leq 1 \leq a_n$, then

$$a_1^4 + a_2^4 + \cdots + a_n^4 - n \geq 6(a_1^2 + a_2^2 + \cdots + a_n^2 - n);$$

(b) if $a_1 \leq \cdots \leq a_{n-2} \leq 1 \leq a_{n-1} \leq a_n$, then

$$a_1^4 + a_2^4 + \cdots + a_n^4 - n \geq \frac{14}{3}(a_1^2 + a_2^2 + \cdots + a_n^2 - n);$$

(c) if $a_1 \leq a_2 \leq 1 \leq a_3 \leq \cdots \leq a_n$, then

$$a_1^4 + a_2^4 + \cdots + a_n^4 - n \geq \frac{2(n^2 - 3n + 3)}{n^2 - 5n + 7}(a_1^2 + a_2^2 + \cdots + a_n^2 - n).$$

2.7. Let $a, b, c, d, e$ be nonnegative real numbers so that $a + b + c + d + e = 5$. Prove that

(a) if $a \geq b \geq 1 \geq c \geq d \geq e$, then

$$21(a^2 + b^2 + c^2 + d^2 + e^2) \geq a^4 + b^4 + c^4 + d^4 + e^4 + 100;$$

(b) if $a \geq b \geq c \geq 1 \geq d \geq e$, then

$$13(a^2 + b^2 + c^2 + d^2 + e^2) \geq a^4 + b^4 + c^4 + d^4 + e^4 + 60.$$

2.8. Let $a_1, a_2, \ldots, a_n \ (n \geq 3)$ be nonnegative numbers so that $a_1 + a_2 + \cdots + a_n = n$. Prove that

(a) if $a_1 \geq \cdots \geq a_{n-1} \geq 1 \geq a_n$, then

$$7(a_1^3 + a_2^3 + \cdots + a_n^3) \geq 3(a_1^4 + a_2^4 + \cdots + a_n^4) + 4n;$$

(b) if $a_1 \geq \cdots \geq a_{n-2} \geq 1 \geq a_{n-1} \geq a_n$, then

$$13(a_1^3 + a_2^3 + \cdots + a_n^3) \geq 4(a_1^4 + a_2^4 + \cdots + a_n^4) + 9n.$$

2.9. If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1 + a_2 + \cdots + a_n = n$ and

$$a_1 \geq \cdots \geq a_m \geq 1 \geq a_{m+1} \geq \cdots \geq a_n, \quad m \in \{1, 2, \ldots, n-1\},$$

then

$$(n-m+1)^2 \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n\right) \geq 4(n-m)(a_1^2 + a_2^2 + \cdots + a_n^2 - n).$$
2.10. If \(a_1, a_2, \ldots, a_n\) are positive real numbers so that \(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = n\) and
\[
a_1 \leq \cdots \leq a_m \leq 1 \leq a_{m+1} \leq \cdots \leq a_n, \quad m \in \{1, 2, \ldots, n-1\},
\]
then
\[
a_1^2 + a_2^2 + \cdots + a_n^2 - n \geq 2 \left(1 + \frac{\sqrt{n-m}}{n-m+1}\right)(a_1 + a_2 + \cdots + a_n - n).
\]

2.11. Let \(a_1, a_2, \ldots, a_n\) \((n \geq 3)\) be nonnegative numbers so that \(a_1 + a_2 + \cdots + a_n = n\). Prove that
(a) if \(a_1 \leq \cdots \leq a_{n-1} \leq 1 \leq a_n\), then
\[
\frac{1}{a_1^2 + 2} + \frac{1}{a_2^2 + 2} + \cdots + \frac{1}{a_n^2 + 2} \geq \frac{n}{3},
\]
(b) if \(a_1 \leq \cdots \leq a_{n-2} \leq 1 \leq a_{n-1} \leq a_n\), then
\[
\frac{1}{2a_1^2 + 3} + \frac{1}{2a_2^2 + 3} + \cdots + \frac{1}{2a_n^2 + 3} \geq \frac{n}{5}.
\]

2.12. If \(a_1, a_2, \ldots, a_{2n}\) are nonnegative real numbers so that
\[
a_1 \geq \cdots \geq a_n \geq 1 \geq a_{n+1} \geq \cdots \geq a_{2n}, \quad a_1 + a_2 + \cdots + a_{2n} = 2n,
\]
then
\[
\frac{1}{na_1^2 + n^2 + n + 1} + \frac{1}{na_2^2 + n^2 + n + 1} + \cdots + \frac{1}{na_{2n}^2 + n^2 + n + 1} \leq \frac{2n}{(n+1)^2}.
\]

2.13. If \(a, b, c, d, e, f\) are nonnegative real numbers so that
\[
a \geq b \geq c \geq 1 \geq d \geq e \geq f, \quad a + b + c + d + e + f = 6,
\]
then
\[
\frac{3a+4}{3a^2+4} + \frac{3b+4}{3b^2+4} + \frac{3c+4}{3c^2+4} + \frac{3d+4}{3d^2+4} + \frac{3e+4}{3e^2+4} + \frac{3f+4}{3f^2+4} \leq 6.
\]
2.14. If \( a, b, c, d, e, f \) are nonnegative real numbers so that
\[
a \geq b \geq 1 \geq c \geq d \geq e \geq f, \quad a + b + c + d + e + f = 6,
\]
then
\[
\frac{a^2 - 1}{(2a + 7)^2} + \frac{b^2 - 1}{(2b + 7)^2} + \frac{c^2 - 1}{(2c + 7)^2} + \frac{d^2 - 1}{(2d + 7)^2} + \frac{e^2 - 1}{(2e + 7)^2} + \frac{f^2 - 1}{(2f + 7)^2} \geq 0.
\]

2.15. If \( a, b, c, d, e, f \) are nonnegative real numbers so that
\[
a \leq b \leq 1 \leq c \leq d \leq e \leq f, \quad a + b + c + d + e + f = 6,
\]
then
\[
\frac{a^2 - 1}{(2a + 5)^2} + \frac{b^2 - 1}{(2b + 5)^2} + \frac{c^2 - 1}{(2c + 5)^2} + \frac{d^2 - 1}{(2d + 5)^2} + \frac{e^2 - 1}{(2e + 5)^2} + \frac{f^2 - 1}{(2f + 5)^2} \leq 0.
\]

2.16. If \( a, b, c \) are nonnegative real numbers so that
\[
a \leq b \leq 1 \leq c, \quad a + b + c = 3,
\]
then
\[
\sqrt{\frac{2a}{b + c}} + \sqrt{\frac{2b}{c + a}} + \sqrt{\frac{2c}{a + b}} \geq 3.
\]

2.17. If \( a_1, a_2, \ldots, a_8 \) are nonnegative real numbers so that
\[
a_1 \geq a_2 \geq a_3 \geq a_4 \geq 1 \geq a_5 \geq a_6 \geq a_7 \geq a_8, \quad a_1 + a_2 + \cdots + a_8 = 8,
\]
then
\[
(a_1^2 + 1)(a_2^2 + 1) \cdots (a_8^2 + 1) \geq (a_1 + 1)(a_2 + 1) \cdots (a_8 + 1).
\]

2.18. If \( a, b, c, d \) are real numbers so that
\[
-\frac{1}{2} \leq a \leq b \leq 1 \leq c \leq d, \quad a + b + c + d = 4,
\]
then
\[
7 \left( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \right) + 3 \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) \geq 40.
\]
2.19. Let $a, b, c, d$ be real numbers. Prove that

(a) if $-1 \leq a \leq b \leq c \leq 1 \leq d$, then

$$3 \left( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \right) \geq \frac{8}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d};$$

(b) if $-1 \leq a \leq b \leq c \leq d$, then

$$2 \left( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \right) \geq \frac{4}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}.$$

2.20. If $a, b, c, d$ are positive real numbers so that

$$a \geq b \geq c \geq d, \quad abcd = 1,$$

then

$$a^2 + b^2 + c^2 + d^2 - 4 \geq 18 \left( a + b + c + d - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} - \frac{1}{d} \right).$$

2.21. If $a, b, c, d$ are positive real numbers so that

$$a \leq b \leq c \leq d, \quad abcd = 1,$$

then

$$\sqrt{a^2 - a + 1} + \sqrt{b^2 - b + 1} + \sqrt{c^2 - c + 1} + \sqrt{d^2 - d + 1} \geq a + b + c + d.$$

2.22. If $a, b, c, d$ are positive real numbers so that

$$a \leq b \leq c \leq d, \quad abcd = 1,$$

then

$$\frac{1}{a^3 + 3a + 2} + \frac{1}{b^3 + 3b + 2} + \frac{1}{c^3 + 3c + 2} + \frac{1}{d^3 + 3d + 2} \geq \frac{2}{3}.$$

2.23. If $a_1, a_2, \ldots, a_n$ are positive real numbers so that

$$a_1 \geq \cdots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \geq a_1 + a_2 + \cdots + a_n.$$
2.24. Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that

$$a_1 \leq \cdots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1 a_2 \cdots a_n = 1.$$ 

If $k \geq 1$, then

$$\frac{1}{1 + ka_1} + \frac{1}{1 + ka_2} + \cdots + \frac{1}{1 + ka_n} \geq \frac{n}{1 + k}.$$

2.25. If $a_1, a_2, \ldots, a_9$ are positive real numbers so that

$$a_1 \leq \cdots \leq a_8 \leq 1 \leq a_9, \quad a_1 a_2 \cdots a_9 = 1,$$

then

$$\frac{1}{(a_1 + 2)^2} + \frac{1}{(a_2 + 2)^2} + \cdots + \frac{1}{(a_9 + 2)^2} \geq 1.$$

2.26. Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that

$$a_1 \leq \cdots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1 a_2 \cdots a_n = 1.$$ 

If $p, q \geq 0$ so that

$$p + q \geq 1 + \frac{2pq}{p + 4q},$$

then

$$\frac{1}{1 + pa_1 + qa_1^2} + \frac{1}{1 + pa_2 + qa_2^2} + \cdots + \frac{1}{1 + pa_n + qa_n^2} \geq \frac{n}{1 + p + q}.$$

2.27. Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that

$$a_1 \leq \cdots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1 a_2 \cdots a_n = 1.$$ 

If $m \geq 1$ and $0 < k \leq m$, then

$$\frac{1}{(a_1 + k)^m} + \frac{1}{(a_2 + k)^m} + \cdots + \frac{1}{(a_n + k)^m} \geq \frac{n}{(1 + k)^m}.$$

2.28. If $a_1, a_2, \ldots, a_n$ are positive real numbers so that

$$a_1 \leq \cdots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{\sqrt{1 + 3a_1}} + \frac{1}{\sqrt{1 + 3a_2}} + \cdots + \frac{1}{\sqrt{1 + 3a_n}} \geq \frac{n}{2}.$$
2.29. Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that
\[
a_1 \leq \cdots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1a_2\cdots a_n = 1.
\]
If $0 < m < 1$ and $0 < k \leq \frac{1}{2^{1/m} - 1}$, then
\[
\frac{1}{(a_1 + k)^{m}} + \frac{1}{(a_2 + k)^{m}} + \cdots + \frac{1}{(a_n + k)^{m}} \geq \frac{n}{(1 + k)^{m}}.
\]

2.30. If $a_1, a_2, \ldots, a_n$ ($n \geq 4$) are positive real numbers so that
\[
a_1 \geq a_2 \geq a_3 \geq 1 \geq a_4 \geq \cdots \geq a_n, \quad a_1a_2\cdots a_n = 1,
\]
then
\[
\frac{1}{3a_1 + 1} + \frac{1}{3a_2 + 1} + \cdots + \frac{1}{3a_n + 1} \geq \frac{n}{4}.
\]

2.31. If $a_1, a_2, \ldots, a_n$ ($n \geq 4$) are positive real numbers so that
\[
a_1 \geq a_2 \geq a_3 \geq 1 \geq a_4 \geq \cdots \geq a_n, \quad a_1a_2\cdots a_n = 1,
\]
then
\[
\frac{1}{(a_1 + 1)^{2}} + \frac{1}{(a_2 + 1)^{2}} + \cdots + \frac{1}{(a_n + 1)^{2}} \geq \frac{n}{4}.
\]

2.32. If $a_1, a_2, \ldots, a_n$ are positive real numbers so that
\[
a_1 \geq \cdots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1a_2\cdots a_n = 1,
\]
then
\[
\frac{1}{(a_1 + 3)^{2}} + \frac{1}{(a_2 + 3)^{2}} + \cdots + \frac{1}{(a_n + 3)^{2}} \leq \frac{n}{16}.
\]

2.33. Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that
\[
a_1 \geq \cdots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1a_2\cdots a_n = 1.
\]
If $p, q \geq 0$ so that $p + q \leq 1$, then
\[
\frac{1}{1 + pa_1 + qa_1^2} + \frac{1}{1 + pa_2 + qa_2^2} + \cdots + \frac{1}{1 + pa_n + qa_n^2} \leq \frac{n}{1 + p + q}.
\]
2.34. Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that

$$a_1 \geq \cdots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1 a_2 \cdots a_n = 1.$$ 

If $m > 1$ and $k \geq \frac{1}{2^{1/m} - 1}$, then

$$\frac{1}{(a_1 + k)^m} + \frac{1}{(a_2 + k)^m} + \cdots + \frac{1}{(a_n + k)^m} \leq \frac{n}{(1 + k)^m}.$$ 

2.35. If $a_1, a_2, \ldots, a_n$ are positive real numbers so that

$$a_1 \geq \cdots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{\sqrt{1 + 2a_1}} + \frac{1}{\sqrt{1 + 2a_2}} + \cdots + \frac{1}{\sqrt{1 + 2a_n}} \leq \frac{n}{\sqrt{3}}.$$ 

2.36. Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that

$$a_1 \geq \cdots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1 a_2 \cdots a_n = 1.$$ 

If $0 < m < 1$ and $k \geq m$, then

$$\frac{1}{(a_1 + k)^m} + \frac{1}{(a_2 + k)^m} + \cdots + \frac{1}{(a_n + k)^m} \leq \frac{n}{(1 + k)^m}.$$ 

2.37. If $a_1, a_2, \ldots, a_n$ ($n \geq 3$) are positive real numbers so that

$$a_1 \geq \cdots \geq a_{n-2} \geq 1 \geq a_{n-1} \geq a_n, \quad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{(a_1 + 5)^2} + \frac{1}{(a_2 + 5)^2} + \cdots + \frac{1}{(a_n + 5)^2} \leq \frac{n}{36}.$$ 

2.38. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that

$$a_1 \geq \cdots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n,$$

then

$$\frac{1}{3 - a_1} + \frac{1}{3 - a_2} + \cdots + \frac{1}{3 - a_n} \leq \frac{n}{2}.$$ 

2.39. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that

$$a_1 \leq \cdots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1 + a_2 + \cdots + a_n = n.$$ 

Prove that

$$a_1^3 + a_2^3 + \cdots + a_n^3 - n \geq (n - 1)^2 \left[ \left( \frac{n - a_1}{n - 1} \right)^3 + \left( \frac{n - a_2}{n - 1} \right)^3 + \cdots + \left( \frac{n - a_n}{n - 1} \right)^3 - n \right].$$
2.3 Solutions

P 2.1. If $a, b, c, d$ are real numbers so that

$$a \leq b \leq 1 \leq c \leq d, \quad a + b + c + d = 4,$$

then

$$(3a^2 - 2)(a - 1)^2 + (3b^2 - 2)(b - 1)^2 + (3c^2 - 2)(c - 1)^2 + (3d^2 - 2)(d - 1)^2 \geq 0.$$  

(Vasile C., 2007)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a + b + c + d}{4} = 1,$$

where

$$f(u) = (3u^2 - 2)(u - 1)^2, \quad u \in \mathbb{I} = \mathbb{R}.$$

From

$$f''(u) = 2(18u^2 - 18u + 1),$$

it follows that $f''(u) > 0$ for $u \geq 1$, hence $f$ is convex on $\mathbb{I}_{\geq 1}$. Therefore, we may apply the RHCF-OV Theorem for $n = 4$ and $m = 2$. Thus, it suffices to show that

$$f(x) + 2f(y) \geq 3f(1)$$

for all real $x, y$ so that $x + 2y = 3$. Using Note 1, we only need to show that

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

We have

$$g(u) = 3(u^3 + u^2 + u + 1) - 6(u^2 + u + 1) + u + 1 = 3u^3 - 3u^2 - 2u - 2,$$

$$h(x, y) = 3(x^2 + xy + y^2) - 3(x + y) - 2 = (3y - 4)^2 \geq 0.$$

From $x + 2y = 3$ and $h(x, y) = 0$, we get $x = 1/3, y = 4/3$. Therefore, in accordance with Note 4, the equality holds for $a = b = c = d = 1$, and also for

$$a = \frac{1}{3}, \quad b = 1, \quad c = \frac{4}{3}.$$

Remark. Similarly, we can prove the following generalization:

- Let $a_1, a_2, \ldots, a_{2n}$ be real numbers so that

$$a_1 \leq \cdots \leq a_n \leq 1 \leq a_{n+1} \leq \cdots \leq a_{2n}, \quad a_1 + a_2 + \cdots + a_{2n} = 2n.$$
If \( k = \frac{n}{n^2 - n + 1} \), then
\[
(a_1^2 - k)(a_1 - 1)^2 + (a_2^2 - k)(a_2 - 1)^2 + \cdots + (a_{2n}^2 - k)(a_{2n} - 1)^2 \geq 0,
\]
with equality for \( a_1 = a_2 = \cdots = a_{2n} = 1 \), and also for \( a_1 = \frac{1}{n^2 - n + 1}, \ a_2 = \cdots = a_n = 1, \ a_{n+1} = \cdots = a_n = \frac{n^2}{n^2 - n + 1} \). \( \Box \)

**P 2.2.** If \( a, b, c, d \) are nonnegative real numbers so that
\[
a \geq b \geq c \geq d, \quad a + b + c + d = 4,
\]
then
\[
\frac{1}{2a^3 + 5} + \frac{1}{2b^3 + 5} + \frac{1}{2c^3 + 5} + \frac{1}{2d^3 + 5} \leq \frac{4}{7}.
\]

*(Vasile C., 2009)*

**Solution.** Write the inequality as
\[
f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a + b + c + d}{4} = 1,
\]
where
\[
f(u) = \frac{-1}{2u^3 + 5}, \quad u \geq 0.
\]
From
\[
f''(u) = \frac{12u(5 - 4u^3)}{(2u^3 + 5)^3},
\]
it follows that \( f''(u) \geq 0 \) for \( u \in [0, 1] \), hence \( f \) is convex on \([0,s]\). Therefore, we may apply the LHCF-OV Theorem for \( n = 4 \) and \( m = 2 \). Using Note 1, we only need to show that \( h(x, y) \geq 0 \) for \( x, y \geq 0 \) so that \( x + 2y = 3 \). We have
\[
g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{2(u^2 + u + 1)}{7(2u^3 + 5)},
\]
\[
h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{2E}{7(2x^3 + 5)(2y^3 + 5)},
\]
where
\[
E = -2x^2y^2 - 2xy(x + y) - 2(x^2 + xy + y^2) + 5(x + y) + 5.
\]
Since
\[
E = (1 - 2y)^2(2 + 3y - 2y^2) = (1 - 2y)^2(2 + xy) \geq 0,
\]
the proof is completed. From \( x + 2y = 3 \) and \( h(x, y) = 0 \), we get \( x = 2, \ y = 1/2 \). Therefore, in accordance with Note 4, the equality holds for \( a = b = c = d = 1 \), and also for

\[
a = 2, \quad b = 1, \quad c = d = \frac{1}{2}.
\]

**Remark.** Similarly, we can prove the following generalization.

- If \( a_1, a_2, \ldots, a_{2n} \) are nonnegative real numbers so that

\[
a_1 \geq \cdots \geq a_n \geq 1 \geq a_{n+1} \geq \cdots \geq a_{2n}, \quad a_1 + a_2 + \cdots + a_{2n} = 2n.
\]

then

\[
\frac{1}{a_1^3 + n + \frac{1}{n}} + \frac{1}{a_2^3 + n + \frac{1}{n}} + \cdots + \frac{1}{a_{2n}^3 + n + \frac{1}{n}} \geq \frac{2n^2}{n^2 + n + 1},
\]

with equality for \( a_1 = a_2 = \cdots = a_{2n} = 1 \), and also for

\[
a_1 = n, \quad a_2 = \cdots = a_n = 1, \quad a_{n+1} = \cdots = a_{2n} = \frac{1}{n}.
\]

\(\square\)

**P 2.3.** If

\[
\frac{-2n - 1}{n - 1} \leq a_1 \leq \cdots \leq a_n \leq 1 \leq a_{n+1} \leq \cdots \leq a_{2n}, \quad a_1 + a_2 + \cdots + a_{2n} = 2n,
\]

then

\[
a_1^3 + a_2^3 + \cdots + a_{2n}^3 \geq 2n.
\]

(Vasile C., 2007)

**Solution.** Write the inequality as

\[
f(a_1) + f(a_2) + \cdots + f(a_{2n}) \geq 2nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_{2n}}{2n} = 1,
\]

where

\[
f(u) = u^3, \quad u \geq \frac{-2n - 1}{n - 1}.
\]

From \( f''(u) = 6u \), it follows that \( f(u) \) is convex for \( u \geq s \). Therefore, we may apply the RHCF-OV Theorem for \( 2n \) numbers and \( m = n \). By Note 1, it suffices to show that \( h(x, y) \geq 0 \) for all \( x, y \geq \frac{-2n - 1}{n - 1} \) so that \( x + ny = 1 + n \). We have

\[
g(u) = \frac{f(u) - f(1)}{u - 1} = u^2 + u + 1,
\]
\[ h(x, y) = \frac{g(x) - g(y)}{x - y} = x + y + 1 = \frac{(n-1)x + 2n + 1}{n-1} \geq 0. \]

From \( x + ny = 1 + n \) and \( h(x, y) = 0 \), we get
\[ x = \frac{-2n - 1}{n-1}, \quad y = \frac{n + 2}{n-1}. \]

In accordance with Note 4, the equality holds for \( a_1 = a_2 = \cdots = a_{2n} = 1 \), and also for
\[ a_1 = \frac{-2n - 1}{n-1}, \quad a_2 = \cdots = a_n = 1, \quad a_{n+1} = \cdots = a_{2n} = \frac{n + 2}{n-1}. \]

**P 2.4.** Let \( a_1, a_2, \ldots, a_n \ (n \geq 3) \) be real numbers so that \( a_1 + a_2 + \cdots + a_n = n \). Prove that

(a) if \(-3 \leq a_1 \leq \cdots \leq a_{n-2} \leq 1 \leq a_{n-1} \leq a_n\), then
\[ a_1^3 + a_2^3 + \cdots + a_n^3 \geq a_1^2 + a_2^2 + \cdots + a_n^2, \]

(b) if \(-\frac{n - 1}{n-3} \leq a_1 \leq a_2 \leq 1 \leq \cdots \leq a_n\), then
\[ a_1^3 + a_2^3 + \cdots + a_n^3 + n \geq 2(a_1^2 + a_2^2 + \cdots + a_n^2). \]

*(Vasile C., 2007)*

**Solution.** (a) Write the inequality as
\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1, \]
where
\[ f(u) = u^3 - u^2, \quad u \geq -3. \]

For \( u \geq 1 \), we have
\[ f''(u) = 6u - 2 > 0, \]

hence \( f(u) \) is convex for \( u \geq s \). Thus, we may apply the RHCF-OV Theorem for \( m = n - 2 \). According to this theorem, it suffices to show that
\[ f(x) + 2f(y) \geq 3f(1) \]

for \(-3 \leq x \leq y\) satisfying \( x + 2y = 3 \). Using Note 1, we only need to show that \( h(x, y) \geq 0 \), where
\[ h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}. \]
We have

\[ g(u) = u^2, \]

\[ h(x, y) = x + y = \frac{x + 3}{2} \geq 0. \]

From \( x + 2y = 3 \) and \( h(x, y) = 0 \), we get \( x = -3 \) and \( y = 3 \). Therefore, in accordance with Note 4, the equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for \( a_1 = -3, \ a_2 = \cdots = a_{n-2} = 1, \ a_{n-1} = a_n = 3 \).

(b) Write the inequality as

\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1, \]

where

\[ f(u) = u^3 - 2u^2, \quad u \geq \frac{n-1}{n-3}. \]

For \( u \geq 1 \), we have

\[ f''(u) = 6u - 4 > 0, \]

hence \( f(u) \) is convex for \( u \geq s \). Thus, we may apply the RHCF-OV Theorem for \( m = 2 \). According to this theorem, it suffices to show that

\[ f(x) + (n-2)f(y) \geq (n-1)f(1) \]

for \( \frac{n-1}{n-3} \leq x \leq y \) satisfying \( x + (n-2)y = n-1 \). Using Note 1, we only need to show that \( h(x, y) \geq 0 \), where

\[ h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}. \]

We have

\[ g(u) = u^2 - u - 1, \]

\[ h(x, y) = x + y - 1 = \frac{(n-3)x + n - 1}{n - 1} \geq 0. \]

From \( x + (n-2)y = n-1 \) and \( h(x, y) = 0 \), we get \( x = \frac{n-1}{n-3} \) and \( y = \frac{n-1}{n-3} \). Therefore, in accordance with Note 4, the equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \).

If \( n \geq 4 \), then the equality holds also for

\[ a_1 = -\frac{n-1}{n-3}, \quad a_2 = 1, \quad a_3 = \cdots = a_n = \frac{n-1}{n-3}. \]
P 2.5. Let \( a_1, a_2, \ldots, a_n \) be nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n = n \) and let \( m \in \{1, 2, \ldots, n-1\} \). Prove that

(a) if \( a_1 \leq a_2 \leq \cdots \leq a_m \leq 1 \), then

\[
(n - m)(a_1^3 + a_2^3 + \cdots + a_n^3 - n) \geq (2n - 2m + 1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n);
\]

(b) if \( a_1 \geq a_2 \geq \cdots \geq a_m \geq 1 \), then

\[
a_1^3 + a_2^3 + \cdots + a_n^3 - n \leq (n - m + 2)(a_1^2 + a_2^2 + \cdots + a_n^2 - n).
\]

(Vasile C., 2007)

**Solution.** (a) Write the inequality as

\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,
\]

where

\[
f(u) = (n - m)u^3 - (2n - 2m + 1)u^2, \quad u \in \mathbb{I} = [0, n].
\]

For \( u \geq 1 \), we have

\[
f''(u) = 6(n - m)u - 2(2n - 2m + 1) \geq 6(n - m) - 2(2n - 2m + 1) = 2(n - m - 1) \geq 0,
\]

hence \( f \) is convex on \( \mathbb{I}_{\geq s} \). Thus, by the RHCF-OV Theorem and Note 1, we only need to show that \( h(x, y) \geq 0 \) for all nonnegative numbers \( x, y \) so that \( x + (n - m)y = n - m + 1 \). We have

\[
g(u) = \frac{f(u) - f(1)}{u - 1} = (n - m)(u^2 + u + 1) - (2n - 2m + 1)(u + 1)
\]

\[
= (n - m)u^2 - (n - m + 1)u - n + m - 1,
\]

\[
h(x, y) = \frac{g(x) - g(y)}{x - y} = (n - m)(x + y) - n + m - 1 = (n - m - 1)x \geq 0.
\]

From \( x + (n-m)y = 1+n-m \) and \( h(x, y) = 0 \), we get \( x = 0, \ y = (n-m+1)/(n-m) \). Therefore, in accordance with Note 4, the equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for

\[
a_1 = 0, \quad a_2 = \cdots = a_m = 1, \quad a_{m+1} = \cdots = a_n = 1 + \frac{1}{n - m}.
\]

(b) Write the inequality as

\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,
\]

where

\[
f(u) = (n - m + 2)u^2 - u^3, \quad u \in \mathbb{I} = [0, n].
\]
For $u \leq 1$, we have
\[
f''(u) = 2(n - m + 2 - 3u) \geq 2(n - m + 2 - 3) = 2(n - m - 1) \geq 0,
\]
hence $f$ is convex on $I_{\leq 3}$. By the LHCF-OV Theorem and Note 1, it suffices to show that $h(x, y) \geq 0$ for all $x, y \geq 0$ so that $x + (n - m)y = 1 + n - m$. We have
\[
g(u) = \frac{f(u) - f(1)}{u - 1} = (n - m + 2)(u + 1) - (u^2 + u + 1)
= -u^2 + (n - m + 1)u + n - m + 1,
\]
\[
h(x, y) = \frac{g(x) - g(y)}{x - y} = -(x + y) + n - m + 1 = (n - m - 1)y \geq 0.
\]
From $x + (n - m)y = 1 + n - m$ and $h(x, y) = 0$, we get $x = n - m + 1$, $y = 0$. Therefore, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for
\[
a_1 = n - m + 1, \quad a_2 = \cdots = a_m = 1, \quad a_{m+1} = \cdots = a_n = 0.
\]

**Remark 1.** For $m = 1$, we get the following results:

- If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then
\[
(n - 1)(a_1^3 + a_2^3 + \cdots + a_n^3 - n) \geq (2n - 1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n),
\]
with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for
\[
a_1 = 0, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n - 1}
\]
(or any cyclic permutation).

- If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then
\[
a_1^3 + a_2^3 + \cdots + a_n^3 - n \leq (n + 1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n),
\]
with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for
\[
a_1 = n, \quad a_2 = a_3 = \cdots = a_n = 0
\]
(or any cyclic permutation).

**Remark 2.** For $m = n - 1$, we get the following statements:

- If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that
\[
a_1 \leq \cdots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1 + a_2 + \cdots + a_n = n,
\]
then
\[
a_1^3 + a_2^3 + \cdots + a_n^3 + 2n \geq 3(a_1^2 + a_2^2 + \cdots + a_n^2),
\]
with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = 0, \quad a_2 = \cdots = a_{n-1} = 1, \quad a_n = 2.$$  

- If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers so that

$$a_1 \geq \cdots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1 + a_2 + \cdots + a_n = n,$$

then

$$a_1^3 + a_2^3 + \cdots + a_n^3 + 2n \leq 3(a_1^2 + a_2^2 + \cdots + a_n^2),$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = 2, \quad a_2 = \cdots = a_{n-1} = 1, \quad a_n = 0.$$

**Remark 3.** Replacing $n$ with $2n$ and choosing then $m = n$, we get the following results:

- If $a_1, a_2, \ldots, a_{2n}$ are nonnegative real numbers so that

$$a_1 \leq \cdots \leq a_n \leq 1 \leq a_{n+1} \leq \cdots \leq a_{2n}, \quad a_1 + a_2 + \cdots + a_{2n} = 2n,$$

then

$$n(a_1^3 + a_2^3 + \cdots + a_{2n}^3 - 2n) \geq (2n+1)(a_1^2 + a_2^2 + \cdots + a_{2n}^2 - 2n),$$

with equality for $a_1 = a_2 = \cdots = a_{2n} = 1$, and also for

$$a_1 = 0, \quad a_2 = \cdots = a_n = 1, \quad a_{n+1} = \cdots = a_{2n} = 1 + \frac{1}{n}.$$

- If $a_1, a_2, \ldots, a_{2n}$ are nonnegative real numbers so that

$$a_1 \geq \cdots \geq a_n \geq 1 \geq a_{n+1} \geq \cdots \geq a_{2n}, \quad a_1 + a_2 + \cdots + a_{2n} = 2n,$$

then

$$a_1^3 + a_2^3 + \cdots + a_{2n}^3 - 2n \leq (n+2)(a_1^2 + a_2^2 + \cdots + a_{2n}^2 - 2n),$$

with equality for $a_1 = a_2 = \cdots = a_{2n} = 1$, and also for

$$a_1 = n+1, \quad a_2 = \cdots = a_n = 1, \quad a_{n+1} = \cdots = a_{2n} = 0.$$  

\[ \square \]

**P 2.6.** Let $a_1, a_2, \ldots, a_n$ ($n \geq 3$) be real numbers so that $a_1 + a_2 + \cdots + a_n = n$. Prove that

(a) if $a_1 \leq \cdots \leq a_{n-1} \leq 1 \leq a_n$, then

$$a_1^4 + a_2^4 + \cdots + a_n^4 - n \geq 6(a_1^2 + a_2^2 + \cdots + a_n^2 - n);$$
(b) if \( a_1 \leq \cdots \leq a_{n-2} \leq 1 \leq a_{n-1} \leq a_n \), then
\[
a_1^4 + a_2^4 + \cdots + a_n^4 - n \geq \frac{14}{3} (a_1^2 + a_2^2 + \cdots + a_n^2 - n);
\]

(c) if \( a_1 \leq a_2 \leq 1 \leq a_3 \leq \cdots \leq a_n \), then
\[
a_1^4 + a_2^4 + \cdots + a_n^4 - n \geq \frac{2(n^2 - 3n + 3)}{n^2 - 5n + 7} (a_1^2 + a_2^2 + \cdots + a_n^2 - n).
\]

**Solution.** Consider the inequality
\[
a_1^4 + a_2^4 + \cdots + a_n^4 - n \geq k(a_1^2 + a_2^2 + \cdots + a_n^2 - n), \quad k \leq 6,
\]
and write it as
\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,
\]
where
\[
f(u) = u^4 - ku^2, \quad u \in \mathbb{R}.
\]
From \( f''(u) = 2(6u^2 - k) \), it follows that \( f \) is convex for \( u \geq 1 \). Therefore, we may apply the RHCF-OV Theorem for \( m = n - 1, m = n - 2 \) and \( m = 2 \), respectively. By Note 1, it suffices to show that \( h(x, y) \geq 0 \) for all real \( x, y \) so that \( x + (n - m)y = 1 + n - m \). We have
\[
\begin{align*}
g(u) &= \frac{f(u) - f(1)}{u - 1} = u^3 + u^2 + u + 1 - k(u + 1), \\
h(x, y) &= \frac{g(x) - g(y)}{x - y} = x^2 + xy + y^2 + x + y + 1 - k.
\end{align*}
\]

(a) We need to show that \( h(x, y) \geq 0 \) for \( k = 6, m = n - 1, x + y = 2 \). Indeed, we have
\[
h(x, y) = 1 - xy = \frac{1}{4}(x - y)^2 \geq 0.
\]
From \( x + y = 2 \) and \( h(x, y) = 0 \), we get \( x = y = 1 \). Therefore, in accordance with Note 4, the equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \).

(b) For \( k = 14/3, m = n - 2 \) and \( x + 2y = 3 \), we have
\[
h(x, y) = \frac{1}{3}(3y - 5)^2 \geq 0.
\]
From \( x + 2y = 3 \) and \( h(x, y) = 0 \), we get \( x = -1/3 \) and \( y = 5/3 \). Therefore, the equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for
\[
a_1 = \frac{-1}{3}, \quad a_2 = \cdots = a_{n-2} = 1, \quad a_{n-1} = a_n = \frac{5}{3}.
\]
(c) We have \( k = \frac{2(n^2 - 3n + 3)}{n^2 - 5n + 7}, \) \( m = 2 \) and \( x + (n-2)y = n-1, \) which involve

\[
h(x, y) = \frac{[(n^2 - 5n + 7)y - n^2 + 3n - 1]^2}{n^2 - 5n + 7} \geq 0.
\]

From \( x + (n-2)y = n-1 \) and \( h(x, y) = 0, \) we get

\[
x = \frac{-n^2 + 5n - 5}{n^2 - 5n + 7}, \quad y = \frac{n^2 - 3n + 1}{n^2 - 5n + 7}.
\]

Therefore, the equality holds for \( a_1 = a_2 = \cdots = a_n = 1, \) and also for

\[
a_1 = \frac{-n^2 + 5n - 5}{n^2 - 5n + 7}, \quad a_2 = 1, \quad a_3 = \cdots = a_n = \frac{n^2 - 3n + 1}{n^2 - 5n + 7}.
\]

\[\square\]

**P 2.7.** Let \( a, b, c, d, e \) be nonnegative real numbers so that \( a + b + c + d + e = 5. \) Prove that

(a) if \( a \geq b \geq 1 \geq c \geq d \geq e, \) then

\[
21(a^2 + b^2 + c^2 + d^2 + e^2) \geq a^4 + b^4 + c^4 + d^4 + e^4 + 100;
\]

(b) if \( a \geq b \geq c \geq 1 \geq d \geq e, \) then

\[
13(a^2 + b^2 + c^2 + d^2 + e^2) \geq a^4 + b^4 + c^4 + d^4 + e^4 + 60.
\]

(Vasile C., 2009)

**Solution.** Consider the inequality

\[
k(a^2 + b^2 + c^2 + d^2 + e^2 - 5) \geq a^4 + b^4 + c^4 + d^4 + e^4 - 5, \quad k \geq 6,
\]

and write it as

\[
f(a) + f(b) + f(c) + f(d) + f(e) \geq 5f(s), \quad s = \frac{a + b + c + d + e}{5} = 1,
\]

where

\[
f(u) = ku^2 - u^4, \quad u \geq 0.
\]

From \( f''(u) = 2(k - 6u^2), \) it follows that \( f \) is convex on [0, 1]. Therefore, we may apply the LHCF-OV Theorem for \( m = 2 \) and \( m = 3, \) respectively. By Note 1, it suffices to show that \( h(x, y) \geq 0 \) for all \( x, y \geq 0 \) so that \( x + (5-m)y = 6 - m. \) We have

\[
g(u) = \frac{f(u) - f(1)}{u - 1} = k(u + 1) - (u^3 + u^2 + u + 1),
\]
\[ h(x, y) = \frac{g(x) - g(y)}{x - y} = k - (x^2 + xy + y^2 + x + y + 1). \]

(a) We need to show that \( h(x, y) \geq 0 \) for \( k = 21, n = 5, m = 2 \) and \( x + 3y = 4 \); indeed, we have
\[ h(x, y) = 21 - (x^2 + xy + y^2 + x + y + 1) = y(22 - 7y) = y(10 + 3x + 2y) \geq 0. \]
From \( x + 3y = 4 \) and \( h(x, y) = 0 \), we get \( x = 4 \) and \( y = 0 \). Therefore, in accordance with Note 4, the equality holds for \( a = b = c = d = e = 1 \), and also for
\[ a = 4, \quad b = 1, \quad c = d = e = 0. \]

(b) We have \( k = 13, n = 5, m = 3 \) and \( x + 2y = 3 \), which involve
\[ h(x, y) = 13 - (x^2 + xy + y^2 + x + y + 1) = y(10 - 3y) = y(4 + 2x + y) \geq 0. \]
From \( x + 2y = 3 \) and \( h(x, y) = 0 \), we get \( x = 3 \) and \( y = 0 \). Therefore, the equality holds for \( a = b = c = d = e = 1 \), and also for
\[ a = 3, \quad b = c = 1, \quad d = e = 0. \]

\[ \Box \]

**P 2.8.** Let \( a_1, a_2, \ldots, a_n \ (n \geq 3) \) be nonnegative numbers so that \( a_1 + a_2 + \cdots + a_n = n \). Prove that

(a) If \( a_1 \geq \cdots \geq a_{n-1} \geq 1 \geq a_n \), then
\[ 7(a_1^3 + a_2^3 + \cdots + a_n^3) \geq 3(a_1^4 + a_2^4 + \cdots + a_n^4) + 4n; \]

(b) If \( a_1 \geq \cdots \geq a_{n-1} \geq 1 \geq a_{n-1} \geq a_n \), then
\[ 13(a_1^3 + a_2^3 + \cdots + a_n^3) \geq 4(a_1^4 + a_2^4 + \cdots + a_n^4) + 9n. \]

*(Vasile C., 2009)*

**Solution.** Consider the inequality
\[ k(a_1^3 + a_2^3 + \cdots + a_n^3 - n) \geq a_1^4 + a_2^4 + \cdots + a_n^4 - n, \quad k \geq 2, \]
and write it as
\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1, \]
where
\[ f(u) = ku^3 - u^4, \quad u \geq 0. \]
From $f''(u) = 6u(k - 2u^2)$, it follows that $f$ is convex on $[0, 1]$. Therefore, we may apply the LHCF-OV Theorem for $m = n - 1$ and $m = n - 2$, respectively. By Note 1, it suffices to show that $h(x, y) \geq 0$ for $x \geq y \geq 0$ so that $x + my = 1 + m$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = k(u^2 + u + 1) - (u^2 + u + 1),$$

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = -(x^2 + xy + y^2) + (k - 1)(x + y + 1).$$

(a) We need to show that $h(x, y) \geq 0$ for $k = 7/3$, $m = n - 1$, $x + y = 2$. Indeed,

$$h(x, y) = xy \geq 0.$$  

From $x > y$, $x + y = 2$ and $h(x, y) = 0$, we get $x = 2$ and $y = 0$. Therefore, in accordance with Note 4, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = 2, \ a_2 = \cdots = a_{n-1} = 1, \ a_n = 0$.

(b) We have $k = 13/4$, $m = n - 2$, $x + 2y = 3$, which involve

$$h(x, y) = 3y(9 - 4y) = 3y(3 + 2x) \geq 0.$$  

From $x + 2y = 3$ and $h(x, y) = 0$, we get $x = 3$ and $y = 0$. Therefore, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = 3, \ a_2 = \cdots = a_{n-2} = 1, \ a_{n-1} = a_n = 0$.

\[\square\]

**P 2.9.** If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1 + a_2 + \cdots + a_n = n$ and

$$a_1 \geq \cdots \geq a_m \geq 1 \geq a_{m+1} \geq \cdots \geq a_n, \quad m \in \{1, 2, \ldots, n-1\},$$

then

$$(n - m + 1)^2 \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n \right) \geq 4(n - m)(a_1^2 + a_2^2 + \cdots + a_n^2 - n).$$

(Vasile C., 2007)

**Solution.** Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{(n - m + 1)^2}{u} - 4(n - m)u^2, \quad u > 0.$$
For $u \in (0, 1]$, we have
\[
f''(u) = \frac{2(n-m+1)^2}{u^3} - 8(n-m) \\
\geq 2(n-m+1)^2 - 8(n-m) = 2(n-m-1)^2 \geq 0.
\]

Since $f$ is convex on $(0, s]$, we may apply the LHCF-OV Theorem. By Note 1, it suffices to show that $h(x, y) \geq 0$ for all $x, y > 0$ so that $x + (n-m)y = 1 + n - m$. We have
\[
g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-(n-m+1)^2}{u} - 4(n-m)(u+1), \\
h(x, y) = \frac{(n-m+1)^2}{xy} - 4(n-m) = \frac{[n-m+1 - 2(n-m)y]^2}{xy} \geq 0.
\]

From $x + (n-m)y = 1 + n - m$ and $h(x, y) = 0$, we get
\[
x = \frac{n-m+1}{2}, \quad y = \frac{n-m+1}{2(n-m)}.
\]

Therefore, in accordance with Note 4, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for
\[
a_1 = \frac{n-m+1}{2}, \quad a_2 = a_3 = \cdots = a_m = 1, \quad a_{m+1} = \cdots = a_n = \frac{n-m+1}{2(n-m)}.
\]

**Remark 1.** For $m = n - 1$, we get the following elegant statement:
- If $a_1, a_2, \ldots, a_n$ are positive real numbers so that
  \[a_1 \geq \cdots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1 + a_2 + \cdots + a_n = n,\]
then
  \[\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \geq a_1^2 + a_2^2 + \cdots + a_n^2,\]
  with equality for $a_1 = a_2 = \cdots = a_n = 1$.

**Remark 2.** Replacing $n$ with $2n$ and choosing then $m = n$, we get the following statement:
- If $a_1, a_2, \ldots, a_{2n}$ are positive real numbers so that
  \[a_1 \geq \cdots \geq a_n \geq 1 \geq a_{n+1} \geq \cdots \geq a_{2n}, \quad a_1 + a_2 + \cdots + a_{2n} = 2n,\]
then
  \[(n+1)^2 \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{2n}} - 2n\right) \geq 4n(a_1^2 + a_2^2 + \cdots + a_{2n}^2 - 2n),\]
  with equality for $a_1 = a_2 = \cdots = a_{2n} = 1$, and also for
  \[a_1 = \frac{n+1}{2}, \quad a_2 = a_3 = \cdots = a_n = 1, \quad a_{n+1} = \cdots = a_{2n} = \frac{n+1}{2n}.\]
\[\square\]
**P 2.10.** If \(a_1, a_2, \ldots, a_n\) are positive real numbers so that \(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = n\) and

\[a_1 \leq \cdots \leq a_m \leq 1 \leq a_{m+1} \leq \cdots \leq a_n, \quad m \in \{1, 2, \ldots, n-1\},\]

then

\[a_1^2 + a_2^2 + \cdots + a_n^2 - n \geq 2 \left(1 + \frac{\sqrt{n-m}}{n-m+1}\right) (a_1 + a_2 + \cdots + a_n - n).\]

*(Vasile C., 2007)*

**Solution.** Replacing each \(a_i\) by \(1/a_i\), we need to prove that

\[a_1 \geq \cdots \geq a_m \geq 1 \geq a_{m+1} \geq \cdots \geq a_n, \quad a_1 + a_2 + \cdots + a_n = n\]

involves

\[f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,\]

where

\[f(u) = \frac{1}{u^2} - \frac{2k}{u}, \quad k = 1 + \frac{\sqrt{m-n}}{n-m+1}, \quad u > 0.\]

For \(u \in (0, 1]\), we have

\[f''(u) = \frac{6 - 4ku}{u^4} \geq \frac{6 - 4k}{u^4} = \frac{2(\sqrt{n-m} - 1)^2}{(n-m+1)u^4} \geq 0.\]

Thus, \(f\) is convex on \((0, 1]\). By the LHCF-OV Theorem and Note 1, it suffices to show that \(h(x, y) \geq 0\) for \(x, y > 0\) so that \(x + (n-m)y = 1 + n-m\), where

\[h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.\]

We have

\[g(u) = \frac{-1}{u^2} + \frac{2k-1}{u}\]

and

\[h(x, y) = \frac{1}{xy} \left(\frac{1}{x} + \frac{1}{y} + 1 - 2k\right).\]

We only need to show that

\[\frac{1}{x} + \frac{1}{y} \geq 1 + \frac{2\sqrt{n-m}}{n-m+1}.\]

Indeed, using the Cauchy-Schwarz inequality, we get

\[\frac{1}{x} + \frac{1}{y} \geq \frac{(1 + \sqrt{n-m})^2}{x + (n-m)y} = \frac{(1 + \sqrt{n-m})^2}{n-m+1} = 1 + \frac{2\sqrt{n-m}}{n-m+1}.\]
From \( x + (n-m)y = 1 + n - m \) and \( h(x, y) = 0 \), we get
\[
x = \frac{n-m+1}{1 + \sqrt{n-m}}, \quad y = \frac{n-m+1}{n-m + \sqrt{n-m}}.
\]
By Note 4, we have
\[
f(a_1) + f(a_2) + \cdots + f(a_n) = nf(1)
\]
for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for
\[
a_1 = \frac{n-m+1}{1 + \sqrt{n-m}}, \quad a_2 = a_3 = \cdots = a_m = 1, \quad a_{m+1} = \cdots = a_n = \frac{n-m+1}{n-m + \sqrt{n-m}}.
\]
Therefore, the original inequality becomes an equality for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for
\[
a_1 = \frac{1 + \sqrt{n-m}}{n-m+1}, \quad a_2 = a_3 = \cdots = a_m = 1, \quad a_{m+1} = \cdots = a_n = \frac{n-m + \sqrt{n-m}}{n-m + 1}.
\]

**Remark.** Replacing \( n \) with \( 2n \) and choosing then \( m = n \), we get the statement below.

- If \( a_1, a_2, \ldots, a_{2n} \) are positive real numbers so that
  \[
a_1 \leq \cdots \leq a_n \leq 1 \leq a_{n+1} \leq \cdots \leq a_{2n}, \quad \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{2n}} = 2n,
\]
then
\[
a_1^2 + a_2^2 + \cdots + a_{2n}^2 - 2n \geq 2\left(1 + \frac{\sqrt{n}}{n+1}\right)(a_1 + a_2 + \cdots + a_{2n} - 2n).
\]
with equality for \( a_1 = a_2 = \cdots = a_{2n} = 1 \), and also for
\[
a_1 = \frac{1 + \sqrt{n}}{n+1}, \quad a_2 = a_3 = \cdots = a_n = 1, \quad a_{n+1} = \cdots = a_{2n} = \frac{n + \sqrt{n}}{n+1}.
\]

**P 2.11.** Let \( a_1, a_2, \ldots, a_n \ (n \geq 3) \) be nonnegative numbers so that \( a_1 + a_2 + \cdots + a_n = n \). Prove that

(a) if \( a_1 \leq \cdots \leq a_{n-1} \leq 1 \leq a_n \), then
\[
\frac{1}{a_1^2 + 2} + \frac{1}{a_2^2 + 2} + \cdots + \frac{1}{a_n^2 + 2} \geq \frac{n}{3};
\]
(b) if \( a_1 \leq \cdots \leq a_{n-2} \leq 1 \leq a_{n-1} \leq a_n \), then
\[
\frac{1}{2a_1^2 + 3} + \frac{1}{2a_2^2 + 3} + \cdots + \frac{1}{2a_n^2 + 3} \geq \frac{n}{5}.
\]

(Vasile C., 2007)
**Solution.** Consider the inequality
\[
\frac{1}{a_1^2 + k} + \frac{1}{a_2^2 + k} + \cdots + \frac{1}{a_n^2 + k} \geq \frac{n}{1+k}, \quad k \in [0,3];
\]
and write it as
\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,
\]
and
\[
f(u) = \frac{1}{u^2 + k}, \quad u \geq 0.
\]
For \(u \geq 1\), we have
\[
f''(u) = \frac{2(3u^2 - k)}{(u^2 + k)^3} \geq \frac{2(3 - k)}{(u^2 + k)^3} \geq 0,
\]
hence \(f(u)\) is convex for \(u \geq s\). Therefore, we may apply the RHCF-OV Theorem for \(m = n-1\) and \(m = n-2\), respectively. By Note 1, it suffices to show that \(h(x, y) \geq 0\) for all \(x, y \geq 0\) so that \(x + (n-m)y = 1 + n - m\). Since
\[
g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-u - 1}{(1 + k)(u^2 + k)},
\]
\[
h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{xy + x + y - k}{(1 + k)(x^2 + k)(y^2 + k)}
\]
we only need to show that \(xy + x + y - k \geq 0\).

(a) We need to show that \(xy + x + y - k \geq 0\) for \(k = 2, m = n-1, x + y = 2\); indeed, we have
\[
xy + x + y - k = xy \geq 0.
\]
From \(x < y, x + y = 2\) and \(xy + x + y - k = 0\), we get \(x = 0\) and \(y = 2\). Therefore, by Note 4, the equality holds for \(a_1 = a_2 = \cdots = a_n = 1\), and also for
\[
a_1 = 0, \quad a_2 = \cdots = a_{n-1} = 1, \quad a_n = 2.
\]

(b) We have \(k = 3/2, m = n-2, x + 2y = 3\), hence
\[
xy + x + y - k = \frac{x(4-x)}{2} = \frac{x(1+2y)}{2} \geq 0.
\]
From \(x + 2y = 3\) and \(xy + x + y - k = 0\), we get \(x = 0\) and \(y = 3/2\). Therefore, the equality holds for \(a_1 = a_2 = \cdots = a_n = 1\), and also for
\[
a_1 = 0, \quad a_2 = \cdots = a_{n-2} = 1, \quad a_{n-1} = a_n = \frac{3}{2}.
\]
\[\square\]
P 2.12. If \( a_1, a_2, \ldots, a_{2n} \) are nonnegative real numbers so that
\[
a_1 \geq \cdots \geq a_n \geq 1 \geq a_{n+1} \geq \cdots \geq a_{2n}, \quad a_1 + a_2 + \cdots + a_{2n} = 2n,
\]
then
\[
\frac{1}{na_1^2 + n^2 + n + 1} + \frac{1}{na_2^2 + n^2 + n + 1} + \cdots + \frac{1}{na_{2n}^2 + n^2 + n + 1} \leq \frac{2n}{(n+1)^2}.
\]

(\text{Vasile C., 2007})

Solution. Write the inequality as
\[
f(a_1) + f(a_2) + \cdots + f(a_{2n}) \geq 2nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_{2n}}{2n} = 1,
\]
where
\[
f(u) = \frac{-1}{nu^2 + n^2 + n + 1}, \quad u \geq 0.
\]
For \( u \in [0,1] \), we have
\[
f''(u) = \frac{2nu(n^2 + n + 1 - 3nu^2)}{(nu^2 + n^2 + n + 1)^3} \geq \frac{2nu(n^2 + n + 1 - 3n)}{(nu^2 + n^2 + n + 1)^3} \geq 0,
\]
hence \( f \) is convex on \([0,s]\). Therefore, we may apply the LHCF-OV Theorem for \( 2n \) numbers and \( m = n \). By Note 1, it suffices to show that \( h(x, y) \geq 0 \) for all \( x, y \geq 0 \) so that \( x + ny = 1 + n \). We have
\[
g(u) = \frac{f(u) - f(1)}{u-1} = \frac{n(u+1)}{(n+1)^2(nu^2 + n^2 + n + 1)},
\]
\[
h(x, y) = \frac{g(x) - g(y)}{x - y}
\]
\[
= \frac{n(n^2 + n + 1 - nx - ny - nxy)}{(n+1)^2(nx^2 + n^2 + n + 1)(ny^2 + n^2 + n + 1)}
\]
\[
= \frac{n(ny - 1)^2}{(n+1)^2(nx^2 + n^2 + n + 1)(ny^2 + n^2 + n + 1)} \geq 0.
\]
From \( x + ny = 1 + n \) and \( h(x, y) = 0 \), we get \( x = n \) and \( y = 1/n \). Therefore, the equality holds for \( a_1 = a_2 = \cdots = a_{2n} = 1 \), and also for
\[
a_1 = n, \quad a_2 = \cdots = a_n = 1, \quad a_{n+1} = \cdots = a_n = \frac{1}{n}.
\]
P 2.13. If \(a, b, c, d, e, f\) are nonnegative real numbers so that
\[a \geq b \geq c \geq d \geq e \geq f, \quad a + b + c + d + e + f = 6,\]
then
\[
\frac{3a + 4}{3a^2 + 4} + \frac{3b + 4}{3b^2 + 4} + \frac{3c + 4}{3c^2 + 4} + \frac{3d + 4}{3d^2 + 4} + \frac{3e + 4}{3e^2 + 4} + \frac{3f + 4}{3f^2 + 4} \leq 6.
\]

(Solution) Write the inequality as
\[f(a) + f(b) + f(c) + f(d) + f(e) + f(f) \geq 6f(s), \quad s = \frac{a + b + c + d + e + f}{6} = 1,
\]
where
\[f(u) = \frac{-3u - 4}{3u^2 + 4}, \quad u \geq 0.
\]
For \(u \in [0, 1]\), we have
\[f''(u) = \frac{6(16 - 9u^3) + 216u(1-u)}{(3u^2 + 4)^3} > 0,
\]
hence \(f\) is convex on \([0, s]\). Therefore, we may apply the LHCF-OV Theorem for \(n = 6\) and \(m = 3\). By Note 1, it suffices to show that \(h(x, y) \geq 0\) for all \(x, y \geq 0\) so that \(x + 3y = 4\). We have
\[
g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{3u}{3u^2 + 4},
\]
\[
h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{3(4 - 3xy)}{(3x^2 + 4)(3y^2 + 4)},
\]
\[= \frac{3(x - 2)^2}{(3x^2 + 4)(3y^2 + 4)} \geq 0.
\]
From \(x + 3y = 4\) and \(h(x, y) = 0\), we get \(x = 2\) and \(y = 2/3\). Therefore, in accordance with Note 4, the equality holds for \(a = b = c = d = e = f = 1\), and also for
\[a = 2, \quad b = c = 1, \quad d = e = f = \frac{2}{3}.
\]
\[\square\]

P 2.14. If \(a, b, c, d, e, f\) are nonnegative real numbers so that
\[a \geq b \geq 1 \geq c \geq d \geq e \geq f, \quad a + b + c + d + e + f = 6,
\]
then
\[
\frac{a^2 - 1}{(2a + 7)^2} + \frac{b^2 - 1}{(2b + 7)^2} + \frac{c^2 - 1}{(2c + 7)^2} + \frac{d^2 - 1}{(2d + 7)^2} + \frac{e^2 - 1}{(2e + 7)^2} + \frac{f^2 - 1}{(2f + 7)^2} \geq 0.
\]

(Vasile C., 2009)
Solution. Write the inequality as
\[ f(a) + f(b) + f(c) + f(d) + f(e) + f(f) \geq 6f(s), \quad s = \frac{a + b + c + d + e + f}{6} = 1, \]
where
\[ f(u) = \frac{u^2 - 1}{(2u + 7)^2}, \quad u \geq 0. \]
For \( u \in [0, 1] \), we have
\[ f''(u) = \frac{2(37 - 28u)}{(2u + 7)^4} > 0, \]
hence \( f \) is convex on \([0, s]\). Therefore, we may apply the LHCF-OV Theorem for \( n = 6 \) and \( m = 2 \). By Note 1, it suffices to show that \( h(x, y) \geq 0 \) for all \( x, y \geq 0 \) so that \( x + 4y = 5 \). We have
\[ g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{u + 1}{(2u + 7)^2}, \]
\[ h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{21 - 4x - 4y - 4xy}{(2x + 7)^2(2y + 7)^2} \]
\[ = \frac{(x - 4)^2}{(2x + 7)^2(2y + 7)^2} \geq 0. \]
From \( x + 4y = 5 \) and \( h(x, y) = 0 \), we get \( x = 4 \) and \( y = 1/4 \). Therefore, the equality holds only for \( a = b = c = d = e = f = 1 \), and also for
\[ a = 4, \quad b = 1, \quad c = d = e = f = \frac{1}{4}. \]

\[ \square \]

P 2.15. If \( a, b, c, d, e, f \) are nonnegative real numbers so that
\[ a \leq b \leq 1 \leq c \leq d \leq e \leq f, \quad a + b + c + d + e + f = 6, \]
then
\[ \frac{a^2 - 1}{(2a + 5)^2} + \frac{b^2 - 1}{(2b + 5)^2} + \frac{c^2 - 1}{(2c + 5)^2} + \frac{d^2 - 1}{(2d + 5)^2} + \frac{e^2 - 1}{(2e + 5)^2} + \frac{f^2 - 1}{(2f + 5)^2} \leq 0. \]

(Vasile C., 2009)
Solution. Write the inequality as
\[ f(a) + f(b) + f(c) + f(d) + f(e) + f(f) \geq 6f(s), \quad s = \frac{a + b + c + d + e + f}{6} = 1, \]
where
\[ f(u) = \frac{1 - u^2}{(2u + 5)^2}, \quad u \geq 0. \]
For \( u \geq 1 \), we have
\[ f''(u) = \frac{2(20u - 13)}{(2u + 5)^4} > 0, \]
hence \( f(u) \) is convex for \( u \geq s \). Therefore, we may apply the RHCF-OV Theorem for \( n = 6 \) and \( m = 2 \). By Note 1, it suffices to show that \( h(x, y) \geq 0 \) for all \( x, y \geq 0 \) so that \( x + 4y = 5 \). We have
\[ g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-u - 1}{(2u + 5)^2}, \]
\[ h(x, y) = \frac{g(x) - g(y)}{x - y} \]
\[ = \frac{4xy + 4x + 4y - 5}{(2x + 5)^2(2y + 5)^2} \]
\[ = \frac{4xy + 3x}{(2x + 5)^2(2y + 5)^2} \geq 0. \]
From \( x + 4y = 5 \) and \( h(x, y) = 0 \), we get \( x = 0 \) and \( y = 5/4 \). Therefore, in accordance with Note 4, the equality holds only for \( a = b = c = d = e = f = 1 \), and also for
\[ a = 0, \quad b = 1, \quad c = d = e = f = \frac{5}{4}. \]
\[ \square \]

P 2.16. If \( a, b, c \) are nonnegative real numbers so that
\[ a \leq b \leq 1 \leq c, \quad a + b + c = 3, \]
then
\[ \sqrt{\frac{2a}{b + c}} + \sqrt{\frac{2b}{c + a}} + \sqrt{\frac{2c}{a + b}} \geq 3. \]

(Vasile C., 2008)
Solution. Write the inequality as

\[ f(a) + f(b) + f(c) \geq 3f(s), \quad s = \frac{a + b + c}{3} = 1, \]

where

\[ f(u) = \sqrt{\frac{u}{3-u}}, \quad u \in [0,3). \]

From

\[ f''(u) = \frac{3(4u-3)}{4u^{3/2}(3-u)^{5/2}}, \]

it follows that \( f(u) \) is convex for \( u \geq s \). Therefore, we may apply the RHCF-OV Theorem for \( n = 3 \) and \( m = 2 \). So, it suffices to show that

\[ f(x) + f(y) \geq 2f(1) \]

for \( x + y = 2, 0 \leq x \leq 1 \leq y \). This inequality is true if \( g(x) \geq 0 \), where

\[ g(x) = f(x) + f(y) - 2f(1), \quad y = 2 - x, \quad x \in [0,1]. \]

Since \( y' = -1 \), we have

\[
g'(x) = f'(x) - f'(y) = \frac{3}{2} \left[ \frac{1}{\sqrt{x(3-x)^3}} - \frac{1}{\sqrt{y(3-y)^3}} \right].
\]

The derivative \( f'(x) \) has the same sign as \( h(x) \), where

\[
h(x) = y(3-y)^3 - x(3-x)^3 = (2-x)(1+x)^3 - x(3-x)^3 = 2(1 - 11x + 15x^2 - 5x^3) = 2(1-x)(1 - 10x + 5x^2).
\]

Let

\[
x_1 = 1 - \frac{2}{\sqrt{5}}.
\]

Since \( h(x_1) = 0, h(x) > 0 \) for \( x \in [0,x_1) \) and \( h(x) < 0 \) for \( x \in (x_1,1) \), it follows that \( g \) is increasing on \([0,x_1]\) and decreasing on \([x_1,1]\). From

\[
g(0) = f(0) + f(2) - 2f(1) = 0,
\]

\[
g(1) = f(1) + f(1) - 2f(1) = 0,
\]

it follows that \( g(x) \geq 0 \) for \( x \in [0,1] \).

The equality holds for \( a = b = c = 1 \), and also for \( a = 0, b = 1 \) and \( c = 2 \).
**P 2.17.** If $a_1, a_2, \ldots, a_8$ are nonnegative real numbers so that
\[ a_1 \geq a_2 \geq a_3 \geq a_4 \geq 1 \geq a_5 \geq a_6 \geq a_7 \geq a_8, \quad a_1 + a_2 + \cdots + a_8 = 8, \]
then
\[ (a_1^2 + 1)(a_2^2 + 1) \cdots (a_8^2 + 1) \geq (a_1 + 1)(a_2 + 1) \cdots (a_8 + 1). \]

*(Vasile C., 2008)*

**Solution.** Write the inequality as
\[ f(a_1) + f(a_2) + \cdots + f(a_8) \geq 8f(s), \quad s = \frac{a_1 + a_2 + \cdots + a_8}{8} = 1, \]
where
\[ f(u) = \ln(u^2 + 1) - \ln(u + 1), \quad u \geq 0. \]

For $u \in [0, 1]$, we have
\[ f''(u) = \frac{2(1-u^2)}{(u^2+1)^2} + \frac{1}{(u+1)^2} = \frac{(u^2-u^4) + 4u(1-u^2) + u^2 + 3}{(u^2+1)^2(u+1)^2} > 0. \]

Therefore, $f$ is convex on $[0,s]$. According to the LHCF-OV Theorem applied for $n = 8$ and $m = 4$, it suffices to show that $f(x) + 4f(y) \geq 5f(1)$ for $x, y \geq 0$ so that $x + 4y = 5$. Using Note 2, we only need to show that $H(x, y) \geq 0$ for $x, y \geq 0$ so that $x + 4y = 5$, where
\[ H(x, y) = \frac{f'(x) - f'(y)}{x - y} = \frac{2(1-xy)}{(x^2+1)(y^2+1)} + \frac{1}{(x+1)(y+1)}. \]

The inequality $H(x, y) \geq 0$ is equivalent to
\[ 2(1-xy)(x+1)(y+1) + (x^2+1)(y^2+1) \geq 0. \]

Since $2(x^2 + 1) \geq (x + 1)^2$ and $2(y^2 + 1) \geq (y + 1)^2$, it suffices to prove that
\[ 8(1-xy) + (x+1)(y+1) \geq 0. \]

Indeed,
\[ 8(1-xy) + (x+1)(y+1) = 28x^2 - 38x + 14 = 28(x - 19/28)^2 + 31/28 > 0. \]

The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_8$. \[ \square \]
P 2.18. If \( a, b, c, d \) are real numbers so that
\[
-\frac{1}{2} \leq a \leq b \leq 1 \leq c \leq d, \quad a + b + c + d = 4,
\]
then
\[
7 \left( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \right) + 3 \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) \geq 40.
\]

*(Vasile C., 2011)*

**Solution.** We have
\[
d = 4 - a - b - c \leq 4 + \frac{1}{2} + \frac{1}{2} - 1 = 4.
\]
Write the inequality as
\[
f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a + b + c + d}{4} = 1,
\]
where
\[
f(u) = \frac{7}{u^2} + \frac{3}{u}, \quad u \in \mathbb{I} = \left[ -\frac{1}{2}, 4 \right] \setminus \{0\}.
\]
Clearly, \( f(u) \) is convex for \( u \geq 1 \) (because \( \frac{7}{u^2} \) and \( \frac{3}{u} \) are convex). According to Note 3, we may apply the RHCF-OV Theorem for \( n = 4 \) and \( m = 2 \). By Note 1, we only need to show that \( h(x, y) \geq 0 \) for \( x, y \in \mathbb{I} \) so that \( x + 2y = 3 \), where
\[
h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.
\]
We have
\[
g(u) = -\frac{7}{u^2} - \frac{10}{u},
\]
\[
h(x, y) = \frac{7(x + y) + 10xy}{x^2y^2} = \frac{(2x + 1)(-5x + 21)}{2x^2y^2} \geq 0.
\]
From \( x + 2y = 3 \) and \( h(x, y) = 0 \), we get \( x = -1/2 \), \( y = 7/3 \). Therefore, in accordance with Note 4, the equality holds for \( a = b = c = d = 1 \), and also for
\[
a = -\frac{1}{2}, \quad b = 1, \quad c = d = \frac{7}{4}.
\]
\( \square \)
P 2.19. Let $a, b, c, d$ be real numbers. Prove that

(a) if $-1 \leq a \leq b \leq c \leq 1 \leq d$, then

$$3 \left( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \right) \geq 8 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d};$$

(b) if $-1 \leq a \leq b \leq 1 \leq c \leq d$, then

$$2 \left( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \right) \geq 4 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}. $$

(Vasile C., 2011)

Solution. (a) We have

$$d = 4 - a - b - c \leq 4 + 1 + 1 + 1 = 7.$$

Write the desired inequality as

$$f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a + b + c + d}{4} = 1,$$

where

$$f(u) = \frac{3}{u^2} - \frac{1}{u}, \quad u \in \mathbb{I} = [-1, 7] \setminus \{0\}.$$

From

$$f''(u) = \frac{2(9-u)}{u^4} > 0,$$

it follows that $f$ is convex on $\mathbb{I}_{\geq}$. According to Note 3, we may apply the RHCF-OV Theorem for $n = 4$ and $m = 3$. By Note 1, it suffices to show that $h(x, y) \geq 0$ for all $x, y \in \mathbb{I}$ so that $x + y = 2$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{2}{u} - \frac{3}{u^2},$$

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{3(x + y) + 2xy}{x^2y^2} = \frac{2(x + 1)(3 - x)}{x^2y^2} = \frac{2(x + 1)(y + 1)}{x^2y^2} \geq 0.$$

From $x < y$, $x + y = 2$ and $h(x, y) = 0$, we get $x = -1$ and $y = 3$. Therefore, in accordance with Note 4, the equality holds for $a = b = c = d = 1$, and also for

$$a = -1, \quad b = c = 1, \quad d = 3.$$

(b) We have

$$d = 4 - a - b - c \leq 4 + 1 + 1 - 1 = 5.$$
Write the desired inequality as

\[ f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a + b + c + d}{4} = 1, \]

where

\[ f(u) = \frac{2}{u^2} - \frac{1}{u}, \quad u \in \mathbb{I} = [-1, 5] \setminus \{0\}. \]

From

\[ f''(u) = \frac{2(6-u)}{u^4} > 0, \]

it follows that \( f \) is convex on \( \mathbb{I} \geq s \). According to Note 3, we may apply the RHCF-OV Theorem for \( n = 4 \) and \( m = 2 \). By Note 1, it suffices to show that \( h(x, y) \geq 0 \) for all \( x, y \in \mathbb{I} \) so that \( x + 2y = 3 \). We have

\[ g(u) = \frac{f(u) - f(1)}{u-1} = -\frac{1}{u} - \frac{2}{u^2}, \]

\[ h(x, y) = \frac{g(x) - g(y)}{x-y} = \frac{2(x+y)+xy}{x^2y^2} \]

\[ = \frac{(x+1)(6-x)}{2x^2y^2} \geq 0. \]

From \( x + 2y = 3 \) and \( h(x, y) = 0 \), we get \( x = -1 \) and \( y = 2 \). Therefore, the equality holds for \( a = b = c = d = 1 \), and also for \( a = -1, \quad b = 1, \quad c = d = 2 \).

\[ \square \]

**P 2.20.** If \( a, b, c, d \) are positive real numbers so that \( a \geq b \geq 1 \geq c \geq d \), \( abcd = 1 \), then

\[ a^2 + b^2 + c^2 + d^2 - 4 \geq 18 \left( a + b + c + d - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} - \frac{1}{d} \right). \]

\( (\text{Vasile C.}, \ 2008) \)

**Solution.** Using the substitution

\[ a = e^x, \quad b = e^y, \quad c = e^z, \quad d = e^w, \]

we need to show that

\[ f(x) + f(y) + f(z) + f(w) \geq 4f(s), \]
where
\[ x \geq y \geq 0 \geq z \geq w, \quad s = \frac{x + y + z + w}{4} = 0, \]
\[ f(u) = e^{2u} - 1 - 18(e^u - e^{-u}), \quad u \in \mathbb{R}. \]

For \( u \leq 0 \), we have
\[ f''(u) = 4e^{2u} + 18(e^{-u} - e^u) > 0, \]

hence \( f \) is convex on \((-\infty, s]\). By the LHCF-OV Theorem applied for \( n = 4 \) and \( m = 2 \), it suffices to show that \( f(x) + 2f(y) \geq 3f(0) \) for all real \( x, y \) so that \( x + 2y = 0 \); that is, to show that
\[ a^2 + 2b^2 - 3 - 18\left(\frac{a + 2b - 1}{a} - \frac{2}{b}\right) \geq 0 \]

for all \( a, b > 0 \) so that \( ab^2 = 1 \). This inequality is equivalent to
\[ \frac{(b^2 - 1)^2(2b^2 + 1)}{b^4} + \frac{18(b - 1)^3(b + 1)}{b^2} \geq 0, \]
\[ \frac{(b^2 - 1)^2(2b - 1)^2(b + 1)(5b + 1)}{b^4} \geq 0. \]

The proof is completed. The equality holds for \( a = b = c = d = 1 \), and also for \( a = 4, \quad b = 1, \quad c = d = 1/2 \).

\[ \square \]

**P 2.21.** If \( a, b, c, d \) are positive real numbers so that
\[ a \leq b \leq 1 \leq c \leq d, \quad abcd = 1, \]

then
\[ \sqrt{a^2 - a + 1} + \sqrt{b^2 - b + 1} + \sqrt{c^2 - c + 1} + \sqrt{d^2 - d + 1} \geq a + b + c + d. \]

*(Vasile C., 2008)*

**Solution.** Using the substitution
\[ a = e^x, \quad b = e^y, \quad c = e^z, \quad d = e^w, \]

we need to show that
\[ f(x) + f(y) + f(z) + f(w) \geq 4f(s), \]

where
\[ x \leq y \leq 0 \leq z \leq w, \quad s = \frac{x + y + z + w}{4} = 0, \]
\[ f(u) = \sqrt{e^{2u} - e^u + 1} - e^u, \quad u \in \mathbb{R}. \]

We claim that \( f \) is convex for \( u \geq 0 \). Since
\[
e^{-u}f''(u) = \frac{4e^{3u} - 6e^{2u} + 9e^u - 2}{4(e^{2u} - e^u + 1)^{3/2}} - 1,
\]
we need to show that
\[ 4t^3 - 6t^2 + 9t - 2 \geq 0 \]
and
\[ (4t^3 - 6t^2 + 9t - 2)^2 \geq 16(t^2 - t + 1)^3, \]
where \( t = e^u \geq 1 \). Indeed, we have
\[ 4t^3 - 6t^2 + 9t - 2 \geq 4t^3 - 6t^2 + 7t \geq 2t(t - 1)(2t - 1) \geq 0 \]
and
\[ (4t^3 - 6t^2 + 9t - 2)^2 - 16(t^2 - t + 1)^3 = 12t^3(t - 1) + 9t^2 + 12(t - 1) > 0. \]

By the RHCF-OV Theorem applied for \( n = 4 \) and \( m = 2 \), it suffices to show that
\[ f(x) + 2f(y) \geq 3f(0) \]
for all \( x \) and \( y \) so that \( x + 2y = 0 \); that is, to show that
\[ \sqrt{a^2 - a + 1} + 2\sqrt{b^2 - b + 1} \geq a + 2b \]
for all \( a, b > 0 \) so that \( ab^2 = 1 \). This inequality is equivalent to
\[ \frac{\sqrt{b^4 - b^2 + 1}}{b^2} + 2\sqrt{b^2 - b + 1} \geq \frac{1}{b^2} + 2b, \]
\[ \frac{\sqrt{b^4 - b^2 + 1} - 1}{b^2} + 2(\sqrt{b^2 - b + 1} - 1) \geq 0, \]
\[ \frac{b^2 - 1}{\sqrt{b^4 - b^2 + 1} + 1} + \frac{2(1 - b)}{\sqrt{b^2 - b + 1} + b} \geq 0. \]

Since
\[ \frac{b^2 - 1}{\sqrt{b^4 - b^2 + 1} + 1} \geq \frac{b^2 - 1}{b^2 + 1}, \]
it suffices to show that
\[ \frac{b^2 - 1}{b^2 + 1} + \frac{2(1 - b)}{\sqrt{b^2 - b + 1} + b} \geq 0, \]
which is equivalent to
\[ (b - 1) \left[ \frac{b + 1}{b^2 + 1} - \frac{2}{\sqrt{b^2 - b + 1} + b} \right] \geq 0, \]
\[
(b - 1) \left[ (b + 1) \sqrt{b^2 - b + 1 - b^2 + b - 2} \right] \geq 0,
\]
\[
\frac{(b - 1)^2(3b^2 - 2b + 3)}{(b + 1) \sqrt{b^2 - b + 1 + b^2 - b + 2}} \geq 0.
\]
The last inequality is clearly true. The equality holds for \(a = b = c = d = 1\).

\[
\text{P 2.22. If } a, b, c, d \text{ are positive real numbers so that } a \leq b \leq c \leq 1 \leq d, \quad abcd = 1,
\]
then
\[
\frac{1}{a^3 + 3a + 2} + \frac{1}{b^3 + 3b + 2} + \frac{1}{c^3 + 3c + 2} + \frac{1}{d^3 + 3d + 2} \geq \frac{2}{3}.
\]

(Vasile C., 2007)

**Solution.** Using the substitution
\[
a = e^x, \quad b = e^y, \quad c = e^z, \quad d = e^w,
\]
we need to show that
\[
f(x) + f(y) + f(z) + f(w) \geq 4f(s),
\]
where
\[
x \leq y \leq z \leq 0 \leq w, \quad s = \frac{x + y + z + w}{4} = 0,
\]
\[
f(u) = \frac{1}{e^{3u} + 3e^u + 2}, \quad u \in \mathbb{R}.
\]
We claim that \(f\) is convex for \(u \geq 0\). Indeed, denoting \(t = e^u, t \geq 1\), we have
\[
f''(u) = \frac{3t(3t^5 + 2t^3 - 6t^2 + 3t - 2)}{(t^3 + 3t + 2)^3}
\]
\[
= \frac{3t(t - 1)(3t^4 + 3t^3 + 5t^2 - t + 2)}{(t^3 + 3t + 2)^3} \geq 0.
\]
By the RHCF-OV Theorem applied for \(n = 4\) and \(m = 3\), it suffices to show that
\[
f(x) + f(y) \geq 2f(0) \text{ for all real } x, y \text{ so that } x + y = 0; \text{ that is, to show that}
\]
\[
\frac{1}{a^3 + 3a + 2} + \frac{1}{b^3 + 3b + 2} \geq \frac{1}{3}
\]
for all \(a, b > 0\) so that \(ab = 1\). This inequality is equivalent to
\[
(a - 1)^4(a^2 + a + 1) \geq 0.
\]
The equality holds for \(a = b = c = d = 1\).
P 2.23. If \( a_1, a_2, \ldots, a_n \) are positive real numbers so that
\[
a_1 \geq \cdots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1a_2 \cdots a_n = 1,
\]
then
\[
\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \geq a_1 + a_2 + \cdots + a_n.
\]

(Vasile C., 2007)

Solution. Using the substitution
\[
a_i = e^{x_i}, \quad i = 1, 2, \ldots, n,
\]
we need to show that
\[
f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s),
\]
where
\[
x_1 \geq \cdots \geq x_{n-1} \geq 0 \geq x_n, \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,
\]
\[
f(u) = e^{-u} - e^u, \quad u \in \mathbb{R}.
\]
For \( u \leq 0 \), we have
\[f''(u) = e^{-u} - e^u \geq 0,
\]
therefore \( f(u) \) is convex for \( u \leq s \). By the LHCF-OV Theorem applied for \( m = n - 1 \), it suffices to show that \( f(x) + f(y) \geq 2f(0) \) for all real \( x, y \) so that \( x + y = 0 \); that is, to show that
\[
\frac{1}{a} - a + \frac{1}{b} - b \geq 0
\]
for all \( a, b > 0 \) so that \( ab = 1 \). This is true since
\[
\frac{1}{a} - a + \frac{1}{b} - b = \frac{1}{a} - a + a - \frac{1}{a} = 0.
\]
The equality holds for
\[
a_1 \geq 1, \quad a_2 = \cdots = a_{n-1} = 1, \quad a_n = 1/a_1.
\]
\( \square \)

P 2.24. Let \( a_1, a_2, \ldots, a_n \) be positive real numbers so that
\[
a_1 \leq \cdots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1a_2 \cdots a_n = 1.
\]
If \( k \geq 1 \), then
\[
\frac{1}{1 + ka_1} + \frac{1}{1 + ka_2} + \cdots + \frac{1}{1 + ka_n} \geq \frac{n}{1 + k}.
\]

(Vasile C., 2007)
**Solution.** Using the substitution

\[ a_i = e^{x_i}, \quad i = 1, 2, \ldots, n, \]

we need to show that

\[ f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s), \]

where

\[ x_1 \leq \cdots \leq x_{n-1} \leq 0 \leq x_n, \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0, \]

\[ f(u) = \frac{1}{1 + ke^u}, \quad u \in \mathbb{R}. \]

For \( u \geq 0 \), we have

\[ f''(u) = \frac{ke^u(ke^u - 1)}{(1 + ke^u)^3} \geq 0, \]

therefore \( f(u) \) is convex for \( u \geq s \). By the RHCF-OV Theorem applied for \( m = n-1 \), it suffices to show that \( f(x) + f(y) \geq 2f(0) \) for all real \( x, y \) so that \( x + y = 0 \); that is, to show that

\[ \frac{1}{1 + ka} + \frac{1}{1 + kb} \geq \frac{2}{1 + k} \]

for all \( a, b > 0 \) so that \( ab = 1 \). This is true since

\[ \frac{1}{1 + ka} + \frac{1}{1 + kb} - \frac{2}{1 + k} = \frac{k(k-1)(a-1)^2}{(1 + ka)(a + k)} \geq 0. \]

The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \). If \( k = 1 \), then the equality holds for

\[ a_1 = 1, \quad a_2 = \cdots = a_{n-1} = 1, \quad a_n = 1/a_1. \]

\[ \square \]

**P 2.25.** If \( a_1, a_2, \ldots, a_9 \) are positive real numbers so that

\[ a_1 \leq \cdots \leq a_8 \leq 1 \leq a_9, \quad a_1a_2 \cdots a_9 = 1, \]

then

\[ \frac{1}{(a_1 + 2)^2} + \frac{1}{(a_2 + 2)^2} + \cdots + \frac{1}{(a_9 + 2)^2} \geq 1. \]

(Vasile C., 2007)
**Solution.** Using the substitution

\[ a_i = e^{x_i}, \quad i = 1, 2, \ldots, 9, \]

we can write the inequality as

\[ f(x_1) + f(x_2) + \cdots + f(x_9) \geq 9f(s), \]

where

\[ x_1 \leq \cdots \leq x_8 \leq 0 \leq x_9, \quad s = \frac{x_1 + x_2 + \cdots + x_9}{9} = 0, \]

\[ f(u) = \frac{1}{(e^u + 2)^2}, \quad u \in \mathbb{R}. \]

For \( u \in [0, \infty) \), we have

\[ f''(u) = \frac{4e^u(e^u - 1)}{(e^u + 2)^4} \geq 0, \]

hence \( f \) is convex on \([s, \infty)\). According to the RHCF-OV Theorem (case \( n = 9 \) and \( m = 8 \)), it suffices to show that \( f(x) + f(y) \geq 2f(0) \) for all real \( x, y \) so that \( x + y = 0 \); that is, to show that

\[ \frac{1}{(a + 2)^2} + \frac{1}{(b + 2)^2} \geq \frac{2}{9} \]

for all \( a, b > 0 \) so that \( ab = 1 \). Write this inequality as

\[ \frac{b^2}{(2b + 1)^2} + \frac{1}{(b + 2)^2} \geq \frac{2}{9}, \]

which is equivalent to the obvious inequality

\[ (b - 1)^4 \geq 0. \]

The equality holds for \( a_1 = a_2 = \cdots = a_9 = 1 \). \( \square \)

**P 2.26.** Let \( a_1, a_2, \ldots, a_n \) be positive real numbers so that

\[ a_1 \leq \cdots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1a_2\cdots a_n = 1. \]

If \( p, q \geq 0 \) so that

\[ p + q \geq 1 + \frac{2pq}{p + 4q}, \]

then

\[ \frac{1}{1 + pa_1 + qa_1^2} + \frac{1}{1 + pa_2 + qa_2^2} + \cdots + \frac{1}{1 + pa_n + qa_n^2} \geq \frac{n}{1 + p + q}. \]

(Vasile C., 2007)
**Solution.** Using the substitution

\[ a_i = e^{x_i}, \quad i = 1, 2, \ldots, n, \]

we can write the inequality as

\[ f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s), \]

where

\[ x_1 \leq \cdots \leq x_{n-1} \leq 0 \leq x_n, \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0, \]

\[ f(u) = \frac{1}{1 + pe^u + qe^{2u}}, \quad u \in \mathbb{R}. \]

We have

\[ f''(u) = \frac{e^u f_1(u)}{(1 + pe^u + qe^{2u})^3}, \]

where

\[ f_1(u) = 4q^2e^{3u} + 3pqe^{2u} + (p^2 - 4q)e^u - p. \]

The hypothesis \( p + q \geq 1 + \frac{2pq}{p + 4q} \) is equivalent to

\[ p^2 + 3pq + 4q^2 \geq p + 4q. \]

For \( u \in [0, \infty) \), we have

\[ f_1(u) \geq 4q^2e^u + 3pqe^u + (p^2 - 4q)e^u - p \geq p(e^u - 1) \geq 0, \]

hence \( f \) is convex on \([s, \infty)\). According to the RHCF-OV Theorem (case \( m = n-1 \)), it suffices to show that \( f(x) + f(y) \geq 2f(0) \) for all real \( x, y \) so that \( x + y = 0 \); that is, to show that

\[ \frac{1}{1 + pa + qa^2} + \frac{1}{1 + pb + qb^2} \geq \frac{2}{1 + p + q} \]

for all \( a, b > 0 \) so that \( ab = 1 \). Write this inequality as

\[ \frac{1}{1 + pa + qa^2} + \frac{a^2}{a^2 + pa + q} \geq \frac{2}{1 + p + q} \]

which is equivalent to

\[ (a - 1)^2h(a) \geq 0, \]

where

\[
\begin{align*}
h(a) &= q(p + q - 1)(a^2 + 1) + (p^2 + pq + 2q^2 - p - 2q)a \\
    &\geq 2q(p + q - 1)a + (p^2 + pq + 2q^2 - p - 2q)a \\
    &= (p^2 + 3pq + 4q^2 - p - 4q)a \geq 0.
\end{align*}
\]

The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \).

**Remark.** For \( p = 1, q = 1/4 \) and \( n = 9 \), we get the preceding P 2.25. \( \square \)
P 2.27. Let \(a_1, a_2, \ldots, a_n\) be positive real numbers so that
\[
a_1 \leq \cdots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1 a_2 \cdots a_n = 1.
\]
If \(m \geq 1\) and \(0 < k \leq m\), then
\[
\frac{1}{(a_1 + k)^m} + \frac{1}{(a_2 + k)^m} + \cdots + \frac{1}{(a_n + k)^m} \geq \frac{n}{(1 + k)^m}.
\]
(Vasile C., 2007)

Solution. Using the substitution
\[
a_i = e^{x_i}, \quad i = 1, 2, \ldots, n,
\]
we can write the inequality as
\[
f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s),
\]
where
\[
x_1 \leq \cdots \leq x_{n-1} \leq 0 \leq x_n, \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,
\]
\[
f(u) = \frac{1}{(e^u + k)^m}, \quad u \in \mathbb{R}.
\]
For \(u \in [0, \infty)\), we have
\[
f''(u) = \frac{m e^u (m e^u - k)}{(e^u + k)^{m+2}} \geq 0,
\]
hence \(f\) is convex on \([s, \infty)\). According to the RHCF-OV Theorem (case \(m = n-1\)), it suffices to show that \(f(x) + f(y) \geq 2f(0)\) for all real \(x, y\) so that \(x \leq y\) and \(x + y = 0\); that is, to show that
\[
\frac{1}{(a + k)^m} + \frac{1}{(b + k)^m} \geq \frac{2}{(1 + k)^m}
\]
for all \(a, b > 0\) so that \(a \in (0, 1]\) and \(ab = 1\). Write this inequality as \(g(a) \geq 0\), where
\[
g(a) = \frac{1}{(a + k)^m} + \frac{a^m}{(ka + 1)^m} - \frac{2}{(1 + k)^m},
\]
with
\[
g'(a) = \frac{a^{m-1}(a + k)^{m+1} - (ka + 1)^{m+1}}{(a + k)^{m+1}(ka + 1)^{m+1}}.
\]
If \(g'(a) \leq 0\) for \(a \in (0, 1]\), then \(g\) is decreasing, hence \(g(a) \geq g(1) = 0\). Thus, it suffices to show that
\[
a^{m-1} \leq \left(\frac{ka + 1}{a + k}\right)^{m+1}.
\]
Since
\[
\frac{ka + 1}{a + k} - \frac{ma + 1}{a + m} = \frac{(m - k)(1 - a^2)}{(a + k)(a + m)} \geq 0,
\]
we only need to show that
\[
a^{m-1} \leq \left( \frac{ma + 1}{a + m} \right)^{m+1},
\]
which is equivalent to \( h(a) \leq 0 \) for \( a \in (0, 1] \), where
\[
h(a) = (m - 1) \ln a + (m + 1) \ln(a + m) - (m + 1) \ln(ma + 1),
\]
with
\[
h'(a) = \frac{m - 1}{a} + \frac{m + 1}{a + m} - \frac{m(m + 1)}{ma + 1} = \frac{m(m - 1)(a - 1)^2}{a(a + m)(ma + 1)}.
\]
Since \( h'(a) \geq 0 \), \( h(a) \) is increasing for \( a \in (0, 1] \), therefore \( h(a) \leq h(1) = 0 \). The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \).

**Remark.** For \( k = m = 2 \) and \( n = 9 \), we get the inequality in P 2.25.

\[\square\]

**P 2.28.** If \( a_1, a_2, \ldots, a_n \) are positive real numbers so that
\[
a_1 \leq \cdots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1a_2\cdots a_n = 1,
\]
then
\[
\frac{1}{\sqrt{1 + 3a_1}} + \frac{1}{\sqrt{1 + 3a_2}} + \cdots + \frac{1}{\sqrt{1 + 3a_n}} \geq \frac{n}{2}.
\]

*(Vasile C., 2007)*

**Solution.** Using the substitution
\[
a_i = e^{x_i}, \quad i = 1, 2, \ldots, n,
\]
we can write the inequality as
\[
f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s),
\]
where
\[
x_1 \leq \cdots \leq x_{n-1} \leq 0 \leq x_n, \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,
\]
\[
f(u) = \frac{1}{\sqrt{1 + 3e^u}}, \quad u \in \mathbb{R}.
\]
For \( u \geq 0 \), we have
\[
f''(u) = \frac{3e^u(3e^u - 2)}{4(1 + 3e^u)^{5/2}} > 0,
\]
hence \( f \) is convex on \([s, \infty)\). According to the RHCF-OV Theorem (case \( m = n-1 \)), it suffices to show that \( f(x) + f(y) \geq 2f(0) \) for all real \( x, y \) so that \( x + y = 0 \); that is, to show that

\[
\frac{1}{\sqrt{1 + 3a}} + \frac{1}{\sqrt{1 + 3b}} \geq 1
\]

for all \( a, b > 0 \) so that \( ab = 1 \). Write this inequality as

\[
\frac{1}{\sqrt{1 + 3a}} + \sqrt{\frac{a}{a + 3}} \geq 1.
\]

Substituting \( \frac{1}{\sqrt{1 + 3a}} = t, \ 0 < t < 1 \), the inequality becomes

\[
\sqrt{\frac{1 - t^2}{8t^2 + 1}} \geq 1 - t.
\]

By squaring, we get

\[
t(1-t)(2t-1)^2 \geq 0,
\]

which is true. The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \).

\( \square \)

\[ \textbf{P 2.29.} \text{ Let } a_1, a_2, \ldots, a_n \text{ be positive real numbers so that } a_1 \leq \cdots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1 a_2 \cdots a_n = 1. \]

If \( 0 < m < 1 \) and \( 0 < k \leq \frac{1}{2^{1/m} - 1} \), then

\[
\frac{1}{(a_1 + k)^m} + \frac{1}{(a_2 + k)^m} + \cdots + \frac{1}{(a_n + k)^m} \geq \frac{n}{(1 + k)^m}.
\]

(Vasile C., 2007)

\[ \textbf{Solution.} \text{ By Bernoulli's inequality, we have } 2^{1/m} > 1 + \frac{1}{m}, \]

hence

\[
k \leq \frac{1}{2^{1/m} - 1} < m < 1.
\]

Using the substitution

\[ a_i = e^{x_i}, \quad i = 1, 2, \ldots, n, \]

we can write the inequality as

\[ f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s), \]
where
\[ x_1 \leq \cdots \leq x_{n-1} \leq 0 \leq x_n, \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0, \]
\[ f(u) = \frac{1}{(e^u + k)^m}, \quad u \in \mathbb{R}. \]
For \( u \in [0, \infty) \), we have
\[ f''(u) = \frac{m e^u (m e^u - k)}{(e^u + k)^{m+2}} \geq 0, \]
hence \( f \) is convex on \([s, \infty)\). According to the RHCF-OV Theorem (case \( m = n-1 \)), it suffices to show that \( f(x) + f(y) \geq 2f(0) \) for all real \( x, y \) so that \( x + y = 0 \); that is, to show that
\[ \frac{1}{(a+k)^m} + \frac{1}{(b+k)^m} \geq \frac{2}{(1+k)^m} \]
for all \( a, b > 0 \) so that \( ab = 1 \). Write this inequality as \( g(a) \geq 0 \) for \( a \geq 1 \), where
\[ g(a) = \frac{1}{(a+k)^m} + \frac{a^m}{(ka+1)^m} - \frac{2}{(1+k)^m}. \]
The derivative
\[ \frac{g'(a)}{m} = \frac{a^{m-1}(a+k)^{m+1} - (ka+1)^{m+1}}{(a+k)^{m+1}(ka+1)^{m+1}} \]
has the same sign as the function
\[ h(a) = (m-1)\ln a + (m+1)\ln(a+k) - (m+1)\ln(ka+1). \]
We have
\[ h'(a) = \frac{m-1}{a} + (m+1)\left(\frac{1}{a+k} - \frac{k}{ka+1}\right) = \frac{kh_1(a)}{a(a+k)(ka+1)}, \]
where
\[ h_1(a) = (m-1)(a^2 + 1) - 2\left(k - \frac{m}{k}\right)a. \]
The discriminant \( D \) of the quadratic function \( h_1(a) \) is
\[ \frac{D}{4} = \left(k - \frac{m}{k}\right)^2 - (m-1)^2 = (1-k^2)\left(\frac{m^2}{k^2} - 1\right). \]
Since \( D > 0 \), the roots \( a_1 \) and \( a_2 \) of \( h_1(a) \) are real and unequal. If \( a_1 < a_2 \), then \( h_1(a) \geq 0 \) for \( a \in [a_1, a_2] \) and \( h_1(a) \leq 0 \) for \( a \in (-\infty, a_1] \cup [a_2, \infty) \). Since
\[ h_1(1) = \frac{2(k+1)(m-k)}{k} > 0, \]
it follows that $a_1 < 1 < a_2$, therefore $h_1(a)$ and $h'(a)$ are positive for $a \in [1, a_2)$ and negative for $a \in (a_2, \infty)$, $h$ is increasing on $[1, a_2]$ and decreasing on $[a_2, \infty)$. From $h(1) = 0$ and
\[
\lim_{a \to \infty} h(a) = -\infty,
\]
it follows that there is $a_3 > a_2$ so that $h(a)$ and $g'(a)$ are positive for $a \in (1, a_3)$ and negative for $a \in (a_3, \infty)$. As a result, $g$ is increasing on $[1, a_3]$ and decreasing on $[a_3, \infty)$. Since $g(1) = 0$ and
\[
\lim_{a \to \infty} g(a) = \frac{1}{km} - \frac{2}{(1+k)^m} \geq 0,
\]
it follows that $g(a) \geq 0$ for $a \geq 1$. This completes the proof. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

**Remark.** For $k = \frac{1}{3}$ and $m = \frac{1}{2}$, we get the preceding P 2.28.

\[\square\]

**P 2.30.** If $a_1, a_2, \ldots, a_n$ ($n \geq 4$) are positive real numbers so that
\[
a_1 \geq a_2 \geq a_3 \geq 1 \geq a_4 \geq \cdots \geq a_n, \quad a_1a_2\cdots a_n = 1,
\]
then
\[
\frac{1}{3a_1+1} + \frac{1}{3a_2+1} + \cdots + \frac{1}{3a_n+1} \geq \frac{n}{4}.
\]
(Vasile C., 2007)

**Solution.** Using the substitution
\[
a_i = e^{x_i}, \quad i = 1, 2, \ldots, n,
\]
we can write the inequality as
\[
f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s),
\]
where
\[
x_1 \geq x_2 \geq x_3 \geq 0 \geq x_4 \geq \cdots \geq x_n, \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,
\]
\[
f(u) = \frac{1}{3e^u + 1}, \quad u \in \mathbb{R}.
\]
For $u \in [0, \infty)$, we have
\[
f''(u) = \frac{3e^u(3e^u-1)}{(3e^u+1)^3} > 0,
\]
hence $f$ is convex on $[s, \infty)$. According to the RHCF-OV Theorem (case $m = n - 3$), it suffices to show that $f(x) + 3f(y) \geq 4f(0)$ for all real $x, y$ so that $x + 3y = 0$; that is, to show that
\[
\frac{1}{3a + 1} + \frac{3}{3b + 1} \geq 1
\]
for all $a, b > 0$ so that $ab^3 = 1$. The inequality is equivalent to
\[
\frac{b^3}{b^3 + 3} + \frac{3}{3b + 1} \geq 1,
\]
\[
(b - 1)(b + 2) \geq 0.
\]
The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

\[
\text{P 2.31.} \text{ If } a_1, a_2, \ldots, a_n \ (n \geq 4) \text{ are positive real numbers so that}
\]
\[
a_1 \geq a_2 \geq a_3 \geq 1 \geq a_4 \geq \cdots \geq a_n, \quad a_1a_2\cdots a_n = 1,
\]
then
\[
\frac{1}{(a_1 + 1)^2} + \frac{1}{(a_2 + 1)^2} + \cdots + \frac{1}{(a_n + 1)^2} \geq \frac{n}{4}.
\]

(Vasile C., 2007)

\textbf{Solution.} Using the substitution
\[
a_i = e^{x_i}, \quad i = 1, 2, \ldots, n,
\]
we can write the inequality as
\[
f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s),
\]
where
\[
x_1 \geq x_2 \geq x_3 \geq 0 \geq x_4 \geq \cdots \geq x_n, \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,
\]
\[
f(u) = \frac{1}{(e^u + 1)^2}, \quad u \in \mathbb{R}.
\]
For $u \in [0, \infty)$, we have
\[
f''(u) = \frac{2e^u(2e^u - 1)}{(e^u + 1)^4} > 0,
\]
hence $f$ is convex on $[s, \infty)$. According to the RHCF-OV Theorem (case $m = 3$), it suffices to show that $f(x) + 3f(y) \geq 4f(0)$ for all real $x, y$ so that $x + 3y = 0$; that is, to show that
\[
\frac{1}{(a + 1)^2} + \frac{3}{(b + 1)^2} \geq 1
\]
for all \( a, b > 0 \) so that \( ab^3 = 1 \). The inequality is equivalent to
\[
\frac{b^6}{(b^3 + 1)^3} + \frac{3}{(b + 1)^2} \geq 1.
\]
Using the Cauchy-Schwarz inequality, it suffices to show that
\[
\frac{(b^3 + 3)^2}{(b^3 + 1)^2 + 3(b + 1)^2} \geq 1,
\]
which is equivalent to the obvious inequality
\[
(b - 1)^2(4b + 5) \geq 0.
\]
The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \).

\[\square\]

\textbf{P 2.32.} If \( a_1, a_2, \ldots, a_n \) are positive real numbers so that
\[a_1 \geq \cdots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1a_2\cdots a_n = 1,\]
then
\[
\frac{1}{(a_1 + 3)^2} + \frac{1}{(a_2 + 3)^2} + \cdots + \frac{1}{(a_n + 3)^2} \leq \frac{n}{16}.
\]
(\textit{Vasile C., 2007})

\textbf{Solution.} Using the substitution
\[a_i = e^{x_i}, \quad i = 1, 2, \ldots, n,\]
we can write the inequality as
\[f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s),\]
where
\[x_1 \geq \cdots \geq x_{n-1} \geq 0 \geq x_n, \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,\]
\[f(u) = \frac{-1}{(e^u + 3)^2}, \quad u \in \mathbb{R}.
\]
For \( u \in (-\infty, 0] \), we have
\[f''(u) = \frac{2e^u(3 - 2e^u)}{(e^u + 3)^4} > 0,\]
hence \( f \) is convex on \((-\infty, s]\). According to the LHCF-OV Theorem (case \( m = n-1 \)), it suffices to show that \( f(x) + f(y) \geq 2f(0) \) for all real \( x, y \) so that \( x + y = 0 \); that is, to show that
\[
\frac{1}{(a + 3)^2} + \frac{1}{(b + 3)^2} \leq \frac{1}{8}
\]
for all $a, b > 0$ so that $ab = 1$. Write this inequality as

$$\frac{b^2}{(3b + 1)^2} + \frac{1}{(b + 3)^2} \leq \frac{1}{8},$$

which is equivalent to the obvious inequality

$$(b^2 - 1)^2 + 12b(b - 1)^2 \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

**Remark.** Similarly, we can prove the following generalization:

- Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that

  $$a_1 \geq \cdots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1a_2\cdots a_n = 1,$$

  If $k \geq 1 + \sqrt{2}$, then

  $$\frac{1}{(a_1 + k)^2} + \frac{1}{(a_2 + k)^2} + \cdots + \frac{1}{(a_n + k)^2} \leq \frac{n}{(1 + k)^2},$$

  with equality for $a_1 = a_2 = \cdots = a_n = 1$.

\[\square\]

**P 2.33.** Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that

$$a_1 \geq \cdots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1a_2\cdots a_n = 1.$$

If $p, q \geq 0$ so that $p + q \leq 1$, then

$$\frac{1}{1 + pa_1 + qa_1^2} + \frac{1}{1 + pa_2 + qa_2^2} + \cdots + \frac{1}{1 + pa_n + qa_n^2} \leq \frac{n}{1 + p + q}.$$

*(Vasile C., 2007)*

**Solution.** Using the substitution

$$a_i = e^{x_i}, \quad i = 1, 2, \ldots, n,$$

we can write the inequality as

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s),$$

where

$$x_1 \geq \cdots \geq x_{n-1} \geq 0 \geq x_n, \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$
\[ f(u) = \frac{-1}{1 + pe^u + qe^{2u}}, \quad u \in \mathbb{R}. \]

For \( u \leq 0 \), we have
\[
f''(u) = e^u\left[ -4q^2 e^{3u} - 3pq e^{2u} + (4q - p^2)e^u + p \right]
\[
\geq e^{2u}\left[ -4q^2 - 3pq + (4q - p^2) + p \right]
\]
\[
= e^{2u}\left[ (p + 4q)(1 - p - q) + 2pq \right]
\]
\[
\geq 0,
\]
therefore \( f(u) \) is convex for \( u \leq s \). According to the LHCF-OV Theorem (case \( m = n-1 \)), it suffices to show that \( f(x) + f(y) \geq 2f(0) \) for all real \( x, y \) so that \( x + y = 0 \); that is, to show that
\[
\frac{1}{1 + pa + qa^2} + \frac{1}{1 + pb + qb^2} \leq \frac{2}{1 + p + q}
\]
for all \( a, b > 0 \) so that \( ab = 1 \). Write this inequality as
\[
(a - 1)^2[q(1 - p - q)a^2 + (p + 2q - p^2 - pq - 2q^2)a + q(1 - p - q)] \geq 0,
\]
which is true because
\[
p + 2q - p^2 - pq - 2q^2 \geq (p + 2q)(p + q) - p^2 - pq - 2q^2 = 2pq \geq 0.
\]
The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \).

**P 2.34.** Let \( a_1, a_2, \ldots, a_n \) be positive real numbers so that
\[
a_1 \geq \cdots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1a_2\cdots a_n = 1.
\]
If \( m > 1 \) and \( k \geq \frac{1}{2^{1/m} - 1} \), then
\[
\frac{1}{(a_1 + k)^m} + \frac{1}{(a_2 + k)^m} + \cdots + \frac{1}{(a_n + k)^m} \leq \frac{n}{(1 + k)^m}.
\]

*(Vasile C., 2007)*

**Solution.** By Bernoulli’s inequality, we have
\[
2^{1/m} < 1 + \frac{1}{m},
\]
hence
\[ k \geq \frac{1}{2^{1/m} - 1} > m > 1. \]

Using the substitution
\[ a_i = e^{x_i}, \quad i = 1, 2, \ldots, n, \]
we can write the inequality as
\[ f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s), \]
where
\[ x_1 \geq \cdots \geq x_{n-1} \geq 0 \geq x_n, \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0, \]
\[ f(u) = \frac{-1}{(e^u + k)^m}, \quad u \in \mathbb{R}. \]

For \( u \leq 0 \), we have
\[ f''(u) = \frac{me^u(k-me^u)}{(e^u + k)^{m+2}} \geq 0, \]
hence \( f \) is convex \( u \leq s \). By the LHCF-OV Theorem (case \( m = n-1 \)), it suffices to show that \( f(x) + f(y) \geq 2f(0) \) for all real \( x, y \) so that \( x + y = 0 \); that is, to show that
\[ \frac{1}{(a+k)^m} + \frac{1}{(b+k)^m} \leq \frac{2}{(1+k)^m} \]
for all \( a, b > 0 \) so that \( ab = 1 \). Write this inequality as \( g(a) \leq 0 \) for \( a \geq 1 \), where
\[ g(a) = \frac{1}{(a+k)^m} + \frac{a^m}{(ka+1)^m} - \frac{2}{(1+k)^m}. \]

The derivative
\[ \frac{g'(a)}{m} = \frac{a^{m-1}(a+k)^{m+1}-(ka+1)^{m+1}}{(a+k)^{m+1}(ka+1)^{m+1}} \]
has the same sign as the function
\[ h(a) = (m-1)\ln a + (m+1)\ln(a+k) - (m+1)\ln(ka+1). \]

We have
\[ h'(a) = \frac{m-1}{a} + (m+1)\left(\frac{1}{a+k} - \frac{k}{ka+1}\right) = \frac{kh_1(a)}{a(a+k)(ka+1)}, \]
where
\[ h_1(a) = (m-1)(a^2 + 1) - 2\left(k - \frac{m}{k}\right)a. \]

The discriminant \( D \) of the quadratic function \( h_1(a) \) is
\[ D = \left(k - \frac{m}{k}\right)^2 - (m-1)^2 = (k^2 - 1)\left(1 - \frac{m^2}{k^2}\right). \]
Since $D > 0$, the roots $a_1$ and $a_2$ of $h_1(a)$ are real and unequal. If $a_1 < a_2$, then $h_1(a) \leq 0$ for $a \in [a_1, a_2]$ and $h_1(a) \geq 0$ for $a \in (-\infty, a_1] \cup [a_2, \infty)$. Since
\[ h_1(1) = \frac{2(k + 1)(m - k)}{k} < 0, \]
it follows that $a_1 < 1 < a_2$, therefore $h_1(a)$ and $h'(a)$ are negative for $a \in [1, a_2)$ and positive for $a \in (a_2, \infty)$, $h(a)$ is decreasing for $a \in [1, a_2)$ and increasing for $a \in [a_2, \infty)$. From $h(1) = 0$ and
\[ \lim_{a \to \infty} h(a) = \infty, \]
it follows that there is $a_3 > a_2$ so that $h(a)$ and $g'(a)$ are negative for $a \in (1, a_3)$ and positive for $a \in (a_3, \infty)$. As a result, $g$ is decreasing on $[1, a_3]$ and increasing on $[a_3, \infty)$. Since $g(1) = 0$ and
\[ \lim_{a \to \infty} g(a) = \frac{1}{k^m} - \frac{2}{(1 + k)^m} \leq 0, \]
it follows that $g(a) \leq 0$ for $a \geq 1$. This completes the proof. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

\[ \Box \]

**P 2.35.** If $a_1, a_2, \ldots, a_n$ are positive real numbers so that
\[ a_1 \geq \cdots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1 a_2 \cdots a_n = 1, \]
then
\[ \frac{1}{\sqrt{1 + 2a_1}} + \frac{1}{\sqrt{1 + 2a_2}} + \cdots + \frac{1}{\sqrt{1 + 2a_n}} \leq \frac{n}{\sqrt{3}}, \]

(Vasile C., 2007)

**Solution.** Using the substitution
\[ a_i = e^{x_i}, \quad i = 1, 2, \ldots, n, \]
we can write the inequality as
\[ f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s), \]
where
\[ x_1 \geq \cdots \geq x_{n-1} \geq 0 \geq x_n, \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0, \]
\[ f(u) = \frac{-1}{\sqrt{1 + 2e^u}}, \quad u \in \mathbb{R}. \]
For \( u \leq 0 \), we have
\[
 f''(u) = \frac{e^u(1-e^u)}{(1+2e^u)^{5/2}} > 0,
\]
hence \( f \) is convex on \((-\infty, s]\). According to the LHCF-OV Theorem (case \( m = n-1 \)), it suffices to show that \( f(x)+f(y) \geq 2f(0) \) for all \( x, y \) so that \( x+y=0 \); that is, to show that
\[
\sqrt{\frac{3}{1+2a}} + \sqrt{\frac{3}{1+2b}} \leq 2
\]
for all \( a, b > 0 \) so that \( ab = 1 \). By the Cauchy-Schwarz inequality, we get
\[
\sqrt{\frac{3}{1+2a}} + \sqrt{\frac{3}{1+2b}} \leq \sqrt{\left(\frac{3}{1+2a} + 1\right)\left(\frac{3}{1+2b} + 1\right)} = 2.
\]
The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \). \( \square \)

**P 2.36.** Let \( a_1, a_2, \ldots, a_n \) be positive real numbers so that
\[
a_1 \geq \cdots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1a_2\cdots a_n = 1.
\]
If \( 0 < m < 1 \) and \( k \geq m \), then
\[
\frac{1}{(a_1+k)^m} + \frac{1}{(a_2+k)^m} + \cdots + \frac{1}{(a_n+k)^m} \leq \frac{n}{(1+k)^m}.
\]
(Vasile C., 2007)

**Solution.** Using the substitution
\[
a_i = e^{x_i}, \quad i = 1, 2, \ldots, n,
\]
we can write the inequality as
\[
f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s),
\]
where
\[
x_1 \geq \cdots \geq x_{n-1} \geq 0 \geq x_n, \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,
\]
\[
f(u) = \frac{-1}{(e^u+k)^m}, \quad u \in \mathbb{R}.
\]
For \( u \leq 0 \), we have
\[
f''(u) = \frac{me^u(k-me^u)}{(e^u+k)^{m+2}} \geq 0,
\]
hence $f$ is convex on $(-\infty,s]$. According to the LHCF-OV Theorem (case $m = n - 1$), it suffices to show that $f(x) + f(y) \geq 2f(0)$ for all real $x, y$ so that $x + y = 0$; that is, to show that

$$\frac{1}{(a+k)^m} + \frac{1}{(b+k)^m} \leq \frac{2}{(1+k)^m}$$

for all $a, b > 0$ so that $ab = 1$. Write this inequality as $g(a) \leq 0$ for $a \geq 1$, where

$$g(a) = \frac{1}{(a+k)^m} + \frac{a^m}{(ka+1)^m} - \frac{2}{(1+k)^m},$$

with

$$g'(a) = \frac{a^{m-1}(a+k)^{m+1} - (ka+1)^{m+1}}{(a+k)^{m+1}(ka+1)^{m+1}}.$$

If $g'(a) \leq 0$ for $a \geq 1$, then $g$ is decreasing, hence $g(a) \leq g(1) = 0$. Thus, it suffices to show that

$$a^{m-1} \leq \left(\frac{ka+1}{a+k}\right)^{m+1}.$$

Since

$$\frac{ka+1}{a+k} - \frac{ma+1}{a+m} = \frac{(k-m)(a^2-1)}{(a+k)(a+m)} \geq 0,$$

we only need to show that

$$a^{m-1} \leq \left(\frac{ma+1}{a+m}\right)^{m+1},$$

which is equivalent to $h(a) \leq 0$ for $a \geq 1$, where

$$h(a) = (m-1)\ln a + (m+1)\ln(a+m) - (m+1)\ln(ma+1),$$

$$h'(a) = \frac{m-1}{a} + \frac{m+1}{a+m} - \frac{m(m+1)}{ma+1} = \frac{m(m-1)(a-1)^2}{a(a+m)(ma+1)}.$$

Since $h'(a) \leq 0$, $h(a)$ is decreasing for $a \geq 1$, hence

$$h(a) \leq h(1) = 0.$$

This completes the proof. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

**Remark.** For $k = \frac{1}{2}$ and $m = \frac{1}{2}$, we get the preceding P 2.35.

**P 2.37.** If $a_1, a_2, \ldots, a_n \ (n \geq 3)$are positive real numbers so that

$$a_1 \geq \cdots \geq a_{n-2} \geq 1 \geq a_{n-1} \geq a_n, \quad a_1a_2\cdots a_n = 1,$$

then

$$\frac{1}{(a_1+5)^2} + \frac{1}{(a_2+5)^2} + \cdots + \frac{1}{(a_n+5)^2} \leq \frac{n}{36}.$$

(Vasile C., 2007)
Solution. Using the substitution
\[ a_i = e^{x_i}, \quad i = 1, 2, \ldots, n, \]
we can write the inequality as
\[ f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s), \]
where
\[ x_1 \geq \cdots \geq x_{n-2} \geq 0 \geq x_{n-1} \geq x_n, \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0, \]
\[ f(u) = \frac{-1}{(e^u + 5)^2}, \quad u \in \mathbb{R}. \]

For \( u \in (-\infty, 0] \), we have
\[ f''(u) = \frac{2e^u(5 - 2e^u)}{(e^u + 5)^4} > 0, \]
hence \( f \) is convex on \((-\infty, s] \). According to the LHCF-OV Theorem (case \( m = n - 2 \)), it suffices to show that \( f(x) + 2f(y) \geq 3f(0) \) for all real \( x, y \) so that \( x + 2y = 0 \); that is, to show that
\[ \frac{1}{(a + 5)^2} + \frac{2}{(b + 5)^2} \leq \frac{1}{12} \]
for all \( a, b > 0 \) so that \( ab^2 = 1 \). Since
\[ \frac{1}{(a + 5)^2} = \frac{b^4}{(5b^2 + 1)^2} \leq \frac{b^4}{(4b^2 + 2b)^2} = \frac{b^2}{4(2b + 1)^2}, \]
it suffices to show that
\[ \frac{b^2}{4(2b + 1)^2} + \frac{2}{(b + 5)^2} \leq \frac{1}{12}, \]
which is equivalent to the obvious inequality
\[ (b - 1)^2(b^2 + 16b + 1) \geq 0. \]
The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \).

Remark. Similarly, we can prove the following refinement:

- Let \( a_1, a_2, \ldots, a_n \) be positive real numbers so that
\[ a_1 \geq \cdots \geq a_{n-2} \geq 1 \geq a_{n-1} \geq a_n, \quad a_1a_2 \cdots a_n = 1. \]
If \( k \geq 2 + \sqrt{6} \), then
\[ \frac{1}{(a_1 + k)^2} + \frac{1}{(a_2 + k)^2} + \cdots + \frac{1}{(a_n + k)^2} \leq \frac{n}{(1 + k)^2}, \]
with equality for \( a_1 = a_2 = \cdots = a_n = 1 \).\[\square\]
P 2.38. If \(a_1, a_2, \ldots, a_n\) are nonnegative real numbers so that
\[
a_1 \geq \cdots \geq a_{n-1} \geq 1 \geq a_n, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n,
\]
then
\[
\frac{1}{3-a_1} + \frac{1}{3-a_2} + \cdots + \frac{1}{3-a_n} \leq \frac{n}{2}.
\]
(Vasile C., 2007)

Solution. From
\[
n = a_1^2 + (a_2^2 + \cdots + a_{n-1}^2) + a_n^2 \geq a_1^2 + (n-2) + 0,
\]
we get
\[
a_1 \leq \sqrt{2}.
\]
Replacing \(a_1, a_2, \ldots, a_n\) by \(\sqrt{a_1}, \sqrt{a_2}, \ldots, \sqrt{a_n}\), we have to prove that
\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s),
\]
where
\[
2 \geq a_1 \geq \cdots \geq a_{n-1} \geq 1 \geq a_n, \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,
\]
\[
f(u) = \frac{1}{\sqrt{u} - 3}, \quad u \in [0, 2].
\]
For \(u \in [0, 1]\), we have
\[
f''(u) = \frac{3(1-\sqrt{u})}{4u\sqrt{u}(3-\sqrt{u})^2} \geq 0.
\]
Therefore, \(f\) is convex on \([0, s]\). According to the LHCF-OV Theorem and Note 1 (case \(m = n-1\)), it suffices to show that \(h(x, y) \geq 0\) for \(x, y \geq 0\) so that \(x + y = 2\). Since
\[
g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-1}{2(3-\sqrt{u})(1+\sqrt{u})}
\]
and
\[
h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{2 - \sqrt{x} - \sqrt{y}}{2(\sqrt{x} + \sqrt{y})(1 + \sqrt{x})(1 + \sqrt{y})(3 - \sqrt{x})(3 - \sqrt{y})},
\]
we need to show that
\[
\sqrt{x} + \sqrt{y} \leq 2.
\]
Indeed, we have
\[
\sqrt{x} + \sqrt{y} \leq \sqrt{2(x + y)} = 2.
\]
This completes the proof. The equality holds for \(a_1 = a_2 = \cdots = a_n = 1\). \(\square\)
**P 2.39.** Let \( a_1, a_2, \ldots, a_n \) be nonnegative real numbers so that
\[
 a_1 \leq \cdots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1 + a_2 + \cdots + a_n = n.
\]
Prove that
\[
a_1^3 + a_2^3 + \cdots + a_n^3 - n \geq (n-1)^2 \left[ \left( \frac{n-a_1}{n-1} \right)^3 + \left( \frac{n-a_2}{n-1} \right)^3 + \cdots + \left( \frac{n-a_n}{n-1} \right)^3 - 1 \right].
\]

*(Vasile C., 2010)*

**Solution.** Write the inequality as
\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,
\]
where
\[
f(u) = u^3 - (n-1)^2 \left( \frac{n-u}{n-1} \right)^3, \quad u \geq 0.
\]
For \( u \geq 1 \), we have
\[
f''(u) = \frac{6n(u-1)}{n-1} \geq 0.
\]
Therefore, \( f(u) \) is convex for \( u \geq s \). Thus, by the RHCF-OV Theorem (case \( m = n-1 \)), it suffices to show that \( f(x) + f(y) \geq 2f(1) \) for \( x, y \geq 0 \) so that \( x + y = 2 \). We have
\[
f(x) + f(y) - 2f(1) = x^3 + y^3 - 2 - (n-1)^2 \left[ \left( \frac{n-x}{n-1} \right)^3 + \left( \frac{n-y}{n-1} \right)^3 - 2 \right]
\[
= 6(1-xy) - 6(n-1)^2 \left[ 1 - \frac{(n-x)(n-y)}{(n-1)^2} \right] = 0.
\]
This completes the proof. The equality holds for
\[
a_1 \leq 1, \quad a_2 = \cdots = a_{n-1} = 1, \quad a_n = 2 - a_1.
\]
Chapter 3

Partially Convex Function Method

3.1 Theoretical Basis

The following statement is known as the Right Partially Convex Function Theorem (RPCF-Theorem).

**Right Partially Convex Function Theorem** (Vasile Cîrtoaje, 2012). Let $f$ be a real function defined on an interval $\mathbb{I}$ and convex on $[s, s_0]$, where $s, s_0 \in \mathbb{I}$, $s < s_0$. In addition, $f$ is decreasing on $\mathbb{I}_{\leq s_0}$ and $f(u) \geq f(s_0)$ for $u \in \mathbb{I}$. The inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying

$$a_1 + a_2 + \cdots + a_n = ns$$

if and only if

$$f(x) + (n - 1)f(y) \geq nf(s)$$

for all $x, y \in \mathbb{I}$ so that $x \leq s \leq y$ and $x + (n - 1)y = ns$.

**Proof.** For

$$a_1 = x, \quad a_2 = a_3 = \cdots = a_n = y,$$

the inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq f(s)$$

becomes

$$f(x) + (n - 1)f(y) \geq nf(s);$$

therefore, the necessity is obvious.

The proof of sufficiency is based on Lemma below. According to this lemma, it suffices to consider that $a_1, a_2, \ldots, a_n \in \mathbb{J}$, where

$$\mathbb{J} = \mathbb{I}_{\leq s_0}.$$
Because \(f(u)\) is convex on \(J_{s_0}\), the desired inequality follows from the RHCF Theorem (see Chapter 1) applied to the interval \(J\).

**Lemma.** Let \(f\) be a real function defined on an interval \(I\). In addition, \(f\) is decreasing on \(I_{s_0}\) and \(f(u) \geq f(s_0)\) for \(u \in I\), where \(s, s_0 \in I\), \(s < s_0\). If the inequality

\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s)
\]

holds for all \(a_1, a_2, \ldots, a_n \in I_{s_0}\) so that \(a_1 + a_2 + \cdots + a_n = ns\), then it holds for all \(a_1, a_2, \ldots, a_n \in I\) so that \(a_1 + a_2 + \cdots + a_n = ns\).

**Proof.** For \(i = 1, 2, \ldots, n\), define the numbers

\[
b_i = \begin{cases} 
a_i, & a_i \leq s_0 \\
s_0, & a_i > s_0.
\end{cases}
\]

Clearly, \(b_i \in I_{s_0}\) and \(b_i \leq a_i\). Since \(f(u) \geq f(s_0)\) for \(u \in I_{s_0}\), it follows that \(f(b_i) \leq f(a_i)\) for \(i = 1, 2, \ldots, n\). Therefore,

\[
b_1 + b_2 + \cdots + b_n \leq a_1 + a_2 + \cdots + a_n = ns
\]

and

\[
f(b_1) + f(b_2) + \cdots + f(b_n) \leq f(a_1) + f(a_2) + \cdots + f(a_n).
\]

Thus, it suffices to show that

\[
f(b_1) + f(b_2) + \cdots + f(b_n) \geq nf(s)
\]

for all \(b_1, b_2, \ldots, b_n \in I_{s_0}\) so that \(b_1 + b_2 + \cdots + b_n \leq ns\). By hypothesis, this inequality is true for \(b_1, b_2, \ldots, b_n \in I_{s_0}\) and \(b_1 + b_2 + \cdots + b_n = ns\). Since \(f(u)\) is decreasing on \(I_{s_0}\), the more we have \(f(b_1) + f(b_2) + \cdots + f(b_n) \geq nf(s)\) for \(b_1, b_2, \ldots, b_n \in I_{s_0}\) and \(b_1 + b_2 + \cdots + b_n \leq ns\).

Similarly, we can prove the Left Partially Convex Function Theorem (LPCF-Theorem).

**Left Partially Convex Function Theorem** (Vasile Cîrtoaje, 2012). Let \(f\) be a real function defined on an interval \(I\) and convex on \([s_0, s]\), where \(s_0, s \in I\), \(s_0 < s\). In addition, \(f\) is increasing on \(I_{s_0}\) and \(f(u) \geq f(s_0)\) for \(u \in I\). The inequality

\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)
\]

holds for all \(a_1, a_2, \ldots, a_n \in I\) satisfying

\[
a_1 + a_2 + \cdots + a_n = ns
\]

if and only if

\[
f(x) + (n-1)f(y) \geq nf(s)
\]

for all \(x, y \in I\) so that \(x \geq s \geq y\) and \(x + (n-1)y = ns\).
From the RPCF-Theorem and the LPCF-Theorem, we find the PCF-Theorem (Partially Convex Function Theorem).

**Partially Convex Function Theorem** (Vasile Cîrtoaje, 2012). Let $f$ be a real function defined on an interval $\mathbb{I}$ and convex on $[s_0, s]$ or $[s, s_0]$, where $s_0, s \in \mathbb{I}$. In addition, $f$ is decreasing on $\mathbb{I}_{\leq s_0}$ and increasing on $\mathbb{I}_{\geq s_0}$. The inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying

$$a_1 + a_2 + \cdots + a_n = ns$$

if and only if

$$f(x) + (n-1)f(y) \geq nf(s)$$

for all $x, y \in \mathbb{I}$ so that $x + (n-1)y = ns$.

**Note 1.** Let us denote

$$g(u) = \frac{f(u) - f(s)}{u - s}, \quad h(x, y) = \frac{g(x) - g(y)}{x - y}.$$  

As shown in Note 1 from Chapter 1, we may replace the hypothesis condition in the RPCF-Theorem and the LPCF-Theorem, namely

$$f(x) + (n-1)f(y) \geq nf(s),$$

by the condition

$$h(x, y) \geq 0 \text{ for all } x, y \in \mathbb{I} \text{ so that } x + (n-1)y = ns.$$

**Note 2.** Assume that $f$ is differentiable on $\mathbb{I}$, and let

$$H(x, y) = \frac{f'(x) - f'(y)}{x - y}.$$  

As shown in Note 2 from Chapter 1, the inequalities in the RPCF-Theorem and the LPCF-Theorem hold true by replacing the hypothesis

$$f(x) + (n-1)f(y) \geq nf(s)$$

with the more restrictive condition

$$H(x, y) \geq 0 \text{ for all } x, y \in \mathbb{I} \text{ so that } x + (n-1)y = ns.$$

**Note 3.** The desired inequalities in the RPCF-Theorem and the LPCF-Theorem become equalities for

$$a_1 = a_2 = \cdots = a_n = s.$$
In addition, if there exist \( x, y \in \mathbb{I} \) so that
\[
x + (n-1)y = ns, \quad f(x) + (n-1)f(y) = nf(s), \quad x \neq y,
\]
then the equality holds also for
\[
a_1 = x, \quad a_2 = \cdots = a_n = y
\]
(or any cyclic permutation). Notice that these equality conditions are equivalent to
\[
x + (n-1)y = ns, \quad h(x, y) = 0
\]
\((x < y \text{ for the RPCF-Theorem, and } x > y \text{ for the LPCF-Theorem}).\)

**Note 4.** From the proof of the RPCF-Theorem, it follows that this theorem is also valid in the case when \( f \) is defined on \( \mathbb{I} \setminus \{u_0\} \), where \( u_0 \in \mathbb{I}_{>s_0} \). Similarly, the LPCF-Theorem is also valid in the case when \( f \) is defined on \( \mathbb{I} \setminus \{u_0\} \), where \( u_0 \in \mathbb{I}_{<s_0} \).

**Note 5.** The RPCF-Theorem holds true by replacing the condition
\[
\text{f is decreasing on } \mathbb{I}_{\leq s_0}
\]
with
\[
ns - (n-1)s_0 \leq \inf \mathbb{I}.
\]
More precisely, the following theorem holds:

**Theorem 1.** Let \( f \) be a function defined on a real interval \( \mathbb{I} \), convex on \([s, s_0]\) and satisfying
\[
\min_{u \in \mathbb{I}_s} f(u) = f(s_0),
\]
where
\[
s, s_0 \in \mathbb{I}, \quad s < s_0, \quad ns - (n-1)s_0 \leq \inf \mathbb{I}.
\]
If
\[
f(x) + (n-1)f(y) \geq nf(s)
\]
for all \( x, y \in \mathbb{I} \) so that \( x \leq s \leq y \) and \( x + (n-1)y = ns \), then
\[
f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf \left( \frac{x_1 + x_2 + \cdots + x_n}{n} \right)
\]
for all \( x_1, x_2, \ldots, x_n \in \mathbb{I} \) satisfying \( x_1 + x_2 + \cdots + x_n = ns \).

In order to prove Theorem 1, we define the function
\[
f_0(u) = \begin{cases} 
  f(u), & u \leq s_0, \ u \in \mathbb{I} \\
  f(s_0), & u \geq s_0, \ u \in \mathbb{I},
\end{cases}
\]
which is convex on \( \mathbb{I}_{\geq s} \). Taking into account that \( f_0(s) = f(s) \) and \( f_0(u) \leq f(u) \) for all \( u \in \mathbb{I} \), it suffices to prove that
\[
f_0(x_1) + f_0(x_2) + \cdots + f_0(x_n) \geq nf_0(s)
\]
for all \(x_1, x_2, \ldots, x_n \in \mathbb{I}\) satisfying \(x_1 + x_2 + \cdots + x_n = ns\). According to the HCF-Theorem and Note 5 from Chapter 1, we only need to show that

\[ f_0(x) + (n-1)f_0(y) \geq nf_0(s) \]

for all \(x, y \in \mathbb{I}\) so that \(x \leq s \leq y\) and \(x + (n-1)y = ns\). Since

\[ y - s_0 = \frac{ns - x}{n - 1} - s_0 = \frac{ns - (n-1)s_0 - x}{n - 1} \leq \frac{ns - (n-1)s_0 - \inf \mathbb{I}}{n - 1} \leq 0, \]

the inequality \(f_0(x) + (n-1)f_0(y) \geq nf_0(s)\) turns into \(f(x) + (n-1)f(y) \geq nf(s)\), which holds (by hypothesis) for all \(x, y \in \mathbb{I}\) so that \(x \leq s \leq y\) and \(x + (n-1)y = ns\).

Similarly, the LPCF-Theorem holds true by replacing the condition

\[ f \text{ is increasing on } \mathbb{I}_{\geq s_0} \]

with

\[ ns - (n-1)s_0 \geq \sup \mathbb{I}. \]

More precisely, the following theorem holds:

**Theorem 2.** Let \(f\) be a function defined on a real interval \(\mathbb{I}\), convex on \([s_0, s]\) and satisfying

\[ \min_{u \in \mathbb{I}_{\leq s_0}} f(u) = f(s_0), \]

where

\[ s, s_0 \in \mathbb{I}, \ s > s_0, \ ns - (n-1)s_0 \geq \sup \mathbb{I}. \]

If

\[ f(x) + (n-1)f(y) \geq nf(s) \]

for all \(x, y \in \mathbb{I}\) so that \(x \geq s \geq y\) and \(x + (n-1)y = ns\), then

\[ f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf \left( \frac{x_1 + x_2 + \cdots + x_n}{n} \right) \]

for all \(x_1, x_2, \ldots, x_n \in \mathbb{I}\) satisfying \(x_1 + x_2 + \cdots + x_n = ns\).

The proof of Theorem 2 is similar to the proof of Theorem 1.

**Note 6.** From the proof of Theorem 1, it follows that Theorem 1 is also valid in the case in which \(f\) is defined on \(\mathbb{I} \setminus \{u_0\}\), where \(u_0\) is an interior point of \(\mathbb{I}\) so that \(u_0 \notin [s, s_0]\). Similarly, Theorem 2 is also valid in the case in which \(f\) is defined on \(\mathbb{I} \setminus \{u_0\}\), where \(u_0\) is an interior point of \(\mathbb{I}\) so that \(u_0 \notin [s_0, s]\).

**Note 7.** In the same manner, we can extend weighted Jensen’s inequality to right and left partially convex functions establishing the WRPCF-Theorem, the WLPFC-Theorem and the WPCF-Theorem (Vasile Cîrtoaje, 2014).
**WRPCF-Theorem.** Let \( p_1, p_2, \ldots, p_n \) be positive real numbers so that
\[
p_1 + p_2 + \cdots + p_n = 1, \quad p = \min\{p_1, p_2, \ldots, p_n\},
\]
and let \( f \) be a real function defined on an interval \( \mathbb{I} \) and convex on \([s, s_0]\), where \( s, s_0 \in \mathbb{I}, s < s_0 \). In addition, \( f \) is decreasing on \( \mathbb{I}_{\leq s_0} \) and \( f(u) \geq f(s_0) \) for \( u \in \mathbb{I} \). The inequality
\[
p_1 f(a_1) + p_2 f(a_2) + \cdots + p_n f(a_n) \geq f(p_1 a_1 + p_2 a_2 + \cdots + p_n a_n)
\]
holds for all \( a_1, a_2, \ldots, a_n \in \mathbb{I} \) satisfying
\[
p_1 a_1 + p_2 a_2 + \cdots + p_n a_n = s,
\]
if and only if
\[
p f(x) + (1 - p) f(y) \geq f(s)
\]
for all \( x, y \in \mathbb{I} \) so that \( x \leq s \leq y \) and \( px + (1 - p)y = s \).

**WLPCF-Theorem.** Let \( p_1, p_2, \ldots, p_n \) be positive real numbers so that
\[
p_1 + p_2 + \cdots + p_n = 1, \quad p = \min\{p_1, p_2, \ldots, p_n\},
\]
and let \( f \) be a real function defined on an interval \( \mathbb{I} \) and convex on \([s_0, s]\), where \( s_0, s \in \mathbb{I}, s_0 < s \). In addition, \( f \) is increasing on \( \mathbb{I}_{\geq s_0} \) and \( f(u) \geq f(s_0) \) for \( u \in \mathbb{I} \). The inequality
\[
p_1 f(a_1) + p_2 f(a_2) + \cdots + p_n f(a_n) \geq f(p_1 a_1 + p_2 a_2 + \cdots + p_n a_n)
\]
holds for all \( a_1, a_2, \ldots, a_n \in \mathbb{I} \) satisfying
\[
p_1 a_1 + p_2 a_2 + \cdots + p_n a_n = s,
\]
if and only if
\[
p f(x) + (1 - p) f(y) \geq f(s)
\]
for all \( x, y \in \mathbb{I} \) so that \( x \geq s \geq y \) and \( px + (1 - p)y = s \).
3.2 Applications

3.1. If \(a, b, c\) are real numbers so that \(a + b + c = 3\), then
\[
\frac{16a - 5}{32a^2 + 1} + \frac{16b - 5}{32b^2 + 1} + \frac{16c - 5}{32c^2 + 1} \leq 1.
\]

3.2. If \(a, b, c, d\) are real numbers so that \(a + b + c + d = 4\), then
\[
\frac{18a - 5}{12a^2 + 1} + \frac{18b - 5}{12b^2 + 1} + \frac{18c - 5}{12c^2 + 1} + \frac{18d - 5}{12d^2 + 1} \leq 4.
\]

3.3. If \(a, b, c, d, e, f\) are real numbers so that \(a + b + c + d + e + f = 6\), then
\[
\frac{5a - 1}{5a^2 + 1} + \frac{5b - 1}{5b^2 + 1} + \frac{5c - 1}{5c^2 + 1} + \frac{5d - 1}{5d^2 + 1} + \frac{5e - 1}{5e^2 + 1} + \frac{5f - 1}{5f^2 + 1} \leq 4.
\]

3.4. If \(a_1, a_2, \ldots, a_n\) \((n \geq 3)\) are real numbers so that \(a_1 + a_2 + \cdots + a_n = n\), then
\[
\frac{n(n+1)-2a_1}{n^2+(n-2)a_1^2} + \frac{n(n+1)-2a_2}{n^2+(n-2)a_2^2} + \cdots + \frac{n(n+1)-2a_n}{n^2+(n-2)a_n^2} \leq n.
\]

3.5. If \(a, b, c, d\) are real numbers so that \(a + b + c + d = 4\), then
\[
\frac{a(a - 1)}{3a^2 + 4} + \frac{b(b - 1)}{3b^2 + 4} + \frac{c(c - 1)}{3c^2 + 4} + \frac{d(d - 1)}{3d^2 + 4} \geq 0.
\]

3.6. If \(a, b, c\) are real numbers so that \(a + b + c = 3\), then
\[
\frac{1}{9a^2 - 10a + 9} + \frac{1}{9b^2 - 10b + 9} + \frac{1}{9c^2 - 10c + 9} \leq \frac{3}{8}.
\]

3.7. If \(a, b, c, d\) are real numbers so that \(a + b + c + d = 4\), then
\[
\frac{1}{4a^2 - 5a + 4} + \frac{1}{4b^2 - 5b + 4} + \frac{1}{4c^2 - 5c + 4} + \frac{1}{4d^2 - 5d + 4} \leq \frac{4}{3}.
\]
3.8. Let \( a_1, a_2, \ldots, a_n \neq -k \) be real numbers so that \( a_1 + a_2 + \cdots + a_n = n \), where
\[
k \geq \frac{n}{2\sqrt{n-1}}.
\]
Then,
\[
\frac{a_1(a_1 - 1)}{(a_1 + k)^2} + \frac{a_2(a_2 - 1)}{(a_2 + k)^2} + \cdots + \frac{a_n(a_n - 1)}{(a_n + k)^2} \geq 0.
\]

3.9. Let \( a_1, a_2, \ldots, a_n \neq -k \) be real numbers so that \( a_1 + a_2 + \cdots + a_n = n \). If
\[
k \geq 1 + \frac{n}{\sqrt{n-1}},
\]
then
\[
\frac{a_1^2 - 1}{(a_1 + k)^2} + \frac{a_2^2 - 1}{(a_2 + k)^2} + \cdots + \frac{a_n^2 - 1}{(a_n + k)^2} \geq 0.
\]

3.10. Let \( a_1, a_2, a_3, a_4, a_5 \) be real numbers so that \( a_1 + a_2 + a_3 + a_4 + a_5 \geq 5 \). If
\[
k \in \left[ \frac{1}{6}, \frac{25}{14} \right],
\]
then
\[
\sum \frac{1}{ka_1^2 + a_2 + a_3 + a_4 + a_5} \leq \frac{5}{k + 4}.
\]

3.11. Let \( a_1, a_2, \ldots, a_5 \) be nonnegative numbers so that \( a_1 + a_2 + a_3 + a_4 + a_5 \geq 5 \). If \( k \in [k_1, k_2] \), where
\[
k_1 = \frac{29 - \sqrt{761}}{10} \approx 0.1414, \quad k_2 = \frac{25}{14} \approx 1.7857,
\]
then
\[
\sum \frac{1}{ka_1^2 + a_2 + a_3 + a_4 + a_5} \leq \frac{5}{k + 4}.
\]

3.12. Let \( a_1, a_2, \ldots, a_n \) be nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n \geq n \). If \( k > 1 \), then
\[
\frac{1}{a_1^k + a_2 + \cdots + a_n} + \frac{1}{a_1 + a_2^k + \cdots + a_n} + \cdots + \frac{1}{a_1 + a_2 + \cdots + a_n^k} \leq 1.
\]
3.13. Let $a_1, a_2, \ldots, a_5$ be nonnegative numbers so that $a_1 + a_2 + a_3 + a_4 + a_5 \geq 5$. If 
\[ k \in \left[ \frac{4}{9}, \frac{61}{5} \right], \]
then 
\[ \sum \frac{a_1}{ka_1^2 + a_2 + a_3 + a_4 + a_5} \leq \frac{5}{k + 4}. \]

3.14. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n \geq n$. If $k > 1$, then 
\[ \frac{a_1}{a_1^k + a_2 + \cdots + a_n} + \frac{a_2}{a_1 + a_2^k + \cdots + a_n} + \cdots + \frac{a_n}{a_1 + a_2 + \cdots + a_n^k} \leq 1. \]

3.15. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n \leq n$. If $k \geq 1 - \frac{1}{n}$, then 
\[ \frac{1 - a_1}{ka_1^2 + a_2 + \cdots + a_n} + \frac{1 - a_2}{a_1 + ka_2^2 + \cdots + a_n} + \cdots + \frac{1 - a_n}{a_1 + a_2 + \cdots + ka_n^2} \geq 0. \]

3.16. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n \leq n$. If $k \geq 1 - \frac{1}{n}$, then 
\[ \frac{1 - a_1}{1 - a_1 + ka_1^2} + \frac{1 - a_2}{1 - a_2 + ka_2^2} + \cdots + \frac{1 - a_n}{1 - a_n + ka_n^2} \geq 0. \]

3.17. Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If $0 < k \leq \frac{n}{n - 1}$, then 
\[ a_1^{k/a_1} + a_2^{k/a_2} + \cdots + a_n^{k/a_n} \leq n. \]

3.18. If $a, b, c, d, e$ are nonzero real numbers so that $a + b + c + d + e = 5$, then 
\[ \left( 7 - \frac{5}{a} \right)^2 + \left( 7 - \frac{5}{b} \right)^2 + \left( 7 - \frac{5}{c} \right)^2 + \left( 7 - \frac{5}{d} \right)^2 + \left( 7 - \frac{5}{e} \right)^2 \geq 20. \]
3.19. If \( a_1, a_2, \ldots, a_7 \) are real numbers so that \( a_1 + a_2 + \cdots + a_7 = 7 \), then
\[
(a_1^2 + 2)(a_2^2 + 2) \cdots (a_7^2 + 2) \geq 3^7.
\]

3.20. Let \( a_1, a_2, \ldots, a_n \) be real numbers so that \( a_1 + a_2 + \cdots + a_n = n \). If \( k \geq \frac{n^2}{4(n-1)} \), then
\[
(a_1^2 + k)(a_2^2 + k) \cdots (a_n^2 + k) \geq (1 + k)^n.
\]

3.21. If \( a_1, a_2, \ldots, a_{10} \) are real numbers so that \( a_1 + a_2 + \cdots + a_{10} = 10 \), then
\[
(1 - a_1 + a_1^2)(1 - a_2 + a_2^2) \cdots (1 - a_{10} + a_{10}^2) \geq 1.
\]

3.22. If \( a, b, c \) are nonnegative real numbers so that \( a + b + c = 3 \), then
\[
(1 - a + a^4)(1 - b + b^4)(1 - c + c^4) \geq 1.
\]

3.23. If \( a, b, c, d \) are nonnegative real numbers so that \( a + b + c + d = 4 \), then
\[
(1 - a + a^3)(1 - b + b^3)(1 - c + c^3)(1 - d + d^3) \geq 1.
\]

3.24. If \( a, b, c, d, e \) are nonzero real numbers so that \( a + b + c + d + e = 5 \), then
\[
5 \left( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{e^2} \right) + 45 \geq 14 \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} \right).
\]

3.25. If \( a, b, c \) are positive real numbers so that \( abc = 1 \), then
\[
\frac{7 - 6a}{2 + a^2} + \frac{7 - 6b}{2 + b^2} + \frac{7 - 6c}{2 + c^2} \geq 1.
\]

3.26. If \( a, b, c \) are positive real numbers so that \( abc = 1 \), then
\[
\frac{1}{a + 5bc} + \frac{1}{b + 5ca} + \frac{1}{c + 5ab} \leq \frac{1}{2}.
\]
3.27. If $a, b, c$ are positive real numbers so that $abc = 1$, then
\[
\frac{1}{4 - 3a + 4a^2} + \frac{1}{4 - 3b + 4b^2} + \frac{1}{4 - 3c + 4c^2} \leq \frac{3}{5}.
\]

3.28. If $a, b, c$ are positive real numbers so that $abc = 1$, then
\[
\frac{1}{(3a + 1)(3a^2 - 5a + 3)} + \frac{1}{(3b + 1)(3b^2 - 5b + 3)} + \frac{1}{(3c + 1)(3c^2 - 5c + 3)} \leq \frac{3}{4}.
\]

3.29. Let $a_1, a_2, \ldots, a_n$ ($n \geq 3$) be positive real numbers so that $a_1a_2 \cdots a_n = 1$. If $p, q \geq 0$ so that $p + 4q \geq n - 1$, then
\[
\frac{1 - a_1}{1 + pa_1 + qa_1^2} + \frac{1 - a_2}{1 + pa_2 + qa_2^2} + \cdots + \frac{1 - a_n}{1 + pa_n + qa_n^2} \geq 0.
\]

3.30. If $a, b, c$ are positive real numbers so that $abc = 1$, then
\[
\frac{1 - a}{17 + 4a + 6a^2} + \frac{1 - b}{17 + 4b + 6b^2} + \frac{1 - c}{17 + 4c + 6c^2} \geq 0.
\]

3.31. If $a_1, a_2, \ldots, a_8$ are positive real numbers so that $a_1a_2 \cdots a_8 = 1$, then
\[
\frac{1 - a_1}{(1 + a_1)^2} + \frac{1 - a_2}{(1 + a_2)^2} + \cdots + \frac{1 - a_8}{(1 + a_8)^2} \geq 0.
\]

3.32. Let $a, b, c$ be positive real numbers so that $abc = 1$. If $k \in \left[\frac{-13}{3\sqrt{3}}, \frac{13}{3\sqrt{3}}\right]$, then
\[
\frac{a + k}{a^2 + 1} + \frac{b + k}{b^2 + 1} + \frac{c + k}{c^2 + 1} \leq \frac{3(1 + k)}{2}.
\]

3.33. If $a, b, c$ are positive real numbers and $0 < k \leq 2 + 2\sqrt{2}$, then
\[
\frac{a^3}{ka^2 + bc} + \frac{b^3}{kb^2 + ca} + \frac{c^3}{kc^2 + ab} \geq \frac{a + b + c}{k + 1}.
\]
3.34. If \( a, b, c, d, e \) are positive real numbers so that \( abcde = 1 \), then 
\[
2 \left( \frac{1}{a + 1} + \frac{1}{b + 1} + \cdots + \frac{1}{e + 1} \right) \geq 3 \left( \frac{1}{a + 2} + \frac{1}{b + 2} + \cdots + \frac{1}{e + 2} \right).
\]

3.35. If \( a_1, a_2, \ldots, a_{14} \) are positive real numbers so that \( a_1a_2 \cdots a_{14} = 1 \), then 
\[
3 \left( \frac{1}{2a_1 + 1} + \frac{1}{2a_2 + 1} + \cdots + \frac{1}{2a_{14} + 1} \right) \geq 2 \left( \frac{1}{a_1 + 1} + \frac{1}{a_2 + 1} + \cdots + \frac{1}{a_{14} + 1} \right).
\]

3.36. Let \( a_1, a_2, \ldots, a_8 \) be positive real numbers so that \( a_1a_2 \cdots a_8 = 1 \). If \( k > 1 \), then 
\[
(k + 1) \left( \frac{1}{ka_1 + 1} + \frac{1}{ka_2 + 1} + \cdots + \frac{1}{ka_8 + 1} \right) \geq 2 \left( \frac{1}{a_1 + 1} + \frac{1}{a_2 + 1} + \cdots + \frac{1}{a_8 + 1} \right).
\]

3.37. If \( a_1, a_2, \ldots, a_9 \) are positive real numbers so that \( a_1a_2 \cdots a_9 = 1 \), then 
\[
\frac{1}{2a_1 + 1} + \frac{1}{2a_2 + 1} + \cdots + \frac{1}{2a_9 + 1} \geq \frac{1}{a_1 + 2} + \frac{1}{a_2 + 2} + \cdots + \frac{1}{a_9 + 2}.
\]

3.38. If \( a_1, a_2, \ldots, a_n \) are real numbers so that 
\[
a_1, a_2, \ldots, a_n \leq \pi, \quad a_1 + a_2 + \cdots + a_n = \pi,
\]
then 
\[
\cos a_1 + \cos a_2 + \cdots + \cos a_n \leq n \cos \frac{\pi}{n}.
\]

3.39. If \( a_1, a_2, \ldots, a_n \) (\( n \geq 3 \)) are real numbers so that 
\[
a_1, a_2, \ldots, a_n \geq \frac{-1}{n-2}, \quad a_1 + a_2 + \cdots + a_n = n,
\]
then 
\[
\frac{a_1^2}{a_1^2 - a_1 + 1} + \frac{a_2^2}{a_2^2 - a_2 + 1} + \cdots + \frac{a_n^2}{a_n^2 - a_n + 1} \leq n.
\]
3.40. If \( a_1, a_2, \ldots, a_n \) \((n \geq 3)\) are nonzero real numbers so that
\[
a_1, a_2, \ldots, a_n \geq \frac{-n}{n-2}, \quad a_1 + a_2 + \cdots + a_n = n,
\]
then
\[
\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2} \geq \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}.
\]

3.41. If \( a_1, a_2, \ldots, a_n \geq -1 \) so that \( a_1 + a_2 + \cdots + a_n = n \), then
\[
(n+1) \left( \frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2} \right) \geq 2n + (n-1) \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right).
\]

3.42. If \( a_1, a_2, \ldots, a_n \) \((n \geq 3)\) are real numbers so that
\[
a_1, a_2, \ldots, a_n \geq \frac{-(3n-2)}{n-2}, \quad a_1 + a_2 + \cdots + a_n = n,
\]
then
\[
\frac{1-a_1}{(1+a_1)^2} + \frac{1-a_2}{(1+a_2)^2} + \cdots + \frac{1-a_n}{(1+a_n)^2} \geq 0.
\]

3.43. Let \( a_1, a_2, \ldots, a_n \) be nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n = n \). If \( n \geq 3 \) and \( k \geq 2 - \frac{2}{n} \), then
\[
\frac{1-a_1}{(1-ka_1)^2} + \frac{1-a_2}{(1-ka_2)^2} + \cdots + \frac{1-a_n}{(1-ka_n)^2} \geq 0.
\]
3.3 Solutions

**P 3.1.** If $a, b, c$ are real numbers so that $a + b + c = 3$, then

$$\frac{16a - 5}{32a^2 + 1} + \frac{16b - 5}{32b^2 + 1} + \frac{16c - 5}{32c^2 + 1} \leq 1.$$  

(Vasile C., 2012)

**Solution.** Write the inequality as

$$f(a) + f(b) + f(c) \geq 3f(s), \quad s = \frac{a + b + c}{3} = 1,$$

where

$$f(u) = \frac{5 - 16u}{32u^2 + 1}, \quad u \in \mathbb{R}.$$  

From

$$f'(u) = \frac{16(32u^2 - 20u - 1)}{(32u^2 + 1)^2},$$

it follows that $f$ is increasing on

$$\left(-\infty, \frac{5 - \sqrt{33}}{16}\right] \cup [s_0, \infty),$$

and decreasing on

$$\left[\frac{5 - \sqrt{33}}{16}, s_0\right],$$

where

$$s_0 = \frac{5 + \sqrt{33}}{16} \approx 0.6715.$$  

Also, from

$$\lim_{u \to -\infty} f(u) = 0$$

and

$$f(s_0) < 0,$$

it follows that $f(u) \geq f(s_0)$ for $u \in \mathbb{R}$. In addition, for $u \in [s_0, 1]$, we have

$$\frac{1}{64}f''(u) = \frac{-512u^3 + 480u^2 + 48u - 5}{(32u^2 + 1)^3} = \frac{512u^2(1-u) + 32u(1-u) + (16u - 5)}{(32u^2 + 1)^3} > 0,$$
hence $f$ is convex on $[s_0, s]$. According to the LPCF-Theorem, we only need to show that $f(x) + 2f(y) \geq 3f(1)$ for all real $x, y$ so that $x + 2y = 3$. Using Note 1, it suffices to prove that $h(x, y) \geq 0$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$ 

Indeed, we have

$$g(u) = \frac{32(2u - 1)}{32u^2 + 1},$$

$$h(x, y) = \frac{64(1 + 16x + 16y - 32xy)}{3(32x^2 + 1)(32y^2 + 1)} = \frac{64(4x - 5)^2}{3(32x^2 + 1)(32y^2 + 1)} \geq 0.$$ 

Thus, the proof is completed. From $x + 2y = 3$ and $h(x, y) = 0$, we get

$$x = \frac{5}{4}, \quad y = \frac{7}{8}.$$ 

Therefore, in accordance with Note 3, the equality holds for $a = b = c = 1$, and also for

$$a = \frac{5}{4}, \quad b = c = \frac{7}{8}$$

(or any cyclic permutation). 

\[\square\]

**P 3.2.** If $a, b, c, d$ are real numbers so that $a + b + c + d = 4$, then

$$\frac{18a - 5}{12a^2 + 1} + \frac{18b - 5}{12b^2 + 1} + \frac{18c - 5}{12c^2 + 1} + \frac{18d - 5}{12d^2 + 1} \leq 4.$$ 

(Vasile C., 2012)

**Solution.** Write the inequality as

$$f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a + b + c + d}{4} = 1,$$

where

$$f(u) = \frac{5 - 18u}{12u^2 + 1}, \quad u \in \mathbb{R}.$$ 

From

$$f'(u) = \frac{6(36u^2 - 20u - 3)}{(12u^2 + 1)^2},$$

it follows that $f$ is increasing on

$$\left(-\infty, \frac{5 - \sqrt{52}}{18}\right] \cup [s_0, \infty).$$
and decreasing on 
\[ \left[ \frac{5 - \sqrt{52}}{18}, s_0 \right], \quad s_0 = \frac{5 + \sqrt{52}}{18} \approx 0.678. \]

Also, from 
\[ \lim_{u \to -\infty} f(u) = 0 \]
and 
\[ f(s_0) < 0, \]
it follows that 
\[ f(u) \geq f(s_0) \]
for 
\[ u \in \mathbb{R}. \]
In addition, for 
\[ u \in [s_0, 1], \]
we have 
\[ \frac{1}{24} f''(u) = \frac{-216u^3 + 180u^2 + 54u - 5}{(12u^2 + 1)^3} \]
\[ = \frac{216u^2(1-u) + 36u(1-u) + (18u - 5)}{(32u^2 + 1)^3} > 0, \]

hence \( f \) is convex on \([s_0, s]\). According to the LPCF-Theorem and Note 1, we only need to show that 
\[ h(x, y) \geq 0 \]
for 
\[ x, y \in \mathbb{R} \]
so that 
\[ x + 3y = 4. \]
We have 
\[ g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{6(2u - 1)}{12u^2 + 1}, \]
\[ h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{12(1 + 6x + 6y - 12xy)}{(12x^2 + 1)(12y^2 + 1)} = \frac{12(2x - 3)^2}{(12x^2 + 1)(12y^2 + 1)} \geq 0. \]

Thus, the proof is completed. From 
\[ x + 3y = 4 \]
and 
\[ y = \frac{5}{6} \]
Therefore, in accordance with Note 3, the equality holds for 
\[ a = b = c = d = 1, \]
and also for 
\[ a = \frac{3}{2}, \quad b = c = d = \frac{5}{6} \]
(or any cyclic permutation).

\[ \square \]

**P 3.3.** If \( a, b, c, d, e, f \) are real numbers so that 
\[ a + b + c + d + e + f = 6, \]
then 
\[ \frac{5a - 1}{5a^2 + 1} + \frac{5b - 1}{5b^2 + 1} + \frac{5c - 1}{5c^2 + 1} + \frac{5d - 1}{5d^2 + 1} + \frac{5e - 1}{5e^2 + 1} + \frac{5f - 1}{5f^2 + 1} \leq 4. \]

*(Vasile C., 2012)*

**Solution.** Write the inequality as 
\[ f(a) + f(b) + f(c) + f(d) + f(e) + f(f) \geq 4f(s), \quad s = \frac{a + b + c + d + e + f}{6} = 1, \]
where
\[ f(u) = \frac{1 - 5u}{5u^2 + 1}, \quad u \in \mathbb{R}. \]

From
\[ f'(u) = \frac{5(5u^2 - 2u - 1)}{(5u^2 + 1)^2}, \]

it follows that \( f \) is increasing on
\[ \left(-\infty, \frac{1 - \sqrt{5}}{5}\right) \cup \left[s_0, \infty\right) \]

and decreasing on
\[ \left[\frac{1 - \sqrt{5}}{5}, s_0\right], \quad s_0 = \frac{1 + \sqrt{5}}{5} \approx 0.69. \]

Also, from
\[ \lim_{u \to -\infty} f(u) = 0 \]

and
\[ f(s_0) < 0, \]

it follows that \( f(u) \geq f(s_0) \) for \( u \in \mathbb{R} \). In addition, for \( u \in [s_0, 1] \), we have
\[
\frac{1}{24} f''(u) = \frac{-216u^3 + 180u^2 + 54u - 5}{(12u^2 + 1)^3} = \frac{216u^2(1-u) + 36u(1-u) + (18u - 5)}{(32u^2 + 1)^3} > 0,
\]

hence \( f \) is convex on \([s_0, s]\). According to the LPCF-Theorem and Note 1, we only need to show that \( h(x, y) \geq 0 \) for \( x, y \in \mathbb{R} \) so that \( x + 5y = 6 \). We have
\[
g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{5(2u - 1)}{3(5u^2 + 1)},
\]
\[
h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{5(2 + 5x + 5y - 10xy)}{3(5x^2 + 1)(5y^2 + 1)} = \frac{10(x - 2)^2}{3(5x^2 + 1)(5y^2 + 1)} \geq 0.
\]

In accordance with Note 3, the equality holds for \( a = b = c = d = e = f = 1 \), and also for
\[ a = 2, \quad b = c = d = e = f = \frac{4}{5} \]

(or any cyclic permutation). \( \square \)
**P 3.4.** If \(a_1, a_2, \ldots, a_n\) (\(n \geq 3\)) are real numbers so that \(a_1 + a_2 + \cdots + a_n = n\), then
\[
\frac{n(n+1) - 2a_1}{n^2 + (n-2)a_1^2} + \frac{n(n+1) - 2a_2}{n^2 + (n-2)a_2^2} + \cdots + \frac{n(n+1) - 2a_n}{n^2 + (n-2)a_n^2} \leq n.
\]

(Vasile C., 2008)

**Solution.** The desired inequality is true for \(a_1 > \frac{n(n+1)}{2}\) since
\[
\frac{n(n+1) - 2a_1}{n^2 + (n-2)a_1^2} < 0
\]
and
\[
\frac{n(n+1) - 2a_i}{n^2 + (n-2)a_i^2} < \frac{n}{n-1}, \quad i = 2, 3, \ldots, n.
\]
The last inequalities are equivalent to
\[
n(n-2)a_i^2 + 2(n-1)a_i + n > 0,
\]
which are true because
\[
n(n-2)a_i^2 + 2(n-1)a_i + n \geq (n-1)a_i^2 + 2(n-1)a_i + n > (n-1)(a_i + 1)^2 \geq 0.
\]
Consider further that
\[
a_1, a_2, \ldots, a_n \leq \frac{n(n+1)}{2},
\]
and rewrite the desired inequality as
\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,
\]
where
\[
f(u) = \frac{2u - n(n+1)}{(n-2)u^2 + n^2}, \quad u \in \mathbb{I} = \left(-\infty, \frac{n(n+1)}{2}\right].
\]
We have
\[
f'(u) = \frac{n^2 + n(n+1)u - u^2}{2(n-2)u^2 + n^2}
\]
and
\[
f''(u) = \frac{f_1(u)}{[(n-2)u^2 + n^2]^2},
\]
where
\[
f_1(u) = 2(n-2)u^3 - 3n(n+1)(n-2)u^2 - 2n^2(2n-3)u + n^3(n+1).
\]
From the expression of \(f'\), it follows that \(f\) is decreasing on \((-\infty, s_0]\) and increasing on \(\left[s_0, \frac{n(n+1)}{2}\right]\), where
\[
s_0 = \frac{n}{2} \left(n + 1 - \sqrt{n^2 + 2n + 5}\right) \in (-1, 0);
\]
therefore, 
\[ \min_{u \in I} f(u) = f(s_0). \]

On the other hand, for \(-1 \leq u \leq 1\), we have
\[
f_1(u) > -2(n-2) - 3n(n+1)(n-2) - 2n^2(2n-3) + n^3(n+1)
= n^2(n-3)^2 + 4(n+1) > 0,
\]
hence \(f''(u) > 0\). Since \([s_0, s] \subset [-1, 1]\), \(f\) is convex on \([s_0, s]\). By the LPCF-Theorem and Note 1, we only need to show that 
\[ h(x, y) \geq 0 \]
for \(x, y \in \mathbb{R}\) and 
\[ x + (n-1)y = n, \]
where
\[
h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.
\]
Indeed, we have
\[
g(u) = \frac{(n-2)u + n}{(n-2)u^2 + n^2}
\]
and
\[
\frac{h(x, y)}{n-2} = \frac{n^2 - n(x + y) - (n-2)xy}{[(n-2)x^2 + n^2][(n-2)y^2 + n^2]}
= \frac{(n-1)(n-2)y^2}{[(n-2)x^2 + n^2][(n-2)y^2 + n^2]} \geq 0.
\]
The proof is completed. By Note 3, the equality holds for \(a_1 = a_2 = \cdots = a_n = 1\), and also for 
\[ a_1 = n, \quad a_2 = \cdots = a_n = 0 \]
(or any cyclic permutation). \(\square\)

**P 3.5.** If \(a, b, c, d\) are real numbers so that \(a + b + c + d = 4\), then
\[
\frac{a(a - 1)}{3a^2 + 4} + \frac{b(b - 1)}{3b^2 + 4} + \frac{c(c - 1)}{3c^2 + 4} + \frac{d(d - 1)}{3d^2 + 4} \geq 0.
\]

(Vasile C., 2012)

**Solution.** Write the inequality as
\[
f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a + b + c + d}{4} = 1,
\]
where
\[
f(u) = \frac{u^2 - u}{3u^2 + 4}, \quad u \in \mathbb{R}.
\]
From
\[ f'(u) = \frac{3u^2 + 8u - 4}{(3u^2 + 4)^2}, \]
it follows that \( f \) is increasing on \( \left( -\infty, \frac{-4 - 2\sqrt{7}}{3} \right] \cup \left[ s_0, \infty \right) \) and decreasing on \( \left[ \frac{-4 - 2\sqrt{7}}{3}, s_0 \right] \), where
\[ s_0 = \frac{-4 + 2\sqrt{7}}{3} \approx 0.43. \]

Since
\[ \lim_{u \to -\infty} f(u) = \frac{1}{3} \]
and \( f(s_0) < 0 \), it follows that
\[ \min_{u \in \mathbb{R}} f(u) = f(s_0). \]

For \( u \in [0, 1] \), we have
\[ \frac{1}{2} f''(u) = \frac{-9u^3 - 36u^2 + 36u + 14}{(3u^2 + 4)^3} = \frac{9u^2(1 - u) + 45u(1 - u) + (16 - 9u)}{(3u^2 + 4)^3} > 0. \]
Therefore, \( f \) is convex on \([0, 1]\), hence on \([s_0, s]\). According to the LPCF-Theorem and Note 1, we only need to show that \( h(x, y) \geq 0 \) for \( x, y \in \mathbb{R} \) so that \( x + 3y = 4 \). We have
\[ g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{u}{3u^2 + 4}, \]
\[ h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{4 - 3xy}{(3x^2 + 4)(3y^2 + 4)} = \frac{(x - 2)^2}{(3x^2 + 4)(3y^2 + 4)} \geq 0. \]

The proof is completed. From \( x + 3y = 4 \) and \( h(x, y) = 0 \), we get \( x = 2 \) and \( y = 2/3 \). By Note 3, the equality holds for \( a = b = c = d = 1 \), and also for
\[ a = 2, \quad b = c = d = \frac{2}{3} \]
(or any cyclic permutation).

**Remark.** In the same manner, we can prove the following generalization:
- If \( a_1, a_2, \ldots, a_n \) are real numbers so that \( a_1 + a_2 + \cdots + a_n = n \), then
\[ \frac{a_1(a_1 - 1)}{4(n - 1)a_1^2 + n^2} + \frac{a_2(a_2 - 1)}{4(n - 1)a_2^2 + n^2} + \cdots + \frac{a_n(a_n - 1)}{4(n - 1)a_n^2 + n^2} \geq 0, \]
with equality for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for
\[
a_1 = \frac{n}{2}, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{2(n - 1)}
\]
(or any cyclic permutation).

\[\Box\]

**P 3.6.** If \( a, b, c \) are real numbers so that \( a + b + c = 3 \), then
\[
\frac{1}{9a^2 - 10a + 9} + \frac{1}{9b^2 - 10b + 9} + \frac{1}{9c^2 - 10c + 9} \leq \frac{3}{8}.
\]

(Vasile C., 2015)

**Solution.** Write the inequality as
\[
f(a) + f(b) + f(c) \geq 3f(s), \quad s = \frac{a + b + c}{3} = 1,
\]
where
\[
f(u) = \frac{-1}{9u^2 - 10u + 9}, \quad u \in \mathbb{R}.
\]
From
\[
f''(u) = \frac{2(9u - 5)}{(9u^2 - 10u + 9)^2},
\]
it follows that \( f \) is decreasing on \((-\infty, s_0]\) and increasing on \([s_0, \infty)\) and , where
\[
s_0 = \frac{5}{9} < 1 = s.
\]

For \( u \in [s_0, s] = [5/9, 1] \), we have
\[
f''(u) = \frac{2(-243u^2 + 270u - 19)}{(9u^2 - 10u + 9)^3} > \frac{2(-243u^2 + 270u - 27)}{(9u^2 - 10u + 9)^3} = \frac{54(-9u^2 + 10u - 1)}{(9u^2 - 10u + 9)^3} = \frac{54(1-u)(9u-1)}{(9u^2 - 10u + 9)^3} \geq 0.
\]

Therefore, \( f \) is convex on \([s_0, s]\). According to the LPCF-Theorem and Note 1, we only need to show that \( h(x, y) \geq 0 \) for \( x, y \in \mathbb{R} \) so that \( x + 2y = 3 \). We have
\[
g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{9u - 1}{8(9u^2 - 10u + 9)},
\]
\[
h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{9(x + y) - 81xy + 71}{8(9x^2 - 10x + 9)(9y^2 - 10y + 9)} = \frac{2(9y - 7)^2}{8(9x^2 - 10x + 9)(9y^2 - 10y + 9)} \geq 0.
\]
The proof is completed. From \(x + 2y = 3\) and \(h(x, y) = 0\), we get
\[
x = \frac{13}{9}, \quad y = \frac{7}{9}.
\]
Thus, the equality holds for \(a = b = c = d = 1\), and also for
\[
a = \frac{13}{9}, \quad b = c = \frac{7}{9}
\]
(or any cyclic permutation).

\[\square\]

**P 3.7.** If \(a, b, c, d\) are real numbers so that \(a + b + c + d = 4\), then
\[
\frac{1}{4a^2 - 5a + 4} + \frac{1}{4b^2 - 5b + 4} + \frac{1}{4c^2 - 5c + 4} + \frac{1}{4d^2 - 5d + 4} \leq \frac{4}{3}.
\]

*(Vasile C., 2015)*

**Solution.** Write the inequality as
\[
f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a + b + c + d}{4} = 1,
\]
where
\[
f(u) = \frac{-1}{4u^2 - 5u + 4}, \quad u \in \mathbb{R}.
\]
From
\[
f'(u) = \frac{2(8u - 5)}{(4u^2 - 5u + 4)^2},
\]
it follows that \(f\) is decreasing on \((-\infty, s_0]\) and increasing on \([s_0, \infty)\), where
\[
s_0 = \frac{5}{8} < 1 = s.
\]
For \(u \in [s_0, s] = [5/8, 1]\), we have
\[
f''(u) = \frac{4(-48u^2 + 60u - 9)}{(4u^2 - 5u + 4)^3} > \frac{4(-48u^2 + 60u - 12)}{(4u^2 - 5u + 4)^3} = \frac{48(-4u^2 + 5u - 1)}{(4u^2 - 5u + 4)^3} = \frac{48(1-u)(4u-1)}{(4u^2 - 5u + 4)^3} \geq 0.
\]
Therefore, \(f\) is convex on \([s_0, s]\). According to the LPCF-Theorem and Note 1, we only need to show that \(h(x, y) \geq 0\) for \(x, y \in \mathbb{R}\) so that \(x + 3y = 4\). We have
\[
g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{4u - 1}{3(4u^2 - 5u + 4)}.
\]
\[ h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{4(x + y) - 16xy + 11}{3(4x^2 - 5x + 4)(4y^2 - 5y + 4)} \]
\[
= \frac{(4y - 3)^2}{(4x^2 - 5x + 4)(4y^2 - 5y + 4)} \geq 0.
\]

From \( x + 3y = 4 \) and \( h(x, y) = 0 \), we get
\[
x = \frac{7}{4}, \quad y = \frac{3}{4}.
\]
In accord with Note 3, the equality holds for \( a = b = c = 1 \), and also for
\[
a = \frac{7}{4}, \quad b = c = d = \frac{3}{4}
\]
(or any cyclic permutation).

**Remark.** Similarly, we can prove the following generalization:

- Let \( a_1, a_2, \ldots, a_n \) be real numbers so that \( a_1 + a_2 + \cdots + a_n = n \). If
\[
k = 1 - \frac{2(n-1)}{n^2},
\]
then
\[
\frac{1}{a_1^2 - 2ka_1 + 1} + \frac{1}{a_2^2 - 2ka_2 + 1} + \cdots + \frac{1}{a_n^2 - 2ka_n + 1} \geq \frac{n}{2(1-k)},
\]
with equality for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for
\[
a_1 = \frac{3n^2 - 6n + 4}{n^2}, \quad a_2 = a_3 = \cdots = a_n = \frac{n^2 - 2n + 4}{n^2}
\]
(or any cyclic permutation).

\[ \square \]

**P 3.8.** Let \( a_1, a_2, \ldots, a_n \neq -k \) be real numbers so that \( a_1 + a_2 + \cdots + a_n = n \), where
\[
k \geq \frac{n}{2\sqrt{n} - 1}.
\]
Then,
\[
\frac{a_1(a_1 - 1)}{(a_1 + k)^2} + \frac{a_2(a_2 - 1)}{(a_2 + k)^2} + \cdots + \frac{a_n(a_n - 1)}{(a_n + k)^2} \geq 0.
\]

(Vasile C., 2008)
Solution. Write the inequality as
\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1, \]
where
\[ f(u) = \frac{u(u-1)}{(u+k)^2}, \quad u \in \mathbb{I} = \mathbb{R} \setminus \{-k\}. \]
From
\[ f'(u) = \frac{(2k+1)u-k}{(u+k)^3}, \]
it follows that \( f \) is increasing on \((-\infty, -k) \cup [s_0, \infty)\) and decreasing on \((-k, s_0]\), where
\[ s_0 = \frac{k}{2k+1} < 1 = s. \]
Since
\[ \lim_{u \to -\infty} f(u) = 1 \]
and \( f(s_0) < 0 \), we have
\[ \min_{u \in \mathbb{I}} f(u) = f(s_0). \]
From
\[ \frac{1}{2} f''(u) = \frac{k(k+2) - (2k+1)u}{(u+k)^4}, \]
it follows that \( f \) is convex on \( \left[ 0, \frac{k(k+2)}{2k+1} \right] \), hence on \([s_0, 1]\). According to the LPCF-Theorem, Note 4 and Note 1, it suffices to show that \( h(x, y) \geq 0 \) for all \( x, y \in \mathbb{I} \) which satisfy \( x + (n-1)y = n \), where
\[ h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u-1}. \]
Indeed, we have
\[ g(u) = \frac{u}{(u+k)^2} \]
and
\[ h(x, y) = \frac{k^2 - xy}{(x+k)^2(y+k)^2} \geq \frac{n^2}{4(n-1)} - xy \]
\[ = \frac{[2(n-1)y-n]^2}{4(n-1)(x+k)^2(y+k)^2} \geq 0. \]
The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \). If \( k = \frac{n}{2\sqrt{n-1}} \), then the equality holds also for
\[ a_1 = \frac{n}{2}, \quad a_2 = \cdots = a_n = \frac{n}{2(n-1)} \]
(or any cyclic permutation). \( \square \)
P 3.9. Let \( a_1, a_2, \ldots, a_n \neq -k \) be real numbers so that \( a_1 + a_2 + \cdots + a_n = n \). If

\[
k \geq 1 + \frac{n}{\sqrt{n-1}},
\]

then

\[
\frac{a_1^2 - 1}{(a_1 + k)^2} + \frac{a_2^2 - 1}{(a_2 + k)^2} + \cdots + \frac{a_n^2 - 1}{(a_n + k)^2} \geq 0.
\]

(Vasile C., 2008)

**Solution.** Write the inequality as

\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,
\]

where

\[
f(u) = \frac{u^2 - 1}{(u + k)^2}, \quad u \in \mathbb{I} = \mathbb{R} \setminus \{-k\}.
\]

From

\[
f'(u) = \frac{2(ku + 1)}{(u + k)^3},
\]

it follows that \( f \) is increasing on \((-\infty, -k) \cup [s_0, \infty)\) and decreasing on \((-k, s_0]\), where

\[
s_0 = \frac{-1}{k} < 0 = s, \quad s_0 > -1.
\]

Since

\[
\lim_{u \to -\infty} f(u) = 1
\]

and \( f(s_0) < 0 \), we have

\[
\min_{u \in \mathbb{I}} f(u) = f(s_0).
\]

For \( u \in [-1, 1] \), we have

\[
f''(u) = \frac{2(k^2 - 3 - 2ku)}{(u + k)^4} \geq \frac{2(k^2 - 3 - 2k)}{(u + k)^4} = \frac{2(k + 1)(k - 3)}{(u + k)^4} \geq 0,
\]

hence \( f \) is convex on \([s_0, 1]\). According to the LPCF-Theorem, Note 4 and Note 1, it suffices to show that \( h(x, y) \geq 0 \) for \( x, y \in \mathbb{I} \) which satisfy \( x + (n - 1)y = n \). We have

\[
g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{u + 1}{(u + k)^2},
\]

\[
h(x, y) = g(x) - g(y) = \frac{(k - 1)^2 - 1 - x - y - xy}{(x + k)^2(y + k)^2} \geq 0,
\]

since

\[
(k - 1)^2 - 1 - x - y - xy \geq \frac{n^2}{n - 1} - 1 - x - y - xy = \frac{(n - 1)(y - 1)^2}{n - 1} \geq 0.
\]
The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \). If \( k = 1 + \frac{n}{\sqrt{n-1}} \), then the equality holds also for
\[
a_1 = n - 1, \quad a_2 = \cdots = a_n = \frac{1}{n-1}
\]
(or any cyclic permutation).

\[\square\]

**P 3.10.** Let \( a_1, a_2, a_3, a_4, a_5 \) be real numbers so that \( a_1 + a_2 + a_3 + a_4 + a_5 \geq 5 \). If
\[
k \in \left[ \frac{1}{6}, \frac{25}{14} \right],
\]
then
\[
\sum \frac{1}{ka_i^2 + a_2 + a_3 + a_4 + a_5} \leq \frac{5}{k + 4}.
\]

*(Vasile C., 2006)*

**Solution.** We see that
\[
ka_i^2 - a_i + (a_1 + a_2 + a_3 + a_4 + a_5) > \frac{1}{6}a_i^2 - a_i + \frac{3}{2} = \frac{(a_1 - 3)^2}{6} \geq 0
\]
for all \( i \in \{1, 2, \ldots, n\} \). Since each term of the left hand side of the inequality decreases by increasing any number \( a_i \), it suffices to consider the case
\[
a_1 + a_2 + a_3 + a_4 + a_5 = 5,
\]
when the desired inequality can be written as
\[
f(a_1) + f(a_2) + f(a_3) + f(a_4) + f(a_5) \geq 5f(s), \quad s = \frac{a_1 + a_2 + a_3 + a_4 + a_5}{5} = 1,
\]
where
\[
f(u) = \frac{-1}{ku^2 - u + 5}, \quad u \in \mathbb{R}.
\]
From
\[
f'(u) = \frac{2ku - 1}{(ku^2 - u + 5)^2},
\]
it follows that \( f \) is decreasing on \((-\infty, s_0]\) and increasing on \([s_0, \infty)\), where
\[
s_0 = \frac{1}{2k}.
\]
We have
\[
f''(u) = \frac{2g(u)}{(ku^2 - u + 5)^2}, \quad g(u) = -3k^2u^2 + 3ku + 5k - 1.
\]
For \( \frac{1}{2} \leq k \leq \frac{25}{14} \), we have
\[ s_0 = \frac{1}{2k} \leq 1 = s, \]
and for \( u \in [s_0, s] \), that is \( \frac{1}{2k} \leq u \leq 1 \), we have
\[
(1 - u)(2ku - 1) \geq 0, \\
-2ku^2 \geq (2k + 1)u + 1, \\
-2k^2u^2 \geq k(2k + 1)u + k,
\]
therefore
\[
g(u) \geq \frac{3}{2}[k(2k + 1)u + k] + 3ku + 5k - 1 = \frac{-3k(2k - 1)u + 13k - 2}{2} \\
\geq \frac{-3k(2k - 1) + 13k - 2}{2} = -3k^2 + 8k - 1 = 3k(2 - k) + (2k - 1) > 0.
\]
Consequently, \( f \) is convex on \([s_0, s]\).

For \( \frac{1}{6} \leq k \leq \frac{1}{2} \), we have
\[ s_0 = \frac{1}{2k} \geq 1 = s, \]
and for \( u \in [s, s_0], \) that is \( 1 \leq u \leq \frac{1}{2k}, \)
we have
\[
g(u) = -3k^2u^2 + 3ku + 5k - 1 \geq 3ku(1 - k) + 5k - 1 \\
\geq 3k(1 - k) + 5k - 1 = -3k^2 + 8k - 1 \\
> -6k^2 + 7k - 1 = (1 - k)(6k - 1) \geq 0.
\]
Consequently, \( f \) is convex on \([s, s_0]\).

In both cases, by the PCF-Theorem, it suffices to show that
\[
\frac{1}{kx^2 - x + 5} + \frac{4}{ky^2 - y + 5} \leq \frac{5}{k + 4}
\]
for \( x + 4y = 5, \quad x, y \in \mathbb{R}. \)
Write this inequality as follows:

\[
\frac{1}{k+4} - \frac{1}{kx^2 - x + 5} + 4 \left[ \frac{1}{k+4} - \frac{1}{ky^2 - y + 5} \right] \geq 0,
\]

\[
\frac{(x-1)(kx+k-1)}{kx^2 - x + 5} + \frac{4(y-1)(ky+k-1)}{ky^2 - y + 5} \geq 0.
\]

Since

\[
4(y-1) = 1-x,
\]

the inequality is equivalent to

\[
(x-1)\left(\frac{kx+k-1}{kx^2-x+5} - \frac{ky+k-1}{ky^2-y+5}\right) \geq 0,
\]

\[
\frac{5(x-1)^2h(x,y)}{4(kx^2-x+5)(ky^2-y+5)} \geq 0,
\]

where

\[
h(x,y) = -k^2xy - k(k-1)(x+y) + 6k - 1
\]

\[
= 4k^2y^2 - k(2k+3)y - 5k^2 + 11k - 1
\]

\[
= \left(2ky - \frac{2k+3}{4}\right)^2 + \frac{(25-14k)(6k-1)}{16} \geq 0.
\]

The equality holds for \(a_1 = a_2 = a_3 = a_4 = a_5 = 1\). If \(k = \frac{1}{6}\), then the equality holds also for

\[
a_1 = -5, \quad a_2 = a_3 = a_4 = a_5 = \frac{5}{2}
\]

(or any cyclic permutation). If \(k = \frac{25}{14}\), then the equality holds also for

\[
a_1 = \frac{79}{25}, \quad a_2 = a_3 = a_4 = a_5 = \frac{23}{50}
\]

(or any cyclic permutation).

**Remark.** In the same manner, we can prove the following generalization:

- Let \(a_1, a_2, \ldots, a_n\) be real numbers so that \(a_1 + a_2 + \cdots + a_n \leq n\). If \(k \in [k_1, k_2]\), where

\[
k_1 = \frac{(n-1)(\sqrt{53n^2 - 54n + 101} - 5n + 11)}{2(7n^2 + 14n - 5)},
\]

\[
k_2 = \frac{2n^2 - 2n + 1 + \sqrt{(n-1)(3n^3 - 4n^2 + 3n - 1)}}{2(n^2 - n + 1)}.
\]
then
\[ \sum_{i=1}^{n} \frac{1}{ka_i^2 + a_2 + \cdots + a_n} \leq \frac{n}{k + n - 1}, \]
with equality for \( a_1 = a_2 = \cdots = a_n = 1 \). If \( k = k_1 \), then the equality holds also for
\[ a_1 = -n, \quad a_2 = \cdots = a_n = \frac{2n}{n - 1} \]
(or any cyclic permutation). If \( k = k_2 \), then the equality holds also for
\[ a_1 = \frac{(2k-1)(n-1)+1}{2k}, \quad a_2 = \cdots = a_n = \frac{2k+n-2}{2k(n-1)} \]
(or any cyclic permutation).

\[ = \boxed{\text{Proof}} \]

**P 3.11.** Let \( a_1, a_2, \ldots, a_5 \) be nonnegative numbers so that \( a_1 + a_2 + a_3 + a_4 + a_5 \geq 5 \). If \( k \in [k_1, k_2] \), where
\[ k_1 = \frac{29 - \sqrt{761}}{10} \approx 0.1414, \quad k_2 = \frac{25}{14} \approx 1.7857, \]
then
\[ \sum_{i=1}^{5} \frac{1}{ka_i^2 + a_2 + a_3 + a_4 + a_5} \leq \frac{5}{k + 4}. \]

**(Vasile C., 2006)**

**Solution.** Since all terms of the left hand side of the inequality decrease by increasing any number \( a_i \), it suffices to consider the case
\[ a_1 + a_2 + a_3 + a_4 + a_5 = 5. \]

The proof is similar to the one of the preceding P 3.10. Having in view P 3.10, it suffices to consider the case
\[ k \in \left[ k_1, \frac{1}{6} \right], \]
when
\[ s_0 = \frac{1}{2k} > 1 = s. \]

For \( u \in [s, s_0] \), that is
\[ 1 \leq u \leq \frac{1}{2k}, \]
\( f \) is convex because
\[ g(u) = -3k^2u^2 + 3ku + 5k - 1 \geq 3ku(1-k) + 5k - 1 \]
\[ \geq 3k(1-k) + 5k - 1 = -3k^2 + 8k - 1 \]
\[ > -\frac{15}{4}k^2 + 8k - 1 = \frac{(2-k)(15k-2)}{4} > 0. \]
Thus, by the RPCF-Theorem, it suffices to show that
\[
\frac{1}{kx^2 - x + 5} + \frac{4}{ky^2 - y + 5} \leq \frac{5}{k + 4}
\]
for
\[
x + 4y = 5, \quad 0 \leq x \leq 1 \leq y \leq \frac{5}{4}.
\]
As shown at P 3.10, this inequality is true if \( h(x, y) \geq 0 \), where
\[
h(x, y) = -k^2 xy - k(k - 1)(x + y) + 6k - 1.
\]
We have
\[
h(x, y) = 4k^2 y^2 - k(2k + 3)y - 5k^2 + 11k - 1
\]
\[
= (5 - 4y)(A - k^2 y) + B = x(A - k^2 y) + B,
\]
where
\[
A = \frac{3k(1 - k)}{4}, \quad B = \frac{-5k^2 + 29k - 4}{4}.
\]
Since \( B \geq 0 \), it suffices to show that \( A - k^2 y \geq 0 \). Indeed, we have
\[
A - k^2 y \geq \frac{3k(1 - k)}{4} - \frac{5k^2}{4} = \frac{k(3 - 8k)}{4} > 0.
\]
The equality holds for \( a_1 = a_2 = a_3 = a_4 = a_5 = 1 \). If \( k = k_1 \), then the equality holds also for
\[
a_1 = 0, \quad a_2 = a_3 = a_4 = a_5 = \frac{5}{4}
\]
(or any cyclic permutation). If \( k = k_2 \), then the equality holds also for
\[
a_1 = \frac{79}{25}, \quad a_2 = a_3 = a_4 = a_5 = \frac{23}{50}
\]
(or any cyclic permutation).

**Remark.** Similarly, we can prove the following generalization:

- Let \( a_1, a_2, \ldots, a_n \) be nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n \leq n \). If \( k \in [k_1, k_2] \), where

\[
k_1 = \frac{n^2 + n - 1 - \sqrt{n^4 + 2n^3 - 5n^2 + 2n + 1}}{2n},
\]

\[
k_2 = \frac{2n^2 - 2n + 1 + \sqrt{(n - 1)(3n^3 - 4n^2 + 3n - 1)}}{2(n^2 - n + 1)},
\]

then
\[
\sum \frac{1}{ka_1^2 + a_2 + \cdots + a_n} \leq \frac{n}{k + n - 1}.
\]
with equality for \( a_1 = a_2 = \cdots = a_n = 1 \). If \( k = k_1 \), then the equality holds also for

\[
a_1 = 0, \quad a_2 = \cdots = a_n = \frac{n}{n-1}
\]

(or any cyclic permutation). If \( k = k_2 \), then the equality holds also for

\[
a_1 = \frac{(2k-1)(n-1)+1}{2k}, \quad a_2 = \cdots = a_n = \frac{2k+n-2}{2k(n-1)}
\]

(or any cyclic permutation).

\( \square \)

P 3.12. Let \( a_1, a_2, \ldots, a_n \) be nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n \geq n \). If \( k > 1 \), then

\[
\frac{1}{a_1^k + a_2 + \cdots + a_n} + \frac{1}{a_1 + a_2^k + \cdots + a_n} + \cdots + \frac{1}{a_1 + a_2 + \cdots + a_n^k} \leq 1.
\]

(Vasile C., 2006)

Solution. It suffices to consider the case \( a_1 + a_2 + \cdots + a_n = n \), when the desired inequality can be written as

\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,
\]

where

\[
f(u) = \frac{-1}{u^k - u + n}, \quad u \in [0,n].
\]

From

\[
f'(u) = \frac{k u^{k-1} - 1}{(u^k - u + n)^2},
\]

it follows that \( f \) is decreasing on \([0,s_0]\) and increasing on \([s_0,n]\), where

\[
s_0 = k^{\frac{1}{k-1}} < 1 = s.
\]

We will show that \( f \) is convex on \([s_0,1]\). For \( u \in [s_0,1] \), we have

\[
f''(u) = \frac{-k(k+1)u^{2k-2} + k(k+3)u^{k-1} + nk(k-1)u^{k-2} - 2}{(u^k - u + n)^3} > \frac{g(u)}{(u^k - u + n)^3},
\]

where

\[
g(u) = -k(k+1)u^{2k-2} + k(k+3)u^{k-1} - 2.
\]

Denoting

\[
t = ku^{k-1}, \quad 1 \leq t \leq k,
\]
we get

\[ kg(u) = -(k + 1)t^2 + k(k + 3)t - 2k = (k + 1)(t - 1)(k - t) + (k - 1)(t + k) > 0. \]

By the LPCF-Theorem, it suffices to show that

\[ \frac{1}{x^k - x + n} + \frac{n - 1}{y^k - y + n} \leq 1 \]

for \( x \geq 1 \geq y \geq 0 \) and \( x + (n-1)y = n \). Since this inequality is trivial for \( x = y = 1 \), assume next that \( x > 1 > y \geq 0 \), and write the desired inequality as follows:

\[ x^k - x + n \geq \frac{y^k - y + n}{y^k - y + 1}, \]

\[ x^k - x \geq \frac{(n - 1)(y - y^k)}{y^k - y + 1}, \]

\[ \frac{x^k - x}{x - 1} \geq \frac{y - y^k}{(1 - y)(y^k - y + 1)}. \]

Let \( h(x) = \frac{x^k - x}{x - 1}, x > 1 \). By the weighted AM-GM inequality, we have

\[ h'(x) = \frac{(k - 1)x^k + 1 - kx^{k-1}}{(x - 1)^2} > 0. \]

Therefore, \( h \) is increasing. Since

\[ x - 1 = (n - 1)(1 - y) \geq 1 - y, \quad x \geq 2 - y > 1, \]

we get

\[ h(x) \geq h(2 - y) = \frac{(2 - y)^k + y - 2}{1 - y}. \]

Thus, it suffices to show that

\[ (2 - y)^k + y - 2 \geq \frac{y - y^k}{y^k - y + 1}, \]

which is equivalent to

\[ (2 - y)^k + y - 1 \geq \frac{1}{y^k - y + 1}. \]

Using the substitution

\[ t = 1 - y, \quad 0 < t \leq 1, \]
the inequality becomes
\[(1 + t)^k - t \geq \frac{1}{(1-t)^k + t},\]
\[(1 - t^2)^k + t(1 + t)^k \geq 1 + t^2 + t(1 - t)^k.\]

By Bernoulli’s inequality,
\[(1 - t^2)^k + t(1 + t)^k \geq 1 - kt^2 + t(1 + kt) = 1 + t.\]

So, we only need to show that
\[1 + t \geq 1 + t^2 + t(1 - t)^k,\]
which is equivalent to the obvious inequality
\[t(1 - t)[1 - (1 - t)^{k-1}] \geq 0.\]

The equality holds for \(a_1 = a_2 = \cdots = a_n = 1.\)

**Remark.** Using this result, we can formulate the following statement:

- Let \(x_1, x_2, \ldots, x_n\) be nonnegative real numbers so that \(x_1 + x_2 + \cdots + x_n \geq n.\) If \(k > 1,\) then

\[\frac{x_1^k - x_1}{x_1^k + x_2 + \cdots + x_n} + \frac{x_2^k - x_2}{x_1 + x_2^k + \cdots + x_n} + \cdots + \frac{x_n^k - x_n}{x_1 + x_2 + \cdots + x_n^k} \geq 0.\]

This inequality is equivalent to
\[\frac{1}{x_1^k + x_2 + \cdots + x_n} + \frac{1}{x_1 + x_2^k + \cdots + x_n} + \cdots + \frac{1}{x_1 + x_2 + \cdots + x_n^k} \leq \frac{n}{x_1 + x_2 + \cdots + x_n}.\]

Using the substitutions
\[s = \frac{x_1 + x_2 + \cdots + x_n}{n}, \quad s \geq 1,\]
and
\[a_i = \frac{x_i}{s}, \quad i = 1, 2, \ldots, n,\]
which yields \(a_1 + a_2 + \cdots + a_n = n,\) the desired inequality becomes
\[\sum s^{k-1}a_i^k - a_1 + a_2 + \cdots + a_n \leq 1.\]

Since \(s^{k-1} \geq 1,\) it suffices to show that
\[\sum a_i^k - a_1 + a_2 + \cdots + a_n \leq 1,\]
which follows immediately from the inequality in P 3.12.

Since \( x_1 x_2 \cdots x_n \geq 1 \) involves \( x_1 + x_2 + \cdots + x_n \geq n \), the inequality is also true under the more restrictive condition \( x_1 x_2 \cdots x_n \geq 1 \). For \( n = 3 \) and \( k = 5/2 \), we get the inequality from IMO-2005:

- If \( x, y, z \) are nonnegative real numbers so that \( xyz \geq 1 \), then
  \[
  \frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0.
  \]

\[ \square \]

**P 3.13.** Let \( a_1, a_2, \ldots, a_5 \) be nonnegative numbers so that \( a_1 + a_2 + a_3 + a_4 + a_5 \geq 5 \). If

\[
k \in \left[ \frac{4}{9}, \frac{61}{5} \right],
\]

then

\[
\sum \frac{a_1}{ka_1^2 + a_2 + a_3 + a_4 + a_5} \leq \frac{5}{k + 4}.
\]

(Vasile C., 2006)

**Solution.** Using the substitution

\[
x_1 = \frac{a_1}{s}, \ x_2 = \frac{a_2}{s}, \ x_3 = \frac{a_3}{s}, \ x_4 = \frac{a_4}{s}, \ x_5 = \frac{a_5}{s},
\]

where

\[
s = \frac{a_1 + a_2 + a_3 + a_4 + a_5}{5} \geq 1,
\]

we need to show that \( x_1 + x_2 + x_3 + x_4 + x_5 = 5 \) involves

\[
\frac{x_1}{ksx_1^2 + x_2 + x_3 + x_4 + x_5} + \cdots + \frac{x_5}{x_1 + x_2 + x_3 + x_4 + ksx_5^2} \leq \frac{5}{k + 4}.
\]

Since \( s \geq 1 \), it suffices to prove the inequality for \( s = 1 \); that is, to show that

\[
\frac{a_1}{ka_1^2 - a_1 + 5} + \frac{a_2}{ka_2^2 - a_1 + 5} + \cdots + \frac{a_5}{ka_5^2 - a_5 + 5} \leq \frac{5}{k + 4}
\]

for

\[
a_1 + a_2 + a_3 + a_4 + a_5 = 5.
\]

Write the desired inequality as

\[
f(a_1) + f(a_2) + f(a_3) + f(a_4) + f(a_5) \geq 5f(s),
\]
where
\[ s = \frac{a_1 + a_2 + a_3 + a_4 + a_5}{5} = 1 \]
and
\[ f(u) = \frac{-u}{ku^2 - u + 5}, \quad u \in [0, 5]. \]
From
\[ f'(u) = \frac{ku^2 - 5}{(ku^2 - u + 5)^2}, \]
it follows that \( f \) is decreasing on \([0, s_0]\) and increasing on \([s_0, 5]\), where
\[ s_0 = \sqrt{\frac{5}{k}}. \]

We have
\[ f''(u) = \frac{2g(u)}{(u^2 - u + 5)^3}, \quad g(u) = -k^2u^3 + 15ku - 5, \quad g'(u) = 3k(5 - ku^2). \]

Case 1: \( \frac{4}{9} \leq k \leq 5 \). We have
\[ s_0 = \sqrt{\frac{5}{k}} \geq 1 = s. \]

For \( u \in [1, s_0] \), the derivative \( g' \) is nonnegative, \( g \) is increasing, hence
\[ g(u) \geq g(1) = -k^2 + 15k - 5 = (k - \frac{4}{9})(5 - k) + \frac{86k - 25}{9} > 0. \]
Consequently, \( f''(u) > 0 \) for \( u \in [1, s_0] \), hence \( f \) is convex on \([s, s_0]\).

Case 2: \( 5 \leq k \leq \frac{61}{5} \). We have
\[ s_0 = \sqrt{\frac{5}{k}} < 1 = s. \]

For \( u \in [s_0, 1] \), we have \( g'(u) \leq 0, g(u) \) is decreasing, hence
\[ g(u) \geq g(1) = -k^2 + 15k - 5 = (k - 1)(13 - k) + k + 8 > 0. \]
Consequently, \( f''(u) > 0 \) for \( u \in [s_0, 1] \), hence \( f \) is convex on \([s_0, s]\).

In both cases, by the PCF-Theorem, it suffices to show that
\[ \frac{x}{kx^2 - x + 5} + \frac{4y}{ky^2 - y + 5} \leq \frac{5}{k + 4} \]
for
\[ x + 4y = 5, \quad x, y \geq 0. \]

Write this inequality as follows:
\[
\frac{1}{k+4} - \frac{x}{kx^2 - x + 5} + 4\left[ \frac{1}{k+4} - \frac{y}{ky^2 - y + 5} \right] \geq 0,
\]
\[
\frac{(x-1)(kx-5)}{kx^2 - x + 5} + \frac{4(y-1)(ky-5)}{ky^2 - y + 5} \geq 0.
\]

Since
\[ 4(y - 1) = 1 - x, \]

the inequality is equivalent to
\[
(x - 1)\left( \frac{kx - 5}{kx^2 - x + 5} - \frac{ky - 5}{ky^2 - y + 5} \right) \geq 0,
\]
\[
\frac{(x - 1)^2 h(x, y)}{(kx^2 - x + 5)(ky^2 - y + 5)} \geq 0,
\]

where
\[
h(x, y) = -k^2xy + 5k(x + y) + 5k - 5
\]
\[
= 4k^2y^2 - 5k(k + 3)y + 5(6k - 1).
\]

We need to show that \( h(x, y) \geq 0 \) for \( k \in \left[ \frac{4}{9}, \frac{61}{5} \right] \). For \( k \in \left[ \frac{4}{9}, 1 \right] \), we have
\[
h(x, y) = (5 - 4y)\left( -k^2y + \frac{15k}{4} \right) + \frac{5(9k - 4)}{4}
\]
\[
= kx(15 - 4ky) + \frac{5(9k - 4)}{4}
\]
\[
= \frac{kx(kx + 15 - 5k)}{4} + \frac{5(9k - 4)}{4} \geq 0,
\]

while for \( k \in \left[ 1, \frac{61}{5} \right] \), we have
\[
h(x, y) = \left( 2ky - \frac{5k + 15}{4} \right)^2 + \frac{(61 - 5k)(k - 1)}{16} \geq 0.
\]

The equality holds for \( a_1 = a_2 = a_3 = a_4 = a_5 = 1 \). If \( k = \frac{4}{9} \), then the equality holds also for
\[ a_1 = 0, \quad a_2 = a_3 = a_4 = a_5 = \frac{5}{4}. \]
(or any cyclic permutation). If \( k = \frac{61}{5} \), then the equality holds also for
\[
a_1 = \frac{115}{61}, \quad a_2 = a_3 = a_4 = a_5 = \frac{95}{122}
\]
(or any cyclic permutation).

**Remark.** In the same manner, we can prove the following generalization:

- Let \( a_1, a_2, \ldots, a_n \) be real numbers so that \( a_1 + a_2 + \cdots + a_n \leq n \). If \( k \in [k_1, k_2] \), where
\[
k_1 = \frac{n - 1}{2n - 1},
\]
\[
k_2 = \frac{n^2 + 2n - 2 + 2\sqrt{(n - 1)(2n^2 - 1)}}{n},
\]
then
\[
\sum \frac{a_1}{ka_1^2 + a_2 + \cdots + a_n} \leq \frac{n}{k + n - 1},
\]
with equality for \( a_1 = a_2 = \cdots = a_n = 1 \). If \( k = k_1 \), then the equality holds also for
\[
a_1 = 0, \quad a_2 = a_3 = a_4 = a_5 = \frac{n}{n - 1}
\]
(or any cyclic permutation). If \( k = k_2 \), then the equality holds also for
\[
a_1 = \frac{n(k - n + 2)}{2k}, \quad a_2 = \cdots = a_n = \frac{n(k + n - 2)}{2k(n - 1)}
\]
(or any cyclic permutation).

\[\square\]

**P 3.14.** Let \( a_1, a_2, \ldots, a_n \) be nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n \geq n \). If \( k > 1 \), then
\[
\frac{a_1}{a_1^k + a_2 + \cdots + a_n} + \frac{a_2}{a_1 + a_2^k + \cdots + a_n} + \cdots + \frac{a_n}{a_1 + a_2 + \cdots + a_n^k} \leq 1.
\]

*(Vasile C., 2006)*

**Solution.** Using the substitution
\[
x_1 = \frac{a_1}{s}, \quad x_2 = \frac{a_2}{s}, \ldots, \quad x_n = \frac{a_n}{s},
\]
where
\[
s = \frac{a_1 + a_2 + \cdots + a_n}{n} \geq 1,
\]

we need to show that \( x_1 + x_2 + \cdots + x_n = n \) involves
\[
\frac{x_1}{s^{k-1}x_1^k + x_2 + \cdots + x_n} + \cdots + \frac{x_n}{x_1 + x_2 + \cdots + s^{k-1}x_n^k} \leq 1.
\]
Since \( s^{k-1} \geq 1 \), it suffices to prove the inequality for \( s = 1 \); that is, to show that
\[
\frac{a_1}{a_1^k - a_1 + n} + \frac{a_2}{a_2^k - a_2 + n} + \cdots + \frac{a_n}{a_n^k - a_n + n} \leq 1
\]
for
\[
a_1 + a_2 + \cdots + a_n = n.
\]

Case 1: \( 1 < k \leq n + 1 \). By Bernoulli’s inequality, we have
\[
a_1^k \geq 1 + k(a_1 - 1), \quad a_1^k - a_1 + n \geq (k - 1)a_1 + n - k + 1.
\]
Thus, it suffices to show that
\[
\sum \frac{a_i}{(k-1)a_1 + n - k + 1} \leq 1.
\]
This is an equality for \( k = n - 1 \). If \( 1 < k < n + 1 \), then the inequality is equivalent to
\[
\sum \frac{1}{(k-1)a_1 + n - k + 1} \geq 1,
\]
which follows from the the AM-HM inequality
\[
\sum \frac{1}{(k-1)a_1 + n - k + 1} \geq \frac{n^2}{\sum[(k-1)a_1 + n - k + 1]}.
\]

Case 2: \( k > n + 1 \). Write the desired inequality as
\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,
\]
where
\[
f(u) = \frac{-u}{u^k - u + n}, \quad u \in [0, n].
\]
We have
\[
f'(u) = \frac{(k-1)u^k - n}{(u^k - u + n)^2}
\]
and
\[
f''(u) = \frac{f_1(u)}{(u^k - u + n)^3},
\]
where
\[
f_1(u) = k(k-1)u^{k-1}(u^k - u + n) - 2(ku^{k-1} - 1)[(k-1)u^k - n].
\]
From the expression of $f'$, it follows that $f$ is decreasing on $[0, s_0]$ and increasing on $[s_0, n]$, where

$$s_0 = \left(\frac{n}{k-1}\right)^{1/k} < 1 = s.$$  

For $u \in [s_0, 1]$, we have

$$(k - 1)u^k - n \geq (k - 1)s_0^k - n = 0,$$  

hence

$$f_1(u) \geq k(k - 1)u^{k-1}(u^k - u + n) - 2ku^{k-1}[(k - 1)u^k - n]$$  

$$= ku^{k-1}[-(k - 1)(u^k + u) + n(k + 1)]$$  

$$\geq ku^{k-1}[-(2(k - 1) + 2k + 1)] = 4ku^{k-1} > 0.$$  

Since $f''(u) > 0$, it follows that $f$ is convex on $[s_0, s]$. By the LPCF-Theorem, we need to show that

$$f(x) + (n - 1)f(y) \geq nf(1)$$  

for

$$x \geq 1 \geq y \geq 0, \quad x + (n - 1)y = n.$$  

Consider the nontrivial case where $x > 1 > y \geq 0$ and write the required inequality as follows:

$$\frac{x}{x^k - x + n} + \frac{(n - 1)y}{y^k - y + n} \leq 1,$$

$$x^k - x + n \geq \frac{x(y^k - y + n)}{y^k - ny + n},$$

$$x^k - x \geq \frac{(n - 1)y(y - y^k)}{y^k - ny + n}.$$  

Since $y < 1$ and $y^k - ny + n > y^k - y + 1$, it suffices to show that

$$x^k - x \geq \frac{(n - 1)(y - y^k)}{y^k - y + 1},$$  

which has been proved at P 3.12.

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

\[\square\]

**P 3.15.** Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers so that $a_1 + a_2 + \cdots + a_n \leq n$.

If $k \geq 1 - \frac{1}{n}$, then

$$\frac{1 - a_1}{ka_1^2 + a_2 + \cdots + a_n} + \frac{1 - a_2}{a_1 + ka_2^2 + \cdots + a_n} + \cdots + \frac{1 - a_n}{a_1 + a_2 + \cdots + ka_n^2} \geq 0.$$  

(Vasile C., 2006)
Solution. Let 
\[ s = \frac{a_1 + a_2 + \cdots + a_n}{n}, \quad s \leq 1. \]

We have three cases to consider.

Case 1: \( s \leq \frac{1}{n} \). The inequality is trivial because 
\[ a_i \leq a_1 + a_2 + \cdots + a_n = ns \leq 1 \]
for \( i = 1, 2, \ldots, n \).

Case 2: \( \frac{1}{n} < s < 1 \). Without loss of generality, assume that 
\[ a_1 \leq \cdots \leq a_j < 1 \leq a_{j+1} \cdots \leq a_n, \quad j \in \{1, 2, \ldots, n\}. \]
Clearly, there are \( b_1, b_2, \ldots, b_n \) so that \( b_1 + b_2 + \cdots + b_n = n \) and 
\[ a_1 \leq b_1 \leq 1, \quad \ldots, \quad a_j \leq b_j \leq 1, \quad b_{j+1} = a_{j+1}, \quad \ldots, \quad b_n = a_n. \]
Write the desired inequality as 
\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq 0, \]
where 
\[ f(u) = \frac{1 - u}{ku^2 - u + ns}, \quad u \in [0, ns]. \]
For \( u \in [0, 1] \), we have 
\[ f'(u) = \frac{k[(1-u)^2 - 1] + (1-ns)}{(ku^2 - u + ns)^2} < 0, \]
hence \( f \) is strictly decreasing on \([0, 1]\) and 
\[ f(b_1) \leq f(a_1), \quad \ldots, \quad f(b_j) \leq f(a_j), \quad f(b_{j+1}) = f(a_{j+1}), \quad \ldots, \quad f(b_n) = f(a_n). \]
Since 
\[ f(b_1) + f(b_2) + \cdots + f(b_n) \leq f(a_1) + f(a_2) + \cdots + f(a_n), \]
it suffices to show that \( f(b_1) + f(b_2) + \cdots + f(b_n) \geq 0 \) for \( b_1 + b_2 + \cdots + b_n = n \).
This inequality is proved at Case 3.

Case 3: \( s = 1 \). Write the inequality as 
\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1, \]
where 
\[ f(u) = \frac{1 - u}{ku^2 - u + n}, \quad u \in [0, n]. \]
From
\[ f'(u) = \frac{k[(u-1)^2-1]-(n-1)}{(ku^2-u+n)^2}, \]
it follows that \( f \) is decreasing on \([0,s_0]\) and increasing on \([s_0,n]\), where
\[ s_0 = 1 + \sqrt{1 + \frac{n-1}{k}} > 1 = s, \quad s_0 < n. \]

We will show that \( f \) is convex on \([1,s_0]\). We have
\[ f''(u) = \frac{2g(u)}{(ku^2-u+n)^3}, \]
where
\[ g(u) = -k^2u^3 + 3k^2u^2 + 3k(n-1)u - kn - n + 1, \quad g'(u) = 3k(-k^2u^2 + 2ku + n - 1). \]

For \( u \in [1,s_0] \), we have \( g'(u) \geq 0 \), \( g \) is increasing, therefore
\[ g(u) \geq g(1) = 2k^2 + (2n-3)k - n + 1 \]
\[ \geq \frac{2(n-1)^2}{n^2} + \frac{(2n-3)(n-1)}{n} - n + 1 \]
\[ = \frac{(n^2-1)(n-2)}{n^2} \geq 0, \]

so \( f''(u) \geq 0 \), \( f(u) \) is convex for \( u \in [s,s_0] \). By the RPCF-Theorem, it suffices to show that
\[ \frac{1-x}{kx^2-x+n} + \frac{(n-1)(1-y)}{ky^2-y+n} \geq 0 \]
for \( 0 \leq x \leq 1 \leq y \) and \( x + (n-1)y = n \). Since \((n-1)(1-y) = x-1\), we have
\[ \frac{1-x}{kx^2-x+n} + \frac{(n-1)(1-y)}{ky^2-y+n} = (x-1) \left( -\frac{1}{kx^2-x+n} + \frac{1}{ky^2-y+n} \right) \]
\[ = \frac{(x-1)(x-y)(kx+ky-1)}{(kx^2-x+n)(ky^2-y+n)} \]
\[ = \frac{n(x-1)^2(kx+ky-1)}{(n-1)(kx^2-x+n)(ky^2-y+n)} \geq 0, \]

because
\[ k(x+y) - 1 \geq \frac{n-1}{n}(x+y) - 1 = \frac{(n-2)x}{n} \geq 0. \]

The proof is completed. The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \). If \( k = 1 - \frac{1}{n} \), then the equality holds also for
\[ a_1 = 0, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n-1}. \]
Partially Convex Function Method

Remark. For \( k = 1 \), we get the following statement:

- If \( a_1, a_2, \ldots, a_n \) are nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n \leq n \), then
  \[
  \frac{1 - a_1}{a_1^2 + a_2 + \cdots + a_n} + \frac{1 - a_2}{a_1 + a_2^2 + \cdots + a_n} + \cdots + \frac{1 - a_n}{a_1 + a_2 + \cdots + a_n^2} \geq 0.
  \]
  with equality for \( a_1 = a_2 = \cdots = a_n = 1 \).

\[ \square \]

**P 3.16.** Let \( a_1, a_2, \ldots, a_n \) be nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n \leq n \). If \( k \geq 1 - \frac{1}{n} \) then

\[
\frac{1 - a_1}{1 - a_1 + ka_1^2} + \frac{1 - a_2}{1 - a_2 + ka_2^2} + \cdots + \frac{1 - a_n}{1 - a_n + ka_n^2} \geq 0.
\]

(Vasile C., 2006)

**Solution.** The proof is similar to the one of the preceding P 3.15. For the case 3, we need to show that

\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,
\]

where

\[
f(u) = \frac{1 - u}{1 - u + ku^2}, \quad u \in [0, n].
\]

From

\[
f'(u) = \frac{ku(u - 2)}{(1 - u + ku^2)^2},
\]

it follows that \( f \) is decreasing on \([0, s_0]\) and increasing on \([s_0, n]\), where

\[
s_0 = 2 > s.
\]

We will show that \( f \) is convex on \([1, s_0]\). For \( u \in [1, s_0] \), we have

\[
f''(u) = \frac{2kg(u)}{(1 - u + ku^2)^3}, \quad g(u) = -ku^3 + 3ku^2 - 1.
\]

Since

\[
g'(u) = 3ku(2 - u) \geq 0,
\]

\( g \) is increasing, \( g(1) = 2k - 1 \geq 0 \), hence \( f''(u) \geq 0 \) for \( u \in [1, s_0] \). By the RPCF-Theorem, it suffices to show that

\[
\frac{1 - x}{1 - x + kx^2} + \frac{(n - 1)(1 - y)}{1 - y + ky^2} \geq 0
\]
for $0 \leq x \leq 1 \leq y$ and $x + (n-1)y = n$. Since $(n-1)(1-y) = x-1$, we have

$$
\frac{1-x}{1-x+kx^2} + \frac{(n-1)(1-y)}{1-y+ky^2} = (1-x) \left( \frac{1}{1-x+kx^2} - \frac{1}{1-y+ky^2} \right)
= \frac{(1-x)(y-x)(kx+ky-1)}{(1-x+kx^2)(1-y+ky^2)}
= \frac{n(x-1)^2(kx+ky-1)}{(n-1)(1-x+kx^2)(1-y+ky^2)}.
$$

Since

$$
k(x+y)-1 \geq \frac{n-1}{n}(x+y)-1 = \frac{(n-2)x}{n} \geq 0,
$$

the conclusion follows. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $k = 1 - \frac{1}{n}$, then the equality holds also for

$$
a_1 = 0, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n-1}
$$

(or any cyclic permutation).

\[\square\]

**P 3.17.** Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that $a_1 + a_2 + \cdots + a_n = n$. If $0 < k \leq \frac{n}{n-1}$, then

$$
a_1^{k/a_1} + a_2^{k/a_2} + \cdots + a_n^{k/a_n} \leq n.
$$

(Vasile C., 2006)

**Solution.** According to the power mean inequality, we have

$$
\left( \frac{a_1^{p/a_1} + a_2^{p/a_2} + \cdots + a_n^{p/a_n}}{n} \right)^{1/p} \geq \left( \frac{a_1^{q/a_1} + a_2^{q/a_2} + \cdots + a_n^{q/a_n}}{n} \right)^{1/q}
$$

for all $p \geq q > 0$. Thus, it suffices to prove the desired inequality for

$$
k = \frac{n}{n-1}, \quad 1 < k \leq 2.
$$

Rewrite the desired inequality as

$$
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,
$$

where

$$
f(u) = -u^{k/u}, \quad u \in \mathbb{I} = (0,n).
$$
We have
\[
\begin{align*}
f'(u) &= ku^{k-2} (\ln u - 1), \\
f''(u) &= ku^{k-4} [u + (1 - \ln u)(2u-k+k\ln u)].
\end{align*}
\]
For \(n = 2\), when \(k = 2\) and \(I = (0, 2)\), \(f\) is convex on \([1, 2)\) because
\[
1 - \ln u > 0, \quad 2u - k + k\ln u = 2u - 2 + 2\ln u \geq 2u - 2 \geq 0.
\]
Therefore, we may apply the RHCF-Theorem. Consider now that \(n \geq 3\). From the expression of \(f'\), it follows that \(f\) is decreasing on \((0, s_0]\) and increasing on \([s_0, n)\), where
\[
s_0 = e > 1 = s.
\]
In addition, we claim that \(f\) is convex on \([1, s_0]\). Indeed, since
\[
1 - \ln u \geq 0, \quad 2u - k + k\ln u \geq 2 - k > 0,
\]
we have \(f'' > 0\) for \(u \in [1, s_0]\). Therefore, by the RHCF-Theorem (for \(n = 2\)) and the RPCF-Theorem (for \(n \geq 3\)), we only need to show that
\[
x^{k/x} + (n - 1)y^{k/y} \leq n
\]
for
\[
0 < x \leq 1 \leq y, \quad x + (n - 1)y = n.
\]
We have
\[
k > 1.
\]
Also, from
\[
\frac{k}{y} = \frac{n}{x + (n-1)y} = 1, \quad \frac{k}{y} = \frac{n}{(n-1)y} \leq \frac{2}{y} \leq 2,
\]
we get
\[
0 < \frac{k}{y} - 1 \leq 1.
\]
Therefore, by Bernoulli's inequality, we have
\[
x^{k/x} + (n - 1)y^{k/y} - n = \frac{1}{x^{k/x}} + (n - 1)y \cdot y^{k/y-1} - n
\]
\[
\leq \frac{1}{1 + \frac{k}{x} \left(\frac{1}{x} - 1\right)} + (n - 1)y \left[1 + \left(\frac{k}{y} - 1\right)(y-1)\right] - n
\]
\[
= \frac{x^2}{x^2 - kx + k} - (k - 1)x^2 - (2 - k)x
\]
\[
= \frac{-x(x-1)^2[(k-1)x + k(2-k)]}{x^2 - kx + k} \leq 0.
\]
The proof is completed. The equality holds for \(a_1 = a_2 = \cdots = a_n = 1\). \(\square\)
P 3.18. If \( a, b, c, d, e \) are nonzero real numbers so that \( a + b + c + d + e = 5 \), then
\[
\left( 7 - \frac{5}{a} \right)^2 + \left( 7 - \frac{5}{b} \right)^2 + \left( 7 - \frac{5}{c} \right)^2 + \left( 7 - \frac{5}{d} \right)^2 + \left( 7 - \frac{5}{e} \right)^2 \geq 20.
\]

(Vasile C., 2012)

**Solution.** Write the inequality as
\[
f(a) + f(b) + f(c) + f(d) + f(e) \geq 5f(s), \quad s = \frac{a + b + c + d + e}{5} = 1,
\]
where
\[
f(u) = \left( 7 - \frac{5}{u} \right)^2, \quad u \in I = \mathbb{R} \setminus \{0\}.
\]

From
\[
f'(u) = \frac{10(7u - 5)}{u^3},
\]

it follows that \( f \) is increasing on \((-\infty, 0) \cup [s_0, \infty)\) and decreasing on \((0, s_0]\), where
\[
s_0 = \frac{5}{7} < 1 = s.
\]

Since
\[
\lim_{u \to -\infty} f(u) = 49
\]
and \( f(s_0) = 0 \), we have
\[
\min_{u \in I} f(u) = f(s_0).
\]

Also, \( f \) is convex on \([s_0, s] = [5/7, 1]\) because
\[
f''(u) = \frac{10(15 - 14u)}{u^4} > 0.
\]

According to the LPCF-Theorem and Note 4, we only need to show that
\[
f(x) + 4f(y) \geq 5f(1)
\]
for all nonzero real \( x, y \) so that \( x + 4y = 5 \). Using Note 1, it suffices to prove that \( h(x, y) \geq 0 \), where
\[
h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.
\]

We have
\[
g(u) = 5 \left( \frac{9}{u} - \frac{5}{u^2} \right),
\]
\[
h(x, y) = \frac{5(5x + 5y - 9xy)}{x^2y^2} = \frac{5(6y - 5)^2}{x^2y^2} \geq 0.
\]
In accordance with Note 3, the equality holds for \( a = b = c = d = e = 1 \), and also for
\[
a = \frac{5}{3}, \quad b = c = d = e = \frac{5}{6}
\]
(or any cyclic permutation).

**Remark.** Similarly, we can prove the following generalization:

- Let \( a_1, a_2, \ldots, a_n \) be nonzero real numbers so that \( a_1 + a_2 + \cdots + a_n = n \). If
\[
k = \frac{n}{n + \sqrt{n - 1}},
\]
then
\[
\left( 1 - \frac{k}{a_1} \right)^2 + \left( 1 - \frac{k}{a_2} \right)^2 + \cdots + \left( 1 - \frac{k}{a_n} \right)^2 \geq n(1 - k)^2,
\]
with equality for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for
\[
a_1 = \frac{n}{1 + \sqrt{n - 1}}, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n - 1 + \sqrt{n - 1}}
\]
(or any cyclic permutation).

\[\square\]

**P 3.19.** If \( a_1, a_2, \ldots, a_7 \) are real numbers so that \( a_1 + a_2 + \cdots + a_7 = 7 \), then
\[
(a_1^2 + 2)(a_2^2 + 2) \cdots (a_7^2 + 2) \geq 3^7.
\]

*(Vasile C., 2007)*

**Solution.** Write the inequality as
\[
f(a_1) + f(a_2) + \cdots + f(a_7) \geq 7f(s), \quad s = \frac{a_1 + a_2 + \cdots + a_7}{7} = 1,
\]
where
\[
f(u) = \ln(u^2 + 2), \quad u \in \mathbb{R}.
\]
From
\[
f'(u) = \frac{2u}{u^2 + 2},
\]
it follows that \( f \) is decreasing on \(( -\infty, s_0 ] \) and increasing on \([ s_0, \infty \) ], where
\[
s_0 = 0.
\]
From
\[
f''(u) = \frac{2(2 - u^2)}{(u^2 + 2)^2},
\]
it follows that \( f''(u) > 0 \) for \( u \in [0, 1] \), therefore \( f \) is convex on \([s_0, s]\). By the LPCF-Theorem, it suffices to prove that

\[
f(x) + 6f(y) \geq 7f(1)
\]

for \( x, y \in \mathbb{R} \) so that \( x + 6y = 7 \). The inequality can be written as \( g(y) \geq 0 \), where

\[
g(y) = \ln \left[ \frac{(7 - 6y)^2 + 2}{y^2 + 2} \right] - 7 \ln 3, \quad y \in \mathbb{R}.
\]

From

\[
g'(y) = \frac{4(6y - 7)}{12y^2 - 28y + 17} + \frac{12y}{y^2 + 2}
\]

\[
= \frac{28(6y^3 - 13y^2 + 9y - 2)}{(12y^2 - 28y + 17)(y^2 + 2)}
\]

\[
= \frac{28(3y - 2)(y - 1)}{(12y^2 - 28y + 17)(y^2 + 2)},
\]

it follows that \( g \) is decreasing on \( (-\infty, \frac{1}{2}] \cup \left[ \frac{2}{3}, 1 \right] \) and increasing on \( \left[ \frac{1}{2}, \frac{2}{3} \right] \cup [1, \infty) \); therefore,

\[
g \geq \min\{g(1/2), g(1)\}.
\]

Since \( g(1) = 0 \), we only need to show that \( g(1/2) \geq 0 \); that is, to show that \( x = 4 \) and \( y = 1/2 \) involve

\[
(x^2 + 2)(y^2 + 2)^6 \geq 3^7.
\]

Indeed, we have

\[
(x^2 + 2)(y^2 + 2)^6 - 3^7 = 3^7 \left( \frac{3^7}{2^{11}} - 1 \right) = \frac{139 \cdot 3^7}{2^{11}} > 0.
\]

The equality holds for \( a_1 = a_2 = \cdots = a_7 = 1 \). \( \square \)

**P 3.20.** Let \( a_1, a_2, \ldots, a_n \) be real numbers so that \( a_1 + a_2 + \cdots + a_n = n \). If \( k \geq \frac{n^2}{4(n-1)} \), then

\[
(a_1^2 + k)(a_2^2 + k) \cdots (a_n^2 + k) \geq (1 + k)^n.
\]

*(Vasile C., 2007)*

**Solution.** Write the inequality as

\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,
\]
where
\[ f(u) = \ln(u^2 + k), \quad u \in \mathbb{R}. \]

From
\[ f'(u) = \frac{2u}{u^2 + k}, \]
it follows that \( f \) is decreasing on \((-\infty, s_0]\) and increasing on \([s_0, \infty]\), where
\[ s_0 = 0. \]

From
\[ f''(u) = \frac{2(k-u^2)}{(u^2 + k)^2}, \]
it follows that \( f''(u) \geq 0 \) for \( u \in [0, 1] \), therefore \( f \) is convex on \([s_0, s]\). By the LPCF-Theorem and Note 2, it suffices to prove that
\[
H(x, y) = \frac{f'(x) - f'(y)}{x - y}.
\]

We have
\[
\frac{1}{2} H(x, y) = \frac{k - xy}{(x^2 + k)(y^2 + k)} \geq \frac{n^2 - 4(n-1)xy}{4(n-1)(x^2 + k)(y^2 + k)},
\]
\[
= \frac{[x + (n-1)y]^2 - 4(n-1)xy}{4(n-1)(x^2 + k)(y^2 + k)} \geq 0.
\]
The equality holds for \( a_1 = a_2 = \cdots = a_n = 1. \)

**P 3.21.** If \( a_1, a_2, \ldots, a_{10} \) are real numbers so that \( a_1 + a_2 + \cdots + a_{10} = 10 \), then
\[
(1 - a_1^2)(1 - a_2^2)(1 - a_3^2) \cdots (1 - a_{10}^2) \geq 1.
\]

*(Vasile C., 2006)*

**Solution.** Write the inequality as
\[
f(a_1) + f(a_2) + \cdots + f(a_{10}) \geq 10f(s), \quad s = \frac{a_1 + a_2 + \cdots + a_{10}}{10} = 1
\]
where 
\[ f(u) = \ln(1 - u + u^2), \quad u \in \mathbb{R}. \]

From 
\[ f'(u) = \frac{2u - 1}{1 - u + u^2}, \]
it follows that \( f \) is decreasing on \((-\infty, s_0]\) and increasing on \([s_0, \infty)\), where 
\[ s_0 = \frac{1}{2} < 1 = s. \]

In addition, from 
\[ f''(u) = \frac{1 + 2u(1-u)}{(1-u+u^2)^2}, \]
it follows that \( f''(u) > 0 \) for \( u \in [s_0, 1] \), hence \( f \) is convex on \([s_0, s]\). According to the LPCF-Theorem, we only need to show that 
\[ f(x) + 9f(y) \geq 10f(1) \]
for all real \( x, y \) so that \( x + 9y = 10 \). By Note 2, it suffices to prove that \( H(x, y) \geq 0 \), where 
\[ H(x, y) = \frac{f'(x) - f'(y)}{x - y}. \]

Since 
\[ H(x, y) = \frac{1 + x + y - 2xy}{(1 - x + x^2)(1 - y + y^2)}, \]
and 
\[ 1 + x + y - 2xy = 18y^2 - 28y + 11 = 2\left(3y - \frac{7}{3}\right)^2 + \frac{1}{9} > 0, \]
the conclusion follows. The equality holds for \( a_1 = a_2 = \cdots = a_{10} = 1 \).

**Remark.** By replacing \( a_1, a_2, \ldots, a_{10} \) respectively with \( 1 - a_1, 1 - a_2, \ldots, 1 - a_{10} \), we get the following statement:

- If \( a_1, a_2, \ldots, a_{10} \) are real numbers so that \( a_1 + a_2 + \cdots + a_{10} = 0 \), then 
  \[ (1 - a_1 + a_1^2)(1 - a_2 + a_2^2) \cdots (1 - a_{10} + a_{10}^2) \geq 1, \]
with equality for \( a_1 = a_2 = \cdots = a_n = 0 \). \( \square \)

**P 3.22.** If \( a, b, c \) are nonnegative real numbers so that \( a + b + c = 3 \), then 
\[ (1 - a + a^4)(1 - b + b^4)(1 - c + c^4) \geq 1. \]
Solution. Write the inequality as

\[ f(a) + f(b) + f(c) \geq 3f(s), \quad s = \frac{a + b + c}{3} = 1, \]

where

\[ f(u) = \ln(1 - u + u^4), \quad u \in [0, 3]. \]

From

\[ f'(u) = \frac{4u^3 - 1}{1 - u + u^4}, \]

it follows that \( f \) is decreasing on \([0, s_0]\) and increasing on \([s_0, 3]\), where

\[ s_0 = \frac{1}{\sqrt[4]{4}} < 1 = s. \]

Also, \( f \) is convex on \([s_0, 1]\) because

\[ f''(u) = \frac{-4u^6 - 4u^3 + 12u^2 - 1}{(1 - u + u^4)^2} \geq \frac{-4u^6 - 4u^2 + 12u^2 - 1}{(1 - u + u^4)^2} = \frac{4u^2 - 1}{(1 - u + u^4)^2} > 0. \]

According to the LPCF-Theorem, we only need to show that

\[ f(x) + 2f(y) \geq 3f(1) \]

for all \( x, y \geq 0 \) so that \( x + 2y = 3 \). Using Note 2, it suffices to prove that \( H(x, y) \geq 0 \), where

\[ H(x, y) = \frac{f'(x) - f'(y)}{x - y}. \]

We have

\[ H(x, y) = \frac{(x + y)(x - y)^2 - 1 + 4(x^2 + y^2 + xy) - 2xy(x + y) - 4x^3y^3}{(1 - x + x^4)(1 - y + y^4)} \]

\[ \geq \frac{-1 + 4(x^2 + y^2 + xy) - 2xy(x + y) - 4x^3y^3}{(1 - x + x^4)(1 - y + y^4)} \]

\[ = \frac{h(x, y)}{(1 - x + x^4)(1 - y + y^4)}, \]

where

\[ h(x, y) = -1 + 2(x + y)[2(x + y) - xy] - 4xy - 4x^3y^3. \]

From \( 3 = x + 2y \geq 2\sqrt{2xy} \) and \((1 - x)(1 - y) \leq 0\), we get

\[ xy \leq \frac{9}{8}, \quad x + y \geq 1 + xy. \]

Therefore,

\[ h(x, y) \geq -1 + 2(1 + xy)[2(1 + xy) - xy] - 4xy - 4x^3y^3 \]

\[ = 3 + 2xy + 2x^2y^2 - 4x^3y^3 \geq 3 + 2xy + 2x^2y^2 - 5x^2y^2 \]

\[ = 3 + 2xy - 3x^2y^2 \geq 3 + 2xy - 4xy = 3 - 2xy > 0. \]

The proof is completed. The equality holds for \( a = b = c = 1 \). \( \square \)
P 3.23. If $a, b, c, d$ are nonnegative real numbers so that $a + b + c + d = 4$, then

$$(1 - a + a^3)(1 - b + b^3)(1 - c + c^3)(1 - d + d^3) \geq 1.$$ 

(Vasile C., 2012)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a + b + c + d}{4} = 1,$$

where

$$f(u) = \ln(1 - u + u^3), \quad u \in [0, 4].$$

From

$$f'(u) = \frac{3u^2 - 1}{1 - u + u^3},$$

it follows that $f$ is decreasing on $[0, s_0]$ and increasing on $[s_0, 4]$, where

$$s_0 = \frac{1}{\sqrt{3}} < 1 = s.$$

In addition, $f$ is convex on $[s_0, 1]$ because

$$f''(u) = \frac{-3u^4 + 6u - 1}{(1 - u + u^3)^2} \geq \frac{-3u + 6u - 1}{(1 - u + u^3)^2} = \frac{3u - 1}{(1 - u + u^3)^2} > 0.$$

According to the LPCF-Theorem, we only need to show that

$$f(x) + 3f(y) \geq 4f(1)$$

for all $x, y \geq 0$ so that $x + 3y = 4$. Using Note 2, it suffices to prove that $H(x, y) \geq 0$, where

$$H(x, y) = \frac{f'(x) - f'(y)}{x - y}.$$

We have

$$H(x, y) = \frac{(x - y)^2 + 3(x + y) - 1 - 3x^2y^2}{(1 - x + x^3)(1 - y + y^3)} \geq \frac{3(x + y) - 1 - 3x^2y^2}{(1 - x + x^3)(1 - y + y^3)}.$$

From $4 = x + 3y \geq 2\sqrt{3xy}$ and $(1 - x)(1 - y) \leq 0$, we get

$$xy \leq \frac{4}{3}, \quad x + y \geq 1 + xy.$$

Therefore,

$$3(x + y) - 1 - 3x^2y^2 \geq 3(1 + xy) - 1 - 3x^2y^2 \geq 3(1 + xy) - 1 - 4xy = 2 - xy > 0,$$

hence $H(x, y) > 0$. The equality holds for $a = b = c = d = 1$. 

\[ \square]
**P 3.24.** If $a, b, c, d, e$ are nonzero real numbers so that $a + b + c + d + e = 5$, then

$$5 \left( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{e^2} \right) + 45 \geq 14 \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} \right).$$

(Vasile C., 2013)

**Solution.** Write the desired inequality as

$$f(a) + f(b) + f(c) + f(d) + f(e) \geq 5f(s), \quad s = \frac{a + b + c + d + e}{5} = 1,$$

where

$$f(u) = \frac{5}{u^2} - \frac{14}{u} + 9, \quad u \in \mathbb{I} = \mathbb{R} \setminus \{0\}.$$

From

$$f'(u) = \frac{2(7u - 5)}{u^3},$$

it follows that $f$ is increasing on $(-\infty, 0) \cup [s_0, \infty)$ and decreasing on $(0, s_0]$, where

$$s_0 = \frac{5}{7} < 1 = s.$$

Since

$$\lim_{u \to -\infty} f(u) = 9$$

and $f(s_0) < f(1) = 0$, we have

$$\min_{u \in \mathbb{I}} f(u) = f(s_0).$$

From

$$f''(u) = \frac{2(15 - 14u)}{u^4},$$

it follows that $f$ is convex on $[s_0, 1]$. By the LPCF-Theorem, Note 4 and Note 1, it suffices to show that $h(x, y) \geq 0$ for all $x, y \in \mathbb{I}$ which satisfy $x + 4y = 5$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

Indeed, we have

$$g(u) = \frac{9}{u} - \frac{5}{u^2},$$

$$h(x, y) = \frac{5x + 5y - 9xy}{x^2y^2} = \frac{(6y - 5)^2}{x^2y^2} \geq 0.$$

In accordance with Note 3, the equality holds for $a = b = c = d = e = 1$, and also for

$$a = \frac{5}{3}, \quad b = c = d = e = \frac{5}{6}$$

(or any cyclic permutation).
P 3.25. If $a$, $b$, $c$ are positive real numbers so that $abc = 1$, then

$$\frac{7 - 6a}{2 + a^2} + \frac{7 - 6b}{2 + b^2} + \frac{7 - 6c}{2 + c^2} \geq 1.$$  

(Vasile C., 2008)

Solution. Using the substitution

$$a = e^x, \ b = e^y, \ c = e^z,$$

we need to show that

$$f(x) + f(y) + f(z) \geq 3f(s),$$

where

$$s = \frac{x + y + z}{3} = 0$$

and

$$f(u) = \frac{7 - 6e^u}{2 + e^{2u}}, \ u \in \mathbb{R}.$$  

From

$$f'(u) = \frac{2(3e^u + 2)(e^u - 3)}{(2 + e^{2u})^2},$$

it follows that $f$ is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \ln 3 > s.$$  

We have

$$f''(u) = \frac{2t \cdot h(t)}{(2 + t^2)^3}, \quad h(t) = -3t^4 + 14t^3 + 36t^2 - 28t - 12, \quad t = e^u.$$  

We will show that $h(t) > 0$ for $t \in [1, 3]$, hence $f$ is convex on $[0, s_0]$. We have

$$h(t) = 3(t^2 - 1)(9 - t^2) + 14t^3 + 6t^2 - 28t + 15$$

$$\geq 14t^3 + 6t^2 - 28t + 15$$

$$= 14t^2(t - 1) + 14(t - 1)^2 + 6t^2 + 1 > 0.$$  

By the RPCF-Theorem, we only need to prove that

$$f(x) + 2f(y) \geq 3f(0)$$

for all real $x$, $y$ so that $x + 2y = 0$. That is, to show that the original inequality holds for $b = c$ and $a = 1/c^2$. Write this inequality as

$$\frac{c^2(7c^2 - 6)}{2c^4 + 1} + \frac{2(7 - 6c)}{2 + c^2} \geq 1,$$
\[(c - 1)^2(c - 2)^2(5c^2 + 6c + 3) \geq 0.\]

By Note 3, the equality holds for \(a = b = c = 1\), and also for

\[a = \frac{1}{4}, \quad b = c = 2\]

(or any cyclic permutation).

\[\square\]

**P 3.26.** If \(a, b, c\) are positive real numbers so that \(abc = 1\), then

\[\frac{1}{a + 5bc} + \frac{1}{b + 5ca} + \frac{1}{c + 5ab} \leq \frac{1}{2}.\]

*(Vasile C., 2008)*

**Solution.** Write the inequality as

\[\frac{a}{a^2 + 5} + \frac{b}{b^2 + 5} + \frac{c}{c^2 + 5} \leq \frac{1}{2}.\]

Using the substitution

\[a = e^x, \quad b = e^y, \quad c = e^z,\]

we need to show that

\[f(x) + f(y) + f(z) \geq 3f(s),\]

where

\[s = \frac{x + y + z}{3} = 0\]

and

\[f(u) = \frac{-e^u}{e^{2u} + 5}, \quad u \in \mathbb{R}.\]

From

\[f'(u) = \frac{e^u(e^{2u} - 5)}{(e^{2u} + 5)^2},\]

it follows that \(f\) is decreasing on \((-\infty, s_0]\) and increasing on \([s_0, \infty)\), where

\[s_0 = \frac{\ln 5}{2} > 0 = s.\]

Also, from

\[f''(u) = \frac{e^u(-e^{4u} + 30e^{2u} - 25)}{(e^{2u} + 5)^3},\]

it follows that \(f\) is convex on \([s, s_0]\), because \(u \in [0, s_0]\) involves \(e^u \in [1, \sqrt{5}]\) and \(e^{2u} \in [1, 5]\), hence

\[-e^{4u} + 30e^{2u} - 25 = e^{2u}(5 - e^{2u}) + 25(e^{2u} - 1) > 0.\]
By the RPCF-Theorem, we only need to prove the original inequality for $b = c$ and $a = 1/c^2$. Write this inequality as

$$\frac{c^2}{5c^4 + 1} + \frac{2c}{c^2 + 5} \leq \frac{1}{2},$$

$$(c - 1)^2(5c^4 - 10c^3 - 2c^2 + 6c + 5) \geq 0,$$

$$(c - 1)^2[5(c - 1)^4 + 2c(5c^2 - 16c + 13)] \geq 0.$$

The equality holds for $a = b = c = 1$.

\[\Box\]

**P 3.27.** If $a, b, c$ are positive real numbers so that $abc = 1$, then

$$\frac{1}{4 - 3a + 4a^2} + \frac{1}{4 - 3b + 4b^2} + \frac{1}{4 - 3c + 4c^2} \leq \frac{3}{5}.$$

*(Vasile Cirtoaje, 2008)*

**Solution.** Let $a = e^x, \ b = e^y, \ c = e^z$.

We need to show that

$$f(x) + f(y) + f(z) \geq 3f(s),$$

where

$$s = \frac{x + y + z}{3} = 0$$

and

$$f(u) = \frac{-1}{4 - 3e^u + 4e^{2u}}, \quad u \in \mathbb{R}.$$  

From

$$f'(u) = \frac{e^u(8e^u - 3)}{(4 - 3e^u + 4e^{2u})^2},$$

it follows that $f$ is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \ln \frac{3}{8} < 0 = s.$$  

We claim that $f$ is convex on $[s_0, 0]$. Since

$$f''(u) = \frac{e^u(-64e^{3u} + 36e^{2u} + 55e^u - 12)}{(4 - 3e^u + 4e^{2u})^3},$$

we need to show that

$$-64t^3 + 36t^2 + 55t - 12 \geq 0,$$
where

\[ t = e^u \in \left[ \frac{3}{8}, 1 \right]. \]

Indeed, we have

\[
-64t^3 + 36t^2 + 55t - 12 > -72t^3 + 36t^2 + 48t - 12 = 12(1 - t)(6t^2 + 3t - 1) \geq 0.
\]

By the LPCF-Theorem, we only need to prove the original inequality for \( b = c \) and 
\( a = 1/c^2 \). Write this inequality as follows:

\[
\frac{c^4}{4c^4 - 3c^2 + 4} + \frac{2}{4 - 3c + 4c^2} \leq \frac{3}{5},
\]

\[
28c^6 - 21c^5 - 48c^4 + 27c^3 + 42c^2 - 36c + 8 \geq 0,
\]

\[
(c - 1)^2(28c^4 + 35c^3 - 6c^2 - 20c + 8) \geq 0.
\]

It suffices to show that

\[ 7(4c^4 + 5c^3 - c^2 - 3c + 1) \geq 0. \]

Indeed,

\[ 4c^4 + 5c^3 - c^2 - 3c + 1 = c^2(2c - 1)^2 + 9c^3 - 2c^2 - 3c + 1 \]

and

\[ 9c^3 - 2c^2 - 3c + 1 = c(3c - 1)^2 + (2c - 1)^2 > 0. \]

The equality holds for \( a = b = c = 1 \).

Remark. Since

\[
\frac{1}{4 - 3a + 4a^2} \geq \frac{1}{4 - 3a + 4a^2 + (1 - a)^2} = \frac{1}{5(1 - a + a^2)},
\]

we get the following known inequality

\[
\frac{1}{1 - a + a^2} + \frac{1}{1 - b + b^2} + \frac{1}{1 - c + c^2} \leq 3.
\]

\[ \square \]

**P 3.28.** If \( a, b, c \) are positive real numbers so that \( abc = 1 \), then

\[
\frac{1}{(3a + 1)(3a^2 - 5a + 3)} + \frac{1}{(3b + 1)(3b^2 - 5b + 3)} + \frac{1}{(3c + 1)(3c^2 - 5c + 3)} \leq \frac{3}{4}.
\]
Solution. Let

\[ a = e^x, \quad b = e^y, \quad c = e^z. \]

We need to show that

\[ f(x) + f(y) + f(z) \geq 3f(s), \]

where

\[ s = \frac{x + y + z}{3} = 0 \]

and

\[ f(u) = \frac{-1}{(3e^u + 1)(3e^{2u} - 5e^u + 3)}, \quad u \in \mathbb{R}. \]

From

\[ f'(u) = \frac{(3e^u - 2)(9e^u - 2)}{(3e^u + 1)^2(3e^{2u} - 5e^u + 3)^2}, \]

it follows that \( f \) is increasing on \((-\infty, s_1] \cup [s_0, \infty)\) and decreasing on \([s_1, s_0]\), where

\[ s_1 = \ln 2 - \ln 9, \quad s_0 = \ln 2 - \ln 3, \quad s_1 < s_0 < 0 = s. \]

Since

\[ \lim_{u \to -\infty} f(u) = f(s_0) = \frac{-1}{3}, \]

we get

\[ \min_{u \in \mathbb{R}} f(u) = f(s_0). \]

We claim that \( f \) is convex on \([s_0, 0]\). We have

\[ f''(u) = \frac{t \cdot h(t)}{(3t + 1)^3(3t^2 - 5t + 3)^3}, \]

where

\[ t = e^u \in \left[ \frac{2}{3}, 1 \right], \quad h(t) = -729t^5 + 1188t^4 - 648t^3 + 387t^2 - 160t + 12. \]

Since the polynomial \( h(t) \) has the real roots

\[ t_1 \approx 0.0933, \quad t_2 \approx 0.5072, \quad t_3 \approx 1.11008, \]

it follows that \( h(t) > 0 \) for \( t \in [2/3, 1] \subset [t_2, t_3] \), hence \( f \) is convex on \([s_0, 0]\). By the LPCF-Theorem, we only need to prove the original inequality for \( b = c \leq 1 \) and \( a = 1/c^2 \). Write this inequality as follows:

\[ \frac{c^6}{(c^2 + 3)(3c^4 - 5c^2 + 3)} + \frac{2}{(3c + 1)(3c^2 - 5c + 3)} \leq \frac{3}{4}. \]

Since

\[ c^2 + 3 \geq 2(c + 1) \]
and
\[3c^4 - 5c^2 + 3 \geq c(3c^2 - 5c + 3),\]
it suffices to prove that
\[\frac{c^5}{2(c+1)(3c^2 - 5c + 3)} + \frac{2}{(3c + 1)(3c^2 - 5c + 3)} \leq \frac{3}{4}.\]
This is equivalent to the obvious inequality
\[(1 - c)^2(1 + 15c + 5c^2 - 14c^3 - 6c^4) \geq 0.\]
The equality holds for \(a = b = c = 1.\)

\[\square\]

**P 3.29.** Let \(a_1, a_2, \ldots, a_n (n \geq 3)\) be positive real numbers so that \(a_1 a_2 \cdots a_n = 1.\) If \(p, q \geq 0\) so that \(p + 4q \geq n - 1,\) then
\[
\frac{1 - a_1}{1 + pa_1 + qa_1^2} + \frac{1 - a_2}{1 + pa_2 + qa_2^2} + \cdots + \frac{1 - a_n}{1 + pa_n + qa_n^2} \geq 0.
\]

*(Vasile C., 2008)*

**Solution.** For \(q = 0,\) we get a known inequality (see Remark 2 from the proof of P 1.62). Consider further that \(q > 0.\) Using the substitutions \(a_i = e^{x_i}\) for \(i = 1, 2, \ldots, n,\) we need to show that
\[f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s),\]
where
\[s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0\]
and
\[f(u) = \frac{1 - e^u}{1 + pe^u + qe^{2u}}, \quad u \in \mathbb{R}.
\]
From
\[f'(t) = \frac{e^u(qe^{2u} - 2qe^u - p - 1)}{(1 + pe^u + qe^{2u})^2},\]
it follows that \(f\) is decreasing on \((-\infty, s_0]\) and increasing on \([s_0, \infty),\) where
\[s_0 = \ln r_0 > 0 = s, \quad r_0 = 1 + \sqrt{1 + \frac{p + 1}{q}}.
\]
Also, we have
\[f''(u) = \frac{t \cdot h(t)}{(1 + pt + qt^2)^3},\]
where
\[ h(t) = -q^2 t^4 + q(p + 4q)t^3 + 3q(p + 2)t^2 + (p - 4q + p^2)t - p - 1, \quad t = e^u. \]
We will show that \( h(t) \geq 0 \) for \( t \in [1, r_0] \), hence \( f \) is convex on \([0, s_0]\). We have
\[ h'(t) = -4q^2 t^3 + 3q(p + 4q)t^2 + 6q(p + 2)t + p - 4q + p^2, \]
\[ h''(t) = 6q[-2qt^2 + (p + 4q)t + p + 2]. \]
Since
\[ h''(t) = 6q[2(-qt^2 + 2qt + p + 1) + p(t - 1)] \geq 12q(-qt^2 + 2qt + p + 1) \geq 0, \]
\( h'(t) \) is increasing,
\[ h'(t) \geq h'(1) = p^2 + 9pq + 8q^2 + p + 8q > 0, \]
h is increasing, hence
\[ h(t) \geq h(1) = p^2 + 4pq + 3q^2 + 2q - 1 = (p + 2q)^2 - (q - 1)^2 = (p + q + 1)(p + 3q - 1). \]
Since
\[ p + 3q - 1 \geq p + 3q - \frac{p + 4q}{n - 1} = \frac{p + 2q}{2} > 0, \]
\( f''(u) > 0 \) for \( u \in [0, s_0] \), therefore \( f \) is convex on \([s, s_0]\). By the RPCF-Theorem, we only need to prove the original inequality for
\[ a_2 = \cdots = a_n := t, \quad a_1 = 1/t^{n-1}, \quad t \geq 1. \]
Write this inequality as
\[ \frac{t^{n-1}(t^{n-1} - 1)}{t^{2n-2} + pt^{n-1} + q} + \frac{(n-1)(1-t)}{1 + pt + qt^2} \geq 0, \]
or
\[ pA + qB \geq C, \]
where
\[ A = t^{n-1}(t^n - nt + n - 1), \]
\[ B = t^{2n} - t^{n+1} - (n-1)(t - 1), \]
\[ C = t^{n-1}[(n-1)t^n + 1 - nt^{n-1}]. \]
Since \( p + 4q \geq n - 1 \) and \( C \geq 0 \) (by the AM-GM inequality applied to \( n \) positive numbers), it suffices to show that
\[ pA + qB \geq \frac{(p + 4q)C}{n - 1}, \]
which is equivalent to
\[ p[(n - 1)A - C] + q[(n - 1)B - 4C] \geq 0. \]

This is true if
\[ (n - 1)A - C \geq 0 \]
and
\[ (n - 1)B - 4C \geq 0 \]
for \( t \geq 1. \) By the AM-GM inequality, we have
\[ (n - 1)A - C = nt^{n-1}[t^{n-1} + n - 2 - (n - 1)t] \geq 0. \]

For \( n = 3, \) we have
\[
B = (t - 1)^2(t^4 + 2t^3 + 2t^2 + 2t + 2), \\
C = t^2(t - 1)^2(2t + 1), \\
B - 2C = (t - 1)^2(t^4 - 2t^3 + 2t + 2) \\
\quad = (t - 1)^2[(t - 1)^2(t^2 - 1) + 3] \geq 0.
\]

Consider further that \( n \geq 4. \)

Since
\[ t - 1 \leq t^{n-1}(t - 1), \]
we have
\[
B \geq t^{2n} - t^{n+1} - (n - 1)t^{n-1}(t - 1) \\
\quad = t^{n-1}[t^{n+1} - t^2 - (n - 1)t + n - 1].
\]

Thus, the inequality \( (n - 1)B - 4C \geq 0 \) is true if
\[ (n - 1)[t^{n+1} - t^2 - (n - 1)t + n - 1] - 4(n - 1)t^n - 4 - 4nt^{n-1} \geq 0, \]
which is equivalent to \( g(t) \geq 0, \) where
\[
g(t) = (n - 1)t^{n+1} - 4(n - 1)t^n + 4nt^{n-1} - (n - 1)t^2 - (n - 1)^2t + n^2 - 2n - 3.
\]

We have
\[
g'(t) = (n - 1)g_1(t), \quad g_1(t) = (n + 1)t^n - 4nt^{n-1} + 4nt^{n-2} - 2t - n + 1, \\
g_1'(t) = n(n + 1)t^{n-1} - 4n(n - 1)t^{n-2} + 4n(n - 2)t^{n-3} - 2.
\]

Since
\[ n(n + 1)t^{n-1} + 4n(n - 2)t^{n-3} \geq 4n\sqrt{(n + 1)(n - 2)}t^{n-2}, \]
we get
\[ g'(t) \geq 4n \left[ \sqrt{(n+1)(n-2)} - n + 1 \right] t^{n-2} - 2 \]
\[ \geq 4n \left[ \sqrt{(n+1)(n-2)} - n + 1 \right] - 2 \]
\[ = \frac{4n(n-3)}{\sqrt{(n+1)(n-2) + n - 1}} - 2 \]
\[ > \frac{4n(n-3)}{(n+1) + n - 1} - 2 = 2(n-4) \geq 0. \]

Therefore, \( g_1(t) \) is increasing for \( t \geq 1 \), \( g_1(t) \geq g_1(1) = 0 \), \( g(t) \) is increasing for \( t \geq 1 \), hence
\[ g(t) \geq g(1) = 0. \]

The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \).

**Remark.** For \( p = 0 \) and \( q = 1 \), we get the inequality (Vasile C., 2006)
\[ \frac{1 - a}{1 + a^2} + \frac{1 - b}{1 + b^2} + \frac{1 - c}{1 + c^2} + \frac{1 - d}{1 + d^2} + \frac{1 - e}{1 + e^2} \geq 0, \]
where \( a, b, c, d, e \) are positive real numbers so that \( abcde = 1 \). Replacing \( a, b, c, d, e \) by \( 1/a, 1/b, 1/c, 1/d, 1/e \), we get
\[ \frac{1 + a}{1 + a^2} + \frac{1 + b}{1 + b^2} + \frac{1 + c}{1 + c^2} + \frac{1 + d}{1 + d^2} + \frac{1 + e}{1 + e^2} \leq 5, \]
where \( a, b, c, d, e \) are positive real numbers so that \( abcde = 1 \).

Notice that the inequality
\[ \frac{1 - a_1}{1 + a_1^2} + \frac{1 - a_2}{1 + a_2^2} + \frac{1 - a_3}{1 + a_3^2} + \frac{1 - a_4}{1 + a_4^2} + \frac{1 - a_5}{1 + a_5^2} + \frac{1 - a_6}{1 + a_6^2} \geq 0 \]
is not true for all positive numbers \( a_1, a_2, a_3, a_4, a_5, a_6 \) satisfying \( a_1a_2a_3a_4a_5a_6 = 1 \). Indeed, for \( a_2 = a_3 = a_4 = a_5 = a_6 = 2 \), the inequality becomes
\[ \frac{1 - a_1}{1 + a_1^2} - 1 \geq 0, \]
which is false for \( a_1 > 0 \).

\( \square \)

**P 3.30.** If \( a, b, c \) are positive real numbers so that \( abc = 1 \), then
\[ \frac{1 - a}{17 + 4a + 6a^2} + \frac{1 - b}{17 + 4b + 6b^2} + \frac{1 - c}{17 + 4c + 6c^2} \geq 0. \]

*(Vasile C., 2008)*
**Solution.** Using the substitution

\[ a = e^x, \ b = e^y, \ c = e^z, \]

we need to show that

\[ f(x) + g(y) + g(z) \geq 3f(s), \]

where

\[ s = \frac{x + y + z}{3} = 0 \]

and

\[ f(u) = \frac{1 - e^u}{1 + pe^u + qe^{2u}}, \quad u \in \mathbb{R}, \]

with

\[ p = \frac{4}{17}, \quad q = \frac{6}{17}. \]

As we have shown in the proof of the preceding P 3.29, \( f \) is decreasing on \( (-\infty, s_0] \) and increasing on \([s_0, \infty)\), where

\[ s_0 = \ln r_0 > 0 = s, \quad r_0 = 1 + \sqrt{1 + \frac{p + 1}{q}} = 1 + \sqrt{\frac{9}{2}}. \]

In addition, since \( p + 3q - 1 = \frac{5}{17} > 0 \) (see the proof of P 3.29), \( f \) is convex on \([0, s_0]\). By the RPCF-Theorem, we only need to prove the original inequality for \( b = c \geq 1 \) and \( a = 1/c^2 \). Write this inequality as follows:

\[ \frac{c^2(c^2 - 1)}{c^4 + pc^2 + q} + \frac{2(1 - c)}{1 + pc + qc^2} \geq 0, \]

\[ pA + qB \geq C, \]

where

\[ A = c^2(c - 1)^2(c + 2), \]

\[ B = (c - 1)^2(c^4 + 2c^3 + 2c^2 + 2c + 2), \]

\[ C = c^2(c - 1)^2(2c + 1). \]

Indeed, we have

\[ pA + qB - C = \frac{3(c - 1)^2(c - 2)^2(2c^2 + 2c + 1)}{17} \geq 0. \]

In accordance with Note 3, the equality holds for \( a = b = c = 1 \), and also for

\[ a = \frac{1}{4}, \quad b = c = 2 \]

(or any cyclic permutation). 

☐
**P 3.31.** If \(a_1, a_2, \ldots, a_8\) are positive real numbers so that \(a_1a_2\cdots a_8 = 1\), then

\[
\frac{1-a_1}{(1+a_1)^2} + \frac{1-a_2}{(1+a_2)^2} + \cdots + \frac{1-a_8}{(1+a_8)^2} \geq 0.
\]

*(Vasile C., 2006)*

**Solution.** Using the substitutions \(a_i = e^{x_i}\) for \(i = 1, 2, \ldots, 8\), we need to show that

\[
f(x_1) + f(x_2) + \cdots + f(x_8) \geq 8f(s),
\]

where

\[
s = \frac{x_1 + x_2 + \cdots + x_8}{8} = 0
\]

and

\[
f(u) = \frac{1-e^u}{(1+e^u)^2}, \quad u \in \mathbb{R}.
\]

From

\[
f'(t) = \frac{e^u(e^u-3)}{(1+e^u)^3},
\]

it follows that \(f\) is decreasing on \((-\infty, s_0]\) and increasing on \([s_0, \infty)\), where

\[
s_0 = \ln 3 > 1 = s.
\]

We have

\[
f''(u) = \frac{e^u(8e^u - e^{2u} - 3)}{(1+e^u)^4}.
\]

For \(u \in [0, \ln 3]\), that is \(e^u \in [1, 3]\), we have

\[
8e^u - e^{2u} - 3 > 8e^u - 3e^u - 7 = (e^u - 1)(7 - e^u) \geq 0;
\]

therefore, \(f\) is convex on \([s, s_0]\). By the RPCF-Theorem, we only need to prove the original inequality for \(a_2 = \cdots = a_8 := t\) and \(a_1 = 1/t^7\), where \(t \geq 1\). For the nontrivial case \(t > 1\), write this inequality as follows:

\[
t^7(t^7 - 1) \geq 7(t-1) \quad \frac{(t^7+1)^2}{(t+1)^2},
\]

\[
t^7(t^7-1)(t+1)^2 \geq 7, \quad \frac{t^7(t^7+1)^2}{(t-1)(t^7+1)} \geq 7,
\]

\[
\frac{t^7(t^6 + t^5 + t^4 + t^3 + t^2 + t + 1)}{(t^6 - t^5 + t^4 - t^3 + t^2 - t + 1)^2} \geq 7.
\]

Since

\[
t^6 - t^5 + t^4 - t^3 + t^2 - t + 1 = t^4(t^2-t+1)-(t-1)(t^2+1) < t^4(t^2-t+1),
\]

\[
\frac{t^7(t^6 + t^5 + t^4 + t^3 + t^2 + t + 1)}{(t^6 - t^5 + t^4 - t^3 + t^2 - t + 1)^2} \geq 7.
\]
it suffices to show that
\[
\frac{t^6 + t^5 + t^4 + t^3 + t^2 + t + 1}{t(t^2 - t + 1)^2} \geq 7,
\]
which is equivalent to the obvious inequality
\[(t - 1)^6 \geq 0.
\]
Thus, the proof is completed. The equality holds for \(a_1 = a_2 = \cdots = a_8 = 1\).

Remark. The inequality
\[
\frac{1 - a_1}{(1 + a_1)^2} + \frac{1 - a_2}{(1 + a_2)^2} + \ldots + \frac{1 - a_9}{(1 + a_9)^2} \geq 0
\]
is not true for all positive numbers \(a_1, a_2, \ldots, a_9\) satisfying \(a_1 a_2 \cdots a_9 = 1\). Indeed, for \(a_2 = a_3 = \cdots = a_9 = 3\), the inequality becomes
\[
\frac{1 - a_1}{(1 + a_1)^2} - 1 \geq 0,
\]
which is false for \(a_1 > 0\).

\[\square\]

**P 3.32.** Let \(a, b, c\) be positive real numbers so that \(abc = 1\). If \(k \in \left[\frac{-13}{3\sqrt{3}}, \frac{13}{3\sqrt{3}}\right]\), then
\[
\frac{a + k}{a^2 + 1} + \frac{b + k}{b^2 + 1} + \frac{c + k}{c^2 + 1} \leq \frac{3(1 + k)}{2}.
\]

(\textit{Vasile C., 2012})

**Solution.** The inequality is equivalent to
\[
k \left( \sum \frac{1}{a^2 + 1} - \frac{3}{2} \right) \leq \sum \left( \frac{1}{2} - \frac{a}{a^2 + 1} \right),
\]
\[
\sum \frac{(a - 1)^2}{a^2 + 1} \geq k \left( \sum \frac{2}{a^2 + 1} - 3 \right). \quad (*)
\]
Thus, it suffices to prove it for \(|k| = \frac{13}{3\sqrt{3}}\). On the other hand, replacing \(a, b, c\) by \(1/a, 1/b, 1/c\), the inequality becomes
\[
\sum \frac{(a - 1)^2}{a^2 + 1} \geq k \left( 3 - \sum \frac{2}{a^2 + 1} \right). \quad (**)
Based on (\(\ast\)) and (\(\ast\ast\)), we only need to prove the desired inequality for

\[ k = \frac{13}{3\sqrt{3}}. \]

Using the substitution

\[ a = e^x, \quad b = e^y, \quad c = e^z, \]

we need to show that

\[ f(x) + g(y) + g(z) \geq 3f(s), \]

where

\[ s = \frac{x + y + z}{3} = 0 \]

and

\[ f(u) = \frac{-e^u - k}{e^{2u} + 1}, \quad u \in \mathbb{R}. \]

From

\[ f'(t) = \frac{e^{2u} + 2ke^u - 1}{(e^{2u} + 1)^2}, \]

it follows that \( f \) is decreasing on \((-\infty, s_0]\) and increasing on \([s_0, \infty)\), where

\[ s_0 = \ln r_0 < 0 = s, \quad r_0 = -k + \sqrt{k^2 + 1} = \frac{1}{3\sqrt{3}}. \]

Also, we have

\[ f''(u) = \frac{t \cdot h(t)}{(1 + t^2)^3}, \]

where

\[ h(t) = -t^4 - 4kt^3 + 6t^2 + 4kt - 1, \quad t = e^u. \]

We will show that \( h(t) > 0 \) for \( t \in [r_0, 1] \), hence \( f \) is convex on \([s_0, s]\). Indeed, since

\[ 4kt = \frac{52t}{3\sqrt{3}} \geq \frac{52}{27} > 1, \]

we have

\[ h(t) = -t^4 + 6t^2 - 1 + 4kt(1 - t^2) \geq -t^4 + 6t^2 - 1 + (1 - t^2) = t^2(5 - t^2) > 0. \]

By the LPCF-Theorem, we only need to prove the original inequality for \( b = c := t \) and \( a = 1/t^2 \), where \( t > 0 \). Write this inequality as

\[ \frac{t^2(kt^2 + 1)}{t^4 + 1} + \frac{2(t + k)}{t^2 + 1} \leq \frac{3(1 + k)}{2}, \]

\[ 3t^6 - 4t^5 + t^4 + t^2 - 4t + 3 - k(1 - t^2)^3 \geq 0, \]

\[ (t - 1)^2[(3 + k)t^4 + 2(1 + k)t^3 + 2t^2 + 2(1 - k)t + 3 - k] \geq 0, \]
\[(t - 1)^2 \left( t - 2 + \sqrt{3} \right)^2 \left[ (27 + 13\sqrt{3})t^2 + 24(2 + \sqrt{3})t + 33 + 17\sqrt{3} \right] \geq 0.\]

The equality holds for \( a = b = c = 1 \). If \( k = \frac{13}{3\sqrt{3}} \), then the equality holds also for
\[a = 7 + 4\sqrt{3}, \quad b = c = 2 - \sqrt{3}\]
(or any cyclic permutation). If \( k = \frac{-13}{3\sqrt{3}} \), then the equality holds also for
\[a = 7 - 4\sqrt{3}, \quad b = c = 2 + \sqrt{3}\]
(or any cyclic permutation).

\[\Box\]

**P 3.33.** If \( a, b, c \) are positive real numbers and \( 0 < k \leq 2 + 2\sqrt{2} \), then
\[\frac{a^3}{ka^2 + bc} + \frac{b^3}{kb^2 + ca} + \frac{c^3}{kc^2 + ab} \geq \frac{a + b + c}{k + 1}.\]

*(Vasile C., 2011)*

**Solution.** Due to homogeneity, we may assume that \( abc = 1 \). On this hypothesis, we write the inequality as follows:
\[\frac{a^4}{ka^3 + 1} + \frac{b^4}{kb^3 + 1} + \frac{c^4}{kc^3 + 1} \geq \frac{a}{k + 1} + \frac{b}{k + 1} + \frac{c}{k + 1},\]
\[\frac{a^4 - a}{ka^3 + 1} + \frac{b^4 - b}{kb^3 + 1} + \frac{c^4 - c}{kc^3 + 1} \geq 0.\]

Using the substitution
\[a = e^x, \quad b = e^y, \quad c = e^z,\]
we need to show that
\[f(x) + g(y) + g(z) \geq 3f(s),\]
where
\[s = \frac{x + y + z}{3} = 0\]
and
\[f(u) = \frac{e^{4u} - e^u}{ke^{3u} + 1}, \quad u \in \mathbb{R}.\]

From
\[f'(t) = \frac{ke^{6u} + 2(k + 2)e^{3u} - 1}{(ke^{3u} + 1)^2},\]
it follows that $f$ is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \ln r_0 < 0, \quad r_0 = \sqrt[k]{\frac{-k - 2 + \sqrt{(k + 1)(k + 4)}}{k}} \in (0, 1).$$

Also, we have

$$f''(u) = \frac{t \cdot h(t)}{(kt^3 + 1)^3},$$

where

$$h(t) = k^2 t^9 - k(4k + 1)t^6 + (13k + 16)t^3 - 1, \quad t = e^u.$$

If $h(t) > 0$ for $t \in [r_0, 1]$, then $f$ is convex on $[s_0, 0]$. We will prove this only for $k = 2 + 2\sqrt{2}$, when $r_0 \approx 0.415$ and $h(t) \geq 0$ for $t \in [t_1, t_2]$, where $t_1 \approx 0.2345$ and $t_2 \approx 1.02$. Since $[r_0, 1] \subset [t_1, t_2]$, the conclusion follows. By the LPCF-Theorem, we only need to prove the original inequality for $b = c$. Due to homogeneity, we may consider that $b = c = 1$. Thus, we need to show that

$$\frac{a^3}{ka^2 + 1} + \frac{2}{a + k} \geq \frac{a + 2}{k + 1},$$

which is equivalent to the obvious inequality

$$(a - 1)^2[a^2 - (k - 2)a + 2] \geq 0.$$  

For $k = 2 + 2\sqrt{2}$, this inequality has the form

$$(a - 1)^2(a - \sqrt{2})^2 \geq 0.$$  

The equality holds for $a = b = c$. If $k = 2 + 2\sqrt{2}$, then the equality holds also for

$$\frac{a}{\sqrt{2}} = b = c$$

(or any cyclic permutation).

\[\square\]

**P 3.34.** If $a, b, c, d, e$ are positive real numbers so that $abcde = 1$, then

$$2 \left( \frac{1}{a + 1} + \frac{1}{b + 1} + \cdots + \frac{1}{e + 1} \right) \geq 3 \left( \frac{1}{a + 2} + \frac{1}{b + 2} + \cdots + \frac{1}{e + 2} \right).$$

(Vasile C., 2012)
Solution. Write the inequality as
\[
\frac{1 - a}{(a + 1)(a + 2)} + \frac{1 - b}{(b + 1)(b + 2)} + \frac{1 - c}{(c + 1)(c + 2)} + \frac{1 - d}{(d + 1)(d + 2)} + \frac{1 - e}{(e + 1)(e + 2)} \geq 0.
\]
Using the substitution
\[a = e^x, \quad b = e^y, \quad c = e^z, \quad d = e^t, \quad e = e^w,\]
we need to show that
\[f(x) + f(y) + f(z) + f(t) + f(w) \geq 5f(s),\]
where
\[s = \frac{x + y + z + t + w}{5} = 0\]
and
\[f(u) = \frac{1 - e^u}{(e^u + 1)(e^u + 2)}, \quad u \in \mathbb{R}.
\]
From
\[f'(u) = \frac{e^u(e^{2u} - 2e^u - 5)}{(e^u + 1)^2(e^u + 2)^2},\]
it follows that \(f\) is decreasing on \((-\infty, s_0]\) and increasing on \([s_0, \infty)\), where
\[s_0 = \ln(1 + \sqrt{6}) < 2, \quad s < s_0.
\]
Also, we have
\[f''(u) = \frac{t \cdot h(t)}{(t + 1)^3(t + 2)^3}, \quad t = e^u,
\]
where
\[h(t) = -t^4 + 7t^3 + 21t^2 + 7t - 10.
\]
We will show that \(h(t) > 0\) for \(t \in [1, 2]\), hence \(f\) is convex on \([0, s_0]\). We have
\[h(t) \geq -2t^3 + 7t^3 + 21t^2 + 7t - 10 = 5t^3 + 21t^2 + 7t - 10 > 0.
\]
By the RPCF-Theorem, we only need to prove the original inequality for
\[b = c = d = e := t, \quad a = 1/t^4, \quad t \geq 1.
\]
Write this inequality as
\[
\frac{t^4(t^4 - 1)}{(t^4 + 1)(2t^4 + 1)} \geq \frac{4(t - 1)}{(t + 1)(t + 2)},
\]
which is true if
\[t^4(t + 1)(t + 2)(t^3 + t^2 + t + 1) \geq 4(t^4 + 1)(2t^4 + 1).\]
Since 
\[(t^4 + 1)(2t^4 + 1) = 2t^8 + 3t^4 + 1 \leq 2t^4(t^4 + 2),\]
it suffices to show that 
\[(t + 1)(t + 2)(t^3 + t^2 + t + 1) \geq 8(t^4 + 2).\]
This inequality is equivalent to 
\[t^5 - 4t^4 + 6t^3 + 6t^2 + 5t - 14 \geq 0,\]
\[t(t - 1)^4 + 10(t^2 - 1) + 4(t - 1) \geq 0.\]
The equality holds for \(a = b = c = d = e = 1.\) 

**P 3.35.** If \(a_1, a_2, \ldots, a_{14}\) are positive real numbers so that \(a_1a_2 \cdots a_{14} = 1,\) then 
\[
3 \left( \frac{1}{2a_1 + 1} + \frac{1}{2a_2 + 1} + \cdots + \frac{1}{2a_{14} + 1} \right) \geq 2 \left( \frac{1}{a_1 + 1} + \frac{1}{a_2 + 1} + \cdots + \frac{1}{a_{14} + 1} \right).
\]

*(Vasile C., 2012)*

**Solution.** Write the inequality as 
\[
\frac{1 - a_1}{(a_1 + 1)(2a_1 + 1)} + \frac{1 - a_2}{(a_2 + 1)(2a_2 + 1)} + \cdots + \frac{1 - a_{14}}{(a_{14} + 1)(2a_{14} + 1)} \geq 0.
\]

Using the substitutions \(a_i = e^{x_i}\) for \(i = 1, 2, \ldots, 14,\) we need to show that 
\[f(x_1) + f(x_2) + \cdots + f(x_{14}) \geq 14f(s),\]
where 
\[s = \frac{x_1 + x_2 + \cdots + x_{14}}{14} = 0\]
and 
\[f(u) = \frac{1 - e^u}{(e^u + 1)(2e^u + 1)}, \quad u \in \mathbb{R}.\]

From 
\[f'(u) = \frac{2e^u(e^{2u} - 2e^u - 2)}{(e^u + 1)^2(2e^u + 1)^2},\]
it follows that \(f\) is decreasing on \((-\infty, s_0]\) and increasing on \([s_0, \infty),\) where 
\[s_0 = \ln(1 + \sqrt{3}) < 2, \quad s < s_0.\]
Also, we have 
\[f''(u) = \frac{2t \cdot h(t)}{(t + 1)^3(2t + 1)^3}, \quad t = e^u,\]
where
\[ h(t) = -2t^4 + 11t^3 + 15t^2 + 2t - 2. \]
We will show that \( h(t) > 0 \) for \( t \in [1, 2] \), hence \( f \) is convex on \([0, s_0]\). We have
\[ h(t) \geq -4t^3 + 11t^3 + 15t^2 + 2t - 2 = 7t^3 + 15t^2 + 2t - 2 > 0. \]
By the RPCF-Theorem, we only need to prove the original inequality for
\[ a_2 = a_3 = \cdots = a_{14} := t, \quad a_1 = 1/t^{13}, \quad t \geq 1. \]
Write this inequality as
\[ \frac{t^{13}(t^{13} - 1)}{(t^{13} + 1)(t^{13} + 2)} \geq \frac{13(t - 1)}{(t + 1)(2t + 1)}. \]
Since
\[ (t^{13} + 1)(t^{13} + 2) = t^{26} + 3t^{13} + 2 \leq t^{13}(t^{13} + 5), \]
it suffices to show that
\[ \frac{t^{13} - 1}{t^{13} + 5} \geq \frac{13(t - 1)}{(t + 1)(2t + 1)}, \]
which is equivalent to
\[ t^{13}(t^2 - 5t + 7) - t^2 - 34t + 32 \geq 0. \]
Substituting
\[ t = 1 + x, \quad x \geq 0, \]
the inequality becomes
\[ (1 + x)^{13}(x^2 - 3x + 3) - x^2 - 36x - 3 \geq 0. \]
Since
\[ (1 + x)^{13} \geq 1 + 13x + 78x^2, \]
it suffices to show that
\[ (78x^2 + 13x + 1)(x^2 - 3x + 3) - x^2 - 36x - 3 \geq 0. \]
This inequality, equivalent to
\[ x^2(78x^2 - 221x + 196) \geq 0, \]
is true since
\[ 78x^2 - 221x + 196 \geq 64x^2 - 224x + 196 = 4(4x - 7)^2 \geq 0. \]
The equality holds for \( a_1 = a_2 = \cdots = a_{14} = 1 \). \( \square \)
**P 3.36.** Let $a_1, a_2, \ldots, a_8$ be positive real numbers so that $a_1a_2\cdots a_8 = 1$. If $k > 1$, then

$$(k + 1) \left( \frac{1}{ka_1 + 1} + \frac{1}{ka_2 + 1} + \cdots + \frac{1}{ka_8 + 1} \right) \geq 2 \left( \frac{1}{a_1 + 1} + \frac{1}{a_2 + 1} + \cdots + \frac{1}{a_8 + 1} \right).$$

**Solution.** Write the inequality as

$$\frac{1 - a_1}{(a_1 + 1)(ka_1 + 1)} + \frac{1 - a_2}{(a_2 + 1)(ka_2 + 1)} + \cdots + \frac{1 - a_8}{(a_8 + 1)(ka_8 + 1)} \geq 0.$$ 

Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \ldots, 8$, we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_8) \geq 8f(s),$$

where

$$s = \frac{x_1 + x_2 + \cdots + x_8}{8} = 0$$

and

$$f(u) = \frac{1 - e^u}{(e^u + 1)(ke^u + 1)}, \quad u \in \mathbb{R}.$$ 

From

$$f'(u) = \frac{e^u(ke^{2u} - 2ke^u - k - 2)}{(e^u + 1)^2(ke^u + 1)^2},$$

it follows that $f$ is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \ln \left( 1 + \sqrt{\frac{2}{k}} \right) < 2, \quad s < s_0.$$ 

Also, we have

$$f''(u) = \frac{t \cdot h(t)}{(t + 1)^3(kt + 1)^3}, \quad t = e^u,$$

where

$$h(t) = -k^2t^4 + k(5k + 1)t^3 + 3k(k + 3)t^2 + (k^2 - k + 2)t - k - 2.$$ 

We will show that $h(t) > 0$ for $t \in [1, 2]$, hence $f$ is convex on $[0, s_0]$. We have

$$h(t) > -2k^2t^3 + k(5k + 1)t^3 + 3k(k + 3)t^2 + (k^2 - k + 2)t - k - 2 = k(3k + 1)t^3 + 3k(k + 3)t^2 + (k^2 - k + 2)t - k - 2 > 3k(k + 3) + (k^2 - k + 2) - k - 2 > 0.$$ 

By the RPCF-Theorem, we only need to prove the original inequality for

$$a_2 = a_3 = \cdots = a_8 := t, \quad a_1 = 1/t^7, \quad t \geq 1.$$
Write this inequality as
\[
\frac{t^7(t^7 - 1)}{(t^7 + 1)(t^7 + k)} \geq \frac{7(t - 1)}{(t + 1)(kt + 1)}.
\]
Since
\[(t^7 + 1)(t^7 + k) = t^{14} + (k + 1)t^7 + k \leq t^7(t^7 + 2k + 1),
\]
it suffices to show that
\[
\frac{t^7 - 1}{t^7 + 2k + 1} \geq \frac{7(t - 1)}{(t + 1)(kt + 1)},
\]
which is equivalent to
\[k(t - 1)P(t) + Q(t) \geq 0,
\]
where
\[P(t) = t(t + 1)(t^6 + t^5 + t^4 + t^3 + t^2 + t + 1) - 14,
\]
\[Q(t) = (t + 1)(t^7 - 1) - 7(t - 1)(t^7 + 1).
\]
Since \((t - 1)P(t) \geq 0\) for \(t \geq 1\), it suffices to consider the case \(k = 1\). So, we need to show that
\[
\frac{t^7 - 1}{t^7 + 3} \geq \frac{7(t - 1)}{(t + 1)^2},
\]
which is equivalent to
\[t^7(t^2 - 5t + 8) - t^2 - 23t + 20 \geq 0.
\]
Substituting
\[t = 1 + x, \quad x \geq 0,
\]
the inequality becomes
\[(1 + x)^7(x^2 - 3x + 4) - x^2 - 25x - 4 \geq 0.
\]
Since
\[(1 + x)^7 \geq 1 + 7x + 21x^2,
\]
it suffices to show that
\[(21x^2 + 7x + 1)(x^2 - 3x + 4) - x^2 - 25x - 4 \geq 0.
\]
This inequality, equivalent to
\[x^2(21x^2 - 56x + 63) \geq 0.
\]
is true since
\[21x^2 - 56x + 63 > 16x^2 - 56x + 49 = (4x - 7)^2 \geq 0.
\]
The equality holds for \(a_1 = a_2 = \cdots = a_8 = 1\).
P 3.37. If \( a_1, a_2, \ldots, a_9 \) are positive real numbers so that \( a_1 a_2 \cdots a_9 = 1 \), then

\[
\frac{1}{2a_1 + 1} + \frac{1}{2a_2 + 1} + \cdots + \frac{1}{2a_9 + 1} \geq \frac{1}{a_1 + 2} + \frac{1}{a_2 + 2} + \cdots + \frac{1}{a_9 + 2}.
\]

(Vasile C., 2012)

Solution. Write the inequality as

\[
\frac{1 - a_1}{(2a_1 + 1)(a_1 + 2)} + \frac{1 - a_2}{(2a_2 + 1)(a_2 + 2)} + \cdots + \frac{1 - a_9}{(2a_9 + 1)(a_9 + 2)} \geq 0.
\]

Using the substitutions \( a_i = e^{x_i} \) for \( i = 1, 2, \ldots, 9 \), we need to show that

\[
f(x_1) + f(x_2) + \cdots + f(x_9) \geq 9f(s),
\]

where

\[
s = \frac{x_1 + x_2 + \cdots + x_9}{9} = 0
\]

and

\[
f(u) = \frac{1 - e^u}{(2e^u + 1)(e^u + 2)}, \quad u \in \mathbb{R}.
\]

From

\[
f'(u) = \frac{e^u(2e^{2u} - 4e^u - 7)}{(2e^u + 1)^2(e^u + 2)^2},
\]

it follows that \( f \) is decreasing on \((-\infty, s_0]\) and increasing on \([s_0, \infty)\), where

\[
s_0 = \ln \left( 1 + \frac{3\sqrt{2}}{2} \right) < 2, \quad s < s_0.
\]

Also, we have

\[
f''(u) = \frac{t \cdot h(t)}{(2t + 1)^3(t + 2)^3}, \quad t = e^u,
\]

where

\[
h(t) = -4t^4 + 26t^3 + 54t^2 + 19t - 14.
\]

We will show that \( h(t) > 0 \) for \( t \in [1, 2] \), hence \( f \) is convex on \([0, s_0]\). We have

\[
h(t) \geq -8t^3 + 26t^2 + 54t^2 + 19t - 14 = 18t^3 + 54t^2 + 19t - 14 > 0.
\]

By the RPCF-Theorem, we only need to prove the original inequality for

\[
a_2 = a_3 = \cdots = a_9 := t, \quad a_1 = 1/t^8, \quad t \geq 1.
\]

Write this inequality as

\[
\frac{t^8(t^8 - 1)}{(t^8 + 2)(2t^8 + 1)} \geq \frac{8(t - 1)}{(2t + 1)(t + 2)}.
\]
Since 
\[(t^8 + 2)(2t^8 + 1) = 2t^{16} + 5t^8 + 2 \leq t^8(2t^8 + 7),\]
it suffices to show that 
\[\frac{t^8 - 1}{2t^8 + 7} \geq \frac{8(t - 1)}{(2t + 1)(t + 2)},\]
which is equivalent to 
\[t^8(2t^2 - 11t + 18) - 2t^2 - 61t + 54 \geq 0.\]
Substituting 
\[t = 1 + x, \quad x \geq 0,\]
the inequality becomes 
\[(1 + x)^8(2x^2 - 7x + 9) - 2x^2 - 65x - 9 \geq 0.\]
Since 
\[(1 + x)^8 \geq 1 + 8x + 28x^2,\]
it suffices to show that 
\[(28x^2 + 8x + 1)(2x^2 - 7x + 9) - 2x^2 - 65x - 9 \geq 0.\]
This inequality, equivalent to 
\[x^2(56x^2 - 180x + 196) \geq 0.\]
is true since 
\[56x^2 - 180x + 196 \geq 49x^2 - 196x + 196 = 49(x - 2)^2 \geq 0.\]
The equality holds for \(a_1 = a_2 = \cdots = a_9 = 1.\)

\(\square\)

**P 3.38.** If \(a_1, a_2, \ldots, a_n\) are real numbers so that 
\[a_1, a_2, \ldots, a_n \leq \pi, \quad a_1 + a_2 + \cdots + a_n = \pi,\]
then 
\[\cos a_1 + \cos a_2 + \cdots + \cos a_n \leq n \cos \frac{\pi}{n}.\]

(Vasile C., 2000)
Solution. Write the inequality as
\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{\pi}{n}, \]
where
\[ f(u) = -\cos u, \quad u \in \mathbb{I} = [-(n-2)\pi, \pi]. \]
Let
\[ s_0 = 0 < s. \]
We see that \( f \) is increasing on \([s_0, \pi] = \mathbb{I} \geq s_0\) and \( f(u) \geq f(s_0) = -1 \) for \( u \in \mathbb{I} \). In addition, \( f \) is convex on \([s_0, s]\). Thus, by the LPCF-Theorem, we only need to prove that \( g(x) \leq 0 \), where
\[ g(x) = \cos x + (n-1)\cos y - n\cos s, \quad x + (n-1)y = \pi, \quad \pi \geq x \geq s \geq y \geq 0. \]
Since \( y' = -\frac{1}{n-1} \), we get
\[ g'(x) = -\sin x + \sin y = -2\sin \frac{x-y}{2} \cos \frac{x+y}{2}. \]
We have \( g'(x) \leq 0 \) because
\[ 0 < \frac{x+y}{2} \leq \frac{x+(n-1)y}{2} = \frac{\pi}{2} \]
and
\[ 0 \leq \frac{x-y}{2} < \frac{\pi}{2}. \]
From \( g' \leq 0 \), it follows that \( g \) is decreasing, hence \( g(x) \leq g(s) = 0 \).

The equality holds for \( a_1 = a_2 = \cdots = a_n = \frac{\pi}{n} \). If \( n = 2 \), then the inequality is an identity.

Remark. In the same manner, we can prove the following generalization:

- If \( a_1, a_2, \ldots, a_n \) are real numbers so that
\[ a_1, a_2, \ldots, a_n \leq \pi, \quad \frac{a_1 + a_2 + \cdots + a_n}{n} = s, \quad 0 < s \leq \frac{\pi}{4}, \]
then
\[ \cos a_1 + \cos a_2 + \cdots + \cos a_n \leq n \cos s, \]
with equality for \( a_1 = a_2 = \cdots = a_n = s \).
P 3.39. If \( a_1, a_2, \ldots, a_n \) (\( n \geq 3 \)) are real numbers so that

\[
a_1, a_2, \ldots, a_n \geq \frac{-1}{n-2}, \quad a_1 + a_2 + \cdots + a_n = n,
\]

then

\[
\frac{a_1^2}{a_1^2 - a_1 + 1} + \frac{a_2^2}{a_2^2 - a_2 + 1} + \cdots + \frac{a_n^2}{a_n^2 - a_n + 1} \leq n.
\]

(Vasile Cirtoaje, 2012)

**Solution.** Write the inequality as

\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,
\]

where

\[
f(u) = \frac{1-u}{u^2-u+1}, \quad u \in I = \left[ \frac{-1}{n-2}, \frac{n^2-n-1}{n-2} \right].
\]

Let \( s_0 = 2 \). We have \( s < s_0 \) and

\[
\min_{u \in I} f(u) = f(s_0)
\]

because

\[
f(u) - f(2) = \frac{1-u}{u^2-u+1} + \frac{1}{3} = \frac{(u-2)^2}{3(u^2-u+1)} \geq 0.
\]

From

\[
f'(u) = \frac{u(u-2)}{(u^2-u+1)^2},
\]

\[
f''(u) = \frac{2(3u^2-u^3-1)}{(u^2-u+1)^3} = \frac{2u^2(2-u) + 2(u^2-1)}{(u^2-u+1)^3},
\]

it follows that \( f \) is convex on \([1, s_0]\). However, we can't apply the RPCF-Theorem in its original form because \( f \) is not decreasing on \( I \leq s_0 \). According to Theorem 1, we may replace this condition with \( ns - (n-1)s_0 \leq \inf I \). Indeed, we have

\[
ns - (n-1)s_0 = n - 2(n-1) = -n + 2 \leq \frac{-1}{n-2} = \inf I.
\]

So, it suffices to show that \( f(x) + (n-1)f(y) \geq nf(1) \) for all \( x, y \in I \) so that

\[
x + (n-1)y = n.
\]

According to Note 1, we only need to show that \( h(x, y) \geq 0 \), where

\[
g(u) = \frac{f(u)-f(1)}{u-1}, \quad h(x, y) = \frac{g(x)-g(y)}{x-y}.
\]
We have
\[ g(u) = \frac{-1}{u^2 - u + 1}, \]
\[ h(x, y) = \frac{x + y - 1}{(x^2 - x + 1)(y^2 - y + 1)} = \frac{(n-2)x + 1}{(n-1)(x^2 - x + 1)(y^2 - y + 1)} \geq 0. \]
The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for
\[ a_1 = \frac{-1}{n-2}, \quad a_2 = a_3 = \cdots = a_n = \frac{n-1}{n-2} \]
(or any cyclic permutation).

\[ \square \]

**P 3.40.** If \( a_1, a_2, \ldots, a_n \) (\( n \geq 3 \)) are nonzero real numbers so that
\[ a_1, a_2, \ldots, a_n \geq -\frac{n}{n-2}, \quad a_1 + a_2 + \cdots + a_n = n, \]
then
\[ \frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2} \geq \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}. \]

(Vasile Cirtoaje, 2012)

**Solution.** According to P 2.25-(a) in Volume 1, the inequality is true for \( n = 3 \). Assume further that \( n \geq 4 \) and write the inequality as
\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1, \]
where
\[ f(u) = \frac{1}{u^2} - \frac{1}{u}, \quad u \in I = \left[ \frac{-n}{n-2}, \frac{n(2n-3)}{n-2} \right] \setminus \{0\}. \]
Let
\[ s_0 = 2, \quad s < s_0. \]
From
\[ f(u) - f(2) = \frac{1}{u^2} - \frac{1}{u} + \frac{1}{4} = \frac{(u-2)^2}{4u^2} \geq 0, \]
it follows that
\[ \min_{u \in I} f(u) = f(s_0), \]
while from
\[ f'(u) = \frac{u-2}{u^3}, \quad f''(u) = \frac{2(3-u)}{u^4}, \]
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it follows that $f$ is convex on $[s, s_0]$. However, we can't apply the RPCF-Theorem because $f$ is not decreasing on $I \leq s_0$. According to Theorem 1 and Note 6, we may replace this condition with $ns - (n - 1)s_0 \leq \inf I$. For $n \geq 4$, we have

$$ns - (n - 1)s_0 = n - 2(n - 1) = -n + 2 \leq \frac{-n}{n - 2} = \inf I.$$ 

So, according to Note 1, it suffices to show that $h(x, y) \geq 0$ for all $x, y \in I$ so that $x + (n - 1)y = n$. We have

$$g(u) = f(u) - f(1) = \frac{-1}{u^2},$$

$$h(x, y) = g(x) - g(y) = \frac{x + y}{x^2y^2} = \frac{(n - 2)x + n}{(n - 1)x^2y^2} \geq 0.$$ 

The proof is completed. By Note 3, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = \frac{-n}{n - 2}, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n - 2}$$

(or any cyclic permutation).

**Remark.** Similarly, we can prove the following generalization:

- Let $a_1, a_2, \ldots, a_n \geq -1$ so that $a_1 + a_2 + \cdots + a_n = n$. If $n \geq 3$ and $k \geq 0$, then

$$\frac{1 - a_1}{k + a_1^2} + \frac{1 - a_2}{k + a_2^2} + \cdots + \frac{1 - a_n}{k + a_n^2} \geq 0,$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = \frac{-n}{n - 2}, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n - 2}$$

(or any cyclic permutation).

**P 3.41.** If $a_1, a_2, \ldots, a_n \geq -1$ so that $a_1 + a_2 + \cdots + a_n = n$, then

$$(n + 1)\left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2}\right) \geq 2n + (n - 1)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

(Vasile C., 2013)
**Solution.** Write the inequality as

\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1, \]

where

\[ f(u) = \frac{n + 1}{u^2} - \frac{n - 1}{u}, \quad u \in I = [-1, 2n-1] \setminus \{0\}. \]

Let

\[ s_0 = \frac{2(n+1)}{n-1} \in I, \quad s < s_0. \]

Since

\[ f(u) - f(s_0) = \frac{[(n-1)u - 2(n+1)]^2}{4(n+1)u^2} \geq 0, \]

we have

\[ \min_{u \in I} f(u) = f(s_0). \]

From

\[ f'(u) = \frac{(n-1)u - 2(n+1)}{u^3}, \quad f''(u) = \frac{6(n+1) - 2(n-1)u}{u^4}, \]

it follows that \( f \) is convex on \([1, s_0]\). Since \( f \) is not decreasing on \( I_{\leq s_0} \), according to Theorem 1 and Note 6, we may replace this condition in RPCF-Theorem with \( ns - (n-1)s_0 \leq \inf I \). We have

\[ ns - (n-1)s_0 = n - 2(n+1) = -n - 2 \leq -1 = \inf I. \]

According to Note 1, we only need to show that \( h(x, y) \geq 0 \) for \(-1 \leq x \leq 1 \leq y \) and \( x + (n-1)y = n \). We have

\[ g(u) = \frac{f(u) - f(1)}{u - 1} = -\frac{2}{u} - \frac{n+1}{u^2} \]

and

\[ h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{2xy + (n+1)(x + y)}{x^2y^2} = \frac{(x + 1)(n^2 + n - 2x)}{(n-1)x^2y^2} \geq 0. \]

According to Note 4, the equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for

\[ a_1 = -1, \quad a_2 = \cdots = a_n = \frac{n+1}{n-1} \]

(or any cyclic permutation). \( \square \)
**P 3.42.** If $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ are real numbers so that

$$a_1, a_2, \ldots, a_n \geq -\frac{(3n-2)}{n-2}, \quad a_1 + a_2 + \cdots + a_n = n,$$

then

$$\frac{1-a_1}{(1+a_1)^2} + \frac{1-a_2}{(1+a_2)^2} + \cdots + \frac{1-a_n}{(1+a_n)^2} \geq 0.$$  

*(Vasile C., 2014)*

**Solution.** According to P 2.25-(b) in Volume 1, the inequality is true for $n = 3$. Assume further that $n \geq 4$ and write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{1-u}{(1+u)^2}, \quad u \in I = \left[ \frac{-(3n-2)}{n-2}, \frac{4n^2-7n+2}{n-2} \right] \setminus \{-1\}.$$

Let

$$s_0 = 3, \quad s < s_0.$$

From

$$f(u) - f(3) = \frac{1-u}{(1+u)^2} + \frac{1}{8} = \frac{(u-3)^2}{8(u+1)^2} \geq 0,$$

it follows that

$$\min_{u \in I} f(u) = f(s_0).$$

From

$$f'(u) = \frac{u-3}{(u+1)^3}, \quad f''(u) = \frac{2(5-u)}{(u+1)^4},$$

it follows that $f$ is convex on $[1,s_0]$. We can’t apply the RPCF-Theorem in its original form because $f$ is not decreasing on $I_{\leq s_0}$. However, according to Theorem 1 and Note 6, we may replace this condition with $ns - (n-1)s_0 \leq \inf I$. Indeed, for $n \geq 4$, we have

$$ns - (n-1)s_0 = n - 3(n-1) = -2n + 3 \leq \frac{-(3n-2)}{n-2} = \inf I.$$

According to Note 1, it suffices to show that $h(x, y) \geq 0$ for all $x, y \in I$ so that $x \leq 1 \leq y$ and $x + (n-1)y = n$. We have

$$g(u) = \frac{f(u) - f(1)}{u-1} = \frac{-1}{(u+1)^2},$$

$$h(x, y) = \frac{g(x) - g(y)}{x-y} = \frac{x + y + 2}{(x+1)^2(y+1)^2} = \frac{(n-2)x + 3n-2}{(n-1)(x+1)^2(y+1)^2} \geq 0.$$
In accordance with Note 3, the equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for
\[
a_1 = \frac{-(3n-2)}{n-2}, \quad a_2 = a_3 = \cdots = a_n = \frac{n+2}{n-2}
\]
(or any cyclic permutation).

**P 3.43.** Let \( a_1, a_2, \ldots, a_n \) be nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n = n \). If \( n \geq 3 \) and \( k \geq 2 - \frac{2}{n} \), then
\[
\frac{1-a_1}{(1-ka_1)^2} + \frac{1-a_2}{(1-ka_2)^2} + \cdots + \frac{1-a_n}{(1-ka_n)^2} \geq 0.
\]

*(Vasile C., 2012)*

**Solution.** According to P 3.97 in Volume 1, the inequality is true for \( n = 3 \). Assume further that \( n \geq 4 \) and write the inequality as
\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,
\]
where
\[
f(u) = \frac{1-u}{(1-ku)^2}, \quad u \in I = [0, n] \setminus \{1/k\}.
\]
Let
\[s_0 = 2 - 1/k, \quad 1 = s < s_0.\]
Since
\[
f(u) - f(s_0) = \frac{1-u}{(1-ku)^2} + \frac{1}{4k(k-1)} = \frac{(ku-2k+1)^2}{4k(k-1)(1-ku)^2} \geq 0,
\]
we have
\[
\min_{u \in I} f(u) = f(s_0).
\]
From
\[
f'(u) = \frac{ku-2k+1}{(ku-1)^3}, \quad f''(u) = \frac{2k(-ku+3k-2)}{(1-ku)^4},
\]
it follows that \( f \) is convex on \([1,s_0]\). We can’t apply the RPCF-Theorem because \( f \) is not decreasing on \( I \leq s_0 \). According to Theorem 1 and Note 6, we may replace this condition with \( ns - (n-1)s_0 \leq \inf I \). Indeed, we have
\[
ns - (n-1)s_0 \leq n - (n-1) \cdot \frac{3n-4}{2(n-1)} \cdot \frac{4-n}{2} \leq 0 = \inf I.
\]
So, it suffices to show that \( f(x) + (n - 1)f(y) \geq nf(1) \) for all \( x, y \in \mathbb{I} \) so that \( x \leq 1 \leq y \) and \( x + (n - 1)y = n \). According to Note 1, we only need to show that \( h(x, y) \geq 0 \), where

\[
g(u) = \frac{f(u) - f(1)}{u - 1}, \quad h(x, y) = \frac{g(x) - g(y)}{x - y}.
\]

Since

\[
g(u) = \frac{-1}{(1 - ku)^2}, \quad h(x, y) = \frac{k[k(x + y) - 2]}{(1 - kx)^2(1 - ky)^2},
\]

we need to show that \( k(x + y) - 2 \geq 0 \). Indeed, we have

\[
\frac{k(x + y) - 2}{2} \geq \frac{(n - 1)(x + y)}{n} - 1 = \frac{(n - 1)(x + y)}{n} - \frac{x + (n - 1)y}{n} = \frac{(n - 2)x}{n} \geq 0.
\]

The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \). If \( k = 2 - \frac{2}{n} \), then the equality also holds for

\[
a_1 = 0, \quad a_2 = a_3 = \cdots = a_n = \frac{n}{n - 1}
\]

(or any cyclic permutation). \( \square \)
Chapter 4
Partially Convex Function Method for Ordered Variables

4.1 Theoretical Basis

The following statement is known as Right Partially Convex Function Theorem for Ordered Variables (RPCF-OV Theorem).

**RPCF-OV Theorem** (Vasile Cirtoaje, 2014). Let $f$ be a real function defined on an interval $I$ and convex on $[s, s_0]$, where $s, s_0 \in I$, $s < s_0$. In addition, $f$ is decreasing on $I \leq s_0$ and $f(u) \geq f(s_0)$ for $u \in I$. The inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in I$ satisfying

$$a_1 + a_2 + \cdots + a_n = ns$$

and

$$a_1 \leq a_2 \leq \cdots \leq a_m \leq s, \quad m \in \{1, 2, \ldots, n-1\},$$

if and only if

$$f(x) + (n-m)f(y) \geq (1+n-m)f(s)$$

for all $x, y \in I$ so that $x \leq s \leq y$ and $x + (n-m)y = (1+n-m)s$.

**Proof.** For

$$a_1 = x, \quad a_2 = \cdots = a_m = s, \quad a_{m+1} = \cdots = a_n = y,$$

the inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s)$$

becomes

$$f(x) + (n-m)f(y) \geq (1+n-m)f(s);$$
therefore, the necessity is obvious. By Lemma from Chapter 3, to prove the sufficiency, it suffices to consider that \(a_1, a_2, \ldots, a_n \in J\), where

\[
J = \mathbb{I}_{s_0}.
\]

Because \(f\) is convex on \(J\), the desired inequality follows from HCF-OV Theorem applied to the interval \(J\).

Similarly, we can prove Left Partially Convex Function Theorem for Ordered Variables (LPCF-OV Theorem).

**LPCF-OV Theorem.** Let \(f\) be a real function defined on an interval \(\mathbb{I}\) and convex on \([s_0, s]\), where \(s_0, s \in \mathbb{I}\), \(s_0 < s\). In addition, \(f\) is increasing on \(\mathbb{I}_{s_0}\) and \(f(u) \geq f(s_0)\) for \(u \in \mathbb{I}\). The inequality

\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf \left( \frac{a_1 + a_2 + \cdots + a_n}{n} \right)
\]

holds for all \(a_1, a_2, \ldots, a_n \in \mathbb{I}\) satisfying

\[
a_1 + a_2 + \cdots + a_n = ns
\]

and

\[
a_1 \geq a_2 \geq \cdots \geq a_m \geq s, \quad m \in \{1, 2, \ldots, n-1\},
\]

if and only if

\[
f(x) + (n-m)f(y) \geq (1 + n - m)f(s)
\]

for all \(x, y \in \mathbb{I}\) so that \(x \geq s \geq y\) and \(x + (n-m)y = (1 + n - m)s\).

The RPCF-OV Theorem and the LPCF-OV Theorems are respectively generalizations of the RPCF Theorem and LPCF Theorem, because the last theorems can be obtained from the first theorems for \(m = 1\).

**Note 1.** Let us denote

\[
g(u) = \frac{f(u) - f(s)}{u - s}, \quad h(x, y) = \frac{g(x) - g(y)}{x - y}.
\]

We may replace the hypothesis condition in the RPCF-OV Theorem and the LPCF-OV Theorem, namely

\[
f(x) + mf(y) \geq (1 + m)f(s),
\]

by the condition

\[
h(x, y) \geq 0 \text{ for all } x, y \in \mathbb{I} \text{ so that } x + my = (1 + m)s.
\]

**Note 2.** Assume that \(f\) is differentiable on \(\mathbb{I}\), and let

\[
H(x, y) = \frac{f'(x) - f'(y)}{x - y}.
\]
The desired inequality of Jensen’s type in the RPCF-OV Theorem and the LPCF-OV Theorem holds true by replacing the hypothesis
\[ f(x) + mf(y) \geq (1 + m)f(s) \]
with the more restrictive condition
\[ H(x, y) \geq 0 \text{ for all } x, y \in \mathbb{I} \text{ so that } x + my = (1 + m)s. \]

**Note 3.** The desired inequalities in the RPCF-OV Theorem and the LPCF-OV Theorem become equalities for
\[ a_1 = a_2 = \cdots = a_n = s. \]
In addition, if there exist \( x, y \in \mathbb{I} \) so that
\[ x + (n - m)y = (1 + n - m)s, \quad f(x) + (n - m)f(y) = (1 + n - m)f(s), \quad x \neq y, \]
then the equality holds also for
\[ a_1 = x, \quad a_2 = \cdots = a_m = s, \quad a_{m+1} = \cdots = a_n = y \]
(or any cyclic permutation). Notice that these equality conditions are equivalent to
\[ x + (n - m)y = (1 + n - m)s, \quad h(x, y) = 0 \]
\( (x < y \text{ for RHCF-OV Theorem, and } x > y \text{ for LHCF-OV Theorem}). \)

**Note 4.** The RPCF-OV Theorem is also valid in the case where \( f \) is defined on \( \mathbb{I} \setminus \{u_0\} \), where \( u_0 \) is an interior point of \( \mathbb{I} \) so that \( u_0 > s_0 \). Similarly, LPCF Theorem is also valid in the case in which \( f \) is defined on \( \mathbb{I} \setminus \{u_0\} \), where \( u_0 \) is an interior point of \( \mathbb{I} \) so that \( u_0 < s_0 \).

**Note 5.** The RPCF-Theorem holds true by replacing the condition
\[ f \text{ is decreasing on } \mathbb{I}_{\leq s_0} \]
with
\[ ns - (n - 1)s_0 \leq \inf \mathbb{I}. \]
More precisely, the following theorem holds:

**Theorem 1.** Let \( f \) be a function defined on a real interval \( \mathbb{I} \), convex on \([s, s_0]\) and satisfying
\[ \min_{u \in \mathbb{I}_{s_0}} f(u) = f(s_0), \]
where
\[ s, s_0 \in \mathbb{I}, \quad s < s_0, \quad (1 + n - m)s - (n - m)s_0 \leq \inf \mathbb{I}. \]
The inequality
\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \]
holds for all \( a_1, a_2, \ldots, a_n \in I \) satisfying
\[
a_1 + a_2 + \cdots + a_n = ns
\]
and
\[
a_1 \leq a_2 \leq \cdots \leq a_m \leq s, \quad m \in \{1, 2, \ldots, n-1\},
\]
if and only if
\[
f(x) + (n-m)f(y) \geq (1+n-m)f(s)
\]
for all \( x, y \in I \) so that \( x \leq s \leq y \) and \( x + (n-m)y = (1+n-m)s \).

The proof of this theorem is similar to the one of Theorem 1 from chapter 3.

Similarly, the LPCF-Theorem holds true by replacing the condition
\[
f \text{ is increasing on } I \geq s_0
\]
with
\[
ns - (n-1)s_0 \geq \sup I.
\]

More precisely, the following theorem holds:

**Theorem 2.** Let \( f \) be a function defined on a real interval \( I \), convex on \([s_0, s]\) and satisfying
\[
\min_{u \in [s_0, s]} f(u) = f(s_0),
\]
where
\[
s, s_0 \in I, \quad s > s_0, \quad (1+n-m)s - (n-m)s_0 \geq \sup I.
\]
The inequality
\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)
\]
holds for all \( a_1, a_2, \ldots, a_n \in I \) satisfying
\[
a_1 + a_2 + \cdots + a_n = ns
\]
and
\[
a_1 \geq a_2 \geq \cdots \geq a_m \geq s, \quad m \in \{1, 2, \ldots, n-1\},
\]
if and only if
\[
f(x) + (n-m)f(y) \geq (1+n-m)f(s)
\]
for all \( x, y \in I \) so that \( x \geq s \geq y \) and \( x + (n-m)y = (1+n-m)s \).

**Note 6.** Theorem 1 is also valid in the case in which \( f \) is defined on \( I \setminus \{u_0\} \), where \( u_0 \) is an interior point of \( I \) so that \( u_0 \notin [s, s_0] \). Similarly, Theorem 2 is also valid in the case in which \( f \) is defined on \( I \setminus \{u_0\} \), where \( u_0 \) is an interior point of \( I \) so that \( u_0 \notin [s_0, s] \).
**Note 7.** We can extend weighted Jensen’s inequality to right and left partially convex functions with ordered variables establishing the WRPCF-OV Theorem and the WLPCF-OV Theorem (Vasile Cirtoaje, 2014).

**WRPCF-OV Theorem.** Let \( p_1, p_2, \ldots, p_n \) be positive real numbers so that
\[
p_1 + p_2 + \cdots + p_n = 1,
\]
and let \( f \) be a real function defined on an interval \( \mathbb{I} \) and convex on \([s, s_0]\), where \( s, s_0 \in \text{int}(\mathbb{I}) \), \( s < s_0 \). In addition, \( f \) is decreasing on \( \mathbb{I}_{\leq s_0} \) and \( f(u) \geq f(s_0) \) for \( u \in \mathbb{I} \). The inequality
\[
p_1 f(x_1) + p_2 f(x_2) + \cdots + p_n f(x_n) \geq f(p_1 x_1 + p_2 x_2 + \cdots + p_n x_n)
\]
holds for all \( x_1, x_2, \ldots, x_n \in \mathbb{I} \) so that \( p_1 x_1 + p_2 x_2 + \cdots + p_n x_n = s \) and
\[
x_1 \leq x_2 \leq \cdots \leq x_n, \quad x_m \leq s, \quad m \in \{1, 2, \ldots, n-1\},
\]
if and only if
\[
f(x) + k f(y) \geq (1 + k) f(s)
\]
for all \( x, y \in \mathbb{I} \) satisfying
\[
x \leq s \leq y, \quad x + ky = (1 + k)s,
\]
where
\[
k = \frac{p_{m+1} + p_{m+2} + \cdots + p_n}{p_1}.
\]

**WLPCF-OV Theorem.** Let \( p_1, p_2, \ldots, p_n \) be positive real numbers so that
\[
p_1 + p_2 + \cdots + p_n = 1,
\]
and let \( f \) be a real function defined on an interval \( \mathbb{I} \) and convex on \([s_0, s]\), where \( s_0, s \in \mathbb{I} \), \( s_0 < s \). In addition, \( f \) is increasing on \( \mathbb{I}_{\geq s_0} \) and \( f(u) \geq f(s_0) \) for \( u \in \mathbb{I} \). The inequality
\[
p_1 f(x_1) + p_2 f(x_2) + \cdots + p_n f(x_n) \geq f(p_1 x_1 + p_2 x_2 + \cdots + p_n x_n)
\]
holds for all \( x_1, x_2, \ldots, x_n \in \mathbb{I} \) so that \( p_1 x_1 + p_2 x_2 + \cdots + p_n x_n = s \) and
\[
x_1 \geq x_2 \geq \cdots \geq x_n, \quad x_m \geq s, \quad m \in \{1, 2, \ldots, n-1\},
\]
if and only if
\[
f(x) + k f(y) \geq (1 + k) f(s)
\]
for all \( x, y \in \mathbb{I} \) satisfying
\[
x \geq s \geq y, \quad x + ky = (1 + k)s,
\]
where

\[ k = \frac{p_{m+1} + p_{m+2} + \cdots + p_n}{p_1}. \]

For the most commonly used case

\[ p_1 = p_2 = \cdots = p_n = \frac{1}{n}, \]

the WRPCF-OV Theorem and the WLPCF-OV Theorem yield the RRPCF-OV Theorem and the LLPCF-OV Theorem, respectively.
### 4.2 Applications

4.1. If $a, b, c, d$ are real numbers so that

\[ a \leq 1 \leq b \leq c \leq d, \quad a + b + c + d = 4, \]

then

\[ \frac{a}{3a^2 + 1} + \frac{b}{3b^2 + 1} + \frac{c}{3c^2 + 1} + \frac{d}{3d^2 + 1} \leq 1. \]

4.2. If $a, b, c, d$ are real numbers so that

\[ a \geq b \geq 1 \geq c \geq d, \quad a + b + c + d = 4, \]

then

\[ \frac{16a - 5}{32a^2 + 1} + \frac{16b - 5}{32b^2 + 1} + \frac{16c - 5}{32c^2 + 1} + \frac{16d - 5}{32d^2 + 1} \leq \frac{4}{3}. \]

4.3. If $a, b, c, d, e$ are real numbers so that

\[ a \geq b \geq 1 \geq c \geq d \geq e, \quad a + b + c + d + e = 5, \]

then

\[ \frac{18a - 5}{12a^2 + 1} + \frac{18b - 5}{12b^2 + 1} + \frac{18c - 5}{12c^2 + 1} + \frac{18d - 5}{12d^2 + 1} + \frac{18e - 5}{12e^2 + 1} \leq 5. \]

4.4. If $a, b, c, d, e$ are real numbers so that

\[ a \geq b \geq 1 \geq c \geq d \geq e, \quad a + b + c + d + e = 5, \]

then

\[ \frac{a(a - 1)}{3a^2 + 4} + \frac{b(b - 1)}{3b^2 + 4} + \frac{c(c - 1)}{3c^2 + 4} + \frac{d(d - 1)}{3d^2 + 4} + \frac{e(e - 1)}{3e^2 + 4} \geq 0. \]

4.5. Let $a_1, a_2, \ldots, a_{2n} \neq -k$ be real numbers so that

\[ a_1 \geq \cdots \geq a_n \geq 1 \geq a_{n+1} \geq \cdots \geq a_{2n}, \quad a_1 + a_2 + \cdots + a_{2n} = 2n. \]

If $k \geq \frac{n + 1}{2\sqrt{n}}$, then

\[ \frac{a_1(a_1 - 1)}{(a_1 + k)^2} + \frac{a_2(a_2 - 1)}{(a_2 + k)^2} + \cdots + \frac{a_{2n}(a_{2n} - 1)}{(a_{2n} + k)^2} \geq 0. \]
4.6. Let \( a_1, a_2, \ldots, a_{2n} \neq -k \) be real numbers so that
\[
a_1 \geq \cdots \geq a_n \geq 1 \geq a_{n+1} \geq \cdots \geq a_{2n}, \quad a_1 + a_2 + \cdots + a_{2n} = 2n.
\]
If \( k \geq 1 + \frac{n + 1}{\sqrt{n}} \), then
\[
\frac{a_1^2 - 1}{(a_1 + k)^2} + \frac{a_2^2 - 1}{(a_2 + k)^2} + \cdots + \frac{a_{2n}^2 - 1}{(a_{2n} + k)^2} \geq 0.
\]

4.7. If \( a_1, a_2, \ldots, a_n \) are positive real numbers so that
\[
a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n, \quad a_1 + a_2 + \cdots + a_n = n,
\]
then
\[
a_1^{3/a_1} + a_2^{3/a_2} + \cdots + a_n^{3/a_n} \leq n.
\]

4.8. If \( a_1, a_2, \ldots, a_{11} \) are real numbers so that
\[
a_1 \geq a_2 \geq 1 \geq a_3 \geq \cdots \geq a_{11}, \quad a_1 + a_2 + \cdots + a_{11} = 11,
\]
then
\[
(1 - a_1 + a_1^2)(1 - a_2 + a_2^2) \cdots (1 - a_{11} + a_{11}^2) \geq 1.
\]

4.9. If \( a_1, a_2, \ldots, a_8 \) are nonzero real numbers so that
\[
a_1 \geq a_2 \geq a_3 \geq a_4 \geq 1 \geq a_5 \geq a_6 \geq a_7 \geq a_8, \quad a_1 + a_2 + \cdots + a_8 = 8,
\]
then
\[
5 \left( \frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_8^2} \right) + 72 \geq 14 \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_8} \right).
\]

4.10. If \( a, b, c, d \) are positive real numbers so that
\[
a \leq b \leq 1 \leq c \leq d, \quad abcd = 1,
\]
then
\[
\frac{7 - 6a}{2 + a^2} + \frac{7 - 6b}{2 + b^2} + \frac{7 - 6c}{2 + c^2} + \frac{7 - 6d}{2 + d^2} \geq \frac{4}{3}.
\]
4.11. If $a, b, c$ are positive real numbers so that
\[ a \leq b \leq 1 \leq c, \quad abc = 1, \]
then
\[ \frac{7 - 4a}{2 + a^2} + \frac{7 - 4b}{2 + b^2} + \frac{7 - 4c}{2 + c^2} \geq 3. \]

4.12. If $a, b, c$ are positive real numbers so that
\[ a \geq 1 \geq b \geq c, \quad abc = 1, \]
then
\[ \frac{23 - 8a}{3 + 2a^2} + \frac{23 - 8b}{3 + 2b^2} + \frac{23 - 8c}{3 + 2c^2} \geq 9. \]

4.13. Let $a_1, a_2, \ldots, a_n$ be positive real numbers so that
\[ a_1 \leq \cdots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1a_2\cdots a_n = 1. \]
If $p, q \geq 0$ so that $p + 3q \geq 1$, then
\[ \frac{1 - a_1}{1 + pa_1 + qa_1^2} + \frac{1 - a_2}{1 + pa_2 + qa_2^2} + \cdots + \frac{1 - a_n}{1 + pa_n + qa_n^2} \geq 0. \]

4.14. If $a, b, c, d, e$ are real numbers so that
\[ -2 \leq a \leq b \leq 1 \leq c \leq d \leq e, \quad a + b + c + d + e = 5, \]
then
\[ \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{e^2} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}. \]
4.3 Solutions

P 4.1. If \(a, b, c, d\) are real numbers so that
\[a \leq 1 \leq b \leq c \leq d, \quad a + b + c + d = 4,\]
then
\[\frac{a}{3a^2 + 1} + \frac{b}{3b^2 + 1} + \frac{c}{3c^2 + 1} + \frac{d}{3d^2 + 1} \leq 1.\]

Solution. Write the inequality as
\[f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a + b + c + d}{4} = 1,\]
where
\[f(u) = \frac{-u}{3u^2 + 1}, \quad u \in \mathbb{R}.\]

From
\[f'(u) = \frac{3u^2 - 1}{(3u^2 + 1)^2},\]
it follows that \(f\) is increasing on \((-\infty, -s_0] \cup [s_0, \infty)\) and decreasing on \([-s_0, s_0]\), where \(s_0 = 1/\sqrt{3}\). Since
\[\lim_{u \to -\infty} f(u) = 0\]
and \(f(s_0) < 0\), it follows that
\[\min_{u \in \mathbb{R}} f(u) = f(s_0).\]

From
\[f''(u) = \frac{18u(1-u^2)}{(3u^2 + 1)^3},\]
it follows that \(f\) is convex on \([0, 1]\), hence on \([s_0, 1]\). Therefore, we may apply the \(\text{LPCF-OV\ Theorem}\) for \(n = 4\) and \(m = 1\). We only need to show that \(f(x) + f(y) \geq 2f(1)\) for all real \(x, y\) so that \(x + y = 2\). Using Note 1, it suffices to prove that \(h(x, y) \geq 0\), where
\[h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.\]

Indeed, we have
\[g(u) = \frac{3u - 1}{4(3u^2 + 1)},\]
\[h(x, y) = \frac{3(1 + x + y - 3xy)}{4(3x^2 + 1)(3y^2 + 1)} = \frac{9(1 - xy)}{4(3x^2 + 1)(3y^2 + 1)} \geq 0,\]
since
\[ 4(1 - xy) = (x + y)^2 - 4xy = (x - y)^2 \geq 0. \]
Thus, the proof is completed. The equality holds for \( a = b = c = d = 1. \)

**Remark.** Similarly, we can prove the following generalization:

- If \( a_1, a_2, \ldots, a_n \) are real numbers so that
  \[ a_1 \leq 1 \leq a_2 \leq \cdots \leq a_n, \quad a_1 + a_2 + \cdots + a_n = n, \]
then
\[ \frac{a_1}{3a_1^2 + 1} + \frac{a_2}{3a_2^2 + 1} + \cdots + \frac{a_n}{3a_n^2 + 1} \leq \frac{n}{4}, \]
with equality for \( a_1 = a_2 = \cdots = a_n = 1. \)

\( \square \)

**P 4.2.** If \( a, b, c, d \) are real numbers so that
\[ a \geq b \geq 1 \geq c \geq d, \quad a + b + c + d = 4, \]
then
\[ \frac{16a - 5}{32a^2 + 1} + \frac{16b - 5}{32b^2 + 1} + \frac{16c - 5}{32c^2 + 1} + \frac{16d - 5}{32d^2 + 1} \leq \frac{4}{3}. \]
(Vasile C., 2012)

**Solution.** Write the inequality as
\[ f(a) + f(b) + f(c) + f(d) \geq 4f(s), \quad s = \frac{a + b + c + d}{4} = 1, \]
where
\[ f(u) = \frac{5 - 16u}{32u^2 + 1}, \quad u \in \mathbb{R}. \]
As shown in the proof of P 3.1, \( f \) is convex on \([s_0, 1]\), increasing for \( u \geq s_0 \) and
\[ \min_{u \in \mathbb{R}} f(u) = f(s_0), \]
where
\[ s_0 = \frac{5 + \sqrt{33}}{16} \approx 0.6715. \]
Therefore, we may apply the LPCF-OV Theorem for \( n = 4 \) and \( m = 2 \). We only need to show that \( f(x) + 2f(y) \geq 3f(1) \) for all real \( x, y \) so that \( x + 2y = 3 \). Using Note 1, it suffices to prove that \( h(x, y) \geq 0 \), where
\[ h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}. \]
Indeed, we have
\[ g(u) = \frac{32(2u - 1)}{3(32u^2 + 1)}, \]
\[ h(x, y) = \frac{64(1 + 16x + 16y - 32xy)}{3(32x^2 + 1)(32y^2 + 1)} = \frac{64(4x - 5)^2}{3(32x^2 + 1)(32y^2 + 1)} \geq 0. \]

From \( x + 2y = 3 \) and \( h(x, y) = 0 \), we get \( x = 5/4 \) and, \( y = 7/8 \). Therefore, in accordance with Note 3, the equality holds for \( a = b = c = d = 1 \), and also for \( a = 5/4, b = 1, c = d = 7/8 \).

**Remark.** Similarly, we can prove the following generalization:

- If \( a_1, a_2, \ldots, a_n \) \( (n \geq 3) \) are real numbers so that
  \[ a_1 \geq \cdots \geq a_{n-2} \geq 1 \geq a_{n-1} \geq a_n, \quad a_1 + a_2 + \cdots + a_n = n, \]

then
  \[ \frac{16a_1 - 5}{32a_1^2 + 1} + \frac{16a_2 - 5}{32a_2^2 + 1} + \cdots + \frac{16a_n - 5}{32a_n^2 + 1} \leq \frac{n}{3}, \]

with equality for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for
  \[ a_1 = \frac{5}{4}, \quad a_2 = \cdots = a_{n-2} = 1, \quad a_{n-1} = a_n = \frac{7}{8}. \]

P 4.3. If \( a, b, c, d, e \) are real numbers so that

\[ a \geq b \geq 1 \geq c \geq d \geq e, \quad a + b + c + d + e = 5, \]

then

\[ \frac{18a - 5}{12a^2 + 1} + \frac{18b - 5}{12b^2 + 1} + \frac{18c - 5}{12c^2 + 1} + \frac{18d - 5}{12d^2 + 1} + \frac{18e - 5}{12e^2 + 1} \leq 5. \]

*(Vasile C., 2012)*

**Solution.** Write the inequality as

\[ f(a) + f(b) + f(c) + f(d) + f(e) \geq 5f(s), \quad s = \frac{a + b + c + d + e}{5} = 1, \]

where

\[ f(u) = \frac{5 - 18u}{12u^2 + 1}, \quad u \in \mathbb{R}. \]

As shown in the proof of P 3.2, \( f \) is convex on \([s_0, 1]\), increasing for \( u \geq s_0 \) and

\[ \min_{u \in \mathbb{R}} f(u) = f(s_0), \]
where
\[ s_0 = \frac{5 + \sqrt{52}}{18} \approx 0.678. \]

Therefore, applying the LPCF-OV Theorem for \( n = 5 \) and \( m = 3 \), we only need to show that \( f(x) + 3f(y) \geq 4f(1) \) for all real \( x, y \) so that \( x + 3y = 4 \). Using Note 1, it suffices to prove that \( h(x, y) \geq 0 \), where
\[
\begin{align*}
  h(x, y) &= \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.
\end{align*}
\]

Indeed, we have
\[
\begin{align*}
  g(u) &= \frac{6(2u - 1)}{12u^2 + 1}, \\
  h(x, y) &= \frac{12(1 + 6x + 6y - 12xy)}{(12x^2 + 1)(12y^2 + 1)} = \frac{12(2x - 3)^2}{(12x^2 + 1)(12y^2 + 1)} \geq 0.
\end{align*}
\]

From \( x + 3y = 4 \) and \( h(x, y) = 0 \), we get \( x = 3/2 \) and \( y = 5/6 \). Therefore, in accordance with Note 3, the equality holds for \( a = b = c = d = e = 1 \), and also for \( a = 3/2, \quad b = 1, \quad c = d = e = 5/6 \).

**Remark.** Similarly, we can prove the following generalization:

- If \( a_1, a_2, \ldots, a_n \) \((n \geq 4)\) are real numbers so that
  \[ a_1 \geq \cdots \geq a_{n-3} \geq 1 \geq a_{n-2} \geq a_{n-1} \geq a_n, \quad a_1 + a_2 + \cdots + a_n = n, \]
  then
  \[
  \frac{18a_1 - 5}{12a_1^2 + 1} + \frac{18a_2 - 5}{12a_2^2 + 1} + \cdots + \frac{18a_n - 5}{12a_n^2 + 1} \leq n,
  \]
  with equality for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for
  \[
  a_1 = \frac{3}{2}, \quad a_2 = \cdots = a_{n-3} = 1, \quad a_{n-2} = a_{n-1} = a_n = \frac{5}{6}.
  \]

\[ \square \]

**P 4.4.** If \( a, b, c, d, e \) are real numbers so that
\[
  a \geq b \geq 1 \geq c \geq d \geq e, \quad a + b + c + d + e = 5,
\]
then
\[
  \frac{a(a - 1)}{3a^2 + 4} + \frac{b(b - 1)}{3b^2 + 4} + \frac{c(c - 1)}{3c^2 + 4} + \frac{d(d - 1)}{3d^2 + 4} + \frac{e(e - 1)}{3e^2 + 4} \geq 0.
\]

(Vasile C., 2012)
**Solution.** Write the inequality as

\[ f(a) + f(b) + f(c) + f(d) + f(e) \geq 5f(s), \quad s = \frac{a + b + c + d + e}{5} = 1, \]

where

\[ f(u) = \frac{u^2 - u}{3u^2 + 4}, \quad u \in \mathbb{R}. \]

As shown in the proof of P 3.5, \( f \) is convex on \([s_0, 1]\), increasing for \( u \geq s_0 \) and

\[ \min_{u \in \mathbb{R}} f(u) = f(s_0), \]

where

\[ s_0 = \frac{-4 + 2\sqrt{7}}{3} \approx 0.43. \]

Therefore, we may apply the LPCF-OV Theorem for \( n = 5 \) and \( m = 2 \). We only need to show that \( f(x) + 3f(y) \geq 4f(1) \) for all real \( x, y \) so that \( x + 3y = 4 \). Using Note 1, it suffices to prove that \( h(x, y) \geq 0 \). Indeed, we have

\[ g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{u}{3u^2 + 4}, \]

\[ h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{4 - 3xy}{(3x^2 + 4)(3y^2 + 4)} = \frac{(x - 2)^2}{(12x^2 + 1)(12y^2 + 1)} \geq 0. \]

From \( x + 3y = 4 \) and \( h(x, y) = 0 \), we get \( x = 2 \) and \( y = 2/3 \). Therefore, in accordance with Note 3, the equality holds for

\[ a = b = c = d = e = 1, \]

and also for

\[ a = 2, \ b = 1, \ c = d = e = \frac{2}{3}. \]

**Remark.** Similarly, we can prove the following generalizations:

- If \( a_1, a_2, \ldots, a_n (n \geq 4) \) are real numbers so that

\[ a_1 \geq \cdots \geq a_{n-3} \geq 1 \geq a_{n-2} \geq a_{n-1} \geq a_n, \quad a_1 + a_2 + \cdots + a_n = n, \]

then

\[ \frac{a_1(a_1 - 1)}{3a_1^2 + 4} + \frac{a_2(a_2 - 1)}{3a_2^2 + 4} + \cdots + \frac{a_n(a_n - 1)}{3a_n^2 + 4} \geq 0, \]

with equality for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for

\[ a_1 = 2, \ a_2 = \cdots = a_{n-3} = 1, \ a_{n-2} = a_{n-1} = a_n = \frac{2}{3}. \]
If $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ are real numbers so that

\[ a_1 \geq a_2 \geq 1 \geq a_3 \geq \cdots \geq a_n, \quad a_1 + a_2 + \cdots + a_n = n, \]

then

\[ \frac{a_1(a_1 - 1)}{4(n-2)a_1^2 + (n-1)^2} + \frac{a_2(a_2 - 1)}{4(n-2)a_2^2 + (n-1)^2} + \cdots + \frac{a_n(a_n - 1)}{4(n-2)a_n^2 + (n-1)^2} \geq 0, \]

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = \frac{n-1}{2}$, $a_2 = 1$, $a_3 = \cdots = a_n = \frac{n-1}{2(n-2)}$.

\[ \square \]

**P 4.5.** Let $a_1, a_2, \ldots, a_{2n} \neq -k$ be real numbers so that

\[ a_1 \geq \cdots \geq a_n \geq 1 \geq a_{n+1} \geq \cdots \geq a_{2n}, \quad a_1 + a_2 + \cdots + a_{2n} = 2n. \]

If $k \geq \frac{n+1}{2\sqrt{n}}$, then

\[ \frac{a_1(a_1 - 1)}{(a_1 + k)^2} + \frac{a_2(a_2 - 1)}{(a_2 + k)^2} + \cdots + \frac{a_{2n}(a_{2n} - 1)}{(a_{2n} + k)^2} \geq 0. \]

(Vasile C., 2012)

**Solution.** Write the inequality as

\[ f(a_1) + f(a_2) + \cdots + f(a_{2n}) \geq 2nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_{2n}}{2n} = 1, \]

where

\[ f(u) = \frac{u(u-1)}{(u+k)^2}, \quad u \in \mathbb{I} = \mathbb{R} \setminus \{-k\}. \]

As shown in the proof of P 3.8, $f$ is convex on $[s_0, 1]$, increasing for $u \geq s_0$ and

\[ \min_{u \in \mathbb{I}} f(u) = f(s_0), \]

where

\[ s_0 = \frac{k}{2k+1} < 1. \]

Having in view Note 4, we may apply the LPCF-OV Theorem for $2n$ real numbers and $m = n$. We only need to show that $f(x) + nf(y) \geq (n+1)f(1)$ for $x, y \in \mathbb{I}$ so that $x + ny = n + 1$. Using Note 1, it suffices to prove that $h(x, y) \geq 0$. We have

\[ g(u) = \frac{f(u) - f(1)}{u-1} = \frac{u}{(u+k)^2}, \]
\[
\begin{align*}
  h(x, y) &= \frac{g(x) - g(y)}{x - y} = \frac{k^2 - xy}{(x + k)^2(y + k)^2} \geq 0,
  \\
  k^2 - xy &\geq \frac{(n + 1)^2}{4n} - xy = \frac{(x + ny)^2}{4n} - xy = \frac{(x - ny)^2}{4n} \geq 0.
\end{align*}
\]

The equality holds for \(a_1 = a_2 = \cdots = a_n = 1\). If \(k = \frac{n + 1}{2n}\), then the equality holds also for

\[
a_1 = \frac{n + 1}{2}, \quad a_2 = \cdots = a_n = 1, \quad a_{n+1} = \cdots = a_{2n} = \frac{n + 1}{2n}.
\]

\(\square\)

**P 4.6.** Let \(a_1, a_2, \ldots, a_{2n} \neq -k\) be real numbers so that

\[
a_1 \geq \cdots \geq a_n \geq 1 \geq a_{n+1} \geq \cdots \geq a_{2n}, \quad a_1 + a_2 + \cdots + a_{2n} = 2n.
\]

If \(k \geq 1 + \frac{n + 1}{\sqrt{n}}\), then

\[
\frac{a_1^2 - 1}{(a_1 + k)^2} + \frac{a_2^2 - 1}{(a_2 + k)^2} + \cdots + \frac{a_{2n}^2 - 1}{(a_{2n} + k)^2} \geq 0.
\]

\((\text{Vasile C., 2012})\)

**Solution.** Write the inequality as

\[
f(a_1) + f(a_2) + \cdots + f(a_{2n}) \geq 2nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_{2n}}{2n} = 1,
\]

where

\[
f(u) = \frac{u^2 - 1}{(u + k)^2}, \quad u \in \mathbb{I} = \mathbb{R} \setminus \{-k\}.
\]

As shown in the proof of P 3.9, \(f\) is convex on \([s_0, 1]\), increasing for \(u \geq s_0\) and

\[
\min_{u \in \mathbb{I}} f(u) = f(s_0),
\]

where

\[
s_0 = \frac{-1}{k} \in (-1, 0).
\]

According to Note 4, we may apply the LPCF-OV Theorem for \(2n\) real numbers and \(m = n\). Thus, we only need to show that \(f(x) + nf(y) \geq (n + 1)f(1)\) for \(x, y \in \mathbb{I}\) so that \(x + ny = n + 1\). Using Note 1, it suffices to prove that \(h(x, y) \geq 0\). We have

\[
g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{u + 1}{(u + k)^2},
\]
\[ h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{(k - 1)^2 - 1 - x - y - xy}{(x + k)^2(y + k)^2} \geq 0, \]
because
\[ (k - 1)^2 - 1 - x - y - xy \geq \frac{(n + 1)^2}{n} - 1 - x - y - xy = \frac{(ny - 1)^2}{n} \geq 0. \]
The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \). If \( k = 1 + \frac{n + 1}{\sqrt{n}} \), then the equality holds also for
\[ a_1 = n, \quad a_2 = \cdots = a_n = 1, \quad a_{n+1} = \cdots = a_{2n} = \frac{1}{n}. \]

\[ \Box \]

**P 4.7.** If \( a_1, a_2, \ldots, a_n \) are positive real numbers so that
\[ a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n, \quad a_1 + a_2 + \cdots + a_n = n, \]
then
\[ a_1^{3/a_1} + a_2^{3/a_2} + \cdots + a_n^{3/a_n} \leq n. \]

*(Vasile C., 2012)*

**Solution.** Rewrite the desired inequality as
\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1, \]
where
\[ f(u) = -u^{3/u}, \quad u \in (0, n). \]
We have
\[ f'(u) = 3u^{3-2}(\ln u - 1), \]
\[ f''(u) = 3u^{3-4} g(t), \quad g(t) = u + (1 - \ln u)(2u - 3 + 3 \ln u). \]
From the expression of \( f' \), it follows that \( f \) is decreasing on \((0, s_0] \) and increasing on \([s_0, n) \), where
\[ s_0 = e. \]
In addition, we claim that \( f''(u) \geq 0 \) for \( u \in [1, e] \). If \( u \in [3/2, e] \), then
\[ g(t) > (1 - \ln u)(2u - 3) \geq 0. \]
Also, for \( u \in [1, 3/2] \), we have
\[ g(t) = 3(u - 1) + (6 - 2u - 3 \ln u) \ln u \geq (6 - 2u - 3 \ln u) \ln u \geq 3 \left( 1 - \ln \frac{3}{2} \right) \ln u > 0. \]
Since $f$ is convex on $[1, s_0]$, we may apply the RPCF-OV Theorem for $m = n - 1$. We only need to show that $f(x) + f(y) \geq 2f(1)$ for all $x, y > 0$ so that $x + y = 2$. The inequality $f(x) + f(y) \geq 2f(1)$ is equivalent to

$$x^{3/x} + y^{3/y} \leq 2,$$

which is just the inequality in P 3.32 from Volume 2. The equality holds for

$$a_1 = a_2 = \cdots = a_n = 1.

\square

**P 4.8.** If $a_1, a_2, \ldots, a_{11}$ are real numbers so that

$$a_1 \geq a_2 \geq 1 \geq a_3 \geq \cdots \geq a_{11}, \quad a_1 + a_2 + \cdots + a_{11} = 11,$$

then

$$(1 - a_1 + a_1^2)(1 - a_2 + a_2^2) \cdots (1 - a_{11} + a_{11}^2) \geq 1.$$

*(Vasile C., 2012)*

**Solution.** Rewrite the desired inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_{11}) \geq 11f(s), \quad s = \frac{a_1 + a_2 + \cdots + a_{11}}{11} = 1,$$

where

$$f(u) = \ln(1 - u + u^2), \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{2u - 1}{1 - u + u^2},$$

it follows that $f$ is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = 1/2.$$

Also, from

$$f''(u) = \frac{1 + 2u(1 - u)}{(1 - u + u^2)^2},$$

it follows that $f''(u) > 0$ for $u \in [s_0, 1]$, hence $f$ is convex on $[s_0, 1]$. Therefore, applying the LPCF-OV Theorem for $n = 11$ and $m = 2$, we only need to show that $f(x) + 9f(y) \geq 9f(1)$ for all real $x, y$ so that $x + 9y = 10$. Using Note 2, it suffices to prove that $H(x, y) \geq 0$, where

$$H(x, y) = \frac{f'(x) - f'(y)}{x - y} = \frac{1 + x + y - 2xy}{(1 - x + x^2)(1 - y + y^2)}.$$
Since
\[ 1 + x + y - 2xy = 18y^2 - 8y + 1 = 2y^2 + (4y - 1)^2 > 0, \]
the conclusion follows. The equality holds for \( a_1 = a_2 = \cdots = a_{11} = 1. \)

**Remark.** By replacing \( a_1, a_2, \ldots, a_{11} \) respectively with \( 1 - a_1, 1 - a_2, \ldots, 1 - a_{11} \), we get the following statement.

- If \( a_1, a_2, \ldots, a_{11} \) are real numbers so that
  \[ a_1 \leq a_2 \leq 0 \leq a_3 \leq \cdots \leq a_{11}, \quad a_1 + a_2 + \cdots + a_{11} = 0, \]
then
\[ (1 - a_1 + a_1^2)(1 - a_2 + a_2^2) \cdots (1 - a_{11} + a_{11}^2) \geq 1, \]
with equality for \( a_1 = a_2 = \cdots = a_n = 0. \)

\[ \square \]

**P 4.9.** If \( a_1, a_2, \ldots, a_8 \) are nonzero real numbers so that
\[ a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 \geq a_7 \geq a_8, \quad a_1 + a_2 + \cdots + a_8 = 8, \]
then
\[ 5 \left( \frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_8^2} \right) + 72 \geq 14 \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_8} \right). \]

(Vasile C., 2012)

**Solution.** Write the desired inequality as
\[ f(a_1) + f(a_2) + \cdots + f(a_8) \geq 8f(s), \quad s = \frac{a_1 + a_2 + \cdots + a_8}{8} = 1, \]
where
\[ f(u) = \frac{5}{u^2} - \frac{14}{u} + 9, \quad u \in \mathbb{I} = \mathbb{R} \setminus \{0\}. \]

As shown in the proof of P 3.24, \( f \) is convex on \([s_0, 1]\), increasing for \( u \geq s_0 \) and
\[ \min_{u \in \mathbb{I}} f(u) = f(s_0), \]
where
\[ s_0 = \frac{5}{7}. \]

Taking into account Note 4, we may apply the LPCF-OV Theorem for \( n = 8 \) and \( m = 4 \). We only need to show that \( f(x) + 4f(y) \geq 5f(1) \) for \( x, y \in \mathbb{I} \) so that \( x + 4y = 5 \). Using Note 1, it suffices to prove that \( h(x, y) \geq 0 \). Indeed, we have
\[ g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{9}{u} - \frac{5}{u^2}. \]
\[ h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{5(x + y) - 9xy}{x^2 y^2} = \frac{(x + 4y)(x + y) - 9xy}{x^2 y^2} = \frac{(x - 2y)^2}{x^2 y^2} \geq 0. \]

In accordance with Note 3, the equality holds for \( a_1 = a_2 = \cdots = a_8 = 1 \), and also for \( a_1 = \frac{5}{3}, \quad a_2 = a_3 = a_4 = 1, \quad a_5 = a_6 = a_7 = a_8 = \frac{5}{6}. \)

\[ \Box \]

**P 4.10.** If \( a, b, c, d \) are positive real numbers so that
\[ a \leq b \leq 1 \leq c \leq d, \quad abcd = 1, \]
then
\[ \frac{7 - 6a}{2 + a^2} + \frac{7 - 6b}{2 + b^2} + \frac{7 - 6c}{2 + c^2} + \frac{7 - 6d}{2 + d^2} \geq \frac{4}{3}. \]

(Vasile C., 2012)

**Solution.** Using the substitution
\[ a = e^x, \quad b = e^y, \quad c = e^z, \quad d = e^w, \]
we need to show that
\[ f(x) + f(y) + f(z) + f(w) \geq 4f(s), \]
where
\[ x \leq y \leq 0 \leq z \leq w, \quad s = \frac{x + y + z + w}{4} = 0, \]
\[ f(u) = \frac{7 - 6e^u}{2 + e^{2u}}, \quad u \in \mathbb{R}. \]

As shown in the proof of P 3.25, \( f \) is convex on \([0, s_0]\), is decreasing on \((-\infty, s_0]\) and increasing on \([s_0, \infty)\), where
\[ s_0 = \ln 3. \]

Therefore, we may apply the RPCF-OV Theorem for \( n = 4 \) and \( m = 2 \). We only need to show that \( f(x) + 2f(y) \geq 3f(0) \) for all real \( x, y \) so that \( x + 2y = 0 \); that is, to prove that
\[ \frac{7 - 6a}{2 + a^2} + \frac{2(7 - 6d)}{2 + d^2} \geq 1 \]
for \( a, d > 0 \) so that \( ad^2 = 1 \). This is equivalent to
\[ (d - 1)^2(d - 2)^2(5d^2 + 6d + 3) \geq 0, \]
which is clearly true. In accordance with Note 3, the equality holds for \(a = b = c = d = 1\), and also for
\[
\frac{1}{4}, \quad b = 1, \quad c = d = 2.
\]

P 4.11. If \(a, b, c\) are positive real numbers so that
\[
a \leq b \leq 1 \leq c, \quad abc = 1,
\]
then
\[
\frac{7 - 4a}{2 + a^2} + \frac{7 - 4b}{2 + b^2} + \frac{7 - 4c}{2 + c^2} \geq 3.
\]

(Vasile C., 2012)

Solution. Using the substitution
\[
a = e^x, \quad b = e^y, \quad c = e^z,
\]
we need to show that
\[
f(x) + f(y) + f(z) \geq 3f(s),
\]
where
\[
x \leq y \leq 0 \leq z, \quad s = \frac{x + y + z}{3} = 0,
\]
\[
f(u) = \frac{7 - 4e^u}{2 + e^{2u}}, \quad u \in \mathbb{R}.
\]
From
\[
f'(u) = \frac{2e^u(2e^u + 1)(e^u - 4)}{(2 + e^{2u})^2},
\]
it follows that \(f\) is decreasing on \((−\infty, s_0]\) and increasing on \([s_0, \infty)\), where
\[
s_0 = \ln 4.
\]
Also, we have
\[
f''(u) = \frac{4t \cdot h(t)}{(2 + t^2)^3}, \quad t = e^u,
\]
where
\[
h(t) = -t^4 + 7t^3 + 12t^2 - 14t - 4.
\]
We will show that \(h(t) \geq 0\) for \(t \in [1, 4]\), hence \(f\) is convex on \([0, s_0]\). Indeed,
\[
h(t) = (t - 1)[t^2(-t + 6) + 18t + 4] \geq 0.
\]
Therefore, we may apply the RPCF-OV Theorem for \( n = 3 \) and \( m = 2 \). We only need to show that \( f(x) + f(y) \geq 2f(0) \) for all real \( x, y \) so that \( x + y = 0 \). That is, to prove that

\[
\frac{7 - 4a}{2 + a^2} + \frac{7 - 4b}{2 + b^2} \geq 2
\]

for all \( a, b > 0 \) so that \( ab = 1 \). This is equivalent to \((a - 1)^4 \geq 0\).

The equality holds for \( a = b = c = 1 \).

\(\square\)

**P 4.12.** If \( a, b, c \) are positive real numbers so that

\[
a \geq 1 \geq b \geq c, \quad abc = 1,
\]

then

\[
\frac{23 - 8a}{3 + 2a^2} + \frac{23 - 8b}{3 + 2b^2} + \frac{23 - 8c}{3 + 2c^2} \geq 9.
\]

(Vasile C., 2012)

**Solution.** Using the substitution

\[
a = e^x, \quad b = e^y, \quad c = e^z,
\]

we need to show that

\[
f(x) + f(y) + f(z) \geq 3f(s),
\]

where

\[
x \geq 1 \geq y \geq z, \quad s = \frac{x + y + z}{3} = 0,
\]

\[
f(u) = \frac{23 - 8e^u}{3 + 2e^{2u}}, \quad u \in \mathbb{R}.
\]

From

\[
f'(u) = \frac{4e^u(4e^u + 1)(e^u - 6)}{(3 + 2e^{2u})^2},
\]

it follows that \( f \) is decreasing on \((-\infty, s_0]\) and increasing on \([s_0, \infty)\), where \( s_0 = \ln 6 \). Also, we have

\[
f''(u) = \frac{8t \cdot h(t)}{(3 + 2t^2)^3}, \quad t = e^u,
\]

where

\[
h(t) = -4t^4 + 46t^3 + 36t^2 - 69t - 9.
\]

We will show that \( h(t) \geq 0 \) for \( t \in [1, 6] \), hence \( f \) is convex on \([0, s_0]\). Indeed,

\[
h(t) = (t - 1)(2t + 3)[2t(-t + 12) + 3] \geq 0.
\]
Therefore, we may apply the RPCF-OV Theorem for $n = 3$ and $m = 2$. We only need to show that $f(x) + f(y) \geq 2f(0)$ for all real $x, y$ so that $x + y = 0$. That is, to prove that
\[
\frac{23 - 8a}{3 + 2a^2} + \frac{23 - 8b}{3 + 2b^2} \geq 6.
\]
for all $a, b > 0$ so that $ab = 1$. This is equivalent to
\[(a - 1)^4 \geq 0.
\]
The equality holds for $a = b = c = 1$.

\[
P 4.13. \text{ Let } a_1, a_2, \ldots, a_n \text{ be positive real numbers so that } a_1 \leq \cdots \leq a_{n-1} \leq 1 \leq a_n, \quad a_1 a_2 \cdots a_n = 1.
\]

If $p, q \geq 0$ so that $p + 3q \geq 1$, then
\[
\frac{1 - a_1}{1 + pa_1 + qa_1^2} + \frac{1 - a_2}{1 + pa_2 + qa_2^2} + \cdots + \frac{1 - a_n}{1 + pa_n + qa_n^2} \geq 0.
\]
(Vasile C., 2012)

\[
\text{Solution.} \text{ For } q = 0, \text{ we need to show that } p \geq 1 \text{ involves }
\[
\frac{1 - a_1}{1 + pa_1} + \frac{1 - a_2}{1 + pa_2} + \cdots + \frac{1 - a_n}{1 + pa_n} \geq 0.
\]
This is just the inequality from P 2.24. Consider next that $q > 0$. Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \ldots, n$, we need to show that
\[
f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s),
\]
where
\[
x_1 \leq \cdots \leq x_{n-1} \leq 0 \leq x_n, \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,
\]
\[
f(u) = \frac{1 - e^u}{1 + pe^u + qe^{2u}}, \quad u \in \mathbb{R}.
\]
As shown in the proof of P 3.29, if $p + 3q - 1 \geq 0$, then $f$ is convex on $[0, s_0]$, where
\[
s_0 = \ln r_0 > 0, \quad r_0 = 1 + \sqrt{1 + \frac{p + 1}{q}}.
\]
In addition, $f$ is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$. Therefore, we may apply the RPCF-OV Theorem for $m = n - 1$. We only need to show that $f(x) + f(y) \geq 2f(0)$ for all real $x, y$ so that $x + y = 0$; that is, to prove that
\[
\frac{1 - a}{1 + pa + qa^2} + \frac{1 - b}{1 + pb + qb^2} \geq 0
\]
for $a, b > 0$ so that $ab = 1$. This is equivalent to

$$(a - 1)^2[(p - 1)a + q(a^2 + a + 1)] \geq 0,$$

which is true because

$$(p - 1)a + q(a^2 + a + 1) \geq (p - 1)a + q(3a) = (p + 3q - 1)a \geq 0.$$ 

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. 

\[ P \text{ 4.14.} \] If $a, b, c, d, e$ are real numbers so that $\begin{align*} -2 \leq a & \leq b \leq 1 \leq c \leq d \leq e, & a + b + c + d + e & = 5, \end{align*}$

then

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{e^2} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}.$$ 

**Solution.** Write the inequality as

$$f(a) + f(b) + f(c) + f(d) + f(e) \geq 5f(s), \quad s = \frac{a + b + c + d + e}{5} = 1,$$

where

$$f(u) = \frac{1}{u^2} - \frac{1}{u}, \quad u \in I = [-2, 7] \setminus \{0\}.$$ 

Let

$$s_0 = 2, \quad s < s_0.$$ 

From

$$f(u) - f(2) = \frac{1}{u^2} - \frac{1}{u} + \frac{1}{4} = \frac{(u - 2)^2}{4u^2} \geq 0,$$

it follows that

$$\min_{u \in I} f(u) = f(s_0),$$

while from

$$f'(u) = \frac{u - 2}{u^3}, \quad f''(u) = \frac{2(3 - u)}{u^4},$$

it follows that $f$ is convex on $[s, s_0]$. We can't apply the the RPCF-OV Theorem because $f$ is not decreasing on $I_{s \leq s_0}$. According to Theorem 1 (applied for $n = 5$ and $m = 2$) and Note 6, we may replace this condition with $(1 + n - m)s - (n - m)s_0 \leq \inf I$. Indeed, we have

$$(1 + n - m)s - (n - m)s_0 = 4 - 6 = -2 = \inf I.$$
So, according to Note 1, it suffices to show that \( h(x, y) \geq 0 \) for all \( x, y \in \mathbb{I} \) so that \( x + 3y = 4 \). We have

\[
g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-1}{u^2},
\]

\[
h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{x + y}{x^2y^2} = \frac{2(x + 2)}{3x^2y^2} \geq 0.
\]

The proof is completed. By Note 3, the equality holds for \( a = b = c = d = e = 1 \), and also for

\[
a = -2, \quad b = 1, \quad c = d = e = 2.
\]
Chapter 5

EV Method for Nonnegative Variables

5.1 Theoretical Basis

The Equal Variables Method is an effective tool for solving some difficult symmetric inequalities.

**EV-Theorem** (Vasile Cirtoaje, 2005). Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be fixed nonnegative real numbers, and let

$$0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$$

so that

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n, \quad x_1^k + x_2^k + \cdots + x_n^k = a_1^k + a_2^k + \cdots + a_n^k,$$

where $k$ is a real number ($k \neq 1$); for $k = 0$, assume that

$$x_1 x_2 \cdots x_n = a_1 a_2 \cdots a_n.$$

Let $f$ be a real-valued function, continuous on $[0, \infty)$ and differentiable on $(0, \infty)$, so that the joined function

$$g(x) = f'\left(x^{\frac{1}{k-1}}\right)$$

is strictly convex on $(0, \infty)$. Then, the sum

$$S_n = f(x_1) + f(x_2) + \cdots + f(x_n)$$

is maximum for

$$x_1 = x_2 = \cdots = x_{n-1} \leq x_n,$$

and is minimum for

$$0 < x_1 \leq x_2 = x_3 = \cdots = x_n$$

or

$$0 = x_1 = \cdots = x_j \leq x_{j+1} = x_{j+2} = \cdots = x_n, \quad j \in \{1, 2, \ldots, n-1\}.$$

To prove the EV-Theorem, we need the EV-Lemma and the EV-Proposition below.
**EV-Lemma.** Let $a, b, c$ be fixed nonnegative real numbers, not all equal and at most one of them equal to zero, and let $x \leq y \leq z$ be nonnegative real numbers so that

$$x + y + z = a + b + c, \quad x^k + y^k + z^k = a^k + b^k + c^k,$$

where $k$ is a real number ($k \neq 1$); for $k = 0$, the second equation is $xyz = abc$. Then, there exist two nonnegative real numbers $m$ and $M$ so that $m < M$ and

1. $y \in [m, M]$;
2. $y = m$ if and only if $x = y < z$;
3. $y = M$ if and only if $0 < x \leq y = z$ or $0 = x < y \leq z$ (only for $k > 0$).

**Proof.** We show first, by contradiction method, that $x < z$. Indeed, if $x = z$, then

$$x = z \Rightarrow x = y = z \Rightarrow x^k + y^k + z^k = 3\left(\frac{x + y + z}{3}\right)^k \Rightarrow a^k + b^k + c^k = 3\left(\frac{a + b + c}{3}\right)^k \Rightarrow a = b = c,$$

which is false. Notice that the last implication follows from Jensen's inequalities

$$a^k + b^k + c^k \geq 3\left(\frac{a + b + c}{3}\right)^k, \quad k \in (-\infty, 0) \cup (1, \infty),$$

$$a^k + b^k + c^k \leq 3\left(\frac{a + b + c}{3}\right)^k, \quad k \in (0, 1),$$

$$abc \leq \left(\frac{a + b + c}{3}\right)^3, \quad k = 0,$$

where the equality holds if and only if $a = b = c$.

According to the relations

$$x + z = a + b + c - y, \quad x^k + z^k = a^k + b^k + c^k - y^k,$$

we may consider $x$ and $z$ as functions of $y$. From

$$x' + z' = -1, \quad x'^{-1}x' + z'^{-1}z' = -y'^{-1},$$

we get

$$x' = \frac{y^{k-1} - z^{k-1}}{z^{k-1} - x^{k-1}}, \quad z' = \frac{y^{k-1} - x^{k-1}}{x^{k-1} - z^{k-1}}. \quad (*)$$

Let us denote

$$f_0(y) = x(y), \quad f_1(y) = x(y) - y, \quad f_2(y) = z(y) - y.$$

From

$$0 \leq x(y) \leq y \leq z(y),$$
it follows that

\[ f_0(y) \geq 0, \quad f_1(y) \leq 0, \quad f_2(y) \geq 0. \]

Using (*), we get

\[ f'_0(y) = \frac{y^{k-1} - z^{k-1}}{z^{k-1} - x^{k-1}}, \]
\[ f'_1(y) = \frac{y^{k-1} - z^{k-1}}{z^{k-1} - x^{k-1}} - 1, \]
\[ f'_2(y) = \frac{y^{k-1} - x^{k-1}}{x^{k-1} - z^{k-1}} - 1. \]

Since \( f'_0(y) \leq 0, \) \( f'_1(y) \leq -1 \) and \( f'_2(y) \leq -1, \) the functions \( f_0, f_1 \) and \( f_2 \) are strictly decreasing. Thus, the inequality \( f_0(y) \geq 0 \) involves \( y \leq y_0, \) where \( y_0 \) is the root of the equation \( x(y) = 0, \) the inequality \( f_1(y) \leq 0 \) involves \( y \geq y_1, \) where \( y_1 \) is the root of the equation \( x(y) = y, \) and the inequality \( f_2(y) \geq 0 \) involves \( y \leq y_2, \) where \( y_2 \) is the root of the equation \( z(y) = y. \) Therefore, we have

\[ y \geq y_1, \quad y \leq \min\{y_0, y_2\}. \]

Denoting

\[ m = y_1, \quad M = \min\{y_0, y_2\}, \]

we get \( y \in [m, M], \) \( y = m \) if and only if \( x = y, \) and \( y = M \) if and only if \( x = 0 \) or \( y = z. \)

**EV-Proposition.** Let \( a, b, c \) be fixed nonnegative real numbers, and let \( 0 \leq x \leq y \leq z \)
so that

\[ x + y + z = a + b + c, \quad x^k + y^k + z^k = a^k + b^k + c^k, \]

where where \( k \) is a real number \( (k \neq 1); \) for \( k = 0, \) the second equation is \( xyz = abc. \)

Let \( f \) be a real-valued function, continuous on \([0, \infty)\) and differentiable on \((0, \infty), \)
so that the joined function

\[ g(x) = f' \left( x^{\frac{1}{k-1}} \right) \]

is strictly convex on \((0, \infty). \) Then, the sum

\[ S_3 = f(x) + f(y) + f(z) \]

(1) is maximum only when \( 0 \leq x = y \leq z; \)
(2) is minimum only when \( 0 \leq x \leq y = z \) or \( 0 = x \leq y \leq z \) (only if \( k > 0). \)

**Proof.** If \( a = b = c, \) then

\[ a^k + b^k + c^k = 3 \left( \frac{a + b + c}{3} \right)^k, \]

hence

\[ x^k + y^k + z^k = 3 \left( \frac{x + y + z}{3} \right)^k, \]
which involves \( x = y = z \). If two of \( a, b, c \) are equal to zero, then
\[
a^k + b^k + c^k = (a + b + c)^k,
\]
hence
\[
x^k + y^k + z^k = (x + y + z)^k,
\]
which involves \( x = y = 0 \). Consider further that \( a, b, c \) are not all equal and at most one of them is equal to zero. As shown in the proof of the EV-Lemma, we have \( x < z \). According to the relations
\[
x + z = a + b + c - y, \quad x^k + z^k = a^k + b^k + c^k - y^k,
\]
we may consider \( x \) and \( z \) as functions of \( y \). Thus, we have
\[
S_3 = f(x(y)) + f(y) + f(z(y)) := F(y).
\]
According to the EV-Lemma, it suffices to show that \( F \) is maximum for \( y = m \) and is minimum for \( y = M \). Using (*), we have
\[
F'(y) = x'f'(x) + f'(y) + z'f'(z)
\]
\[
= \frac{y^{k-1} - z^{k-1}}{z^{k-1} - x^{k-1}} g(x^{k-1}) + g(y^{k-1}) + \frac{y^{k-1} - x^{k-1}}{x^{k-1} - z^{k-1}} g(z^{k-1}),
\]
which, for \( x < y < z \), is equivalent to
\[
\frac{F'(y)}{(y^{k-1} - x^{k-1})(y^{k-1} - z^{k-1})} = \frac{g(x^{k-1})}{(x^{k-1} - y^{k-1})(x^{k-1} - z^{k-1})} + \frac{g(y^{k-1})}{(y^{k-1} - z^{k-1})(y^{k-1} - x^{k-1})} + \frac{g(z^{k-1})}{(z^{k-1} - x^{k-1})(z^{k-1} - y^{k-1})}.
\]
Since \( g \) is strictly convex, the right hand side is positive. Moreover, since
\[
(y^{k-1} - x^{k-1})(y^{k-1} - z^{k-1}) < 0,
\]
we have \( F'(y) < 0 \) for \( y \in (m, M) \), hence \( F \) is strictly decreasing on \( [m, M] \). Therefore, \( F \) is maximum for \( y = m \) and is minimum for \( y = M \).

**Proof of the EV-Theorem.** For \( n = 3 \), the EV-Theorem follows immediately from the EV-Proposition. Consider next that \( n \geq 4 \). Since \( X = \{x_1, x_2, \ldots, x_n\} \) is defined as a compact set in \( \mathbb{R}^+ \), \( S_n \) attains its minimum and maximum values. Using this property and the EV-Proposition, the EV-Theorem can be proved by the contradiction method. Thus, for the sake of contradiction, assume that \( S_n \) attains its maximum at \( (b_1, b_2, \ldots, b_n) \), where \( b_1 \leq b_2 \leq \cdots \leq b_n \) and \( b_1 < b_{n-1} \). Let \( x_1, x_{n-1} \) and \( x_n \) be real numbers so that \( x_1 \leq x_{n-1} \leq x_n \) and
\[
x_1 + x_{n-1} + x_n = b_1 + b_{n-1} + b_n, \quad x_1^k + x_{n-1}^k + x_n^k = b_1^k + b_{n-1}^k + b_n^k.
\]
According to the EV-Proposition, the sum \( f(x_1) + f(x_{n-1}) + f(x_n) \) is maximum for \( x_1 = x_{n-1} \), when
\[
f(x_1) + f(x_{n-1}) + f(x_n) > f(b_1) + f(b_{n-1}) + f(b_n).
\]
This result contradicts the assumption that \( S_n \) attains its maximum value at \((b_1, b_2, \ldots, b_n)\) with \( b_1 < b_{n-1} \). Similarly, we can prove that \( S_n \) is minimum when
\[
0 < x_1 \leq x_2 = \cdots = x_n
\]
or (only if \( k > 0 \))
\[
0 = x_1 = \cdots = x_j \leq x_{j+1} \leq x_{j+2} = \cdots = x_n, \quad j \in \{1, 2, \ldots, n-1\}.
\]

**Note 1.** The EV-Theorem can be also applied for the case in which
\[
a < x_1, x_2, \ldots, x_n < b, \quad a \geq 0,
\]
f is differentiable on \((a, b)\), \( g(x) \) is strictly convex for \( a < x^{\frac{1}{k}} < b \), and the sum \( S_n \) has a global maximum and/or minimum.

**Note 2.** The EV-Theorem can be also applied for the case in which
\[
a \leq x_1, x_2, \ldots, x_n \leq b, \quad a \geq 0
\]
f is continuous on \([a, b]\) and differentiable on \((a, b)\), \( g(x) \) is strictly convex for \( a \leq x^{\frac{1}{k}} \leq b \), and the sum \( S_n \) has a global maximum and/or minimum.

**Note 3.** The EV-Theorem can be also applied for the case in which
\[
a \leq x_1, x_2, \ldots, x_n < b, \quad a \geq 0
\]
f is continuous on \([a, b)\) and differentiable on \((a, b)\), \( g(x) \) is strictly convex for \( a \leq x^{\frac{1}{k}} < b \), and the sum \( S_n \) has a global maximum and/or minimum.

**Note 4.** The EV-Theorem can be also applied for the case in which
\[
a < x_1, x_2, \ldots, x_n \leq b, \quad a \geq 0
\]
f is continuous on \((a, b]\) and differentiable on \((a, b)\), \( g(x) \) is strictly convex for \( a < x^{\frac{1}{k}} \leq b \), and the sum \( S_n \) has a global maximum and/or minimum.

From the EV-Theorem and Notes above, we can obtain some interesting particular results, which are useful in many applications.

**Corollary 1.** Let \( a_1, a_2, \ldots, a_n \) (\( n \geq 3 \)) be fixed nonnegative real numbers, and let
\[
0 \leq x_1 \leq x_2 \leq \cdots \leq x_n
\]
so that
\[ x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n, \]
\[ x_1^2 + x_2^2 + \cdots + x_n^2 = a_1^2 + a_2^2 + \cdots + a_n^2. \]

Let \( f \) be a real-valued function, continuous on \([0, \infty)\) and differentiable on \((0, \infty)\), so that the joined function
\[ g(x) = f'(x) \]
is strictly convex on \((0, \infty)\). The sum
\[ S_n = f(x_1) + f(x_2) + \cdots + f(x_n) \]
is maximum when
\[ x_1 = x_2 = \cdots = x_{n-1} \leq x_n, \]
and is minimum when either
\[ 0 < x_1 \leq x_2 = x_3 = \cdots = x_n \]
or
\[ 0 = x_1 = \cdots = x_j \leq x_{j+1} = x_{j+2} = \cdots = x_n, \quad j \in \{1, 2, \ldots, n-1\}. \]

**Corollary 2.** Let \( a_1, a_2, \ldots, a_n \) \((n \geq 3)\) be fixed positive real numbers, and let
\[ 0 < x_1 \leq x_2 \leq \cdots \leq x_n \]
so that
\[ x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n, \]
\[ \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}. \]

Let \( f \) be a real-valued function, differentiable on \((0, \infty)\), so that the joined function
\[ g(x) = f'\left(\frac{1}{\sqrt{x}}\right) \]
is strictly convex on \((0, \infty)\). The sum
\[ S_n = f(x_1) + f(x_2) + \cdots + f(x_n) \]
is maximum when
\[ x_1 = x_2 = \cdots = x_{n-1} \leq x_n, \]
and is minimum when
\[ x_1 \leq x_2 = x_3 = \cdots = x_n \]

**Corollary 3.** Let \( a_1, a_2, \ldots, a_n \) \((n \geq 3)\) be fixed positive real numbers, and let
\[ 0 < x_1 \leq x_2 \leq \cdots \leq x_n \]
so that
\[ x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n, \quad x_1 x_2 \cdots x_n = a_1 a_2 \cdots a_n. \]

Let \( f \) be a real-valued function, differentiable on \((0, \infty)\), so that the joined function
\[ g(x) = f'(1/x) \]
is strictly convex on \((0, \infty)\). The sum
\[ S_n = f(x_1) + f(x_2) + \cdots + f(x_n) \]
is maximum when
\[ x_1 = x_2 = \cdots = x_{n-1} \leq x_n, \]
and is minimum when
\[ x_1 \leq x_2 = x_3 = \cdots = x_n. \]

**Note 5.** Corollaries 1, 2 and 3 are also valid under the conditions in Note 1, Note 2, Note 3 or Note 4.

**Corollary 4.** Let \( a_1, a_2, \ldots, a_n \ (n \geq 3) \) be fixed nonnegative real numbers, let
\[ 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \]
so that
\[ x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n, \quad x_1^k + x_2^k + \cdots + x_n^k = a_1^k + a_2^k + \cdots + a_n^k, \]
where \( k \) is a real number \((k \neq 0, k \neq 1)\).

1. For \( k < 0 \), the product \( P_n = x_1 x_2 \cdots x_n \) is maximum when
\[ 0 < x_1 \leq x_2 = x_3 = \cdots = x_n, \]
and is minimum when
\[ 0 < x_1 = x_2 = \cdots = x_{n-1} \leq x_n; \]

2. For \( k > 0 \), the product \( P_n = x_1 x_2 \cdots x_n \) is maximum when
\[ x_1 = x_2 = \cdots = x_{n-1} \leq x_n, \]
and is minimum when either
\[ 0 < x_1 \leq x_2 = x_3 = \cdots = x_n \]
or
\[ 0 = x_1 = \cdots = x_j \leq x_{j+1} \leq x_{j+2} = \cdots = x_n, \quad j \in \{1, 2, \ldots, n-1\}. \]
Proof. We apply the EV-Theorem and Note 1 to the function \( f(u) = k \ln u \). We have
\[
g(x) = kx^{\frac{1}{k-1}}, \quad g''(x) = \frac{k^2}{(k-1)^2}x^{\frac{2k-1}{k-1}}.
\]
Since \( g''(x) > 0 \) for \( x > 0 \), \( g \) is strictly convex on \((0, \infty)\).

**Corollary 5.** Let \( a_1, a_2, \ldots, a_n \) \((n \geq 3)\) be fixed nonnegative real numbers, and let

\[
0 \leq x_1 \leq x_2 \leq \cdots \leq x_n
\]

so that

\[
x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n, \quad x_1^k + x_2^k + \cdots + x_n^k = a_1^k + a_2^k + \cdots + a_n^k.
\]

Assume that the sum

\[
S_n = x_1^m + x_2^m + \cdots + x_n^m
\]

has a global extremum (maximum and/or minimum).

**Case 1:** \( k \leq 0 \) (for \( k = 0 \), assume that \( x_1x_2\cdots x_n = a_1a_2\cdots a_n > 0 \)).

(a) If \( m \in (k, 0) \cup (1, \infty) \), then \( S_n \) is maximum for

\[
0 < x_1 = x_2 = \cdots = x_{n-1} \leq x_n,
\]

and is minimum for

\[
0 < x_1 \leq x_2 = x_3 = \cdots = x_n;
\]

(b) If \( m \in (-\infty, k) \cup (0, 1) \), then \( S_n \) is minimum for

\[
0 < x_1 = x_2 = \cdots = x_{n-1} \leq x_n,
\]

and is maximum for

\[
0 < x_1 \leq x_2 = x_3 = \cdots = x_n.
\]

**Case 2:** \( 0 < k < 1 \).

(a) If \( m \in (0, k) \cup (1, \infty) \), then \( S_n \) is maximum for

\[
0 \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n,
\]

and is minimum for either

\[
0 < x_1 \leq x_2 = x_3 = \cdots = x_n
\]

or

\[
0 = x_1 = \cdots = x_j \leq x_{j+1} \leq x_{j+2} = \cdots = x_n, \quad j \in \{1, 2, \ldots, n-1\};
\]
(b) If $m \in (-\infty, 0)$, then $S_n$ is minimum for

$$0 < x_1 = x_2 = \cdots = x_{n-1} \leq x_n,$$

and is maximum for

$$0 < x_1 \leq x_2 = x_3 = \cdots = x_n;$$

(c) If $m \in (k, 1)$, then $S_n$ is minimum for

$$0 \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n,$$

and is maximum for either

$$0 < x_1 \leq x_2 = x_3 = \cdots = x_n$$

or

$$0 = x_1 = \cdots = x_j \leq x_{j+1} \leq x_{j+2} = \cdots = x_n, \quad j \in \{1, 2, \ldots, n-1\}.$$

Case 3: $k > 1$.

(a) If $m \in (0, 1) \cup (k, \infty)$, then $S_n$ is maximum for

$$0 \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n,$$

and is minimum for either

$$0 < x_1 \leq x_2 = x_3 = \cdots = x_n$$

or

$$0 = x_1 = \cdots = x_j \leq x_{j+1} \leq x_{j+2} = \cdots = x_n, \quad j \in \{1, 2, \ldots, n-1\};$$

(b) If $m \in (-\infty, 0)$, then $S_n$ is minimum for

$$0 < x_1 = x_2 = \cdots = x_{n-1} \leq x_n,$$

and is maximum for

$$0 < x_1 \leq x_2 = x_3 = \cdots = x_n;$$

(c) If $m \in (1, k)$, then $S_n$ is minimum for

$$0 \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n,$$

and is maximum for either

$$0 < x_1 \leq x_2 = x_3 = \cdots = x_n$$

or

$$0 = x_1 = \cdots = x_j \leq x_{j+1} \leq x_{j+2} = \cdots = x_n, \quad j \in \{1, 2, \ldots, n-1\}.$$
Proof. We apply the EV-Theorem and Note 1 to the function

\[ f(u) = m(m - 1)(m - k)u^m. \]

We have

\[ f'(u) = m^2(m - 2)(m - k)u^{m-1} \]

and

\[ g(x) = m^2(m - 1)(m - k)x^{m-1}, \quad g''(x) = \frac{m^2(m - 1)^2(m - k)^2}{(k - 1)^2} x^{\frac{1+m-2k}{k-1}}. \]

Since \( g''(x) > 0 \) for \( x > 0 \), \( g \) is strictly convex on \((0, \infty)\).

**Corollary 6.** Let \( a_1, a_2, \ldots, a_n \) \((n \geq 3)\) be fixed nonnegative real numbers, and let

\[ 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \]

so that

\[ x_1^p + x_2^p + \cdots + x_n^p = a_1^p + a_2^p + \cdots + a_n^p, \quad x_1^q + x_2^q + \cdots + x_n^q = a_1^q + a_2^q + \cdots + a_n^q, \]

where

\[ p, q \in \{1, 2, 3\}, \quad p \neq q. \]

The symmetric sum

\[ S_n = \sum_{1 \leq i_1 < i_2 < i_3 \leq n} x_{i_1} x_{i_2} x_{i_3} \]

is maximum for

\[ 0 \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n, \]

and is minimum for either

\[ 0 < x_1 \leq x_2 = x_3 = \cdots = x_n \]

or

\[ 0 = x_1 = \cdots = x_j \leq x_{j+1} \leq x_{j+2} = \cdots = x_n, \quad j \in \{1, 2, \ldots, n-1\}. \]

**Proof.** The statement follows by Corollary 5, taking into account that

\[ 6 \sum_{1 \leq i_1 < i_2 < i_3 \leq n} x_{i_1} x_{i_2} x_{i_3} = \left( \sum x_i \right)^3 - 3 \left( \sum x_i \right) \left( \sum x_i^2 \right) + 2 \sum x_i^3. \]

For \( p = 2 \) and \( q = 3 \), according to this identity, the sum \( \sum_{i\text{sym}} x_1 x_2 x_3 \) is maximum/minimum when \( \sum x_i \) is maximum/minimum. Therefore, we need to show that if

\[ x_1^2 + x_2^2 + \cdots + x_n^2 = \text{constant}, \quad x_1^3 + x_2^3 + \cdots + x_n^3 = \text{constant}, \]

then

\[ x_1^p + x_2^p + \cdots + x_n^p \]

is maximum/minimum when \( x_1 = x_2 = \cdots = x_n \).
then the sum $\sum x_1$ is maximum for

$$0 \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n,$$

and is minimum for either

$$0 < x_1 \leq x_2 = x_3 = \cdots = x_n$$

or

$$0 = x_1 = \cdots = x_j \leq x_{j+1} \leq x_{j+2} = \cdots = x_n, \quad j \in \{1, 2, \ldots, n-1\}.$$

This follows from Corollary 5 (case $k = 3/2$ and $m = 1/2$) by replacing $x_1, x_2, \ldots, x_n$ with $x_1^2, x_2^2, \ldots, x_n^2$. 
5.2 Applications

5.1. If \(a, b, c, d\) are nonnegative real numbers so that
\[
a + b + c + d = a^3 + b^3 + c^3 + d^3 = 2,
\]
then
\[
\frac{7}{4} \leq a^2 + b^2 + c^2 + d^2 \leq 2.
\]

5.2. If \(a_1, a_2, \ldots, a_9\) are nonnegative real numbers so that
\[
a_1 + a_2 + \cdots + a_9 = a_1^2 + a_2^2 + \cdots + a_9^2 = 3,
\]
then
\[
3 \leq a_1^3 + a_2^3 + \cdots + a_9^3 \leq \frac{14}{3}.
\]

5.3. If \(a, b, c, d\) are nonnegative real numbers so that
\[
a + b + c + d = a^2 + b^2 + c^2 + d^2 = \frac{27}{7},
\]
then
\[
\frac{5427}{1372} \leq a^3 + b^3 + c^3 + d^3 \leq \frac{1377}{343}.
\]

5.4. If \(a, b, c\) are positive real numbers so that \(abc = 1\), then
\[
a^5 + b^5 + c^5 \geq \sqrt{3(a^7 + b^7 + c^7)}.
\]

5.5. If \(a, b, c, d\) are positive real numbers so that \(abcd = 1\), then
\[
a^3 + b^3 + c^3 + d^3 \geq \sqrt{4(a^4 + b^4 + c^4 + d^4)}.
\]

5.6. If \(a, b, c, d\) are nonnegative real numbers so that \(a + b + c + d = 4\), then
\[
\frac{bcd}{11a + 16} + \frac{cd}{11b + 16} + \frac{dab}{11c + 16} + \frac{abc}{11d + 16} \leq \frac{4}{27}.
\]
5.7. If \( a, b, c \) are real numbers, then
\[
\frac{bc}{3a^2 + b^2 + c^2} + \frac{ca}{3b^2 + c^2 + a^2} + \frac{ab}{3c^2 + a^2 + b^2} \leq \frac{3}{5}.
\]

5.8. If \( a, b, c \) are nonnegative real numbers so that \( a + b + c = 3 \), then
(a) \[
\frac{bc}{a^2 + 2} + \frac{ca}{b^2 + 2} + \frac{ab}{c^2 + 2} \leq \frac{9}{8};
\]
(b) \[
\frac{bc}{a^2 + 3} + \frac{ca}{b^2 + 3} + \frac{ab}{c^2 + 3} \leq \frac{11\sqrt{33} - 45}{24};
\]
(c) \[
\frac{bc}{a^2 + 4} + \frac{ca}{b^2 + 4} + \frac{ab}{c^2 + 4} \leq \frac{3}{5}.
\]

5.9. If \( a, b, c, d \) are nonnegative real numbers so that
\[
(3a + 1)(3b + 1)(3c + 1)(3d + 1) = 64,
\]
then
\[
abc + bcd + cda + dab \leq 1.
\]

5.10. If \( a_1, a_2, \ldots, a_n \) and \( p, q \) are nonnegative real numbers so that
\[
a_1 + a_2 + \cdots + a_n = p + q, \quad a_1^3 + a_2^3 + \cdots + a_n^3 = p^3 + q^3,
\]
then
\[
a_1^2 + a_2^2 + \cdots + a_n^2 \leq p^2 + q^2.
\]

5.11. If \( a, b, c \) are nonnegative real numbers, then
\[
a \sqrt{a^2 + 4b^2 + 4c^2} + b \sqrt{b^2 + 4c^2 + 4a^2} + c \sqrt{c^2 + 4a^2 + 4b^2} \geq (a + b + c)^2.
\]

5.12. If \( a, b, c \) are nonnegative real numbers so that \( ab + bc + ca = 3 \), then
\[
\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} \leq \frac{3}{2(a + b + c)} + \frac{a + b + c}{3}.
\]
5.13. If \(a, b, c\) are nonnegative real numbers so that \(ab + bc + ca = 3\), then
\[
\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} \geq \frac{3}{a + b + c} + \frac{a + b + c}{6}.
\]

5.14. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. If
\[a^2 + b^2 + c^2 = 3,
\]
then
\[
\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} + \frac{a + b + c}{9} \geq \frac{11}{2(a + b + c)}.
\]

5.15. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. If
\[a + b + c = 4,
\]
then
\[
\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} \geq \frac{15}{8 + ab + bc + ca}.
\]

5.16. If \(a, b, c\) are nonnegative real numbers, no two of which are zero, then
\[
\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} \geq \frac{1}{a + b + c} + \frac{2}{\sqrt{ab + bc + ca}}.
\]

5.17. If \(a, b, c\) are nonnegative real numbers, no two of which are zero, then
\[
\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} \geq \frac{3 - \sqrt{3}}{a + b + c} + \frac{2 + \sqrt{3}}{2\sqrt{ab + bc + ca}}.
\]

5.18. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero, so that
\[ab + bc + ca = 3.
\]
If
\[0 \leq k \leq \frac{9 + 5\sqrt{3}}{6} \approx 2.943,
\]
then
\[
\frac{2}{a + b} + \frac{2}{b + c} + \frac{2}{c + a} \geq \frac{9(1 + k)}{a + b + c + 3k}.
\]
5.19. If \( a, b, c \) are nonnegative real numbers, no two of which are zero, then
\[
\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} \geq \frac{20}{a + b + c + 6\sqrt{ab + bc + ca}}.
\]

5.20. If \( a, b, c \) are nonnegative real numbers so that
\[
7(a^2 + b^2 + c^2) = 11(ab + bc + ca),
\]
then
\[
\frac{51}{28} \leq \frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \leq 2.
\]

5.21. If \( a_1, a_2, \ldots, a_n \) are nonnegative real numbers so that
\[
a_1^2 + a_2^2 + \cdots + a_n^2 = \left( a_1 + a_2 + \cdots + a_n \right)^2,
\]
then
\[
\frac{(n + 1)(2n - 1)}{2} \leq (a_1 + a_2 + \cdots + a_n) \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right) \leq \frac{3n^2(n + 1)}{2(n + 2)}.
\]

5.22. If \( a, b, c, d \) are nonnegative real numbers so that \( a + b + c + d = 3 \), then
\[
abc + bcd + cda + dab \leq 1 + \frac{176}{81} \quad abc \cdot d.
\]

5.23. If \( a, b, c, d \) are nonnegative real numbers so that \( a + b + c + d = 3 \), then
\[
a^2b^2c^2 + b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2 + \frac{3}{4}abcd \leq 1.
\]

5.24. If \( a, b, c, d \) are nonnegative real numbers so that \( a + b + c + d = 3 \), then
\[
a^2b^2c^2 + b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2 + \frac{4}{3} \left( abc \right)^{3/2} \leq 1.
\]

5.25. If \( a, b, c, d \) are nonnegative real numbers so that \( a + b + c + d = 4 \), then
\[
a^2b^2c^2 + b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2 + 2abcd \left( abc \right)^{3/2} \leq 6.
\]
5.26. If \(a, b, c\) are nonnegative real numbers so that \(a + b + c = 3\), then
\[11(ab + bc + ca) + 4(a^2b^2 + b^2c^2 + c^2a^2) \leq 45.\]

5.27. If \(a, b, c\) are nonnegative real numbers so that \(a + b + c = 3\), then
\[a^2b^2 + b^2c^2 + c^2a^2 + a^3b^3 + b^3c^3 + c^3a^3 \geq 6abc.\]

5.28. If \(a, b, c\) are nonnegative real numbers so that \(a + b + c = 3\), then
\[2(a^2 + b^2 + c^2) + 5 \left( \sqrt{a} + \sqrt{b} + \sqrt{c} \right) \geq 21.\]

5.29. If \(a, b, c\) are nonnegative real numbers so that \(ab + bc + ca = 3\), then
\[\sqrt{\frac{1 + 2a}{3}} + \sqrt{\frac{1 + 2b}{3}} + \sqrt{\frac{1 + 2c}{3}} \geq 3.\]

5.30. Let \(a, b, c\) be nonnegative real numbers, no two of which are zero. If \(0 \leq k \leq 15\), then
\[\frac{1}{(a + b)^2} + \frac{1}{(b + c)^2} + \frac{1}{(c + a)^2} + \frac{k}{(a + b + c)^2} \geq \frac{9 + k}{4(ab + bc + ca)}.\]

5.31. If \(a, b, c\) are nonnegative real numbers, no two of which are zero, then
\[\frac{1}{(a + b)^2} + \frac{1}{(b + c)^2} + \frac{1}{(c + a)^2} + \frac{24}{(a + b + c)^2} \geq \frac{8}{ab + bc + ca}.\]

5.32. If \(a, b, c\) are nonnegative real numbers, no two of which are zero, so that
\[k(a^2 + b^2 + c^2) + (2k + 3)(ab + bc + ca) = 9(k + 1), \quad 0 \leq k \leq 6,\]
then
\[\frac{1}{(a + b)^2} + \frac{1}{(b + c)^2} + \frac{1}{(c + a)^2} + \frac{9k}{(a + b + c)^2} \geq \frac{3}{4} + k.\]
5.33. If \(a, b, c\) are nonnegative real numbers, no two of which are zero, then

(a) \[
\frac{2}{a^2 + b^2} + \frac{2}{b^2 + c^2} + \frac{2}{c^2 + a^2} \geq \frac{8}{a^2 + b^2 + c^2} + \frac{1}{ab + bc + ca};
\]

(b) \[
\frac{2}{a^2 + b^2} + \frac{2}{b^2 + c^2} + \frac{2}{c^2 + a^2} \geq \frac{7}{a^2 + b^2 + c^2} + \frac{6}{(a + b + c)^2};
\]

(c) \[
\frac{2}{a^2 + b^2} + \frac{2}{b^2 + c^2} + \frac{2}{c^2 + a^2} \geq \frac{45}{4(a^2 + b^2 + c^2) + ab + bc + ca}.
\]

5.34. If \(a, b, c\) are nonnegative real numbers, no two of which are zero, then

\[
\frac{1}{a^2 + b^2} + \frac{1}{b^2 + c^2} + \frac{1}{c^2 + a^2} + \frac{3}{a^2 + b^2 + c^2} \geq \frac{4}{ab + bc + ca}.
\]

5.35. If \(a, b, c\) are nonnegative real numbers, no two of which are zero, then

(a) \[
\frac{3}{a^2 + ab + b^2} + \frac{3}{b^2 + bc + c^2} + \frac{3}{c^2 + ca + a^2} \geq \frac{5}{ab + bc + ca} + \frac{4}{a^2 + b^2 + c^2};
\]

(b) \[
\frac{3}{a^2 + ab + b^2} + \frac{3}{b^2 + bc + c^2} + \frac{3}{c^2 + ca + a^2} \geq \frac{1}{ab + bc + ca} + \frac{24}{(a + b + c)^2};
\]

(c) \[
\frac{1}{a^2 + ab + b^2} + \frac{1}{b^2 + bc + c^2} + \frac{1}{c^2 + ca + a^2} \geq \frac{21}{2(a^2 + b^2 + c^2) + 5(ab + bc + ca)}.
\]

5.36. If \(a, b, c\) are the lengths of the side of a triangle, then

\[
\frac{1}{(a + b)^2} + \frac{1}{(b + c)^2} + \frac{1}{(c + a)^2} \leq \frac{85}{36(ab + bc + ca)}.
\]

5.37. If \(a, b, c\) are the lengths of the side of a triangle so that \(a + b + c = 3\), then

\[
\frac{1}{(a + b)^2} + \frac{1}{(b + c)^2} + \frac{1}{(c + a)^2} \leq \frac{3(a^2 + b^2 + c^2)}{4(ab + bc + ca)}.
\]

5.38. Let \(a, b, c\) be nonnegative real numbers so that \(a + b + c = 3\). If \(k \geq \frac{8}{5}\), then

\[
\frac{1}{k + a^2 + b^2} + \frac{1}{k + b^2 + c^2} + \frac{1}{k + c^2 + a^2} \leq \frac{3}{k + 2}.
\]
5.39. If \(a, b, c\) are nonnegative real numbers so that \(a + b + c = 3\), then
\[
\frac{2}{2 + a^2 + b^2} + \frac{2}{2 + b^2 + c^2} + \frac{2}{2 + c^2 + a^2} \leq \frac{99}{63 + a^2 + b^2 + c^2}.
\]

5.40. If \(a, b, c\) are nonnegative real numbers so that \(a + b + c = 3\), then
\[
\frac{1}{5 + 2(a^2 + b^2)} + \frac{1}{5 + 2(b^2 + c^2)} + \frac{1}{5 + 2(c^2 + a^2)} \leq \frac{25}{69 + 2(a^2 + b^2 + c^2)}.
\]

5.41. If \(a, b, c\) are nonnegative real numbers so that \(a + b + c = 3\), then
\[
\frac{1}{3 + a^2 + b^2} + \frac{1}{3 + b^2 + c^2} + \frac{1}{3 + c^2 + a^2} \leq \frac{18}{27 + a^2 + b^2 + c^2}.
\]

5.42. If \(a, b, c, d\) are nonnegative real numbers so that \(a + b + c + d = 4\), then
\[
\sum \frac{3}{3 + 2(a^2 + b^2 + c^2)} \leq \frac{296}{218 + a^2 + b^2 + c^2 + d^2}.
\]

5.43. If \(a, b, c\) are nonnegative real numbers so that \(a + b + c = 3\), then
\[
\frac{5}{3 + a^2 + b^2} + \frac{5}{3 + b^2 + c^2} + \frac{5}{3 + c^2 + a^2} \geq \frac{27}{6 + a^2 + b^2 + c^2}.
\]

5.44. If \(a, b, c\) are nonnegative real numbers so that \(ab + bc + ca = 3\), then
\[
\frac{4}{2 + a^2 + b^2} + \frac{4}{2 + b^2 + c^2} + \frac{4}{2 + c^2 + a^2} \geq \frac{21}{4 + a^2 + b^2 + c^2}.
\]

5.45. If \(a, b, c\) are nonnegative real numbers so that \(a^2 + b^2 + c^2 = 3\), then
\[
\frac{1}{10 - (a + b)^2} + \frac{1}{10 - (b + c)^2} + \frac{1}{10 - (c + a)^2} \leq \frac{1}{2}.
\]

5.46. If \(a, b, c\) are nonnegative real numbers, no two of which are zero, so that \(a^4 + b^4 + c^4 = 3\), then
\[
\frac{1}{a^5 + b^5} + \frac{1}{b^5 + c^5} + \frac{1}{c^5 + a^5} \geq \frac{3}{2}.
\]
5.47. If \( a_1, a_2, \ldots, a_n \) are nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n = n \), then
\[
\sqrt{a_1^2 + 1} + \sqrt{a_2^2 + 1} + \cdots + \sqrt{a_n^2 + 1} \geq \sqrt{2 \left( 1 - \frac{1}{n} \right) \left( a_1^2 + a_2^2 + \cdots + a_n^2 \right) + 2(n^2 - n + 1)}.
\]

5.48. If \( a_1, a_2, \ldots, a_n \) are nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n = n \), then
\[
\sum q a_1^2 + 1 + q a_2^2 + 1 + \cdots + q a_n^2 + 1 \geq \sqrt{2 \left( 1 - \frac{1}{n} \right) \left( a_1^2 + a_2^2 + \cdots + a_n^2 \right) + 2(n^2 - n + 1)}.
\]

5.49. If \( a, b, c \) are nonnegative real numbers so that \( a + b + c = 3 \), then
\[
\sqrt{a^2 + 4} + \sqrt{b^2 + 4} + \sqrt{c^2 + 4} \leq \sqrt{\frac{8}{3}(a^2 + b^2 + c^2) + 37}.
\]

5.50. If \( a, b, c \) are nonnegative real numbers so that \( a + b + c = 3 \), then
\[
\sqrt{32a^2 + 3} + \sqrt{32b^2 + 3} + \sqrt{32c^2 + 3} \leq \sqrt{32(a^2 + b^2 + c^2) + 219}.
\]

5.51. If \( a_1, a_2, \ldots, a_n \) are positive real numbers so that \( a_1 + a_2 + \cdots + a_n = n \), then
\[
\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} + \frac{2n\sqrt{n-1}}{a_1^2 + a_2^2 + \cdots + a_n^2} \geq n + 2\sqrt{n-1}.
\]

5.52. If \( a, b, c \in [0, 1] \), then
\[
(1 + 3a^2)(1 + 3b^2)(1 + 3c^2) \geq (1 + ab + bc + ca)^3.
\]

5.53. If \( a, b, c \) are nonnegative real numbers so that \( a + b + c = ab + bc + ca \), then
\[
\frac{1}{4 + 5a^2} + \frac{1}{4 + 5b^2} + \frac{1}{4 + 5c^2} \geq \frac{1}{3}.
\]

5.54. If \( a, b, c, d \) are positive real numbers so that \( a + b + c + d = 4abcd \), then
\[
\frac{1}{1 + 3a} + \frac{1}{1 + 3b} + \frac{1}{1 + 3c} + \frac{1}{1 + 3d} \geq 1.
\]
5.55. If \( a_1, a_2, \ldots, a_n \) are positive real numbers so that

\[
a_1 + a_2 + \cdots + a_n = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n},
\]

then

\[
\frac{1}{1 + (n-1)a_1} + \frac{1}{1 + (n-1)a_2} + \cdots + \frac{1}{1 + (n-1)a_n} \geq 1.
\]

5.56. If \( a, b, c, d, e \) are nonnegative real numbers so that \( a^4 + b^4 + c^4 + d^4 + e^4 = 5 \), then

\[
7(a^2 + b^2 + c^2 + d^2 + e^2) \geq (a + b + c + d + e)^2 + 10.
\]

5.57. If \( a_1, a_2, \ldots, a_n \) are nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n = n \), then

\[
(a_1^2 + a_2^2 + \cdots + a_n^2)^2 - n^2 \geq \frac{n(n-1)}{n^2 - n + 1} (a_1^4 + a_2^4 + \cdots + a_n^4 - n).
\]

5.58. If \( a_1, a_2, \ldots, a_n \) are nonnegative real numbers so that \( a_1^2 + a_2^2 + \cdots + a_n^2 = n \), then

\[
a_1^3 + a_2^3 + \cdots + a_n^3 \geq \sqrt{n^2 - n + 1 + \left(1 - \frac{1}{n}\right)(a_1^6 + a_2^6 + \cdots + a_n^6)}.
\]

5.59. If \( a, b, c \) are positive real numbers so that \( abc = 1 \), then

\[
4\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + \frac{50}{a + b + c} \geq 27.
\]

5.60. If \( a, b, c \) are positive real numbers so that \( abc = 1 \), then

\[
a^3 + b^3 + c^3 + 15 \geq 6\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).
\]

5.61. Let \( a_1, a_2, \ldots, a_n \) be positive numbers so that \( a_1 a_2 \cdots a_n = 1 \). If \( k \geq n - 1 \), then

\[
a_1^k + a_2^k + \cdots + a_n^k + (2k-n)n \geq (2k-n+1)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).
\]
5.62. Let $a_1, a_2, \ldots , a_n$ $(n \geq 3)$ be nonnegative numbers so that $a_1 + a_2 + \cdots + a_n = n$, and let $k$ be an integer satisfying $2 \leq k \leq n + 2$. If

$$r = \left( \frac{n}{n-1} \right)^{k-1} - 1,$$

then

$$a_1^k + a_2^k + \cdots + a_n^k - n \geq nr(1 - a_1a_2 \cdots a_n).$$

5.63. If $a, b, c$ are positive real numbers so that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 3$, then

$$4(a^2 + b^2 + c^2) + 9 \geq 21abc.$$

5.64. If $a_1, a_2, \ldots , a_n$ are positive real numbers so that $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = n$, then,

$$a_1 + a_2 + \cdots + a_n - n \leq e_{n-1}(a_1a_2 \cdots a_n - 1),$$

where

$$e_{n-1} = \left( 1 + \frac{1}{n-1} \right)^{n-1}.$$

5.65. If $a_1, a_2, \ldots , a_n$ are positive real numbers, then

$$\frac{a_1^n + a_2^n + \cdots + a_n^n}{a_1a_2 \cdots a_n} + n(n-1) \geq (a_1 + a_2 + \cdots + a_n) \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right).$$

5.66. If $a_1, a_2, \ldots , a_n$ are nonnegative real numbers, then

$$(n-1)(a_1^n + a_2^n + \cdots + a_n^n) + na_1a_2 \cdots a_n \geq (a_1 + a_2 + \cdots + a_n)(a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1}).$$

5.67. If $a_1, a_2, \ldots , a_n$ are nonnegative real numbers, then

$$(n-1)(a_1^{n+1} + a_2^{n+1} + \cdots + a_n^{n+1}) \geq (a_1 + a_2 + \cdots + a_n)(a_1^n + a_2^n + \cdots + a_n^n - a_1a_2 \cdots a_n).$$

5.68. If $a_1, a_2, \ldots , a_n$ are positive real numbers, then

$$(a_1 + a_2 + \cdots + a_n - n) \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n \right) + a_1a_2 \cdots a_n + \frac{1}{a_1a_2 \cdots a_n} \geq 2.$$


5.69. If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1 a_2 \cdots a_n = 1$, then
\[
\left| \frac{1}{\sqrt{a_1 + a_2 + \cdots + a_n - n}} - \frac{1}{\sqrt{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n}} \right| < 1.
\]

5.70. If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1 a_2 \cdots a_n = 1$, then
\[
a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} + \frac{n^2(n-2)}{a_1 + a_2 + \cdots + a_n} \geq (n-1)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).
\]

5.71. If $a, b, c$ are nonnegative real numbers, then
\[
(a + b + c - 3)^2 \geq \frac{abc - 1}{abc + 1}(a^2 + b^2 + c^2 - 3).
\]

5.72. If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1 + a_2 + \cdots a_n = n$, then
\[
(a_1 a_2 \cdots a_n)^{\frac{1}{n-1}}(a_1^2 + a_2^2 + \cdots + a_n^2) \leq n.
\]

5.73. If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1^3 + a_2^3 + \cdots + a_n^3 = n$, then
\[
a_1 + a_2 + \cdots + a_n \geq n \sqrt[n]{a_1 a_2 \cdots a_n}.
\]

5.74. Let $a, b, c$ be nonnegative real numbers so that $ab + bc + ca = 3$. If
\[
k \geq 2 - \frac{\ln 4}{\ln 3} \approx 0.738,
\]
then
\[
a^k + b^k + c^k \geq 3.
\]

5.75. Let $a, b, c$ be nonnegative real numbers so that $a + b + c = 3$. If
\[
k \geq \frac{\ln 9 - \ln 8}{\ln 3 - \ln 2} \approx 0.29,
\]
then
\[
a^k + b^k + c^k \geq ab + bc + ca.
\]
5.76. If $a_1, a_2, \ldots, a_n$ ($n \geq 4$) are nonnegative numbers so that $a_1 + a_2 + \cdots + a_n = n$, then
\[
\frac{1}{n+1-a_2a_3\cdots a_n} + \frac{1}{n+1-a_3a_4\cdots a_1} + \cdots + \frac{1}{n+1-a_1a_2\cdots a_{n-1}} \leq 1.
\]

5.77. If $a, b, c$ are nonnegative real numbers so that
\[
a + b + c \geq 2, \quad ab + bc + ca \geq 1,
\]
then
\[
\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} \geq 2.
\]

5.78. If $a, b, c, d$ are positive real numbers so that $abcd = 1$, then
\[
(a + b + c + d)^4 \geq 36\sqrt{3}(a^2 + b^2 + c^2 + d^2).
\]

5.79. If $a, b, c, d$ are nonnegative real numbers, then
\[
\left( \sum_{\text{sym}} ab \right) \left( \sum_{\text{sym}} a^2 b^2 \right) \geq 9 \sum a^2 b^2 c^2.
\]

5.80. If $a, b, c$ are nonnegative real numbers so that $ab + bc + ca = 1$, then
\[
\sqrt{33a^2 + 16} + \sqrt{33b^2 + 16} + \sqrt{33c^2 + 16} \leq 9(a + b + c).
\]

5.81. If $a, b, c$ are positive real numbers so that $a + b + c = 3$, then
\[
a^2 b^2 + b^2 c^2 + c^2 a^2 \leq \frac{3}{\sqrt[3]{abc}}.
\]

5.82. If $a_1, a_2, \ldots, a_n$ ($n \leq 81$) are nonnegative real numbers so that
\[
a_1^2 + a_2^2 + \cdots + a_n^2 = a_1^5 + a_2^5 + \cdots + a_n^5,
\]
then
\[
a_1^6 + a_2^6 + \cdots + a_n^6 \leq n.
\]
5.83. If $a, b, c$ are nonnegative real numbers so that $a + b + c = 3$, then
\[ 1 + \sqrt{1 + a^3 + b^3 + c^3} \geq \sqrt{3(a^2 + b^2 + c^2)}. \]

5.84. If $a, b, c$ are nonnegative real numbers so that $a + b + c = 3$, then
\[ \sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a} \leq \sqrt{16 + \frac{2}{3}(ab + bc + ca)}. \]

5.85. If $a, b, c$ are positive real numbers so that $abc = 1$, then
\[ (a) \quad \frac{a + b + c}{3} \geq \sqrt{\frac{2 + a^2 + b^2 + c^2}{5}}; \]
\[ (b) \quad a^3 + b^3 + c^3 \geq \sqrt{3(a^4 + b^4 + c^4)}. \]

5.86. If $a, b, c, d$ are nonnegative real numbers so that $a^2 + b^2 + c^2 + d^2 = 4$, then
\[ (2 - abc)(2 - bcd)(2 - cda)(2 - dab) \geq 1. \]

5.87. If $a, b, c, d$ are nonnegative real numbers so that $a + b + c + d = 4$, then
\[ (a^2 + b^2 + c^2 + d^2 - 4)(a^2 + b^2 + c^2 + d^2 + 18) \leq 10(a^3 + b^3 + c^3 + d^3 - 4). \]

5.88. If $a_1, a_2, \ldots, a_8$ are nonnegative real numbers, then
\[ 19(a_1^2 + a_2^2 + \cdots + a_8^2)^2 \geq 12(a_1 + a_2 + \cdots + a_8)(a_1^3 + a_2^3 + \cdots + a_8^3). \]

5.89. If $a, b, c$ are nonnegative real numbers so that
\[ 5(a^2 + b^2 + c^2) = 17(ab + bc + ca), \]
then
\[ 3\sqrt{\frac{3}{5}} \leq \sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} \leq \frac{1 + \sqrt{7}}{\sqrt{2}}. \]

5.90. If $a, b, c$ are nonnegative real numbers so that
\[ 8(a^2 + b^2 + c^2) = 9(ab + bc + ca), \]
then
\[ \frac{19}{12} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq \frac{141}{88}. \]
5.3 Solutions

P 5.1. If a, b, c, d are nonnegative real numbers so that
\[ a + b + c + d = a^3 + b^3 + c^3 + d^3 = 2, \]
then
\[ \frac{7}{4} \leq a^2 + b^2 + c^2 + d^2 \leq 2. \]

(Vasile C., 2010)

Solution. The right inequality follows from the Cauchy-Schwarz inequality
\[ (a^2 + b^2 + c^2 + d^2)^2 \leq (a + b + c + d)(a^3 + b^3 + c^3 + d^3). \]
The equality holds for \( a = b = 0 \) and \( c = d = 1 \) (or any permutation).
To prove the left inequality, assume that \( a \leq b \leq c \leq d \), then apply Corollary 5 for \( k = 3 \) and \( m = 2 \):
- If \( a, b, c, d \) are nonnegative real numbers so that
  \[ a + b + c + d = 2, \quad a^3 + b^3 + c^3 + d^3 = 2, \quad a \leq b \leq c \leq d, \]
then
  \[ S_4 = a^2 + b^2 + c^2 + d^2 \]
is minimum for \( a = b = c \).
So, we only need to prove that the equations
\[ 3a + d = 3a^3 + d^3 = 2 \]
imply
\[ \frac{7}{4} \leq 3a^2 + d^2. \]
Indeed, from \( 3a + d = 3a^3 + d^3 = 2 \), we get \( a = 1/4 \) and \( d = 5/4 \), when
\[ 3a^2 + d^2 = \frac{7}{4}. \]
The left inequality is an equality for
\[ a = b = c = \frac{1}{4}, \quad d = \frac{5}{4} \]
(or any cyclic permutation).
P 5.2. If \( a_1, a_2, \ldots, a_9 \) are nonnegative real numbers so that
\[
a_1 + a_2 + \cdots + a_9 = a_1^2 + a_2^2 + \cdots + a_9^2 = 3,
\]
then
\[
3 \leq a_3^3 + a_2^3 + \cdots + a_9^3 \leq \frac{14}{3}.
\]

(Vasile C., 2010)

**Solution.** The left inequality follows from the Cauchy-Schwarz inequality
\[
(a_1 + a_2 + \cdots + a_9)(a_1^3 + a_2^3 + \cdots + a_9^3) \geq (a_1^2 + a_2^2 + \cdots + a_9^2)^2.
\]
The equality holds for \( a_1 = a_2 = \cdots = a_6 = 0 \) and \( a_7 = a_8 = a_9 = 1 \) (or any permutation).

To prove the right inequality, assume that
\[
a_1 \leq a_2 \leq \cdots \leq a_9,
\]
then apply Corollary 5 for \( k = 2 \) and \( m = 3 \):
- If \( a_1, a_2, \ldots, a_9 \) are nonnegative real numbers so that
\[
a_1 + a_2 + \cdots + a_9 = 3, \quad a_1^2 + a_2^2 + \cdots + a_9^2 = 3, \quad a_1 \leq a_2 \leq \cdots \leq a_9,
\]
then
\[
S_9 = a_3^3 + a_2^3 + \cdots + a_9^3
\]
is maximum for \( a_1 = a_2 = \cdots = a_8 \leq a_9 \).

Thus, we only need to prove that the equations
\[
8a + b = 3, \quad 8a^2 + b^2 = 3,
\]
involve
\[
8a^3 + b^3 \leq \frac{14}{3}.
\]
Indeed, from the equations above, we get \( a = 1/6 \) and \( b = 5/3 \); therefore
\[
8a^3 + b^3 = \frac{1}{27} + \frac{125}{27} = \frac{14}{3}.
\]
The equality holds for
\[
a_1 = a_2 = \cdots = a_8 = \frac{1}{6}, \quad a_9 = \frac{5}{3}
\]
(or any cyclic permutation). \( \square \)
P 5.3. If $a, b, c, d$ are nonnegative real numbers so that
\[ a + b + c + d = a^2 + b^2 + c^2 + d^2 = \frac{27}{7}, \]
then
\[ \frac{5427}{1372} \leq a^3 + b^3 + c^3 + d^3 \leq \frac{1377}{343}. \]

(Solution) Assume that $a \leq b \leq c \leq d$.

(a) To prove the right inequality, we apply Corollary 5 for $k = 2$ and $m = 3$:

- If $a, b, c, d$ are nonnegative real numbers so that
  \[ a + b + c + d = \frac{27}{7}, \quad a^2 + b^2 + c^2 + d^2 = \frac{27}{7}, \quad a \leq b \leq c \leq d, \]
then
  \[ S_4 = a^3 + b^3 + c^3 + d^3 \]
is maximum for $a = b = c \leq d$.

  Thus, we only need to prove that the equations
  \[ 3a + d = \frac{27}{7}, \quad 3a^2 + d^2 = \frac{27}{7}, \]
involve
  \[ 3a^3 + d^3 \leq \frac{1377}{343}. \]
Indeed, from the equations above, we get $a = 6/7$ and $d = 9/7$; therefore
\[ 3a^3 + d^3 = 3 \left( \frac{6}{7} \right)^3 + \left( \frac{9}{7} \right)^3 = \frac{1377}{343}. \]
The equality holds for
\[ a = b = c = \frac{6}{7}, \quad d = \frac{9}{7} \]
(or any cyclic permutation).

(b) To prove the left inequality, we apply Corollary 5 for $k = 2$ and $m = 3$:

- If $a, b, c, d$ are nonnegative real numbers so that
  \[ a + b + c + d = \frac{27}{7}, \quad a^2 + b^2 + c^2 + d^2 = \frac{27}{7}, \quad a \leq b \leq c \leq d, \]
then
  \[ S_4 = a^3 + b^3 + c^3 + d^3 \]
is minimum for either \(a \leq b = c = d\) or \(a = 0\).

The case \(a = 0\) is not possible because from
\[
b + c + d = \frac{27}{7}, \quad b^2 + c^2 + d^2 = \frac{27}{7},
\]
we get
\[
3(b^2 + c^2 + d^2) - (b + c + d)^2 = \frac{27}{7}(3 - \frac{27}{7}) < 0,
\]
which contradicts the known inequality
\[
3(b^2 + c^2 + d^2) \geq (b + c + d)^2.
\]

For \(a \leq b = c = d\), we need to prove that the equations
\[
a + 3d = \frac{27}{7}, \quad a^2 + 3d^2 = \frac{27}{7},
\]
involve
\[
a^3 + 3d^3 \geq \frac{5427}{1372}.
\]
Indeed, from the equations above, we get \(a = 9/14\) and \(d = 15/14\); therefore
\[
a^3 + 3d^3 = \left(\frac{9}{14}\right)^3 + 3\left(\frac{15}{14}\right)^3 = \frac{5427}{1372}.
\]
The equality holds for
\[
a = \frac{9}{14}, \quad b = c = d = \frac{15}{14}
\]
(or any cyclic permutation).

**Remark.** In the same manner, we can prove the following generalization:
- Let \(k\) be a positive real number \((k > 2)\), and let \(a_1, a_2, \ldots, a_n\) be nonnegative real numbers so that
\[
a_1 + a_2 + \cdots + a_n = a_1^2 + a_2^2 + \cdots + a_n^2 = \frac{(n - 1)^3}{n^2 - 3n + 3}.
\]
The sum
\[
S_n = a_1^k + a_2^k + \cdots + a_n^k
\]
is maximum for
\[
a_1 = \cdots = a_{n-1} = \frac{(n - 1)(n - 2)}{n^2 - 3n + 3}, \quad a_n = \frac{(n - 1)^2}{n^2 - 3n + 3},
\]
and is minimum for
\[
a_1 = \frac{(n - 1)^2(n - 2)}{n(n^2 - 3n + 3)}, \quad a_2 = \cdots = a_n = \frac{(n - 1)(n^2 - 2n + 2)}{n(n^2 - 3n + 3)}.
\]
**P 5.4.** If $a, b, c$ are positive real numbers so that $abc = 1$, then

$$a^5 + b^5 + c^5 \geq \sqrt{3(a^7 + b^7 + c^7)}.\]  

(Vasile C., 2014)

**Solution.** Substituting

$$a = x^{1/5}, \quad b = y^{1/5}, \quad c = z^{1/5},$$

we need to show that $xyz = 1$ involves

$$x + y + z \geq \sqrt{3(x^{7/5} + y^{7/5} + z^{7/5})}.$$  

Assume that $x \leq y \leq z$, then apply Corollary 5 for $k = 0$ and $m = 7/5$:

- If $x, y, z$ are positive real numbers so that $x + y + z = \text{constant}$, $xyz = 1$, $x \leq y \leq z$,

then

$$S_3 = x^{7/5} + y^{7/5} + z^{7/5}$$

is maximum for $x = y$.

So, it suffices to prove the original inequality for $a = b$. Write this inequality in the homogeneous form

$$(a^5 + b^5 + c^5)^2 \geq 3abc(a^7 + b^7 + c^7).$$

We only need to prove this inequality for $a = b = 1$; that is, to show that $f(c) \geq 0$, where

$$f(c) = (c^5 + 2)^2 - 3c(c^7 + 2), \quad c > 0.$$  

We have

$$f'(c) = 10c^4(c^5 + 2) - 24c^7 - 6,$$

$$f''(c) = 2c^3g(t), \quad g(t) = 45c^5 - 84c^3 + 40.$$  

By the AM-GM inequality, we get

$$g(t) = 15c^5 + 15c^5 + 15c^5 + 20 + 20 - 84c^3 \geq 5\sqrt[5]{(15c^5)^3 \cdot 20^2 - 84c^3}$$

$$= 5\sqrt[9]{27 \cdot 16(25 - 14\sqrt{18})c^3} > 0,$$

hence $f''(c) > 0$, $f'(c)$ is increasing. Since $f'(0) = 1$, it follows that $f'(c) \leq 0$ for $c \leq 1$, $f'(c) \geq 0$ for $c \geq 1$, therefore $f$ is decreasing on $(0, 1]$ and increasing on $[1, \infty)$; consequently, $f(c) \geq f(1) = 0$. The equality holds for $a = b = c = 1$. \[\square\]
P 5.5. If \( a, b, c, d \) are positive real numbers so that \( abcd = 1 \), then

\[
a^3 + b^3 + c^3 + d^3 \geq \sqrt[4]{4(a^4 + b^4 + c^4 + d^4)}.
\]

(Vasile C., 2014)

**Solution.** Substituting

\[
a = x^{1/3}, \quad b = y^{1/3}, \quad c = z^{1/3}, \quad d = t^{1/3},
\]
we need to show that \( xyzt = 1 \) involves

\[
x + y + z + t \geq \sqrt[4]{4(x^{4/3} + y^{4/3} + z^{4/3} + t^{4/3})}.
\]

Apply Corollary 5, case \( k = 0 \) and \( m = 4/3 \):

- If \( x, y, z, t \) are positive real numbers so that

\[
x + y + z + t = \text{constant}, \quad xyzt = 1, \quad x \leq y \leq z \leq t,
\]

then

\[
S_4 = x^{4/3} + y^{4/3} + z^{4/3} + t^{4/3}
\]

is maximum for \( x = y = z \).

Therefore, it suffices to prove the original inequality for \( a = b = c \). Write the original inequality in the homogeneous form

\[
(a^3 + b^3 + c^3 + d^3)^2 \geq 4\sqrt{abcd} (a^4 + b^4 + c^4 + d^4).
\]

We only need to prove this inequality for \( a = b = c = 1 \); that is, to show that

\[
(d^3 + 3)^2 \geq 4\sqrt{d} (d^4 + 3).
\]

Putting \( u = \sqrt{d} \), we have

\[
(d^3 + 3)^2 - 4\sqrt{d} (d^4 + 3) = (u^6 + 3)^2 - 4u(u^8 + 3) = (u^3 - 1)^4 + 4(u + 2)(u - 1)^2 \geq 0.
\]

The equality holds for \( a = b = c = d = 1 \).

\[\blacksquare\]

P 5.6. If \( a, b, c, d \) are nonnegative real numbers so that \( a + b + c + d = 4 \), then

\[
\frac{bcd}{11a + 16} + \frac{cda}{11b + 16} + \frac{dab}{11c + 16} + \frac{abc}{11d + 16} \leq \frac{4}{27}.
\]

(Vasile C., 2008)
Solution. For $a = 0$, the inequality becomes

$$bcd \leq \frac{64}{27},$$

where $b, c, d \geq 0$ so that $b + c + d = 4$. Indeed, by the AM-GM inequality, we have

$$bcd \leq \left(\frac{b + c + d}{3}\right)^3 = \left(\frac{4}{3}\right)^3 = \frac{64}{27}.$$

For $abcd \neq 0$, we write the inequality in the form

$$f(a) + f(b) + f(c) + f(d) + \frac{4}{(1+k)abcd} \geq 0,$$

where

$$f(u) = \frac{-1}{u(u+k)}, \quad k = \frac{16}{11}, \quad u > 0.$$  

We have

$$f'(u) = \frac{2u + k}{(u^2 + ku)^2},$$

$$g(x) = f'(1/x) = \frac{kx^4 + 2x^3}{(kx + 1)^2},$$

$$g''(x) = \frac{2x(k^3x^3 + 4k^2x^2 + 6kx + 6)}{(kx + 1)^4}.$$  

Since $g''(x) > 0$ for $x > 0$, $g$ is strictly convex on $(0, \infty)$. By Corollary 3, if $a + b + c + d = 4$, $abcd = constant$, $0 < a \leq b \leq c \leq d$, then the sum

$$S_4 = f(a) + f(b) + f(c) + f(d)$$

is minimum for $b = c = d$. Thus, we only need to prove that

$$\frac{b^3}{11a + 16} + \frac{3ab^2}{11b + 16} \leq \frac{4}{27}$$

for $a + 3b = 4$. The inequality is equivalent to

$$\frac{b^3}{3(20 - 11b)} + \frac{3b^2(4 - 3b)}{11b + 16} \leq \frac{4}{21},$$

$$(b - 1)^2(4 - 3b)(231b + 80) \geq 0,$$

$$(b - 1)^2a(231b + 80) \geq 0.$$  

The equality holds for $a = b = c = d = 1$, and also for

$$a = 0, \quad b = c = d = \frac{4}{3}$$

(or any cyclic permutation).
P 5.7. If \(a, b, c\) are real numbers, then
\[
\frac{bc}{3a^2 + b^2 + c^2} + \frac{ca}{3b^2 + c^2 + a^2} + \frac{ab}{3c^2 + a^2 + b^2} \leq \frac{3}{5}.
\]

(Vasile Cirtoaje and Pham Kim Hung, 2005)

Solution. For \(a = 0\), the inequality is true because
\[
\frac{bc}{b^2 + c^2} \leq \frac{1}{2} < \frac{3}{5}
\]

Consider further that \(a, b, c\) are different from zero. The inequality remains unchanged by replacing \(a, b, c\) with \(-a, -b, -c\), respectively. Thus, we only need to consider the case \(a < 0, b, c > 0\), and the case \(a, b, c > 0\). In the first case, it suffices to show that
\[
\frac{bc}{3a^2 + b^2 + c^2} \leq \frac{3}{5}.
\]
Indeed, we have
\[
\frac{bc}{3a^2 + b^2 + c^2} \leq \frac{bc}{b^2 + c^2} \leq \frac{1}{2} < \frac{3}{5}.
\]
Consider now the case \(a, b, c > 0\). Replacing \(a, b, c\) with \(\sqrt{a}, \sqrt{b}, \sqrt{c}\), the inequality becomes
\[
\frac{1}{\sqrt{a}(3a + b + c)} + \frac{1}{\sqrt{b}(3b + c + a)} + \frac{1}{\sqrt{c}(3c + a + b)} \leq \frac{3}{5\sqrt{abc}}.
\]
Due to homogeneity, we may consider that \(a + b + c = 2\). So, we need to show that
\[
f(a) + f(b + f(c)) + \frac{6}{5\sqrt{abc}} \geq 0,
\]
where
\[
f(u) = \frac{-1}{\sqrt{u}(u + 1)}, \quad u > 0.
\]
We have
\[
f'(u) = \frac{3u + 1}{2u\sqrt{u}(u + 1)^2},
\]
\[
g(x) = f'(1/x) = \frac{x^2\sqrt{x}(x + 3)}{2(x + 1)^2},
\]
\[
g''(x) = \frac{\sqrt{x}(3x^3 + 11x^2 + 5x + 45)}{8(x + 1)^4}.
\]
Since \(g''(x) > 0\) for \(x > 0\), \(g\) is strictly convex on \((0, \infty)\). By Corollary 3, if
\[
a + b + c = 2, \quad abc = \text{constant}, \quad 0 < a \leq b \leq c,
\]

then
\[
f(a) + f(b + f(c)) + \frac{6}{5\sqrt{abc}} \geq 0.
\]
then the sum
\[ S_3 = f(a) + f(b) + f(c) \]
is minimum for \( b = c \). Thus, we only need to prove the original homogeneous inequality for \( b = c = 1 \); that is,
\[ \frac{1}{3a^2 + 2} + \frac{2a}{a^2 + 4} \leq \frac{3}{5}, \]
\[ 9a^4 - 30a^3 + 37a^2 - 20a + 4 \geq 0, \]
\[ (a - 1)^2(3a - 2)^2 \geq 0. \]
The equality holds for \( a = b = c \), and also for
\[ 3a = 2b = 2c \]
(or any cyclic permutation).

\[ \square \]

\textbf{P 5.8.} If \( a, b, c \) are nonnegative real numbers so that \( a + b + c = 3 \), then
\begin{itemize}
  \item[(a)] \( \frac{bc}{a^2 + 2} + \frac{ca}{b^2 + 2} + \frac{ab}{c^2 + 2} \leq \frac{9}{8}; \)
  \item[(b)] \( \frac{bc}{a^2 + 3} + \frac{ca}{b^2 + 3} + \frac{ab}{c^2 + 3} \leq \frac{11\sqrt{33} - 45}{24}; \)
  \item[(c)] \( \frac{bc}{a^2 + 4} + \frac{ca}{b^2 + 4} + \frac{ab}{c^2 + 4} \leq \frac{3}{5}. \)
\end{itemize}
(Vasile C., 2008)

\textbf{Solution.} For the nontrivial case \( abc \neq 0 \), we can write the desired inequalities in the form
\[ f(a) + f(b) + f(c) + \frac{m}{abc} \geq 0, \]
where
\[ f(u) = \frac{-1}{u(u^2 + k)}, \quad k \in \{2, 3, 4\}, \quad u > 0. \]
We have
\[ f'(u) = \frac{3u^2 + k}{u^2(u^2 + k)^2}, \]
\[ g(x) = f'(1/x) = \frac{xx^6 + 3x^4}{(kx^2 + 1)^2}, \]
\[ g''(x) = \frac{2x^2(k^3x^6 + 4k^2x^4 - 3kx^2 + 18)}{(kx^2 + 1)^4}. \]
Since
\[ k^3x^6 + 4k^2x^4 - 3kx^2 + 18 > 4k^2x^4 - 3kx^2 + 18 > 0, \]
we have \( g''(x) > 0 \) for \( x > 0 \), hence \( g \) is strictly convex on \((0, \infty)\). According to Corollary 3, if
\[ a + b + c = 3, \quad abc = \text{constant}, \quad 0 < a \leq b \leq c, \]
then the sum
\[ S_3 = f(a) + f(b) + f(c) \]
is minimum for \( b = c \). Thus, we only need to prove the original inequalities for \( b = c \).

(a) We only need to prove the homogeneous inequality
\[
\frac{bc}{9a^2 + 2(a + b + c)^2} + \frac{ca}{9b^2 + 2(a + b + c)^2} + \frac{ab}{9c^2 + 2(a + b + c)^2} \leq \frac{1}{8}
\]
for \( b = c = 1 \); that is,
\[
\frac{1}{11a^2 + 8a + 8} + \frac{2a}{2a^2 + 8a + 17} \leq \frac{1}{8},
\]
\[
\frac{2a}{2a^2 + 8a + 17} \leq \frac{a(11a + 8)}{8(11a^2 + 8a + 8)},
\]
\[
a(22a^3 - 72a^2 + 123a + 8) \geq 0.
\]
Since
\[ 22a^3 - 72a^2 + 123a + 8 > 20a^3 - 80a^2 + 80a = 20a(a - 2)^2 \geq 0, \]
the conclusion follows. The equality holds for \( a = 0 \) and \( b = c = 3/2 \) (or any cyclic permutation).

(b) Let
\[
m = \frac{11\sqrt{33} - 45}{72} \approx 0.253, \quad r = \frac{\sqrt{33} - 5}{4} \approx 0.186.
\]
We only need to prove the homogeneous inequality
\[
\frac{bc}{3a^2 + (a + b + c)^2} + \frac{ca}{3b^2 + (a + b + c)^2} + \frac{ab}{3c^2 + (a + b + c)^2} \leq m
\]
for \( b = c = 1 \); that is, to show that \( f(a) \leq m \), where
\[
f(a) = \frac{1}{4(a^2 + a + 1)} + \frac{2a}{a^2 + 4a + 7}.
\]
We have

\[ f'(a) = \frac{-8a^5 - 18a^4 + 15a^3 + 28a^2 + 18a - 42a + 7}{4(a^2 + a + 1)^2(a^2 + 4a + 7)^2} \]

Since \( f'(a) \geq 0 \) for \( a \in [0, r] \), and \( f'(a) \leq 0 \) for \( a \in [r, \infty) \), \( f \) is increasing on \([0, r]\) and decreasing on \([r, \infty)\); therefore,

\[ f(a) \geq f(r) = m. \]

The equality holds for

\[ a/r = b = c \]

(or any cyclic permutation).

(c) We only need to prove the homogeneous inequality

\[ \frac{bc}{9a^2 + 4(a + b + c)^2} + \frac{ca}{9b^2 + 4(a + b + c)^2} + \frac{ab}{9c^2 + 4(a + b + c)^2} \leq \frac{1}{15} \]

for \( b = c = 1 \); that is,

\[ \frac{1}{13a^2 + 16a + 16} + \frac{2a}{4a^2 + 16a + 25} \leq \frac{1}{15}, \]

\[ 52a^4 - 118a^3 + 105a^2 - 64a + 25 \geq 0, \]

\[ (a - 1)^2(52a^2 - 14a + 25) \geq 0. \]

Since

\[ 52a^2 - 14a + 25 > 7a^2 - 14a + 7 = 7(a - 1)^2 \geq 0, \]

the conclusion follows. The equality holds for \( a = b = c = 1 \).

\[ \square \]

**P 5.9.** If \( a, b, c, d \) are nonnegative real numbers so that

\[ (3a + 1)(3b + 1)(3c + 1)(3d + 1) = 64, \]

then

\[ abc + bcd + cda + dab \leq 1. \]

(Vasile C., 2014)
**Solution.** For \(d = 0\), we need to show that

\[
(3a + 1)(3b + 1)(3c + 1) = 64
\]

involves \(abc \leq 1\). Indeed, by the AM-GM inequality, we have

\[
64 = (3a + 1)(3b + 1)(3c + 1) \geq \left(4\sqrt[3]{a^3}\right)\left(4\sqrt[3]{b^3}\right)\left(4\sqrt[3]{c^3}\right) = 64\sqrt[3]{(abc)^3},
\]

hence \(abc \leq 1\). Consider further that \(a, b, c, d > 0\) and use the contradiction method. Assume that

\[
abc + bcd + cda + dab > 1,
\]

and prove that

\[
(3a + 1)(3b + 1)(3c + 1) > 64.
\]

It suffices to show that

\[
abc + bcd + cda + dab \geq 1
\]

involves

\[
(3a + 1)(3b + 1)(3c + 1) \geq 64.
\]

Replacing \(a, b, c, d\) by \(1/a, 1/b, 1/c, 1/d\), we need to show that

\[
a + b + c + d = abcd
\]

involves

\[
\left(\frac{3}{a} + 1\right)\left(\frac{3}{b} + 1\right)\left(\frac{3}{c} + 1\right)\left(\frac{3}{d} + 1\right) \geq 64,
\]

which is equivalent to

\[
f(a) + f(b) + f(c) + f(d) \leq -6\ln 2,
\]

where

\[
f(u) = -\ln\left(\frac{3}{u} + 1\right), \quad u > 0.
\]

Apply Corollary 3 for \(n = 4\):

- If \(a, b, c, d\) are positive real numbers so that

\[
a + b + c + d = \text{constant}, \quad abcd = \text{constant}, \quad a \leq b \leq c \leq d,
\]

and \(g(x) = f'(1/x)\) is strictly convex on \((0, \infty)\), then

\[
S_4 = f(a) + f(b) + f(c) + f(d)
\]

is maximum for \(a = b = c\).

We have

\[
g(x) = \frac{3x^2}{3x + 1}, \quad g''(x) = \frac{6}{(3x + 1)^3} > 0,
\]
hence $g$ is strictly convex on $(0, \infty)$. Thus, we only need to prove that

$$3a + d = a^3d, \quad a \leq d$$

implies

$$\left( \frac{3}{a} + 1 \right)^3 \left( \frac{3}{d} + 1 \right) \geq 64.$$

Write this inequality as

$$(3 + a)^3(3 + d) \geq 64a^3d,$$

$$(3 + a)^4(3 + d) \geq 64a^3d(3 + a),$$

$$4 \left( 1 + \frac{a - 1}{4} \right)^4 (3 + d) \geq a^3d(3 + a).$$

By Bernoulli’s inequality, we have

$$\left( 1 + \frac{a - 1}{4} \right)^4 \geq 1 + 4 \cdot \frac{a - 1}{4} = a.$$

Thus, it suffices to show that

$$4(3 + d) \geq a^2d(3 + a),$$

which is equivalent to

$$\frac{12}{d} \geq a^3 + 3a^2 - 4.$$

Since

$$\frac{3}{d} = \frac{a^3 - 1}{a}, \quad a > 1,$$

the inequality becomes

$$\frac{4(a^3 - 1)}{a} \geq a^3 + 3a^2 - 4,$$

$$a^4 - a^3 - 4a + 4 \leq 0,$$

$$(a - 1)(a^3 - 4) \leq 0.$$

This is true if $a^3 \leq 4$. Indeed, we have

$$0 \leq \frac{3}{a} - 3 = \frac{3}{a} - \frac{a^3 - 1}{a} = \frac{4 - a^3}{a}.$$

The proof is completed. The original inequality is an equality for

$$a = b = c = 1, \quad d = 0$$

(or any cyclic permutation).
**P 5.10.** If \( a_1, a_2, \ldots, a_n \) and \( p, q \) are nonnegative real numbers so that
\[
 a_1 + a_2 + \cdots + a_n = p + q, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = p^3 + q^3,
\]
then
\[
 a_1^2 + a_2^2 + \cdots + a_n^2 \leq p^2 + q^2.
\]

*(Vasile C., 2013)*

**Solution.** For \( n = 2 \), the inequality is an equality. Consider now that \( n \geq 3 \) and \( a_1 \leq a_2 \leq \cdots \leq a_n \). We will apply Corollary 5 for \( k = 3 \) and \( m = 2 \):

- If \( a_1, a_2, \ldots, a_n \) are nonnegative real numbers so that \( a_1 \leq a_2 \leq \cdots \leq a_n \) and
\[
 a_1 + a_2 + \cdots + a_n = p + q, \quad a_1^3 + a_2^3 + \cdots + a_n^3 = p^3 + q^3,
\]
then
\[
 S_n = a_1^2 + a_2^2 + \cdots + a_n^2
\]
is maximum for either \( a_1 = 0 \) or \( a_2 = a_3 = \cdots = a_n \).

In the first case \( a_1 = 0 \), the conclusion follows by induction method. In the second case, for
\[
 a_1 = a, \quad a_2 = a_3 = \cdots = a_n = b,
\]
we need to show that
\[
 a^2 + (n-1)b^2 \leq p^2 + q^2
\]
for
\[
 a + (n-1)b = p + q, \quad a^3 + (n-1)b^3 = p^3 + q^3.
\]
Since
\[
 3(p^2 + q^2) = (p + q)^2 + \frac{2(p^3 + q^3)}{p + q},
\]
the inequality can be written as
\[
 3a^2 + 3(n-1)b^2 \leq [a + (n-1)b]^2 + \frac{2[a^3 + (n-1)b^3]}{a + (n-1)b},
\]
which is equivalent to
\[
 (n-1)(n-2)b^2[3a + (n-3)b] \geq 0.
\]
The equality holds when \( n-2 \) of \( a_1, a_2, \ldots, a_n \) are equal to zero.

\[\square\]

**P 5.11.** If \( a, b, c \) are nonnegative real numbers, then
\[
 a\sqrt{a^2 + 4b^2 + 4c^2} + b\sqrt{b^2 + 4c^2 + 4a^2} + c\sqrt{c^2 + 4a^2 + 4b^2} \geq (a + b + c)^2.
\]

*(Vasile C., 2010)
Solution. Due to homogeneity and symmetry, we may assume that

\[ a^2 + b^2 + c^2 = 3, \quad 0 \leq a \leq b \leq c. \]

Under this assumption, we write the desired inequality as

\[ f(a) + f(b) + f(c) + \frac{1}{\sqrt{3}}(a + b + c)^2 \leq 0, \]

where

\[ f(u) = -u\sqrt{4-u^2}, \quad 0 \leq u \leq \sqrt{3}. \]

We will apply Corollary 1 to the function \( f \). We have

\[ g(x) = f'(x) = \frac{2(x^2-2)}{\sqrt{4-x^2}}, \]

\[ g''(x) = \frac{48}{(4-x^2)^{5/2}}. \]

Since \( g''(x) > 0 \) for \( x \in (0,2) \), \( g \) is strictly convex on \([0, \sqrt{3}]\). According to Corollary 1 and Note 5/Note 2, if

\[ a + b + c = \text{constant}, \quad a^2 + b^2 + c^2 = 3, \quad 0 \leq a \leq b \leq c, \]

then the sum

\[ S_3 = f(a) + f(b) + f(c) \]

is maximum for \( a = b \leq c \). Thus, we only need to prove the original inequality for \( a = b \). Since the inequality is an identity for \( a = b = 0 \), we may consider \( a = b = 1 \) and \( c \geq 1 \). We need to prove that

\[ 2\sqrt{4c^2 + 5} + c\sqrt{c^2 + 8} \geq (c + 2)^2. \]

By squaring, the inequality becomes

\[ c\sqrt{(4c^2 + 5)(c^2 + 8)} \geq 2c^3 + 8c - 1. \]

This is true if

\[ c^2(4c^2 + 5)(c^2 + 8) \geq (2c^3 + 8c - 1)^2, \]

which is equivalent to

\[ 5c^4 + 4c^3 - 24c^2 + 16c - 1 \geq 0, \]

\[ (c - 1)^2(5c^2 + 14c - 1) \geq 0. \]

The equality holds for \( a = b = c \), and also for \( a = b = 0 \) (or any cyclic permutation). \( \Box \)
P 5.12. If \(a, b, c\) are nonnegative real numbers so that \(ab + bc + ca = 3\), then

\[
\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} \leq \frac{3}{2(a + b + c)} + \frac{a + b + c}{3}.
\]

(Vasile C., 2010)

**Solution.** Write the inequality in the homogeneous form

\[
\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} \leq \frac{3}{2(a + b + c)} + \frac{a + b + c}{ab + bc + ca}.
\]

Due to homogeneity and symmetry, we may assume that

\[
a + b + c = 1, \quad 0 \leq a \leq b \leq c, \quad ab + bc + ca > 0.
\]

Under this assumption, we write the desired inequality as

\[
f(a) + f(b) + f(c) \leq \frac{3}{2} + \frac{1}{ab + bc + ca},
\]

where

\[f(u) = \frac{1}{1-u}, \quad 0 \leq u < 1.\]

We will apply Corollary 1 to the function \(f\). We have

\[
g(x) = f'(x) = \frac{1}{(1-x)^2},
\]

\[
g''(x) = \frac{6}{(1-x)^4}.
\]

Since \(g''(x) > 0\) for \(x \in [0,1]\), \(g\) is strictly convex on \([0,1]\). According to Corollary 1 and Note 5/Note 3, if

\[
a + b + c = 1, \quad ab + bc + ca = \text{constant}, \quad 0 \leq a \leq b \leq c,
\]

then the sum

\[S_3 = f(a) + f(b) + f(c)\]

is maximum for \(a = b \leq c\). Thus, we only need to prove the homogeneous inequality for \(a = b = 1\) and \(c \geq 1\); that is,

\[
1 + \frac{4}{c + 1} \leq \frac{3}{c + 2} + \frac{2(c + 2)}{2c + 1},
\]

which reduces to

\[(c - 1)^2 \geq 0.
\]

The original inequality is an equality for \(a = b = c = 1\).

\[\Box\]
P 5.13. If \( a, b, c \) are nonnegative real numbers so that \( ab + bc + ca = 3 \), then

\[
\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} \geq \frac{3}{a + b + c} + \frac{a + b + c}{6}.
\]

(Vasile C., 2010)

Solution. Proceeding in the same manner as in the proof of the preceding P 5.12, we only need to prove the homogeneous inequality

\[
\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} \geq \frac{3}{a + b + c} + \frac{a + b + c}{2(ab + bc + ca)}
\]

for \( a = 0 \) and for \( a \leq b = c = 1 \).

Case 1: \( a = 0 \). The homogeneous inequality reduces to

\[
\frac{1}{b + c} \geq \frac{2}{b + c} + \frac{b + c}{2bc},
\]

which is equivalent to

\[
(b - c)^2 \geq 0.
\]

Case 2: \( a \leq b = c = 1 \). The homogeneous inequality becomes

\[
\frac{1}{2} + \frac{2}{a + 1} \geq \frac{3}{a + 2} + \frac{a + 2}{2(2a + 1)},
\]

\[
\frac{1}{2} - \frac{a + 2}{2(2a + 1)} \geq \frac{3}{a + 2} - \frac{2}{a + 1},
\]

\[
\frac{a - 1}{2(2a + 1)} \geq \frac{a - 1}{(a + 1)(a + 2)},
\]

\[
a(a - 1)^2 \geq 0.
\]

The equality holds for \( a = b = c = 1 \), and also for

\[
a = 0, \quad b = c = \sqrt{3}
\]

(or any cyclic permutation).

\[
\square
\]

P 5.14. Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. If

\[
a^2 + b^2 + c^2 = 3,
\]

then

\[
\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} + \frac{a + b + c}{9} \geq \frac{11}{2(a + b + c)}.
\]

(Vasile C., 2010)
**Solution.** Using the same method as in the proof of P 5.12, we only need to prove the homogeneous inequality

\[
\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} + \frac{a+b+c}{3(a^2 + b^2 + c^2)} \geq \frac{11}{2(a+b+c)}
\]

for \(a = 0\) and for \(a \leq b = c = 1\).

**Case 1:** \(a = 0\). The homogeneous inequality reduces to

\[
\frac{1}{b} + \frac{1}{c} + \frac{b+c}{b+c} + \frac{b+c}{3(b^2 + c^2)} \geq \frac{11}{2(b+c)},
\]

\[
\frac{b+c}{bc} + \frac{b+c}{3(b^2 + c^2)} \geq \frac{9}{2(b+c)},
\]

\[
(b+c)^2 \left( \frac{1}{bc} + \frac{1}{3(b^2 + c^2)} \right) \geq \frac{9}{2}.
\]

Using the substitution

\[x = \frac{b^2 + c^2}{bc}, \quad x \geq 2,\]

the inequality becomes

\[(x+2)\left(1 + \frac{1}{3x}\right) \geq \frac{9}{2},\]

which is equivalent to

\[6x^2 - 13x + 4 \geq 0,\]

\[x + 2(x-2)(3x-1) \geq 0.\]

**Case 2:** \(a \leq 1 = b = c\). The homogeneous inequality becomes

\[
\frac{1}{2} + \frac{2}{a+1} + \frac{a+2}{3(a^2 + 2)} \geq \frac{11}{2(a+2)},
\]

\[
\frac{a+2}{3(a^2 + 2)} + \frac{a^2 - 4a - 1}{2(a+1)(a+2)} \geq 0
\]

\[3a^4 - 10a^3 + 13a^2 - 8a + 2 \geq 0,\]

\[(a-1)^2(3a^2 - 4a + 2) \geq 0,\]

\[(a-1)^2[a^2 + 2(a-1)^2] \geq 0.
\]

The equality holds for \(a = b = c = 1\). \(\Box\)
P 5.15. Let $a, b, c$ be nonnegative real numbers, no two of which are zero. If
\[ a + b + c = 4, \]
then
\[ \frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} \geq \frac{15}{8 + ab + bc + ca}. \]
\[(Vasile C., 2010)\]

Solution. Using the same method as in P 5.12, we only need to prove the homogeneous inequality
\[ \frac{2}{a + b} + \frac{2}{b + c} + \frac{2}{c + a} \geq \frac{15(a + b + c)}{(a + b + c)^2 + 2(ab + bc + ca)} \]
for $a = 0$ and for $a \leq b = c = 1$.

Case 1: $a = 0$. The homogeneous inequality reduces to
\[ \frac{2(b + c)}{bc} + \frac{2}{b + c} \geq \frac{15(b + c)}{(b + c)^2 + 2bc}, \]
\[ \frac{2(b + c)^2}{bc} + 2 \geq \frac{15(b + c)^2}{(b + c)^2 + 2bc}. \]
Using the substitution
\[ x = \frac{(b + c)^2}{bc}, \quad x \geq 4, \]
the inequality becomes
\[ 2x + 2 \geq \frac{15x}{x + 2}, \]
which is equivalent to
\[ 2x^2 - 9x + 4 \geq 0, \]
\[ (x - 4)(2x - 1) \geq 0. \]

Case 2: $a \leq 1, b = c = 1$. The homogeneous inequality becomes
\[ 1 + \frac{4}{a + 1} \geq \frac{15(a + 2)}{(a + 2)^2 + 2(2a + 1)}, \]
\[ \frac{a + 5}{a + 1} \geq \frac{15(a + 2)}{a^2 + 8a + 6}, \]
\[ a(a - 1)^2 \geq 0. \]
The equality holds for $a = b = c = 4/3$, and also for
\[ a = 0, \quad b = c = 2 \]
(or any cyclic permutation).
**P 5.16.** If \( a, b, c \) are nonnegative real numbers, no two of which are zero, then

\[
\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{1}{a+b+c} + \frac{2}{\sqrt{ab+bc+ca}}.
\]

(\text{Vasile C., 2010})

**Solution.** Using the same method as in P 5.12, we only need to prove the desired homogeneous inequality for \( a = 0 \) and for \( 0 < a \leq b = c = 1 \).

**Case 1:** \( a = 0 \). The inequality reduces to the obvious form

\[
\frac{1}{b} + \frac{1}{c} \geq \frac{2}{\sqrt{bc}}.
\]

**Case 2:** \( 0 < a \leq 1 = b = c \). The inequality becomes

\[
\frac{1}{2} + \frac{2}{a+1} \geq \frac{1}{a+2} + \frac{2}{\sqrt{2a+1}},
\]

\[
\frac{1}{2} - \frac{1}{a+2} \geq \frac{2}{\sqrt{2a+1}} - \frac{2}{a+1},
\]

\[
\frac{a}{2(a+2)} \geq \frac{2(a+1-\sqrt{2a+1})}{(a+1)\sqrt{2a+1}},
\]

\[
\frac{a}{2(a+2)} \geq \frac{2a^2}{(a+1)\sqrt{2a+1} (a+1+\sqrt{2a+1})}.
\]

Since

\[\sqrt{2a+1} (a+1+\sqrt{2a+1}) \geq \sqrt{2a+1}(\sqrt{2a+1}+\sqrt{2a+1}) = 2(2a+1),\]

it suffices to show that

\[\frac{a}{2(a+2)} \geq \frac{a^2}{(a+1)(2a+1)},\]

which is equivalent to

\[a(1-a) \geq 0.\]

The equality holds for \( a = 0, \quad b = c \)

(or any cyclic permutation).

\[\square\]

**P 5.17.** If \( a, b, c \) are nonnegative real numbers, no two of which are zero, then

\[
\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{3-\sqrt{3}}{a+b+c} + \frac{2+\sqrt{3}}{2\sqrt{ab+bc+ca}}.
\]

(\text{Vasile C., 2010})
Solution. As shown in the proof of P 5.12, it suffices to consider the cases $a = 0$ and $a \leq b = c = 1$.

Case 1: $a = 0$. The inequality reduces to
\[
\frac{1}{b} + \frac{1}{c} \geq \frac{2 - \sqrt{3}}{b + c} + \frac{2 + \sqrt{3}}{2\sqrt{bc}}.
\]
It suffices to show that
\[
\frac{1}{b} + \frac{1}{c} \geq \frac{2 - \sqrt{3}}{2\sqrt{bc}} + \frac{2 + \sqrt{3}}{2\sqrt{bc}},
\]
which is equivalent to the obvious inequality
\[
\frac{1}{b} + \frac{1}{c} \geq \frac{2}{\sqrt{bc}}.
\]

Case 2: $a \leq 1 = b = c$. The inequality reduces to
\[
\frac{1}{2} + \frac{2}{a + 1} \geq \frac{3 - \sqrt{3}}{a + 2} + \frac{2 + \sqrt{3}}{2\sqrt{2a + 1}}.
\]
Using the substitution
\[
2a + 1 = 3x^2, \quad x \geq \frac{\sqrt{3}}{3},
\]
the inequality becomes
\[
\frac{1}{2} + \frac{4}{3x^2 + 1} \geq \frac{6 - 2\sqrt{3}}{3(x^2 + 1)} + \frac{2 + \sqrt{3}}{2\sqrt{3} x},
\]
\[
\frac{1}{2} + \frac{4}{3x^2 + 1} - \frac{2}{x^2 + 1} - \frac{1}{2x} \geq \frac{1}{\sqrt{3} x} - \frac{2}{\sqrt{3} (x^2 + 1)},
\]
\[
\frac{3x^5 - 3x^4 - 4x^2 + 5x - 1}{2x(x^2 + 1)(3x^2 + 1)} \geq \frac{1}{\sqrt{3} x} \left( \frac{1}{x} - \frac{2}{x^2 + 1} \right),
\]
\[
\frac{(x - 1)^2(3x^3 + 3x^2 + 3x - 1)}{2x(x^2 + 1)(3x^2 + 1)} \geq \frac{(x - 1)^2}{\sqrt{3} x(x^2 + 1)}.
\]
This is true if
\[
\frac{3x^3 + 3x^2 + 3x - 1}{2(3x^2 + 1)} \geq \frac{\sqrt{3}}{3},
\]
which is equivalent to
\[
9x^3 + 3(3 - 2\sqrt{3})x^2 + 9x - 3 - 2\sqrt{3} \geq 0,
\]
\[
(3x - \sqrt{3})[3x^2 + (3 - \sqrt{3})x + 2 + \sqrt{3}] \geq 0.
\]
The equality holds for $a = b = c$, and also for
\[
a = 0, \quad b = c
\]
(or any cyclic permutation).
P 5.18. Let \( a, b, c \) be nonnegative real numbers, no two of which are zero, so that
\[
ab + bc + ca = 3.
\]
If
\[
0 \leq k \leq \frac{9 + 5\sqrt{3}}{6} \approx 2.943,
\]
then
\[
\frac{2}{a + b} + \frac{2}{b + c} + \frac{2}{c + a} \geq \frac{9(1 + k)}{a + b + c + 3k}.
\]

(Vasile Cirtoaje and Lorian Saceanu, 2014)

Solution. From
\[
(a + b + c)^2 \geq 3(ab + bc + ca),
\]
we get
\[
a + b + c \geq 3.
\]
Let
\[
m = \frac{9 + 5\sqrt{3}}{6}, \quad m \geq k.
\]
We claim that
\[
\frac{1 + m}{a + b + c + 3m} \geq \frac{1 + k}{a + b + c + 3k}.
\]
Indeed, this inequality is equivalent to the obvious inequality
\[
(m - k)(a + b + c - 3) \geq 0.
\]
Thus, we only need to show that
\[
\frac{2}{a + b} + \frac{2}{b + c} + \frac{2}{c + a} \geq \frac{9(1 + m)}{a + b + c + 3m},
\]
which can be rewritten in the homogeneous form
\[
\frac{2}{a + b} + \frac{2}{b + c} + \frac{2}{c + a} \geq \frac{9(1 + m)}{a + b + c + m\sqrt{3(ab + bc + ca)}}.
\]
As shown in the proof of P 5.12, it suffices to prove this homogeneous inequality for \( a = 0 \) and for \( a \leq b = c = 1 \).

Case 1: \( a = 0 \). The inequality reduces to
\[
\frac{2}{b} + \frac{2}{c} \geq \frac{9(1 + m)}{b + c + m\sqrt{3bc}}.
\]
Substituting
\[
x = \frac{b + c}{\sqrt{bc}}, \quad x \geq 2,
\]
the inequality becomes
\[ 2x + \frac{2}{x} \geq \frac{9(1 + m)}{x + m\sqrt{3}}, \]
\[ 2x^3 + 2\sqrt{3} mx^2 - (7 + 9m)x + 2\sqrt{3} m \geq 0, \]
\[ (x - 2)[2x^2 + 2(\sqrt{3} m + 2)x - \sqrt{3} m] \geq 0. \]

Case 2: \(a \leq 1 = b = c\). The inequality becomes
\[ 1 + \frac{4}{a + 1} \geq \frac{9(1 + m)}{a + 2 + m\sqrt{3}(2a + 1)}. \]

Using the substitution
\[ 2a + 1 = 3x^2, \quad x \geq \frac{\sqrt{3}}{3}, \]
the inequality becomes
\[ \frac{3x^2 + 9}{3x^2 + 1} \geq \frac{6(1 + m)}{x^2 + 2mx + 1}, \]
\[ x^4 + 2mx^3 - 2(3m + 1)x^2 + 6mx + 1 - 2m \geq 0, \]
\[ (x - 1)^2[x^2 + 2(m + 1)x + 1 - 2m] \geq 0. \]

This is true since
\[ x^2 + 2(m + 1)x + 1 - 2m \geq \frac{1}{3} + \frac{2(m + 1)\sqrt{3}}{3} + 1 - 2m \]
\[ = \frac{2[2 + \sqrt{3} - (3 - \sqrt{3})m]}{3} = 0. \]

The equality holds for \(a = b = c = 1\). If \(k = \frac{9 + 5\sqrt{3}}{6}\), then the equality holds also for
\[ a = 0, \quad b = c = \sqrt{3} \]
(or any cyclic permutation).

\[ \square \]

**P 5.19.** If \(a, b, c\) are nonnegative real numbers, no two of which are zero, then
\[ \frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} \geq \frac{20}{a + b + c + 6\sqrt{ab} + bc + ca}. \]

(Vasile C., 2010)
Solution. The proof is similar to the one of P 5.12. Finally, we only need to prove the inequality for $a = 0$ and for $a \leq b = c = 1$.

Case 1: $a = 0$. The inequality reduces to

$$\frac{1}{b} + \frac{1}{c} + \frac{1}{b+c} \geq \frac{20}{b+c + 6 \sqrt{bc}}.$$

Substituting

$$x = \frac{b+c}{\sqrt{bc}}, \quad x \geq 2,$$

the inequality becomes

$$x + \frac{1}{x} \geq \frac{20}{x + 6},$$

$$x^3 + 6x^2 - 19x + 6 \geq 0,$$

$$(x - 2)(x^2 + 8x - 3) \geq 0.$$

Case 2: $a \leq 1 = b = c$. We need to show that

$$\frac{1}{2} + \frac{2}{a+1} \geq \frac{20}{a + 2 + 6 \sqrt{2a + 1}}.$$

Using the substitution

$$2a + 1 = x^2, \quad x \geq 1,$$

the inequality becomes

$$\frac{x^2 + 9}{2(x^2 + 1)} \geq \frac{40}{x^2 + 12x + 3},$$

$$x^4 + 12x^3 - 68x^2 + 108x - 53 \geq 0,$$

$$(x - 1)(x^3 + 13x^2 - 55x + 53) \geq 0.$$

It is true since

$$x^3 + 13x^2 - 55x + 53 = (x - 1)^3 + 16x^2 - 58x + 54$$

$$= (x - 1)^3 + 16 \left( x - \frac{29}{16} \right)^2 + \frac{23}{16} > 0.$$

The equality holds for

$$a = 0, \quad b = c$$

(or any cyclic permutation).
P 5.20. If \(a, b, c\) are nonnegative real numbers so that
\[
7(a^2 + b^2 + c^2) = 11(ab + bc + ca),
\]
then
\[
\frac{51}{28} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq 2.
\]

(Vasile C., 2008)

Solution. Due to homogeneity and symmetry, we may consider that
\[
a + b + c = 1, \quad 0 < a \leq b \leq c < 1.
\]
Thus, we need to show that
\[
a + b + c = 1, \quad a^2 + b^2 + c^2 = \frac{11}{25}, \quad 0 < a \leq b \leq c < 1
\]

involves
\[
\frac{51}{28} \leq \frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} \leq 2.
\]
We apply Corollary 1 to the function
\[
f(u) = \frac{u}{1-u}, \quad 0 \leq u < 1.
\]
We have
\[
g(x) = f'(x) = \frac{1}{(1-x)^2}, \quad g''(x) = \frac{6}{(1-x)^4}.
\]
Since \(g''(x) > 0\) for \(x \in [0, 1]\), \(g\) is strictly convex on \([0, 1]\). According to Corollary 1 and Note 5/Note 3, if
\[
a + b + c = 1, \quad a^2 + b^2 + c^2 = \frac{11}{25}, \quad 0 \leq a \leq b \leq c < 1
\]
then the sum
\[
S_3 = f(a) + f(b) + f(c)
\]
is maximum for \(a = b \leq c\), and is minimum for either \(a = 0\) or \(0 < a \leq b = c\). Note that the case \(a = 0\) is not possible because it involves \(7(b^2 + c^2) = 11bc\), which is false.

(1) To prove the right original inequality for \(a = b \leq c\), let us denote
\[
t = \frac{c}{a}, \quad t \geq 1.
\]
The hypothesis \(7(a^2 + b^2 + c^2) = 11(ab + bc + ca)\) involves \(t = 3\), hence
\[
\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{2a}{a+c} + \frac{c}{2a} = \frac{2}{1+t} + \frac{t}{2} = 2.
\]
The right inequality is an equality for \(a = b = \frac{c}{3}\) (or any cyclic permutation).

(2) To prove the left original inequality for \(0 < a \leq b = c\), let us denote
\[
t = \frac{a}{b}, \quad 0 < t \leq 1.
\]
The hypothesis \(7(a^2 + b^2 + c^2) = 11(ab + bc + ca)\) involves \(t = \frac{1}{7}\), hence
\[
\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{a}{2b} + \frac{2b}{a+b} = \frac{t}{2} + \frac{2}{t + 1} = \frac{51}{28}.
\]
The left inequality is an equality for \(7a = b = c\) (or any cyclic permutation).

\[
\square
\]

**P 5.21.** If \(a_1, a_2, \ldots, a_n\) are nonnegative real numbers so that
\[
\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n+3} = \left(\frac{a_1 + a_2 + \cdots + a_n}{n+1}\right)^2,
\]
then
\[
\frac{(n+1)(2n-1)}{2} \leq (a_1 + a_2 + \cdots + a_n) \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \leq \frac{3n^2(n+1)}{2(n+2)}.
\]

\(\text{(Vasile C., 2008)}\)

**Solution.** For \(n = 2\), both inequalities are identities. For \(n \geq 3\), assume that
\[
a_1 \leq a_2 \leq \cdots \leq a_n.
\]
The case \(a_1 = 0\) is not possible because the hypothesis involves
\[
\frac{a_2^2 + \cdots + a_n^2}{(a_2 + \cdots + a_n)^2} = \frac{n+3}{(n+1)^2} < \frac{1}{n-1},
\]
which contradicts the Cauchy-Schwarz inequality
\[
\frac{a_2^2 + \cdots + a_n^2}{(a_2 + \cdots + a_n)^2} \geq \frac{1}{n-1}.
\]
Due to homogeneity and symmetry, we may consider that
\[
a_1 + a_2 + \cdots + a_n = n + 1,
\]
which implies
\[
a_1^2 + a_2^2 + \cdots + a_n^2 = n + 3.
\]
Thus, we need to show that
\[ a_1 + a_2 + \cdots + a_n = n + 1, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n + 3, \quad 0 < a_1 \leq a_2 \leq \cdots \leq a_n \]

involves
\[ \frac{2n - 1}{2} \leq \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \leq \frac{3n^2}{2(n + 2)}. \]

We apply Corollary 5 for \( k = 2 \) and \( m = -1 \):

- If \( a_1, a_2, \ldots, a_n \) are positive real numbers so that \( 0 < a_1 \leq a_2 \leq \cdots \leq a_n \) and
\[ a_1 + a_2 + \cdots + a_n = n + 1, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n + 3, \]

then
\[ S_n = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \]

is minimum for
\[ 0 < a_1 = a_2 = \cdots = a_{n-1} \leq a_n, \]

and is maximum for
\[ a_1 \leq a_2 = a_3 = \cdots = a_n. \]

(1) To prove the left original inequality, we only need to consider the case
\[ a_1 = a_2 = \cdots = a_{n-1} \leq a_n. \]

The hypothesis
\[ \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n + 3} = \left( \frac{a_1 + a_2 + \cdots + a_n}{n + 1} \right)^2 \]

implies
\[ \frac{(n - 1)a_1^2 + a_n^2}{n + 3} = \left[ \frac{(n - 1)a_1 + a_n}{n + 1} \right]^2, \]
\[ (2a_1 - a_n)[2a_1 - (n + 2)a_n] = 0, \]
\[ a_1 = \frac{a_n}{2}, \]

hence
\[ (a_1 + a_2 + \cdots + a_n) \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right) = [(n - 1)a_1 + a_n] \left( \frac{n - 1}{a_1} + \frac{1}{a_n} \right) \]
\[ = (n - 1)^2 + 1 + (n - 1) \left( \frac{a_1}{a_n} + \frac{a_n}{a_1} \right) \]
\[ = \frac{(n + 1)(2n - 1)}{2}. \]

The equality holds for
\[ a_1 = a_2 = \cdots = a_{n-1} = \frac{a_n}{2}. \]
(or any cyclic permutation).

(2) To prove the right original inequality, we only need to consider the case

\[ a_1 \leq a_2 = a_3 = \cdots = a_n. \]

The hypothesis involves

\[ (a_1 - 2a_n)[(n + 2)a_1 - 2a_n] = 0, \]

\[ a_1 = \frac{2a_n}{n + 2}, \]

hence

\[ (a_1 + a_2 + \cdots + a_n) \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right) = [(n - 1)a_1 + a_n] \left( \frac{n - 1}{a_1} + \frac{1}{a_n} \right) \]

\[ = (n - 1)^2 + 1 + (n - 1) \left( \frac{a_1}{a_n} + \frac{a_n}{a_1} \right) \]

\[ = \frac{3n^2(n + 1)}{2(n + 2)}. \]

The equality holds for

\[ a_1 = a_2 = \cdots = a_{n - 1} = \frac{2a_n}{n + 2} \]

(or any cyclic permutation).

\[ \square \]

**P 5.22.** If \( a, b, c, d \) are nonnegative real numbers so that \( a + b + c + d = 3 \), then

\[ abc + bcd + cda + dab \leq 1 + \frac{176}{81} \quad abc. \]

*(Vasile C., 2005)*

**Solution.** Assume that

\[ a \leq b \leq c \leq d. \]

For \( a = 0 \), we need to show that \( b + c + d = 3 \) implies

\[ bcd \leq 1, \]

which follows immediately from the AM-GM inequality:

\[ bcd \leq \left( \frac{b + c + d}{3} \right)^3 = 1. \]
For $a > 0$, rewrite the inequality in the form

$$abcd \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) \leq 1 + \frac{176}{81} abcd$$

and apply Corollary 5 for $k = 0$ and $m = -1$:

- If $a + b + c + d = 3$, $abcd = \text{constant}$, $0 < a \leq b \leq c \leq d$,

then

$$S_4 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$$

is maximum for $a \leq b = c = d$.

Thus, we only need to prove the homogeneous inequality

$$27(a + b + c + d)(abc + bcd + cda + dab) \leq (a + b + c + d)^4 + 176abcd$$

for $a \leq b = c = d = 1$. The inequality becomes

$$27(a + 3)(3a + 1) \leq (a + 3)^4 + 176a,$$

$$a^4 + 12a^3 - 27a^2 + 14a \geq 0,$$

$$a(a - 1)^2(a + 14) \geq 0.$$

The equality holds for $a = b = c = d = 3/4$, and also for $a = 0, b = c = d = 1$

(or any cyclic permutation).  

\[ \square \]

**P 5.23.** If $a, b, c, d$ are nonnegative real numbers so that $a + b + c + d = 3$, then

$$a^2b^2c^2 + b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2 + \frac{3}{4}abcd \leq 1.$$

*(Gabriel Dospinescu and Vasile Cirtoaje, 2005)*

**Solution.** Assume that $a \leq b \leq c \leq d$.

For $a = 0$, we need to show that

$$b^2c^2d^2 \leq 1,$$
which follows immediately from the AM-GM inequality:
\[ bcd \leq \left( \frac{b + c + d}{3} \right)^3 = 1. \]

For \( a > 0 \), rewrite the inequality in the form
\[ a^2 b^2 c^2 d^2 \left( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \right) + \frac{3}{4}abcd \leq 1, \]
and apply Corollary 5 for \( k = 0 \) and \( m = -2 \):

- If
  \[ a + b + c + d = 3, \quad abcd = \text{constant}, \quad 0 < a \leq b \leq c \leq d, \]
  then
  \[ S_4 = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \]
is maximum for \( a \leq b = c = d \).

Thus, we only need to prove the homogeneous inequality
\[
\left( \frac{a + b + c + d}{3} \right)^6 \geq a^2 b^2 c^2 + b^2 c^2 d^2 + c^2 d^2 a^2 + d^2 a^2 b^2 + \frac{1}{12}abcd(a + b + c + d)^2
\]
for \( a \leq b = c = d = 1 \); that is, to show that \( 0 < a \leq 1 \) implies
\[
\left( 1 + \frac{a}{3} \right)^6 \geq 1 + 3a^2 + \frac{1}{12}a(a + 3)^2.
\]
Since
\[
\left( 1 + \frac{a}{3} \right)^3 = 1 + a + \frac{a^2}{3} + \frac{a^3}{27} > 1 + a + \frac{a^2}{3},
\]
it suffices to show that
\[
\left( 1 + a + \frac{a^2}{3} \right)^2 \geq 1 + 3a^2 + \frac{1}{12}a(a + 3)^2,
\]
which is equivalent to the obvious inequality
\[
4a^4 + 3a(1 - a)(15 - 7a) \geq 0.
\]
The equality holds for
\[ a = 0, \quad b = c = d = 1 \]
(or any cyclic permutation).

\[ \square \]
P 5.24. If $a, b, c, d$ are nonnegative real numbers so that $a + b + c + d = 3$, then
\[ a^2 b^2 c^2 + b^2 c^2 d^2 + c^2 d^2 a^2 + d^2 a^2 b^2 + \frac{4}{3} (abcd)^{3/2} \leq 1. \]

(Vasile C., 2005)

**Solution.** The proof is similar to the one of the preceding P 5.23. We need to prove that
\[ \left(1 + \frac{a}{3}\right)^6 \geq 1 + 3a^2 + \frac{4}{3} a^{3/2} \]
for $0 \leq a \leq 1$. Since
\[ 2a^{3/2} \leq a^2 + a, \]
it suffices to show that
\[ \left(1 + \frac{a}{3}\right)^6 \geq 1 + \frac{2}{3} a + \frac{11}{3} a^2. \]
Since
\[ \left(1 + \frac{a}{3}\right)^3 = 1 + a + \frac{a^2}{3} + \frac{a^3}{27} \geq 1 + a + \frac{a^2}{3} \]
and
\[ \left(1 + \frac{a^2}{3}\right)^2 = 1 + 2a + \frac{5}{3} a^2 + \frac{2}{3} a^3 + \frac{1}{9} a^4 \]
\[ \geq 1 + 2a + \frac{5}{3} a^2 + \frac{11}{3} a^3, \]
it suffices to show that
\[ 1 + 2a + \frac{5}{3} a^2 + \frac{11}{3} a^3 \geq 1 + \frac{2}{3} a + \frac{11}{3} a^2, \]
which is equivalent to the obvious inequality
\[ a(1 - a)(2 - a) \geq 0. \]
The equality holds for
\[ a = 0, \quad b = c = d = 1 \]
(or any cyclic permutation).

P 5.25. If $a, b, c, d$ are nonnegative real numbers so that $a + b + c + d = 4$, then
\[ a^2 b^2 c^2 + b^2 c^2 d^2 + c^2 d^2 a^2 + d^2 a^2 b^2 + 2 (abcd)^{3/2} \leq 6. \]

(Vasile C., 2005)
Solution. The proof is similar to the one of P 5.23. We need to prove that
\[ 6 \left( \frac{a+3}{4} \right)^6 \geq 1 + 3a^2 + 2a^{3/2} \]
for \( 0 \leq a \leq 1 \). Since
\[ 2a^{3/2} \leq a^2 + a, \]
it suffices to show that
\[ 6 \left( \frac{a+3}{4} \right)^6 \geq 1 + a + 4a^2. \]
Using the substitution
\[ x = \frac{1-a}{4}, \quad 0 \leq x \leq \frac{1}{4}, \]
the inequality becomes
\[ 3(1-x)^6 \geq 3 - 18x + 32x^2, \]
\[ x^2(13 - 60x + 45x^2 - 18x^3 + 3x^4) \geq 0. \]
This is true since
\[ 2(13 - 60x + 45x^2 - 18x^3 + 3x^4) > 25 - 120x + 90x^2 - 40x^3 \]
\[ = 5(1-4x)(5-4x+2x^2) \geq 0. \]
The equality holds for \( a = b = c = d = 1 \).

\[ \square \]

P 5.26. If \( a, b, c \) are nonnegative real numbers so that \( a + b + c = 3 \), then
\[ 11(ab + bc + ca) + 4(a^2b^2 + b^2c^2 + c^2a^2) \leq 45. \]
(Vasile C., 2005)

Solution. Assume that \( a \leq b \leq c \). For \( a = 0 \), we need to show that \( b + c = 3 \) involves
\[ 11bc + 4b^2c^2 \leq 45. \]
We have
\[ bc \leq \left( \frac{b+c}{2} \right)^2 = \frac{9}{4}, \]
hence
\[ 11bc + 4b^2c^2 \leq \frac{99}{4} + \frac{81}{4} = 45. \]
For \( a > 0 \), rewrite the desired inequality in the form
\[ 11abc \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) + 4a^2b^2c^2 \left( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) \leq 45. \]
According to Corollary 5 (case $k = 2$ and $m < 0$), if 
\[ a + b + c = 3, \quad abc = \text{constant}, \quad 0 < a \leq b \leq c, \]
then the sums \( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \) and \( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \) are maximum for \( 0 < a \leq b = c \).
Therefore, we only need to prove that \( a + 2b = 3 \) involves
\[ 11(2ab + b^2) + 4(2a^2b^2 + b^4) \leq 45, \]
which is equivalent to
\[ 15 - 22b - 13b^2 + 32b^3 - 12b^4 \geq 0, \]
\[ (3 - 2b)(1 - b)^2(5 + 6b) \geq 0, \]
\[ a(1 - b)^2(5 + 6b) \geq 0. \]
The equality holds for \( a = b = c = 1 \), and also for
\[ a = 0, \quad b = c = \frac{3}{2} \]
(or any cyclic permutation).

**Remark.** In the same manner, we can prove the following statement:

- If \( a, b, c, d \) are nonnegative real numbers so that \( a + b + c + d = 4 \), then
  \[ abc + bcd + cda + dab + a^2b^2c^2 + b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2 \leq 8, \]
with equality for \( a = b = c = d = 1 \).

**P 5.27.** If \( a, b, c \) are nonnegative real numbers so that \( a + b + c = 3 \), then
\[ a^2b^2 + b^2c^2 + c^2a^2 + a^3b^3 + b^3c^3 + c^3a^3 \geq 6abc. \]

*(Vasile C., 2005)*

**Solution.** Assume that \( a \leq b \leq c \). For \( a = 0 \), the inequality is trivial. For \( a > 0 \), rewrite the desired inequality in the form
\[ abc \left( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) + a^2b^2c^2 \left( \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \right) \geq 6. \]
According to Corollary 5 (case \( k = 0 \) and \( m < 0 \)), if
\[ a + b + c = 3, \quad abc = \text{constant}, \quad 0 < a \leq b \leq c, \]
then the sums $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$ and $\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}$ are maximum for $0 < a \leq b = c$.

Thus, we only need to prove that

$$2a^2b^2 + b^4 + 2a^3b^3 + b^6 \geq 6ab^2$$

for

$$a + 2b = 3, \quad 1 \leq b < 3/2.$$ 

The inequality is equivalent to

$$b^3(14 - 33b + 24b^2 - 5b^3) \geq 0,$$

$$b^3(1 - b)^2(14 - 5b) \geq 0.$$ 

The equality holds for $a = b = c = 1$, and also for

$$a = b = 0, \quad c = 3$$

(or any cyclic permutation).

\[\square\]

**P 5.28.** If $a, b, c$ are nonnegative real numbers so that $a + b + c = 3$, then

$$2(a^2 + b^2 + c^2) + 5\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) \geq 21.$$ 

(\textit{Vasile C., 2008})

**Solution.** Apply Corollary 5 for $k = 2$ and $m = 1/2$:

- If

  $$a + b + c = 3, \quad a^2 + b^2 + c^2 = \text{constant}, \quad 0 \leq a \leq b \leq c,$$

  then

  $$S_3 = \sqrt{a} + \sqrt{b} + \sqrt{c}$$

is minimum for either $a = 0$ or $0 < a \leq b = c$.

  

  **Case 1:** $a = 0$. We need to show that $b + c = 3$ involves

  $$2(b^2 + c^2) + 5\left(\sqrt{b} + \sqrt{c}\right) \geq 21,$$

  which is equivalent to

  $$5\sqrt{3 + 2\sqrt{bc}} \geq 3 + 4bc.$$

  Substituting

  $$x = \sqrt{bc}, \quad 0 \leq x \leq \frac{b + c}{2} = \frac{3}{2},$$
the inequality becomes

\[ 5 \sqrt{3 + 2x} \geq 3 + 4x^2, \]
\[ 25(3 + 2x) \geq (3 + 4x^2)^2. \]

For \( 0 < x \leq 3/2 \), this inequality is equivalent to \( f(x) \geq 0 \), where

\[ f(x) = \frac{66}{x} + 50 - 24x - 16x^3. \]

We have

\[ f(x) \geq f(3/2) = 4 > 0. \]

**Case 2:** \( 0 < a \leq b = c \). We need to show that

\[ 2(a^2 + 2b^2) + 5\left(\sqrt{a} + 2\sqrt{b}\right) \geq 21 \]

for

\[ a + 2b = 3, \quad 1 \leq b < \frac{3}{2}. \]

Write the inequality as

\[ 5\sqrt{3 - 2b} + 10\sqrt{b} \geq 3 + 24b - 12b^2. \]

Substituting

\[ x = \sqrt{b}, \quad 1 \leq x < \sqrt{\frac{3}{2}}, \]

the inequality becomes

\[ 5\sqrt{3 - 2x^2} \geq 3 - 10x + 24x^2 - 12x^4, \]
\[ 12(x^2 - 1)^2 \geq 5\left(3 - 2x - \sqrt{3 - 2x^2}\right), \]
\[ 12(x^2 - 1)^2 \geq \frac{30(x - 1)^2}{3 - 2x + \sqrt{3 - 2x^2}}, \]

which is true if

\[ 2(x + 1)^2 \geq \frac{5}{3 - 2x + \sqrt{3 - 2x^2}}. \]

It suffices to show that

\[ 2(x + 1)^2 \geq \frac{5}{3 - 2x}, \]

which is equivalent to

\[ 1 + 8x - 2x^2 - 4x^3 \geq 0, \]
\[ x(5 - 4x)\left(\frac{7}{4} + x\right) + \frac{4 - 3x}{4} \geq 0. \]
Since
\[ x < \sqrt{\frac{3}{2}} < \frac{5}{4} < \frac{4}{3}, \]
the conclusion follows.

The equality holds for \( a = b = c = 1. \)

\[ \Box \]

**P 5.29.** If \( a, b, c \) are nonnegative real numbers so that \( ab + bc + ca = 3 \), then
\[ \sqrt{\frac{1+2a}{3}} + \sqrt{\frac{1+2b}{3}} + \sqrt{\frac{1+2c}{3}} \geq 3. \]

(Vasile C., 2008)

**Solution.** Write the hypothesis \( ab + bc + ca = 3 \) as
\[ (a + b + c)^2 = 6 + a^2 + b^2 + c^2, \]
and apply Corollary 1 to
\[ f(u) = \sqrt{\frac{1+2u}{3}}, \quad u \geq 0. \]

We have
\[ g(x) = f'(x) = \frac{1}{\sqrt{3(1+2x)}}, \]
\[ g''(x) = \frac{\sqrt{3}}{(1+2x)^{3/2}}. \]

Since \( g''(x) > 0 \) for \( x \geq 0 \), \( g \) is strictly convex on \([0, \infty)\). According to Corollary 1, if
\[ a + b + c = \text{constant}, \quad a^2 + b^2 + c^2 = \text{constant}, \quad 0 \leq a \leq b \leq c, \]
then the sum
\[ S_3 = f(a) + f(b) + f(c) \]
is minimum for either \( a = 0 \) or \( 0 < a \leq b = c. \)

**Case 1:** \( a = 0. \) We need to show that \( bc = 3 \) involves
\[ \sqrt{1+2b} + \sqrt{1+2c} \geq 3\sqrt{3} - 1. \]

By squaring, the inequality becomes
\[ b + c + \sqrt{13 + 2(b + c)} \geq 13 - 3\sqrt{3}. \]
We have \( b + c \geq 2\sqrt{bc} = 2\sqrt{3} \), hence
\[
b + c + \sqrt{13 + 2(b + c)} \geq 2\sqrt{3} + \sqrt{13 + 4\sqrt{3}} = 4\sqrt{3} + 1 > 13 - 3\sqrt{3}.
\]

**Case 2:** \( 0 < a \leq b = c \). From \( ab + bc + ca = 3 \), it follows that
\[
a = \frac{3 - b^2}{2b}. \quad 0 < b < \sqrt{3}.
\]
Thus, the inequality can be written as
\[
\sqrt{1 + \frac{3 - b^2}{b} + 2\sqrt{1 + 2b}} \geq 3\sqrt{3}.
\]
Substituting
\[
t = \sqrt{\frac{1 + 2b}{3}}, \quad \frac{1}{\sqrt{3}} < t < \sqrt{\frac{1 + 2\sqrt{3}}{3}} < \frac{5}{4},
\]
the inequality turns into
\[
\sqrt{\frac{3 + 4t^2 - 3t^4}{2(3t^2 - 1)}} \geq 3 - 2t.
\]
By squaring, we need to show that
\[
7 - 8t - 14t^2 + 24t^3 - 9t^4 \geq 0,
\]
which is equivalent to
\[
(1 - t)^2(7 + 6t - 9t^2) \geq 0.
\]
This is true since
\[
7 + 6t - 9t^2 = 8 - (3t - 1)^2 > 8 - \left(\frac{15}{4} - 1\right)^2 = \frac{7}{16} > 0.
\]
The equality holds for \( a = b = c = 1 \).

**P 5.30.** Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. If
\[
0 \leq k \leq 15,
\]
then
\[
\frac{1}{(a + b)^2} + \frac{1}{(b + c)^2} + \frac{1}{(c + a)^2} + \frac{k}{(a + b + c)^2} \geq \frac{9 + k}{4(ab + bc + ca)}.
\]
(\text{Vasile C., 2007})
Solution. Due to homogeneity and symmetry, we may consider that

\[ a + b + c = 1, \quad 0 \leq a \leq b \leq c. \]

On this assumption, the inequality becomes

\[ \frac{1}{(1-a)^2} + \frac{1}{(1-b)^2} + \frac{1}{(1-c)^2} + k \geq \frac{9 + k}{2(1-a^2 - b^2 - c^2)}. \]

To prove it, we apply Corollary 1 to the function

\[ f(u) = \frac{1}{(1-u)^2}, \quad 0 \leq u < 1. \]

We have

\[ g(x) = f'(x) = \frac{2}{(1-x)^3}, \quad g''(x) = \frac{24}{(1-x)^5}. \]

Since \( g''(x) > 0 \) for \( x \in [0, 1] \), \( g \) is strictly convex on \([0, 1]\). According to Corollary 1 and Note 5/Note 3, if

\[ a + b + c = 1, \quad a^2 + b^2 + c^2 = \text{constant}, \quad 0 \leq a \leq b \leq c, \]

then the sum

\[ S_3 = f(a) + f(b) + f(c) \]

is minimum for either \( a = 0 \) or \( 0 < a \leq b = c \).

Case 1: \( a = 0 \). For

\[ x = \frac{b}{c} + \frac{c}{b}, \quad x \geq 2, \]

the original inequality becomes

\[ \frac{1}{b^2} + \frac{1}{c^2} + \frac{1+k}{(b+c)^2} \geq \frac{9+k}{4bc}, \]

\[ x + \frac{1+k}{x+2} \geq \frac{9+k}{4}, \]

\[ (x-2)(4x+7-k) \geq 0. \]

This is true since

\[ 4x + 7 - k \geq 15 - k \geq 0. \]

Case 2: \( 0 < a \leq b = c \). The original inequality becomes

\[ \frac{2}{(a+b)^2} + \frac{1}{4b^2} + \frac{k}{(a+2b)^2} \geq \frac{9+k}{4b(2a+b)}, \]

\[ \frac{a(a-b)^2}{2b^2(a+b)^2(2a+b)} + \frac{ka(4b-a)}{4b(a+2b)^2(2a+b)} \geq 0. \]
The equality holds for 
\[ a = 0, \quad b = c \]
(or any cyclic permutation). If \( k = 0 \) (Iran 1996 inequality), then the equality holds also for \( a = b = c \).

\[ \Box \]

**P 5.31.** If \( a, b, c \) are nonnegative real numbers, no two of which are zero, then

\[
\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} + \frac{24}{(a+b+c)^2} \geq \frac{8}{ab+bc+ca}.
\]

(Vasile C., 2007)

**Solution.** As shown in the proof of the preceding P 5.30, it suffices to prove the inequality for \( a = 0 \), and for \( 0 < a \leq b = c \).

**Case 1:** \( a = 0 \). For 
\[ x = \frac{b}{c} + \frac{c}{b}, \quad x \geq 2, \]
the original inequality becomes

\[
\frac{1}{b^2} + \frac{1}{c^2} + \frac{25}{(b+c)^2} \geq \frac{8}{bc},
\]

\[
x + \frac{25}{x+2} \geq 8,
\]

\[(x-3)^2 \geq 0.\]

**Case 2:** \( 0 < a \leq b = c \). Due to homogeneity, we only need to prove the homogeneous inequality for \( 0 < a \leq b = c = 1 \); that is,

\[
\frac{2}{(a+1)^2} + \frac{1}{4} + \frac{24}{(a+2)^2} \geq \frac{8}{2a+1}.
\]

It suffices to show that

\[
\frac{2}{(a+1)^2} \geq \frac{8}{2a+1} - \frac{24}{(a+2)^2},
\]

which is equivalent to

\[
\frac{1}{(1+a)^2} \geq \frac{4(1-a)^2}{(2a+1)(a+2)^2},
\]

\[ a(2a^2 + 9a + 12) \geq 4a^2(a^2 - 2). \]

This is true since

\[ a(2a^2 + 9a + 12) \geq 0 \geq 4a^2(a^2 - 2). \]
The equality holds for
\[ a = 0, \quad \frac{b}{c} + \frac{c}{b} = 3 \]
(or any cyclic permutation).

**Remark.** Actually, the following generalization holds:

- Let \( a, b, c \) be nonnegative real numbers, no two of which are zero. If \( k \geq 15 \), then
  \[
  \frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} + \frac{k}{(a+b+c)^2} \geq \frac{2(\sqrt{k+1} - 1)}{ab + bc + ca},
  \]
  with equality for
  \[ a = 0, \quad \frac{b}{c} + \frac{c}{b} = \sqrt{k+1} - 2 \]
  (or any cyclic permutation).

\( \Box \)

**P 5.32.** If \( a, b, c \) are nonnegative real numbers, no two of which are zero, so that
\[
k(a^2 + b^2 + c^2) + (2k+3)(ab + bc + ca) = 9(k+1), \quad 0 \leq k \leq 6,
\]
then
\[
\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} + \frac{9k}{(a+b+c)^2} \geq \frac{3}{4} + k.
\]
(Vasile C., 2007)

**Solution.** Write the inequality in the homogeneous form
\[
\frac{4}{(a+b)^2} + \frac{4}{(b+c)^2} + \frac{4}{(c+a)^2} + \frac{36k}{(a+b+c)^2} \geq \frac{9(k+1)(4k+3)}{k(a^2 + b^2 + c^2) + (2k+3)(ab + bc + ca)}.
\]
As shown in the proof of P 5.30, it suffices to prove this inequality for \( a = 0 \), and for \( 0 < a \leq b = c \).

**Case 1:** \( a = 0 \). Let
\[ x = \frac{b}{c} + \frac{c}{b}, \quad x \geq 2. \]
The homogeneous inequality becomes
\[
4 \left( \frac{1}{b^2} + \frac{1}{c^2} \right) + \frac{36k + 4}{(b+c)^2} \geq \frac{9(k+1)(4k+3)}{k(b^2 + c^2) + (2k+3)bc},
\]
\[
4x + \frac{36k + 4}{x + 2} \geq \frac{9(k+1)(4k+3)}{kx + 2k + 3},
\]
\[
4kx^3 + 4(4k+3)x^2 - (43k+3)x - 2(5k+21) \geq 0,
\]
(x - 2)[4kx^2 + 4(6k + 3)x + 5k + 21] \geq 0.

**Case 2:** 0 < a \leq b = c. We only need to prove the homogeneous inequality for b = c = 1. The inequality becomes

\[ \frac{8}{(a + 1)^2} + 1 + \frac{36k}{(a + 2)^2} \geq \frac{9(k + 1)(4k + 3)}{ka^2 + (4k + 6)a + 4k + 3}, \]

ka^6 + (10k + 6)a^5 - (14k - 12)a^4 - (10k + 18)a^3 + (17k - 24)a^2 + (24 - 4k)a \geq 0,

a(a - 1)^2[ka^3 + 6(2k + 1)a^2 + 3(3k + 8)a + 4(6 - k)] \geq 0.

Clearly, the last inequality is true for 0 \leq k \leq 6.

The equality holds for a = b = c, and also for

\[ a = 0, \quad b = c \]

(or any cyclic permutation).

\[ \square \]

**P 5.33.** If a, b, c are nonnegative real numbers, no two of which are zero, then

(a) \[ \frac{2}{a^2 + b^2} + \frac{2}{b^2 + c^2} + \frac{2}{c^2 + a^2} \geq \frac{8}{a^2 + b^2 + c^2} + \frac{1}{ab + bc + ca}; \]

(b) \[ \frac{2}{a^2 + b^2} + \frac{2}{b^2 + c^2} + \frac{2}{c^2 + a^2} \geq \frac{7}{a^2 + b^2 + c^2} + \frac{6}{(a + b + c)^2}; \]

(c) \[ \frac{2}{a^2 + b^2} + \frac{2}{b^2 + c^2} + \frac{2}{c^2 + a^2} \geq \frac{45}{4(a^2 + b^2 + c^2)} + \frac{ab + bc + ca}{ab + bc + ca}. \]

(\text{Vasile C., 2007})

**Solution.** (a) Due to homogeneity and symmetry, we may consider that

\[ a^2 + b^2 + c^2 = 1, \quad 0 \leq a \leq b \leq c. \]

On this assumption, the inequality can be written as

\[ \frac{2}{1 - a^2} + \frac{2}{1 - b^2} + \frac{2}{1 - c^2} \geq 8 + \frac{2}{(a + b + c)^2 - 1}. \]

To prove it, we apply Corollary 1 to the function

\[ f(u) = \frac{1}{1 - u^2}, \quad 0 \leq u < 1. \]

We have

\[ g(x) = f'(x) = \frac{2x}{(1 - x^2)^2}, \quad g''(x) = \frac{24x(1 + x^2)}{(1 - x^2)^4}. \]
Since \(g''(x) > 0\) for \(x \in (0,1)\), \(g\) is strictly convex on \([0,1)\). According to Corollary 1 and Note 5/Note 3, if

\[a + b + c = \text{constant}, \quad a^2 + b^2 + c^2 = 1, \quad 0 \leq a \leq b \leq c,\]

then the sum

\[S_3 = f(a) + f(b) + f(c)\]

is minimum for either \(a = 0\) or \(0 < a \leq b = c\).

Case 1: \(a = 0\). For

\[x = \frac{b}{c} + \frac{c}{b}, \quad x \geq 2,\]

the original inequality becomes

\[\frac{2}{b^2} + \frac{2}{c^2} \geq \frac{6}{b^2 + c^2} + \frac{1}{bc},\]

\[2x \geq \frac{6}{x} + 1,\]

\[(x - 2)(2x + 3) \geq 0.\]

Case 2: \(0 < a \leq b = c\). Due to homogeneity, it suffices to prove the original inequality for \(b = c = 1\). Thus, we need to show that

\[1 + \frac{4}{a^2 + 1} \geq \frac{8}{a^2 + 2} + \frac{1}{2a + 1},\]

which is equivalent to

\[\frac{2a}{2a + 1} \geq \frac{4a^2}{(a^2 + 1)(a^2 + 2)},\]

\[a(a^4 - a^2 - 2a + 2) \geq 0,\]

\[a(a - 1)^2(a^2 + 2a + 2) \geq 0.\]

The equality holds for \(a = b = c\), and also for \(a = 0, b = c\) (or any cyclic permutation).

(b) The proof is similar to the one of the inequality in (a). For \(a = 0\) and

\[x = \frac{b}{c} + \frac{c}{b}, \quad x \geq 2,\]

the original inequality becomes

\[\frac{2}{b^2} + \frac{2}{c^2} \geq \frac{5}{b^2 + c^2} + \frac{6}{(b + c)^2},\]

\[2x \geq \frac{5}{x} + \frac{6}{x + 2}.\]
\[(x - 2)(2x^2 + 8x + 5) \geq 0.\]

For \(b = c = 1\), the original inequality is

\[1 + \frac{4}{a^2 + 1} \geq \frac{7}{a^2 + 2} + \frac{6}{(a + 2)^2},\]

\[a(a^5 + 4a^4 - 2a^3 - 15a + 12) \geq 0,\]

\[a(a - 1)^2(a^3 + 6a^2 + 9a + 12) \geq 0.\]

The equality holds for \(a = b = c\), and also for \(a = 0, b = c\) (or any cyclic permutation).

(c) The proof is also similar to the one of the inequality in (a). For \(a = 0\) and

\[x = \frac{b}{c} + \frac{c}{b}, \quad x \geq 2,\]

the original inequality becomes

\[2 \left( \frac{1}{b^2} + \frac{1}{c^2} \right) + \frac{2}{b^2 + c^2} \geq \frac{45}{4(b^2 + c^2) + bc},\]

\[2x + \frac{2}{x} \geq \frac{45}{4x + 1},\]

\[(x - 2)(8x^2 + 18x - 1) \geq 0.\]

For \(b = c = 1\), the original inequality can be written as

\[1 + \frac{4}{a^2 + 1} \geq \frac{45}{4a^2 + 2a + 9},\]

\[a(2a^3 + a^2 - 8a + 5) \geq 0,\]

\[a(a - 1)^2(2a + 5) \geq 0.\]

The equality holds for \(a = b = c\), and also for \(a = 0, b = c\) (or any cyclic permutation).

\[\square\]

**P 5.34.** If \(a, b, c\) are nonnegative real numbers, no two of which are zero, then

\[\frac{1}{a^2 + b^2} + \frac{1}{b^2 + c^2} + \frac{1}{c^2 + a^2} + \frac{3}{a^2 + b^2 + c^2} \geq \frac{4}{ab + bc + ca}.\]

(Vasile C., 2007)
**Solution.** As shown in the proof of the preceding P 5.33, it suffices to prove the inequality for \( a = 0 \), and for \( 0 < a \leq b = c \).

**Case 1:** \( a = 0 \). For 
\[
x = \frac{b}{c} + \frac{c}{b}, \quad x \geq 2,
\]
the original inequality becomes
\[
\frac{1}{b^2} + \frac{1}{c^2} + \frac{4}{b^2 + c^2} \geq \frac{4}{bc},
\]
\[
x + \frac{4}{x} \geq 4,
\]
\[
(x - 2)^2 \geq 0.
\]

**Case 2:** \( 0 < a \leq b = c \). Due to homogeneity, it suffices to prove the original inequality for \( 0 < a \leq b = c = 1 \). Thus, we need to show that
\[
\frac{1}{2} + \frac{2}{a^2 + 1} + \frac{3}{a^2 + 2} \geq \frac{4}{2a + 1}.
\]
It suffices to show that
\[
\frac{2}{a + 1} + \frac{3}{a + 2} \geq \frac{4}{2a + 1} - \frac{1}{2},
\]
which is equivalent to
\[
\frac{5a + 7}{a^2 + 3a + 2} \geq \frac{7 - 2a}{4a + 2},
\]
\[
a(2a^2 + 19a + 21) \geq 0,
\]
The equality holds for
\[
a = 0, \quad b = c
\]
(or any cyclic permutation).

**Remark.** Actually, the following generalization holds:

- **Let** \( a, b, c \) **be nonnegative real numbers, no two of which are zero.**
  - **(a)** If \(-4 \leq k \leq 3\), then
    \[
    \frac{2}{a^2 + b^2} + \frac{2}{b^2 + c^2} + \frac{2}{c^2 + a^2} + \frac{2k}{a^2 + b^2 + c^2} \geq \frac{k + 5}{ab + bc + ca},
    \]
    with equality for
    \[
a = 0, \quad b = c
    \]
    (or any cyclic permutation).
  - **(b)** If \( k \geq 3 \), then
    \[
    \frac{1}{a^2 + b^2} + \frac{1}{b^2 + c^2} + \frac{1}{c^2 + a^2} + \frac{k}{a^2 + b^2 + c^2} \geq \frac{2\sqrt{k + 1}}{ab + bc + ca},
    \]
with equality for

\[ a = 0, \quad \frac{b}{c} + \frac{c}{b} = \sqrt{k + 1} \]

(or any cyclic permutation).

**P 5.35.** If \( a, b, c \) are nonnegative real numbers, no two of which are zero, then

\[
\begin{align*}
(a) & \quad \frac{3}{a^2 + ab + b^2} + \frac{3}{b^2 + bc + c^2} + \frac{3}{c^2 + ca + a^2} \geq \frac{5}{ab + bc + ca} + \frac{4}{a^2 + b^2 + c^2}; \\
(b) & \quad \frac{3}{a^2 + ab + b^2} + \frac{3}{b^2 + bc + c^2} + \frac{3}{c^2 + ca + a^2} \geq \frac{1}{ab + bc + ca} + \frac{24}{(a + b + c)^2}; \\
(c) & \quad \frac{1}{a^2 + ab + b^2} + \frac{1}{b^2 + bc + c^2} + \frac{1}{c^2 + ca + a^2} \geq \frac{21}{2(a^2 + b^2 + c^2) + 5(ab + bc + ca)}.
\end{align*}
\]

*(Vasile C., 2007)*

**Solution.** (a) Due to homogeneity and symmetry, we may consider that

\[ a + b + c = 1, \quad 0 \leq a \leq b \leq c. \]

Let

\[ p = \frac{1 + a^2 + b^2 + c^2}{2}. \]

Since

\[ \frac{1}{2(b^2 + bc + c^2)} = \frac{1}{(a + b + c)^2 + a^2 + b^2 + c^2 - 2a(a + b + c)} = \frac{1}{2(p - a)}, \]

the inequality can be written as

\[ \frac{3}{p - a} + \frac{3}{p - b} + \frac{3}{p - c} \geq \frac{5}{1 - p} + \frac{4}{2p - 1}. \]

To prove it, we apply Corollary 1 to the function

\[ f(u) = \frac{3}{p - u}, \quad 0 \leq u < p. \]

We have

\[ g(x) = f'(x) = \frac{3}{(p - x)^2}, \quad g''(x) = \frac{18}{(p - x)^4}. \]

Since \( g''(x) > 0 \) for \( x \in [0, p] \), \( g \) is strictly convex on \( [0, p] \). According to Corollary 1 and Note 5/Note 3, if

\[ a + b + c = 1, \quad a^2 + b^2 + c^2 = 2p - 1 = \text{constant}, \quad 0 \leq a \leq b \leq c, \]

then

\[ \frac{3}{p - a} + \frac{3}{p - b} + \frac{3}{p - c} \geq \frac{5}{1 - p} + \frac{4}{2p - 1}. \]
then the sum
\[ S_3 = f(a) + f(b) + f(c) \]
is minimum for either \( a = 0 \) or \( 0 < a \leq b = c \).

**Case 1:** \( a = 0 \). For
\[ x = \frac{b}{c} + \frac{c}{b}, \quad x \geq 2, \]
the original inequality becomes
\[ 3 \left( \frac{1}{b^2} + \frac{1}{c^2} \right) + \frac{3}{b^2 + bc + c^2} \geq \frac{5}{bc} + \frac{4}{b^2 + c^2}, \]
which is equivalent to
\[ 3x + \frac{3}{x + 1} \geq 5 + \frac{4}{x}, \quad (x - 2)(3x^2 + 4x + 2) \geq 0. \]

**Case 2:** \( 0 < a \leq b = c \). Due to homogeneity, it suffices to prove the original inequality for \( b = c = 1 \). Thus, we need to show that
\[ \frac{6}{a^2 + a + 1} + 1 \geq \frac{5}{2a + 1} + \frac{4}{a^2 + 2}, \]
which is equivalent to
\[ a(a^4 - a^3 + 3a^2 - 7a + 4) \geq 0, \]
\[ a(a - 1)^2(a^2 + a + 4) \geq 0. \]
The equality holds for \( a = b = c \), and also for \( a = 0, b = c \) (or any cyclic permutation).

(b) The proof is similar to the one of the inequality in (a). For \( a = 0 \), the original inequality becomes
\[ 3 \left( \frac{1}{b^2} + \frac{1}{c^2} \right) + \frac{3}{b^2 + bc + c^2} \geq \frac{1}{bc} + \frac{24}{(b + c)^2}, \]
which is equivalent to
\[ 3x + \frac{3}{x + 1} \geq 1 + \frac{24}{x + 2}, \quad x = \frac{b}{c} + \frac{c}{b}, \quad (x - 2)(3x^2 + 14x + 10) \geq 0. \]
For \( b = c = 1 \), the original inequality becomes
\[ \frac{6}{a^2 + a + 1} + 1 \geq \frac{1}{2a + 1} + \frac{24}{a^2 + 2}. \]
which is equivalent to

\[ a(a^4 + 5a^3 - 9a^2 - a + 4) \geq 0, \]
\[ a(a - 1)^2(a^2 + 7a + 4) \geq 0. \]

The equality holds for \( a = b = c \), and also for \( a = 0, b = c \) (or any cyclic permutation).

(c) The proof is similar to the one of the inequality in (a). For \( a = 0 \), the original inequality becomes

\[ \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{b^2 + bc + c^2} \geq \frac{21}{2(b^2 + c^2) + 5bc}, \]

which is equivalent to

\[ x \geq \frac{21}{2x + 5}, \quad x = \frac{b + c}{b} \]
\[ (x - 2)(2x^2 + 11x + 8) \geq 0. \]

For \( b = c = 1 \), the original inequality becomes

\[ \frac{2}{a^2 + a + 1} + \frac{1}{3} \geq \frac{21}{2a^2 + 10a + 9}, \]

which is equivalent to

\[ a(a^2 + 6a^2 - 15a + 8) \geq 0, \]
\[ a(a - 1)^2(a + 8) \geq 0. \]

The equality holds for \( a = b = c \), and also for \( a = 0, b = c \) (or any cyclic permutation).

\[ \square \]

**P 5.36.** If \( a, b, c \) are the lengths of the side of a triangle, then

\[ \frac{1}{(a + b)^2} + \frac{1}{(b + c)^2} + \frac{1}{(c + a)^2} \leq \frac{85}{36(a b + b c + c a)}. \]

*(Vasile C., 2007)*

**Solution.** Use the substitution

\[ a = y + z, \quad b = z + x, \quad c = x + y, \]

where \( x, y, z \) are nonnegative real numbers. Due to homogeneity and symmetry, we may consider that

\[ x + y + z = 2, \quad 0 \leq x \leq y \leq z. \]
We need to show that
\[
\frac{1}{(x + 2)^2} + \frac{1}{(y + 2)^2} + \frac{1}{(z + 2)^2} \leq \frac{85}{18(12 - x^2 - y^2 - z^2)},
\]
which can be written as
\[
f(x) + f(y) + f(z) + \frac{85}{18(12 - x^2 - y^2 - z^2)} \geq 0,
\]
where
\[
f(u) = \frac{-1}{(u + 2)^2}, \quad u \geq 0.
\]
We have
\[
g(x) = f'(x) = \frac{2}{(x + 2)^3}, \quad g''(x) = \frac{24}{(x + 2)^5}.
\]
Since \(g''(x) > 0\) for \(x \geq 0\), \(g\) is strictly convex on \([0, \infty)\). According to Corollary 1, if
\[
x + y + z = 2, \quad x^2 + y^2 + z^2 = \text{constant}, \quad 0 \leq x \leq y \leq z,
\]
then the sum
\[
S_3 = f(x) + f(y) + f(z)
\]
is minimum for either \(x = 0\) or \(0 < x \leq y = z\).

**Case 1:** \(x = 0\). This implies \(a = b + c\). Since
\[
\frac{1}{(a + b)^2} + \frac{1}{(c + a)^2} = \frac{5(b^2 + c^2) + 8bc}{(2b^2 + 2c^2 + 5bc)^2}
\]
and
\[
ab + bc + ca = a(b + c) + bc = (b + c)^2 + bc = b^2 + c^2 + 3bc,
\]
we need to show that
\[
\frac{5(b^2 + c^2) + 8bc}{(2b^2 + 2c^2 + 5bc)^2} + \frac{1}{(b + c)^2} \leq \frac{85}{36(b^2 + c^2 + 3bc)}.
\]
For \(bc = 0\), the inequality is true. For \(bc \neq 0\), substituting
\[
t = \frac{b}{c} + \frac{c}{b}, \quad t \geq 2,
\]
the inequality becomes
\[
\frac{5t + 8}{(2t + 5)^2} + \frac{1}{t + 2} \leq \frac{85}{36(t + 3)},
\]
\[
\frac{5t + 8}{(2t + 5)^2} \leq \frac{49t + 62}{36(t + 2)(t + 3)}.
\]
It suffices to show that
\[
\frac{5t + 8}{(2t + 5)^2} \leq \frac{48t + 64}{36(t + 2)(t + 3)},
\]
which is equivalent to
\[
\frac{5t + 8}{(2t + 5)^2} \leq \frac{12t + 16}{9(t + 2)(t + 3)},
\]
\[
3t^3 + 7t^2 - 10t - 32 \geq 0,
\]
\[
(t - 2)(3t^2 + 13t + 16) \geq 0.
\]

Case 2: \(0 < x \leq y = z\). This involves \(b = c\). Since the original inequality is homogeneous, we may consider \(b = c = 1\) and \(0 \leq a \leq b + c = 2\). Thus, we only need to show that
\[
\frac{1}{4} + \frac{2}{(a + 1)^2} \leq \frac{85}{36(2a + 1)},
\]
which is equivalent to
\[
(a - 2)(9a^2 - 2a + 1) \leq 0.
\]
The equality holds for a degenerated isosceles triangle with \(a = b + c\), \(b = c\) (or any cyclic permutation).

\(\square\)

**P 5.37.** If \(a, b, c\) are the lengths of the side of a triangle so that \(a + b + c = 3\), then
\[
\frac{1}{(a + b)^2} + \frac{1}{(b + c)^2} + \frac{1}{(c + a)^2} \leq \frac{3(a^2 + b^2 + c^2)}{4(ab + bc + ca)}.
\]

*(Vasile C., 2007)*

**Solution.** Write the inequality in the homogeneous form
\[
\frac{1}{(a + b)^2} + \frac{1}{(b + c)^2} + \frac{1}{(c + a)^2} \leq \frac{27(a^2 + b^2 + c^2)}{4(a + b + c)^2(ab + bc + ca)}.
\]

As shown in the proof of the preceding P 5.36, it suffices to prove this inequality for \(a = b + c\) and for \(b = c = 1\).

Case 1: \(a = b + c\). Since
\[
\frac{1}{(a + b)^2} + \frac{1}{(c + a)^2} = \frac{5(b^2 + c^2) + 8bc}{(2b^2 + 2c^2 + 5bc)^2}
\]
and
\[
\frac{27(a^2 + b^2 + c^2)}{4(a + b + c)^2(ab + bc + ca)} = \frac{27(b^2 + c^2 + bc)}{8(b + c)^2(b^2 + c^2 + 3bc)},
\]
we need to show that
\[
\frac{5(b^2 + c^2) + 8bc}{(2b^2 + 2c^2 + 5bc)^2} + \frac{1}{(b + c)^2} \leq \frac{27(b^2 + c^2 + bc)}{8(b + c)^2(b^2 + c^2 + 3bc)}.
\]

For \(bc = 0\), the inequality is true. For \(bc \neq 0\), substituting
\[
t = \frac{b}{c} + \frac{c}{b}, \quad t \geq 2,
\]
the inequality becomes
\[
\frac{5t + 8}{(2t + 5)^2} + \frac{1}{t + 2} \leq \frac{27(t + 1)}{8(t + 2)(t + 3)},
\]
\[
\frac{9t^2 + 38t + 41}{(2t + 5)^2} \leq \frac{27(t + 1)}{8(t + 3)}.
\]

It suffices to show that
\[
\frac{9t^2 + 39t + 39}{(2t + 5)^2} \leq \frac{27(t + 1)}{8(t + 3)},
\]
which is equivalent to
\[
\frac{3t^2 + 13t + 13}{(2t + 5)^2} \leq \frac{9(t + 1)}{8(t + 3)},
\]
\[
12t^3 + 40^2 - 11t - 87 \geq 0.
\]

Indeed, we have
\[
12t^3 + 40^2 - 11t - 87 > 12t^3 + 40^2 - 20t - 96
\]
\[
= 12(t^3 - 8) + 20t(2t - 1) > 0.
\]

**Case 2:** \(b = c = 1\), \(a \leq b + c = 2\). The homogeneous inequality becomes
\[
\frac{2}{(a + 1)^2} + \frac{1}{4} \leq \frac{27(a^2 + 2)}{4(2a + 1)(a + 2)^2}.
\]

Since
\[
4(2a + 1)(a + 2) \leq 9(a + 1)^2,
\]
it suffices to show that
\[
\frac{2}{(a + 1)^2} + \frac{1}{4} \leq \frac{3(a^2 + 2)}{(a + 1)^2(a + 2)},
\]
which is equivalent to
\[
(a - 6)(a - 1)^2 \leq 0.
\]

The equality holds for an equilateral triangle.
P 5.38. Let \( a, b, c \) be nonnegative real numbers so that \( a + b + c = 3 \). If \( k \geq \frac{8}{5} \), then

\[
\frac{1}{k + a^2 + b^2} + \frac{1}{k + b^2 + c^2} + \frac{1}{k + c^2 + a^2} \leq \frac{3}{k + 2},
\]

(Vasile C., 2006)

**Solution.** Using the substitution

\[ m = k + a^2 + b^2 + c^2, \]

we have to show that

\[ f(a) + f(b) + f(c) \leq \frac{3}{k + 2} \]

for

\[ a + b + c = 3, \quad a^2 + b^2 + c^2 = m - k, \quad 0 \leq a \leq b \leq c, \]

\[ f(u) = \frac{1}{m - u^2}, \quad 0 \leq u \leq \sqrt{m - k}. \]

From

\[
g(x) = f'(x) = \frac{2x}{(m - x^2)^2}, \quad g''(x) = \frac{24x(m + x^2)}{(m - x^2)^4},
\]

it follows that \( g''(x) > 0 \) for \( 0 < x \leq \sqrt{m - k} \), hence \( g \) is strictly convex on \([0, \sqrt{m - k}]\). By Corollary 1 and Note 5/Note 2, if

\[ a + b + c = 3, \quad a^2 + b^2 + c^2 = \text{constant}, \quad 0 \leq a \leq b \leq c, \]

then the sum

\[ S_3 = f(a) + f(b) + f(c) \]

is maximum for \( 0 \leq a = b \leq c \). Therefore, we only need to show that

\[ \frac{1}{k + 2a^2} + \frac{2}{k + a^2 + c^2} \leq \frac{3}{k + 2} \]

for \( 2a + c = 3 \). Write the inequality as follows

\[
\frac{1}{k + 2a^2} + \frac{2}{k + 9 - 12a + 5a^2} \leq \frac{3}{k + 2},
\]

\[
5a^4 - 12a^3 + (2k + 6)a^2 - 4(k - 1)a + 2k - 3 \geq 0,
\]

\[
(a - 1)^2(5a^2 - 2a + 2k - 3) \geq 0.
\]

Since

\[
5a^2 - 2a + 2k - 3 = 5 \left(a - \frac{1}{5}\right)^2 + 2 \left(k - \frac{8}{5}\right) \geq 0,
\]

the conclusion follows.
The equality holds for \( a = b = c = 1 \). If \( k = 8/5 \), then the equality holds also for
\[
  a = b = \frac{1}{5}, \quad c = \frac{13}{5}
\]
(or any cyclic permutation).

**Remark.** In the same manner, we can prove the following generalization:

- Let \( a_1, a_2, \ldots, a_n \) be nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n = n \). If
\[
k \geq \frac{n^2 - 1}{n^2 - n - 1},
\]
then
\[
\sum \frac{1}{k + a_1^2 + \cdots + a_n^2} \leq \frac{n}{k + n - 1},
\]
with equality for \( a_1 = a_2 = \cdots = a_n = 1 \). If \( k = \frac{n^2 - 1}{n^2 - n - 1} \), then the equality holds also for
\[
a_1 = \cdots = a_{n-1} = \frac{1}{n^2 - n - 1}, \quad a_n = n - \frac{n - 1}{n^2 - n - 1}
\]
(or any cyclic permutation).

\[\Box\]

**P 5.39.** If \( a, b, c \) are nonnegative real numbers so that \( a + b + c = 3 \), then
\[
\frac{2}{2 + a^2 + b^2} + \frac{2}{2 + b^2 + c^2} + \frac{2}{2 + c^2 + a^2} \leq \frac{99}{63 + a^2 + b^2 + c^2}.
\]

*(Vasile C., 2009)*

**Solution.** The proof is similar to the one of P 5.38. Thus, we only need to prove the inequality for \( 0 \leq a = b \leq c \); that is, to show that \( 2a + c = 3 \) involves
\[
\frac{1}{1 + a^2} + \frac{4}{2 + a^2 + c^2} \leq \frac{99}{63 + 2a^2 + c^2}.
\]

Write this inequality as follows
\[
\frac{1}{a^2 + 1} + \frac{4}{5a^2 - 12a + 11} \leq \frac{33}{2(a^2 - 2a + 12)},
\]
\[
49a^4 - 112a^3 + 78a^2 - 16a + 1 \geq 0,
\]
\[
(a - 1)^2(7a - 1)^2 \geq 0.
\]

The equality holds for \( a = b = c = 1 \), and also for
\[
  a = b = \frac{1}{7}, \quad c = \frac{19}{7}
\]
(or any cyclic permutation).

\[\Box\]
**P 5.40.** If \(a, b, c\) are nonnegative real numbers so that \(a + b + c = 3\), then

\[
\frac{1}{5 + 2(a^2 + b^2)} + \frac{1}{5 + 2(b^2 + c^2)} + \frac{1}{5 + 2(c^2 + a^2)} \leq \frac{25}{69 + 2(a^2 + b^2 + c^2)}.
\]

*(Vasile C., 2009)*

**Solution.** The proof is similar to the one of P 5.38. Thus, we only need to prove the inequality for \(0 \leq a = b \leq c\); that is, to show that \(2a + c = 3\) involves

\[
\frac{1}{5 + 4a^2} + \frac{2}{5 + 2(a^2 + c^2)} \leq \frac{25}{69 + 4a^2 + 2c^2}.
\]

Write this inequality as follows

\[
\frac{1}{5 + 4a^2} + \frac{2}{10a^2 - 24a + 23} \leq \frac{25}{12a^2 - 24a + 87},
\]

\[
4(196a^4 - 420a^3 + 253a^2 - 30a + 1) \geq 0,
\]

\[
4(a - 1)^2(14a - 1)^2 \geq 0.
\]

The equality holds for \(a = b = c = 1\), and also for

\[a = b = \frac{1}{14}, \quad c = \frac{20}{7}\]

(or any cyclic permutation).

\[\blacksquare\]

**P 5.41.** If \(a, b, c\) are nonnegative real numbers so that \(a + b + c = 3\), then

\[
\frac{1}{3 + a^2 + b^2} + \frac{1}{3 + b^2 + c^2} + \frac{1}{3 + c^2 + a^2} \leq \frac{18}{27 + a^2 + b^2 + c^2}.
\]

*(Vasile C., 2009)*

**Solution.** The proof is similar to the one of P 5.38. Thus, we only need to prove the inequality for \(0 \leq a = b \leq c\). Therefore, we only need to show that \(2a + c = 3\) involves

\[
\frac{1}{3 + 2a^2} + \frac{2}{3 + a^2 + c^2} \leq \frac{18}{27 + 2a^2 + c^2}.
\]

Write this inequality as follows

\[
\frac{1}{2a^2 + 3} + \frac{2}{5a^2 - 12a + 12} \leq \frac{3}{a^2 - 2a + 6},
\]

\[a^2(a - 1)^2 \geq 0.
\]
The equality holds for \( a = b = c = 1 \), and also for
\[
a = b = 0, \quad c = 3
\]
(or any cyclic permutation).

**Remark.** In the same manner, we can prove the following generalization:

- Let \( a_1, a_2, \ldots, a_n \) be nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n = n \). If \( k \geq \frac{n}{n-2} \), then
\[
\sum \frac{1}{k + a_1^2 + \cdots + a_n^2} \leq \frac{n^2(n + k)}{n(n^2 + kn + k^2) + (kn - n - k)(a_1^2 + a_2^2 + \cdots + a_n^2)},
\]
with equality for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for
\[
a_1 = \cdots = a_{n-1} = 0, \quad a_n = n
\]
(or any cyclic permutation).

\( \square \)

**P 5.42.** If \( a, b, c, d \) are nonnegative real numbers so that \( a + b + c + d = 4 \), then
\[
\sum \frac{3}{3 + 2(a^2 + b^2 + c^2)} \leq \frac{296}{218 + a^2 + b^2 + c^2 + d^2}.
\]

*(Vasile C., 2009)*

**Solution.** The proof is similar to the one of P 5.38. Thus, we only need to prove the inequality for \( 0 \leq a = b = c \leq d \). Therefore, we only need to show that \( 3a + d = 4 \) involves
\[
\frac{1}{1 + 2a^2} + \frac{9}{3 + 4a^2 + 2d^2} \leq \frac{296}{218 + 3a^2 + d^2}.
\]

Write this inequality as follows
\[
\frac{1}{1 + 2a^2} + \frac{9}{35 - 48a + 22a^2} \leq \frac{148}{3(39 - 4a + 2a^2)},
\]
\((a - 1)^2(14a - 1)^2 \geq 0.\)

The equality holds for \( a = b = c = d = 1 \), and also for
\[
a = b = c = \frac{1}{14}, \quad d = \frac{53}{14}
\]
(or any cyclic permutation).

\( \square \)
P 5.43. If \( a, b, c \) are nonnegative real numbers so that \( a + b + c = 3 \), then

\[
\frac{5}{3 + a^2 + b^2} + \frac{5}{3 + b^2 + c^2} + \frac{5}{3 + c^2 + a^2} \geq \frac{27}{6 + a^2 + b^2 + c^2}.
\]

(Vasile C., 2014)

Solution. Using the substitution

\[ m = 3 + a^2 + b^2 + c^2, \]

we have to show that

\[ f(a) + f(b) + f(c) \geq \frac{27}{24 + m} \]

for

\[ a + b + c = 3, \quad a^2 + b^2 + c^2 = m - 3, \quad 0 \leq a \leq b \leq c, \quad f(u) = \frac{5}{m - u^2}, \quad 0 \leq u \leq \sqrt{m - 3}. \]

From

\[ g(x) = f'(x) = \frac{10x}{(m - x^2)^2}, \quad g''(x) = \frac{120x(m + x^2)}{(m - x^2)^4}, \]

it follows that \( g''(x) > 0 \) for \( 0 < x \leq \sqrt{m - 3} \), hence \( g \) is strictly convex on \([0, \sqrt{m - 3}]\). By Corollary 1 and Note 5/Note 2, if

\[ a + b + c = 3, \quad a^2 + b^2 + c^2 = \text{constant}, \quad 0 \leq a \leq b \leq c, \]

then the sum

\[ S_3 = f(a) + f(b) + f(c) \]

is minimum for either \( a = 0 \) or \( 0 < a \leq b = c \). Write the inequality in the homogeneous form

\[ \sum \frac{5}{(a + b + c)^2 + 3(a^2 + b^2)} \geq \frac{27}{2(a + b + c)^2 + 3(a^2 + b^2 + c^2)}. \]

Case 1: \( a = 0 \). The homogeneous inequality becomes

\[
\frac{5}{(b + c)^2 + 3b^2} + \frac{5}{(b + c)^2 + 3c^2} + \frac{5}{(b + c)^2 + 3(b^2 + c^2)} \geq \frac{27}{2(b + c)^2 + 3(b^2 + c^2)},
\]

\[
\frac{5[5(b^2 + c^2) + 4bc]}{4(b^2 + c^2)^2 + 10bc(b^2 + c^2) + 13b^2c^2} + \frac{5}{4(b^2 + c^2) + 2bc} \geq \frac{27}{5(b^2 + c^2) + 4bc}.
\]

For the nontrivial case \( bc \neq 0 \), substituting

\[ \frac{b}{c} + \frac{c}{b} = t, \quad t \geq 2, \]

\[ \frac{5}{(b + c)^2 + 3b^2} + \frac{5}{(b + c)^2 + 3c^2} + \frac{5}{(b + c)^2 + 3(b^2 + c^2)} \geq \frac{27}{2(b + c)^2 + 3(b^2 + c^2)}, \]

\[ \frac{5}{4(b^2 + c^2)^2 + 10bc(b^2 + c^2) + 13b^2c^2} + \frac{5}{4(b^2 + c^2) + 2bc} \geq \frac{27}{5(b^2 + c^2) + 4bc}. \]
we may write the inequality as
\[
\frac{5(5t + 4)}{4t^2 + 10t + 13} + \frac{5}{4t + 2} \geq \frac{27}{5t + 4}.
\]
\[
\frac{5(5t + 4)}{4t^2 + 10t + 13} \geq \frac{83t + 34}{2(2t + 1)(5t + 4)}.
\]
Since
\[
83t + 34 \leq 90t + 20,
\]
it suffices to show that
\[
\frac{5 + 4}{4t^2 + 10t + 13} \geq \frac{9t + 2}{(2t + 1)(5t + 4)},
\]
which is equivalent to
\[
14t^2 + 7t^2 - 65t - 10 \geq 0,
\]
\[
(t - 2)(14t^2 + 35t + 5) \geq 0.
\]

**Case 2:** \( 0 < a \leq b = c \). We only need to prove the homogeneous inequality for \( b = c = 1 \); that is,
\[
\frac{10}{(a + 2)^2 + 3(a^2 + 1)} + \frac{5}{(a + 2)^2 + 6} \geq \frac{27}{2(a + 2)^2 + 3(a^2 + 2)},
\]
\[
\frac{10}{4a^2 + 4a + 7} + \frac{5}{a^2 + 4a + 10} \geq \frac{27}{5a^2 + 8a + 14},
\]
a\(a^3 - 3a + 2) \geq 0,
a\(a - 1)^2(a + 2) \geq 0.
\]
The equality holds for \( a = b = c = 1 \), and also for
\[
a = 0, \quad b = c = \frac{3}{2}
\]
(or any cyclic permutation).

**Remark 1.** Similarly, we can prove the following generalization:

- Let \( a, b, c \) be nonnegative real numbers so that \( a + b + c = 3 \). If \( k \geq 0 \), then
\[
\frac{1}{k + a^2 + b^2} + \frac{1}{k + b^2 + c^2} + \frac{1}{k + c^2 + a^2} \geq \frac{9(4k + 15)}{3(4k^2 + 15k + 9) + (8k + 21)(a^2 + b^2 + c^2)},
\]
with equality for \( a = b = c = 1 \), and also for
\[
a = 0, \quad b = c = \frac{3}{2}
\]
(or any cyclic permutation).
For \( k = 0 \), we get the inequality in P 1.171 from Volume 2:

\[
\frac{1}{a^2 + b^2} + \frac{1}{b^2 + c^2} + \frac{1}{c^2 + a^2} \geq \frac{45}{(a + b + c)^2 + 7(a^2 + b^2 + c^2)}.
\]

**Remark 2.** More general, the following statement holds:

- Let \( a_1, a_2, \ldots, a_n \) be nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n = n \). If \( k \geq 0 \), then

\[
\sum \frac{1}{k + a_1^2 + \cdots + a_n^2} \geq \frac{p}{q + a_1^2 + a_2^2 + \cdots + a_n^2},
\]

where

\[
p = \frac{n^2(n-1)^2k + n^3(n^2-n-1)}{(n-1)^3k + n(n^3-2n^2-n+1)}, \quad q = \frac{n(n-1)^2k^2 + n^2(n^2-n-1)k + n^3}{(n-1)^3k + n(n^3-2n^2-n+1)},
\]

with equality for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for

\( a_1 = 0, \quad a_2 = \cdots = a_n = \frac{n}{n-1} \)

(or any cyclic permutation).

For \( k = 0 \) and \( k = n \), we get the inequalities

\[
\sum \frac{1}{a_2^2 + \cdots + a_n^2} \geq \frac{n^2(n^2-n-1)}{n^2 + (n^3-2n^2-n+1)(a_1^2 + a_2^2 + \cdots + a_n^2)},
\]

\[
\sum \frac{2n-1}{n + a_2^2 + \cdots + a_n^2} \geq \frac{n^2(2n-3)}{n(n-1) + (n-2)(a_1^2 + a_2^2 + \cdots + a_n^2)}.
\]

\( \blacksquare \)

**P 5.44.** If \( a, b, c \) are nonnegative real numbers so that \( ab + bc + ca = 3 \), then

\[
\frac{4}{2 + a^2 + b^2} + \frac{4}{2 + b^2 + c^2} + \frac{4}{2 + c^2 + a^2} \geq \frac{21}{4 + a^2 + b^2 + c^2}.
\]

(\( \text{Vasile C., 2014} \))

**Solution.** The proof is similar to the one of P 5.43. Thus, we only need to prove the inequality for \( a = 0 \) and for \( 0 < a \leq b = c \).

**Case 1: \( a = 0 \).** We need to show that \( bc = 3 \) involves

\[
\frac{1}{2 + b^2} + \frac{1}{2 + c^2} + \frac{1}{2 + b^2 + c^2} \geq \frac{21}{4(4 + b^2 + c^2)}.
\]
Denote \( x = b^2 + c^2, \quad x \geq 2bc = 6. \)

Since
\[
\frac{1}{2 + b^2} + \frac{1}{2 + c^2} = \frac{4 + b^2 + c^2}{b^2c^2 + 2(b^2 + c^2) + 4} = \frac{x + 4}{2x + 13},
\]
we only need to show that
\[
\frac{x + 4}{2x + 13} + \frac{1}{x + 2} \geq \frac{21}{4(x + 4)}.
\]

Since
\[
\frac{x + 4}{2x + 13} + \frac{1}{x + 2} = \frac{x^2 + 8x + 21}{(2x + 13)(x + 2)} \geq \frac{7(2x + 3)}{(2x + 13)(x + 2)},
\]
it suffices to show that
\[
\frac{2x + 3}{(2x + 13)(x + 2)} \geq \frac{3}{4(x + 4)}.
\]

This inequality reduces to
\[(x - 6)(2x + 5) \geq 0.\]

**Case 2:** \(0 < a \leq b = c.\) Let
\[q = ab + bc + ca.\]

We only need to prove the homogeneous inequality
\[
\frac{4}{2q + 3(a^2 + b^2)} + \frac{4}{2q + 3(b^2 + c^2)} + \frac{4}{2q + 3(c^2 + a^2)} \geq \frac{21}{4q + 3(a^2 + b^2 + c^2)}
\]
for \(b = c = 1.\) Thus, we need to show that
\[
\frac{8}{2(2a + 1) + 3(a^2 + 1)} + \frac{4}{2(2a + 1) + 6} \geq \frac{21}{4(2a + 1) + 3(a^2 + 2)},
\]
which is equivalent to
\[
\frac{8}{3a^2 + 4a + 5} + \frac{1}{a + 2} \geq \frac{21}{3a^2 + 8a + 10},
\]
\[
\frac{a^2 + 4a + 7}{(3a^2 + 4a + 5)(a + 2)} \geq \frac{7}{3a^2 + 8a + 10},
\]
\[a(3a^3 - a^2 - 7a + 5) \geq 0,
\]
\[a(a - 1)^2(3a + 5) \geq 0.
\]

The equality holds for \(a = b = c = 1,\) and also for
\[a = 0, \quad b = c = \sqrt{3} \]
(or any cyclic permutation).

**Remark.** In the same manner, we can prove the following generalization:

- Let \(a, b, c\) be nonnegative real numbers so that \(ab + bc + ca = 3\). If \(k \geq 0\), then

\[
\frac{1}{k + a^2 + b^2} + \frac{1}{k + b^2 + c^2} + \frac{1}{k + c^2 + a^2} \geq \frac{9(k + 5)}{3(k^2 + 5k + 2) + 2(k + 4)(a^2 + b^2 + c^2)}.
\]

with equality for \(a = b = c = 1\), and also for

\[
a = 0, \quad b = c = \sqrt{3}
\]

(or any cyclic permutation).

For \(k = 0\), we get the inequality in P 1.171 from Volume 2:

\[
\frac{1}{a^2 + b^2} + \frac{1}{b^2 + c^2} + \frac{1}{c^2 + a^2} \geq \frac{45}{2(ab + bc + ca) + 8(a^2 + b^2 + c^2)}.
\]

\(\square\)

**P 5.45.** If \(a, b, c\) are nonnegative real numbers so that \(a^2 + b^2 + c^2 = 3\), then

\[
\frac{1}{10 - (a + b)^2} + \frac{1}{10 - (b + c)^2} + \frac{1}{10 - (c + a)^2} \leq \frac{1}{2}.
\]

(Vasile C., 2006)

**Solution.** Let

\[s = a + b + c, \quad s \leq 3.\]

We need to show that

\[
\frac{1}{10 - (s - a)^2} + \frac{1}{10 - (s - b)^2} + \frac{1}{10 - (s - c)^2} \leq \frac{1}{2}
\]

for \(a + b + c = s\) and \(a^2 + b^2 + c^2 = 3\). Apply Corollary 1 to the function

\[f(u) = \frac{-1}{10 - (s - u)^2}, \quad 0 \leq u \leq s.\]

We have

\[g(x) = f'(x) = \frac{2(s - x)}{[10 - (s - x)^2]^2},\]

\[g''(x) = \frac{24(s - x)[10 + (s - x)^2]}{[10 - (s - x)^2]^4}.\]
Since \( g''(x) > 0 \) for \( x \in [0,s) \), \( g \) is strictly convex on \( [0,s) \). According to the Corollary 1 and Note 5/Note 2, if
\[
a + b + c = s, \quad a^2 + b^2 + c^2 = 3, \quad 0 \leq a \leq b \leq c,
\]
then
\[
S_3 = f(a) + f(b) + f(c)
\]
is minimum for either \( a = 0 \) or \( 0 < a \leq b = c \). Therefore, we only need to prove the homogeneous inequality
\[
\sum \frac{1}{10(a^2 + b^2 + c^2) - 3(b + c)^2} \leq \frac{1}{2(a^2 + b^2 + c^2)}
\]
for \( a = 0 \) and for \( b = c = 1 \).

**Case 1:** \( a = 0 \). The homogeneous inequality becomes
\[
\frac{1}{7(b^2 + c^2) - 6bc} + \frac{1}{10b^2 + 7c^2} + \frac{1}{7b^2 + 10c^2} \leq \frac{1}{2(b^2 + c^2)}.
\]
This is true since
\[
\frac{1}{7(b^2 + c^2) - 6bc} \leq \frac{1}{4(b^2 + c^2)}
\]
and
\[
\frac{1}{10b^2 + 7c^2} + \frac{1}{7b^2 + 10c^2} = \frac{17(b^2 + c^2)}{70(b^2 + c^2) + 149b^2c^2} \leq \frac{17(b^2 + c^2)}{70(b^2 + c^2) + 140b^2c^2} = \frac{17}{70(b^2 + c^2)} < \frac{1}{4(b^2 + c^2)}.
\]

**Case 2:** \( b = c = 1 \). The homogeneous inequality turns into
\[
\frac{1}{2(5a^2 + 4)} + \frac{2}{7a^2 - 6a + 17} \leq \frac{1}{2(a^2 + 2)},
\]
\[
\frac{2}{7a^2 - 6a + 17} \leq \frac{2a^2 + 1}{(5a^2 + 4)(a^2 + 2)},
\]
\[
4a^4 - 12a^3 + 13a^2 - 6a + 1 \geq 0,
\]
\[
(a - 1)^2(2a - 1)^2 \geq 0.
\]
The equality holds for \( a = b = c = 1 \), and also for
\[
2a = b = c = \frac{2}{\sqrt{3}}
\]
(or any cyclic permutation). 
\[\square\]
P 5.46. If $a, b, c$ are nonnegative real numbers, no two of which are zero, so that $a^4 + b^4 + c^4 = 3$, then

$$\frac{1}{a^5 + b^5} + \frac{1}{b^5 + c^5} + \frac{1}{c^5 + a^5} \geq \frac{3}{2}. \quad (Vasile C., 2010)$$

**Solution.** Using the substitution

$$x = a^4, \quad y = b^4, \quad z = c^4, \quad p = x^{5/4} + y^{5/4} + z^{5/4},$$

we need to show that $x + y + z = 3$ and $x^{5/4} + y^{5/4} + z^{5/4} = p$ involve

$$f(x) + f(y) + f(z) \geq \frac{3}{2},$$

where

$$f(u) = \frac{1}{p - u^{5/4}}, \quad 0 \leq u < p^{4/5}.$$ 

We will apply the EV-Theorem for $k = 5/4$. We have

$$f''(u) = \frac{5u^{1/4}}{4(p - u^{5/4})^2},$$

$$g(x) = f'\left(x^{1/4}\right) = f'(x^4) = \frac{5x}{4(p - x^5)^2},$$

$$g''(x) = \frac{75x^4(2p + 3x^5)}{2(p - x^5)^4}.$$ 

Since $g''(x) > 0$ for $0 < x^4 < p^{4/5}$, $g$ is strictly convex on $[0, p^{1/5})$. According to the EV-Theorem and Note 3, if

$$x + y + z = 3, \quad x^{5/4} + y^{5/4} + z^{5/4} = p = \text{constant}, \quad 0 \leq x \leq y \leq z,$$

then

$$S_3 = f(x) + f(y) + f(z)$$

is minimum for either $x = 0$ or $0 < x \leq y = z$. Thus, we only need to prove the homogeneous inequality

$$\frac{1}{a^5 + b^5} + \frac{1}{b^5 + c^5} + \frac{1}{c^5 + a^5} \geq \frac{3}{2} \left(\frac{3}{a^4 + b^4 + c^4}\right)^{5/4}$$

for $a = 0$ and $0 < a \leq b = c = 1$.

**Case 1:** $a = 0$. The homogeneous inequality becomes

$$\frac{1}{b^5} + \frac{1}{c^5} + \frac{1}{b^5 + c^5} \geq \frac{3}{2} \left(\frac{3}{b^4 + c^4}\right)^{5/4}.$$
\[
\left( \frac{b}{c} \right)^{5/2} + \left( \frac{c}{b} \right)^{5/2} + \frac{1}{\left( \frac{b}{c} \right)^{5/2} + \left( \frac{c}{b} \right)^{5/2}} \geq \left( \frac{3}{2} \right)^{9/4} \left[ \frac{2}{\left( \frac{b}{c} \right)^2 + \left( \frac{c}{b} \right)^2} \right]^{5/4},
\]

\[
t^{5/2} + t^{-5/2} + \frac{1}{t^{5/2} + t^{-5/2}} \geq \left( \frac{3}{2} \right)^{9/4} \left( \frac{2}{t^2 + t^{-2}} \right)^{5/4},
\]

\[
2A^{5/2} + \frac{1}{2A^{5/2}} \geq \left( \frac{3}{2} \right)^{9/4} \frac{1}{B^{5/2}},
\]

where

\[
A = \left( \frac{t^{5/2} + t^{-5/2}}{2} \right)^{2/5}, \quad B = \left( \frac{t^2 + t^{-2}}{2} \right)^{1/2}, \quad t = \frac{b}{c}.
\]

By power mean inequality, we have \( A \geq B \geq 1 \). Since

\[
2A^{5/2} + \frac{1}{2A^{5/2}} - \left( 2B^{5/2} + \frac{1}{2B^{5/2}} \right) = \left( A^{5/2} - B^{5/2} \right) \left( 2 - \frac{1}{2A^{5/2}B^{5/2}} \right) \geq 0,
\]

it suffices to show that

\[
2B^{5/2} + \frac{1}{2B^{5/2}} \geq \left( \frac{3}{2} \right)^{9/4} \frac{1}{B^{5/2}},
\]

\[
4B^5 + 1 \geq \left( \frac{3^9}{2^5} \right)^{1/4},
\]

which is true if

\[
5 \geq \left( \frac{3^9}{2^5} \right)^{1/4},
\]

\[
32 \cdot 5^4 \geq 3^9.
\]

This inequality follows by multiplying the inequalities

\[
5^4 > 23 \cdot 3^3
\]

and

\[
32 \cdot 23 > 3^6.
\]

**Case 2:** \( 0 < a \leq 1 = b = c \). The homogeneous inequality becomes

\[
\frac{a^5 + 5}{a^5 + 1} \geq 3 \left( \frac{3}{a^4 + 2} \right)^{5/4},
\]

which is true if \( g(a) \geq 0 \), where

\[
g(a) = \ln(a^5 + 5) - \ln(a^5 + 1) + \frac{5}{4} \ln(a^4 + 2) - \frac{9 \ln 3}{4},
\]
with
\[
g'(a) = \frac{a}{a^5 + 5} - \frac{a}{a^5 + 1} + \frac{1}{a^4 + 2} = \frac{a^{10} + 2a^5 - 8a + 5}{(a^5 + 5)(a^5 + 1)(a^4 + 2)}
\]
\[
= \frac{(a - 1)(a^9 + a^8 + a^7 + a^6 + a^5 + 3a^4 + 3a^3 + 3a^2 + 3a - 5)}{(a^4 + 5)(a^5 + 1)(a^4 + 2)}.
\]

There exists \( d \in (0, 1) \) so that \( g'(d) = 0 \), \( f'(a) > 0 \) for \( a \in [0, d) \) and \( f'(a) < 0 \) for \( a \in (d, 1) \). Therefore, \( g \) is increasing on \([0, d] \) and is decreasing on \([d, 1] \). Since \( g(1) = 0 \), we only need to show that \( g(0) \geq 0 \). Indeed,
\[
g(0) = \frac{1}{4} \ln \frac{5^4 \cdot 2^5}{3^9} > 0.
\]

The equality holds for \( a = b = c = 1 \).

\[\square\]

**P 5.47.** If \( a_1, a_2, \ldots, a_n \) are nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n = n \), then
\[
\sqrt{a_1^2 + 1} + \sqrt{a_2^2 + 1} + \cdots + \sqrt{a_n^2 + 1} \geq \sqrt{2 \left(1 - \frac{1}{n}\right)(a_1^2 + a_2^2 + \cdots + a_n^2) + 2(n^2 - n + 1)}.
\]

*(Vasile C., 2014)*

**Solution.** For \( n = 2 \), we need to show that \( a_1 + a_2 = 2 \) involves
\[
\sqrt{a_1^2 + 1} + \sqrt{a_2^2 + 1} \geq \sqrt{a_1^2 + a_2^2 + 6}.
\]

By squaring, the inequality becomes
\[
\sqrt{(a_1^2 + 1)(a_2^2 + 1)} \geq 2,
\]
which follows immediately from the Cauchy-Schwarz inequality:
\[
(a_1^2 + 1)(a_2^2 + 1) = (a_1^2 + 1)(1 + a_2^2) \geq (a_1 + a_2)^2 = 4.
\]
Assume further that \( n \geq 3 \) and \( a_1 \leq a_2 \leq \cdots \leq a_n \). We will apply Corollary 1 to the function
\[
f(u) = -\sqrt{u^2 + 4}, \quad u \geq 0.
\]
We have
\[
g(x) = f'(x) = \frac{-x}{\sqrt{x^2 + 4}},
\]
\[
g''(x) = \frac{12x}{(x^2 + 4)^{5/2}}.
\]
Since \( g''(x) > 0 \) for \( x > 0 \), \( g(x) \) is strictly convex for \( x \geq 0 \). By Corollary 1, if \( a_1 \leq a_2 \leq \cdots \leq a_n \) and

\[
a_1 + a_2 + \cdots + a_n = n, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = \text{constant},
\]

then the sum

\[
S_n = f(a_1) + f(a_2) + \cdots + f(a_n)
\]

is maximum for \( a_1 = a_2 = \cdots = a_{n-1} \). Thus, we only need to show that

\[
\sqrt{a^2 + 1} + (n-1)\sqrt{b^2 + 1} \geq \sqrt{2 \left( 1 - \frac{1}{n} \right) [a^2 + (n-1)b^2] + 2(n^2 - n + 1)}.
\]

for

\[
a + (n - 1)b = n.
\]

By squaring, the inequality becomes

\[
2n(n-1)\sqrt{(a^2 + 1)(b^2 + 1)} \geq (n-2)a^2 - (n-2)(n-1)^2 b^2 + n^3,
\]

which is equivalent to

\[
\sqrt{(b^2 + 1)((n-1)^2 b^2 - 2n(n-1)b + n^2 + 1)} \geq n - (n-2)b.
\]

This is true if

\[
(b^2 + 1)((n-1)^2 b^2 - 2n(n-1)b + n^2 + 1) \geq [n - (n-2)b]^2,
\]

which is equivalent to

\[
(n-1)^2 b^4 - 2n(n-1)b^3 + (n^2 + 2n - 2)b^2 - 2nb + 1 \geq 0,
\]

\[
(b - 1)^2 [(n-1)b - 1]^2 \geq 0.
\]

The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for

\[
a_1 = a_2 = \cdots = a_{n-1} = \frac{1}{n-1}, \quad a_n = n - 1
\]

(or any cyclic permutation). \( \square \)

**P 5.48.** If \( a_1, a_2, \ldots, a_n \) are nonnegative real numbers so that \( a_1 + a_2 + \cdots + a_n = n \), then

\[
\sum \sqrt{(3n-4)a_i^2 + n} \geq \sqrt{(3n-4)(a_1^2 + a_2^2 + \cdots + a_n^2) + n(4n^2 - 7n + 4)}.
\]

(Vasile C., 2009)
Solution. The proof is similar to the one of the preceding P 5.47. Thus, it suffices to prove the inequality for \( a_1 = a_2 = \cdots = a_{n-1} \). Write the inequality in the homogeneous form

\[
\sum \sqrt{n(3n-4)a_1^2 + S^2} \geq \sqrt{n(3n-4)(a_1^2 + a_2^2 + \cdots + a_n^2) + (4n^2 - 7n + 4)S^2},
\]

where \( S = a_1 + a_2 + \cdots + a_n \). We only need to prove this inequality for \( a_1 = a_2 = \cdots = a_{n-1} = 1 \); that is,

\[
(n-1)\sqrt{n(3n-4) + (n-1 + a_n)^2} + \sqrt{n(3n-4)a_n^2 + (n-1 + a_n)^2} \geq \sqrt{(n-1)(n-1 + a_n)^2 + (4n^2 - 7n + 4)(n-1 + a_n)^2},
\]

which is equivalent to

\[
\sqrt{(n-1)[a_n^2 + 2(n-1)a_n + 4n^2 - 6n + 1] + (3n-1)a_n^2 + 2a_n + n - 1} \geq \sqrt{(7n-4)a_n^2 + 2(4n^2 - 7n + 4)a_n + 4n^3 - 8n^2 + 7n - 4}.
\]

By squaring, the inequality turns into

\[
2\sqrt{(n-1)[(3n-1)a_n^2 + 2a_n + n - 1][a_n^2 + 2(n-1)a_n + 4n^2 - 6n + 1]} \geq (3n-2)a_n^2 + 2(n-1)(3n-2)a_n + 2n^2 - n - 2.
\]

Squaring again, we get

\[
(a_n - 1)^2(a_n - 2n + 3)^2 \geq 0.
\]

The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for

\[
a_1 = a_2 = \cdots = a_{n-1} = \frac{a_n}{2n-3} = \frac{n}{3n-4}
\]

(or any cyclic permutation).

Remark. For \( n = 3 \), we get the inequality

\[
\sqrt{5a^2 + 3} + \sqrt{5b^2 + 3} + \sqrt{5c^2 + 3} \geq \sqrt{5(a^2 + b^2 + c^2) + 57},
\]

where \( a, b, c \) are nonnegative real numbers so that \( a + b + c = 3 \). By squaring, the inequality turns into

\[
\sqrt{(5a^2 + 3)(5b^2 + 3)} + \sqrt{(5b^2 + 3)(5c^2 + 3)} + \sqrt{(5c^2 + 3)(5a^2 + 3)} \geq 24,
\]

with equality for \( a = b = c = 1 \), and also for

\[
a = b = \frac{c}{3} = \frac{3}{5}
\]

(or any cyclic permutation).
\begin{proof}
\textbf{5.49.} If \(a, b, c\) are nonnegative real numbers so that \(a + b + c = 3\), then

\[
\sqrt{a^2 + 4} + \sqrt{b^2 + 4} + \sqrt{c^2 + 4} \leq \sqrt{\frac{8}{3}(a^2 + b^2 + c^2) + 37}.
\]

(Vasile C., 2009)

\textit{Solution.} Assume that \(a \leq b \leq c\), and apply Corollary 1 to the function \(a f(u) = -\sqrt{u^2 + 4}, u \geq 0\).

We have

\[
g(x) = f'(x) = \frac{-x}{\sqrt{x^2 + 4}},
\]

\[
g''(x) = \frac{12x}{(x^2 + 4)^{5/2}}.
\]

Since \(g''(x) > 0\) for \(x > 0\), \(g(x)\) is strictly convex for \(x \geq 0\). By Corollary 1, if

\[
a + b + c = 3, \quad a^2 + b^2 + c^2 = \text{constant}, \quad a \leq b \leq c,
\]

then the sum

\[
S_3 = f(a) + f(b) + f(c)
\]

is minimum for either \(a = 0\) or \(0 < a \leq b = c\). Thus, we only need to prove the desired inequality for these cases.

Case 1: \(a = 0\). We need to prove that \(b + c = 3\) involves

\[
\sqrt{b^2 + 4} + \sqrt{c^2 + 4} \leq \sqrt{\frac{8}{3}(b^2 + c^2) + 37} - 2.
\]

Substituting

\[
b = \frac{3x}{2}, \quad c = \frac{3y}{2},
\]

we need to prove that \(x + y = 2\) involves

\[
\sqrt{9x^2 + 16} + \sqrt{9y^2 + 16} \leq 2\sqrt{6(x^2 + y^2) + 37} - 4.
\]

By squaring, the inequality becomes

\[
2\sqrt{(9x^2 + 16)(9y^2 + 16)} \leq 15(x^2 + y^2) + 132 - 16\sqrt{6(x^2 + y^2) + 37}.
\]

Denoting \(p = xy, \quad 0 \leq p \leq 1\), we have

\[
x^2 + y^2 = 4 - 2p, \quad (9x^2 + 16)(9y^2 + 16) = 81p^2 - 288p + 832,
\]
and the inequality becomes

$$\sqrt{81p^2 - 288p + 832} \leq -15p + 96 - 8\sqrt{61 - 12p},$$

$$\sqrt{\frac{81}{4}p^2 - 72p + 208} \leq -\frac{15}{2}p + (48 - 4\sqrt{61 - 12p}).$$

By squaring again (the right hand side is positive), the inequality becomes

$$\frac{81}{4}p^2 - 72p + 208 \leq \frac{225}{4}p^2 - 15p(48 - 4\sqrt{61 - 12p}) + (48 - 4\sqrt{61 - 12p})^2,$$

$$3p^2 - 70p + 256 \geq (32 - 5p)\sqrt{61 - 12p}.$$

Since

$$61 - 12p \leq 64 - 15p,$$

it suffices to show that

$$3p^2 - 70p + 256 \geq (32 - 5p)\sqrt{64 - 15p}.$$

Substituting

$$64 - 15p = 64t^2, \quad \frac{7}{8} \leq t \leq 1,$$

hence

$$p = \frac{64}{15}(1 - t^2),$$

the inequality becomes

$$\frac{32}{75}(1 - t^2)^2 - \frac{7}{3}(1 - t^2) + 2 \geq \frac{2}{3}t(2t^2 + 1),$$

$$32t^4 - 100t^3 + 111t^2 - 50t + 7 \geq 0,$$

$$(t - 1)^2(8t - 7)(4t - 1) \geq 0.$$

Case 2: $b = c$. We need to prove that

$$a + 2b = 3$$

implies

$$\sqrt{a^2 + 4} + 2\sqrt{b^2 + 4} \leq \sqrt{\frac{8}{3}(a^2 + 2b^2) + 37}.$$

By squaring, the inequality becomes

$$12\sqrt{(a^2 + 4)(b^2 + 4)} \leq 5a^2 + 4b^2 + 51,$$

which is equivalent to

$$\sqrt{(4b^2 - 12b + 13)(b^2 + 4)} \leq 2b^2 - 5b + 8.$$
By squaring again, the inequality becomes

\[ 2b^3 - 7b^2 + 8b - 3 \leq 0, \]
\[ (b - 1)^2(2b - 3) \leq 0, \]
\[ (b - 1)^2a \geq 0. \]

The equality holds for \( a = b = c = 1 \), and also for \( a = 0, b = c = \frac{3}{2} \)
(or any cyclic permutation).

\[ \Box \]

**P 5.50.** If \( a, b, c \) are nonnegative real numbers so that \( a + b + c = 3 \), then

\[ \sqrt{32a^2 + 3} + \sqrt{32b^2 + 3} + \sqrt{32c^2 + 3} \leq \sqrt{32(a^2 + b^2 + c^2)} + 219. \]

*(Vasile C., 2009)*

**Solution.** The proof is similar to the one of P 5.49. Thus, it suffices to prove the homogeneous inequality

\[ \sum \sqrt{96a^2 + (a + b + c)^2} \leq \sqrt{96(a^2 + b^2 + c^2)} + 73(a + b + c)^2 \]

for \( a = 0 \) and for \( b = c = 1 \).

**Case 1:** \( a = 0 \). We have to show that

\[ b + c + \sqrt{97b^2 + 2bc + c^2} + \sqrt{b^2 + 2bc + 97c^2} \leq \sqrt{169(b^2 + c^2)} + 146bc. \]

Since \( 2bc \leq b^2 + c^2 \), it suffices to prove that

\[ b + c + \sqrt{98b^2 + 2c^2} + \sqrt{2b^2 + 98c^2} \leq \sqrt{169(b^2 + c^2)} + 146bc. \]

By squaring, we get

\[ (b + c)\left( \sqrt{98b^2 + 2c^2} + \sqrt{2b^2 + 98c^2} \right) + 2\sqrt{(49b^2 + c^2)(b^2 + 49c^2)} \leq 34(b^2 + c^2) + 72bc. \]

Since

\[ \sqrt{98b^2 + 2c^2} + \sqrt{2b^2 + 98c^2} \leq \sqrt{2(98b^2 + 2c^2 + 2b^2 + 98c^2)} = 10\sqrt{2(b^2 + c^2)} \]

and

\[ 10(b + c)\sqrt{2(b^2 + c^2)} \leq 20(b + c)^2, \]
it suffices to show that
\[ \sqrt{(49b^2 + c^2)(b^2 + 49c^2)} \leq 7(b^2 + c^2) + 36bc. \]

Squaring again, the inequality becomes
\[ bc(b - c)^2 \geq 0. \]

Case 2: \( b = c = 1 \). The homogeneous inequality turns into
\[ \sqrt{97a^2 + 4a + 4 + 2\sqrt{a^2 + 4a + 100}} \leq \sqrt{169a^2 + 292a + 484}. \]

By squaring, we get
\[ \sqrt{(97a^2 + 4a + 4)(a^2 + 4a + 100)} \leq 17a^2 + 68a + 20. \]

Squaring again, the inequality reduces to
\[ a(a - 1)^2(a + 12) \geq 0. \]

The equality holds for \( a = b = c = 1 \), and also for \( a = 0 \) and \( b = c = 3/2 \) (or any cyclic permutation).

**Remark.** By squaring, we deduce the inequality
\[ \sqrt{(32a^2 + 3)(32b^2 + 3)} + \sqrt{(32b^2 + 3)(32c^2 + 3)} + \sqrt{(32c^2 + 3)(32a^2 + 3)} \leq 105, \]
with equality for \( a = b = c = 1 \), and also for
\[ a = 0, \quad b = c = \frac{3}{2} \]
(or any cyclic permutation).

\[ \square \]

**P 5.51.** If \( a_1, a_2, \ldots, a_n \) are positive real numbers so that \( a_1 + a_2 + \cdots + a_n = n \), then
\[ \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} + \frac{2n\sqrt{n-1}}{a_1^2 + a_2^2 + \cdots + a_n^2} \geq n + 2\sqrt{n-1}. \]

*(Vasile C., 2009)*

**Solution.** For \( n = 2 \), the inequality reduces to
\[ (a_1a_2 - 1)^2 \geq 0. \]
Consider further that $n \geq 3$ and $a_1 \leq a_2 \leq \cdots \leq a_n$. By Corollary 5 (case $k = 2$ and $m = -1$), if $a_1 \leq a_2 \leq \cdots \leq a_n$, then

$$a_1 + a_2 + \cdots + a_n = n, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = \text{constant},$$

then the sum

$$S_n = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

is minimum for $a_1 = \cdots = a_{n-1} \leq a_n$. Therefore, we only need to prove that

$$\frac{n-1}{a_1} + \frac{1}{a_n} + \frac{2n\sqrt{n-1}}{(n-1)a_1^2 + a_n^2} \geq n + 2\sqrt{n-1},$$

for $(n-1)a_1 + a_n = n$. The inequality is equivalent to

$$(a_1 - 1)^2 \left( a_1 - \frac{n}{n-1 + \sqrt{n-1}} \right)^2 \geq 0.$$ 

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = a_2 = \cdots = a_{n-1} = \frac{a_n}{\sqrt{n-1}}$$

(or any cyclic permutation).

\[\square\]

**P 5.52.** If $a, b, c \in [0, 1]$, then

$$(1 + 3a^2)(1 + 3b^2)(1 + 3c^2) \geq (1 + ab + bc + ca)^3.$$ 

**Solution.** Since

$$2(ab + bc + ca) = (a + b + c)^2 - (a^2 + b^2 + c^2),$$

we may apply Corollary 1 to the function

$$f(u) = -\ln(1 + 3u^2), \quad u \in [0, 1]$$

to prove the inequality

$$f(a) + f(b) + f(c) + 3 \ln(1 + ab + bc + ca) \leq 0.$$ 

We have

$$g(x) = f'(x) = \frac{-6x}{1 + 3x^2},$$

$$g''(x) = \frac{108x(1 - x^2)}{(1 + 3x^2)^3}. $$
Since \( g''(x) > 0 \) for \( x \in (0, 1) \), \( g \) is strictly convex on \( [0, 1] \). According to Corollary 1 and Note 5/Note 2, if
\[
a + b + c = \text{constant}, \quad a^2 + b^2 + c^2 = \text{constant}, \quad 0 \leq a \leq b \leq c,
\]
then
\[
S_3 = f(a) + f(b) + f(c)
\]
is maximum for \( a = b \leq c \). Thus, we only need to prove the original inequality for \( a = b \leq c \); that is,
\[
(1 + 3a^2)^2(1 + 3c^2) \geq (1 + a^2 + 2ac)^3.
\]
For \( c = 0 \), the inequality is an equality. For fixed \( c, 0 < c \leq 1 \), we need to show that \( h(a) \geq 0 \), where
\[
h(a) = 2 \ln(1 + 3a^2) + \ln(1 + 3c^2) - 3 \ln(1 + a^2 + 2ac), \quad a \in [0, c].
\]
From
\[
h'(a) = \frac{12a}{1 + 3a^2} - \frac{6(a + c)}{1 + a^2 + 2ac} = \frac{6(1 - a^2)(a - c)}{(1 + 3a^2)(1 + a^2 + 2ac)} \leq 0,
\]
it follows that \( h \) is decreasing on \( [0, c] \), hence \( h(a) \geq h(c) = 0 \). The proof is completed. The equality holds for \( a = b = c \).

\( \square \)

**P 5.53.** If \( a, b, c \) are nonnegative real numbers so that \( a + b + c = ab + bc + ca \), then
\[
\frac{1}{4 + 5a^2} + \frac{1}{4 + 5b^2} + \frac{1}{4 + 5c^2} \geq \frac{1}{3}.
\]

*(Vasile C., 2007)*

**Solution.** By expanding, the inequality becomes
\[
4(a^2 + b^2 + c^2) + 15 \geq 25a^2b^2c^2 + 5(a^2b^2 + b^2c^2 + c^2a^2).
\]

Let \( p = a + b + c \). Since
\[
a^2 + b^2 + c^2 = p^2 - 2p, \quad a^2b^2 + b^2c^2 + c^2a^2 = p^2 - 2pabc,
\]
the inequality becomes
\[
(2p - 4)^2 \geq (p - 5abc)^2,
\]
\[
(3p - 4 - 5abc)(p + 5abc - 4) \geq 0.
\]
We will show that $3p \geq 4 + 5abc$ and $p + 5abc \geq 4$. According to Corollary 4 (case $n = 3, k = 2$) or P 3.57 in Volume 1, if

$$a + b + c = \text{constant}, \quad ab + bc + ca = \text{constant}, \quad 0 \leq a \leq b \leq c \leq d,$$

then the product $abc$ is maximum for $a = b$, and is minimum for $a = 0$ or $b = c$. Thus, we only need to prove that $3p \geq 4 + 5abc$ for $a = b$, and $p + 5abc \geq 4$ for $a = 0$ and for $b = c$.

For $a = b$, from $a + b + c = ab + bc + ca$ we get

$$c = \frac{a(2-a)}{2a-1}, \quad \frac{1}{2} < a \leq 2,$$

hence

$$3p - 4 - 5abc = (3 - 5a^2)c + 6a - 4 = \frac{(a-1)^2(5a^2 + 4)}{2a-1} \geq 0.$$

For $a = 0$, from $a + b + c = ab + bc + ca$ we get

$$c = \frac{b}{b-1}, \quad b > 1,$$

hence

$$p + 5abc - 4 = b + c - 4 = \frac{(b-2)^2}{b-1} \geq 0.$$

For $b = c$, from $a + b + c = ab + bc + ca$ we get

$$a = \frac{b(2-b)}{2b-1}, \quad \frac{1}{2} < b \leq 2,$$

hence

$$p + 5abc - 4 = a(5b^2 + 1) + 2b - 4 = \frac{(2-b)(5b^3 - 3b + 2)}{2b-1} = \frac{(2-b)(4b^3 + (b-1)^2(b+2))}{2b-1} \geq 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 0$ and $b = c = 2$ (or any cyclic permutation).

\[\square\]

**P 5.54.** If $a, b, c, d$ are positive real numbers so that $a + b + c + d = 4abcd$, then

$$\frac{1}{1+3a} + \frac{1}{1+3b} + \frac{1}{1+3c} + \frac{1}{1+3d} \geq 1.$$ 

(Vasile C., 2007)
**Solution.** By expanding, the inequality becomes

\[ 1 + 3 \sum_{sym} ab \geq 19abcd, \]

\[ 2 + 3(a + b + c + d)^2 \geq 3(a^2 + b^2 + c^2 + d^2) + 38abcd. \]

According to Corollary 5 (case \( n = 4, k = 0, m = 2 \)), if

\[ a + b + c + d = \text{constant}, \quad abcd = \text{constant}, \quad 0 < a \leq b \leq c \leq d, \]

then the sum

\[ S_4 = a^2 + b^2 + c^2 + d^2 \]

is maximal for \( a = b = c \leq d \). Thus, we only need to prove that

\[ 3a + d = 4a^3d, \quad d = \frac{3a}{4a^3 - 1}, \quad a > \frac{1}{\sqrt[3]{4}}, \]

involves

\[ \frac{3}{3a + 1} + \frac{1}{3d + 1} \geq 1, \]

\[ \frac{3}{3a + 1} + \frac{4a^3 - 1}{4a^3 + 9a - 1} \geq 1, \]

\[ 4a^3 - 9a^2 + 6a - 1 \geq 0, \]

\[ (a - 1)^2(4a - 1) \geq 0. \]

The equality holds for \( a = b = c = d = 1 \).

**Open problem.** If \( a_1, a_2, \ldots, a_n \) (\( n \geq 3 \)) are positive real numbers so that

\[ a_1 + a_2 + \cdots + a_n = na_1a_2 \cdots a_n, \]

then

\[ \frac{1}{1 + (n-1)a_1} + \frac{1}{1 + (n-1)a_2} + \cdots + \frac{1}{1 + (n-1)a_n} \geq 1. \]

\( \square \)

**P 5.55.** If \( a_1, a_2, \ldots, a_n \) are positive real numbers so that

\[ a_1 + a_2 + \cdots + a_n = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}, \]

then

\[ \frac{1}{1 + (n-1)a_1} + \frac{1}{1 + (n-1)a_2} + \cdots + \frac{1}{1 + (n-1)a_n} \geq 1. \]

(Vasile C., 1996)
**Solution.** For \( n = 2 \), the inequality is an identity. For \( n \geq 3 \), we consider
\[
a_1 \leq a_2 \leq \cdots \leq a_n,
\]
and apply Corollary 2 to the function
\[
f(u) = \frac{1}{1 + (n-1)u}, \quad u > 0.
\]
We have
\[
f'(u) = \frac{-(n-1)}{[1 + (n-1)x]^2},
\]
\[
g(x) = f'(\frac{1}{\sqrt{x}}) = \frac{-(n-1)x}{[\sqrt{x} + n-1]^2},
\]
\[
g''(x) = \frac{3(n-1)^2}{2\sqrt{x}(\sqrt{x} + n-1)^4}.
\]
Since \( g''(x) > 0 \) for \( x > 0 \), \( g \) is strictly convex on \([0, \infty)\). By Corollary 2, if
\[
0 < a_1 \leq a_2 \leq \cdots \leq a_n
\]
and
\[
a_1 + a_2 + \cdots + a_n = constant, \quad \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = constant,
\]
then the sum
\[
S_n = f(a_1) + f(a_2) + \cdots + f(a_n)
\]
is minimum for \( a_2 = \cdots = a_n \). Therefore, we only need to show that
\[
\frac{1}{1 + (n-1)a} + \frac{n-1}{1 + (n-1)b} \geq 1
\]
for
\[
a + (n-1)b = \frac{1}{a} + \frac{n-1}{b}, \quad 0 < a \leq b.
\]
Write the hypothesis as
\[
\frac{1}{a} - a = (n-1)\left( b - \frac{1}{b} \right),
\]
which involves \( a \leq 1 \leq b \) and
\[
\frac{1}{a} - a \geq b - \frac{1}{b}, \quad ab \leq 1.
\]
Write the desired inequality as
\[
\frac{n-1}{1 + (n-1)b} \geq 1 - \frac{1}{1 + (n-1)a},
\]
which is equivalent to
\[
\frac{n-1}{1 + (n-1)b} \geq \frac{(n-1)a}{1 + (n-1)a}.
\]
1 − a ≥ (n − 1)a(b − 1).

For the nontrivial case \( b \neq 1 \), we have

\[
1 - a - (n - 1)a(b - 1) = 1 - a - \frac{b(1 - a^2)}{a(b^2 - 1)}a(b - 1) = \frac{(1 - a)(1 - ab)}{b + 1} \geq 0.
\]

If \( n \geq 3 \), then the equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \).

\( \blacksquare \)

**P 5.56.** If \( a, b, c, d, e \) are nonnegative real numbers so that \( a^4 + b^4 + c^4 + d^4 + e^4 = 5 \), then

\[
7(a^2 + b^2 + c^2 + d^2 + e^2) \geq (a + b + c + d + e)^2 + 10.
\]

(Vasile C., 2008)

**Solution.** According to Corollary 5 (case \( n = 5, k = 4, m = 2 \)), if

\[ a + b + c + d + e = constant, \quad a^4 + b^4 + c^4 + d^4 + e^4 = 5, \quad 0 \leq a \leq b \leq c \leq d \leq e, \]

then the sum

\[ S_4 = a^2 + b^2 + c^2 + d^2 + e^2 \]

is minimum for \( a = b = c = d \leq e \). Thus, we only need to prove the homogeneous inequality

\[
[7(a^2 + b^2 + c^2 + d^2 + e^2) - (a + b + c + d + e)^2] \geq 20(a^4 + b^4 + c^4 + d^4 + e^4)
\]

for \( a = b = c = d = 0 \) and \( a = b = c = d = 1 \). The first case is trivial. In the second case, the inequality becomes

\[
[7(4 + e^2) - (4 + e)^2]^2 \geq 20(4 + e^4),
\]

\[
(3e^2 - 4e + 6)^2 \geq 5e^4 + 20,
\]

\[
e^4 - 6e^3 + 13e^2 - 12e + 4 \geq 0,
\]

\[
(e - 1)^2(e - 2)^2 \geq 0.
\]

The equality holds for \( a = b = c = d = e = 1 \), and also for

\[ a = b = c = d = e = \frac{e}{2} = \frac{5}{6}. \]

**Remark.** Similarly, we can prove the following generalization:

- If \( a_1, a_2, \ldots, a_n \) are nonnegative real numbers so that

\[
a_1^4 + a_2^4 + \cdots + a_n^4 = n,
\]

then

\[
7(a_1^2 + a_2^2 + \cdots + a_n^2) \geq (a_1 + a_2 + \cdots + a_n)^2 + 10.
\]
then
\[(n + \sqrt{n - 1})(a_1^2 + a_2^2 + \cdots + a_n^2 - n) \geq (a_1 + a_2 + \cdots + a_n)^2 - n^2,\]
with equality for \(a_1 = a_2 = \cdots = a_n = 1\), and also for
\[a_1 = \cdots = a_{n-1} = \frac{a_n}{\sqrt{n - 1}} = \frac{n}{n + \sqrt{n - 1}},\]
(or any cyclic permutation).

\[P 5.57.\] If \(a_1, a_2, \ldots, a_n\) are nonnegative real numbers so that \(a_1 + a_2 + \cdots + a_n = n\), then
\[(a_1^2 + a_2^2 + \cdots + a_n^2)^2 - n^2 \geq \frac{n(n-1)}{n^2 - n + 1} (a_1^4 + a_2^4 + \cdots + a_n^4 - n).\]
(Vasile C., 2008)

**Solution.** For \(n = 2\), the inequality reduces to \((a_1 a_2 - 1)^2 \geq 0\). For \(n \geq 3\), we apply Corollary 5 for \(k = 2\) and \(m = 4\): if \(0 \leq a_1 \leq a_2 \leq \cdots \leq a_n\) and
\[a_1 + a_2 + \cdots + a_n = n, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = \text{constant},\]
then
\[S_n = a_1^4 + a_2^4 + \cdots + a_n^4\]
is maximum for \(a_1 = \cdots = a_{n-1} \leq a_n\). Thus, we only need to prove the homogeneous inequality
\[n^2(n^2 - n + 1)(a_1^2 + a_2^2 + \cdots + a_n^2)^2 \geq (n^2 - 2n + 2)(a_1 + a_2 + \cdots + a_n)^4 + n^3(n-1)S_n,\]
for \(a_1 = \cdots = a_{n-1} = 0\) and for \(a_1 = \cdots = a_{n-1} = 1\). For the nontrivial case \(a_1 = \cdots = a_{n-1} = 1\), the inequality becomes
\[n^2(n^2 - n + 1)(n-1 + a_n^2)^2 \geq (n^2 - 2n + 2)(n-1 + a_n)^4 + n^3(n-1)(n-1 + a_n^4),\]
\[(a_n - 1)^2[a_n - (n-1)^2]^2 \geq 0.\]
The equality holds for \(a_1 = a_2 = \cdots = a_n = 1\), and also for
\[a_1 = \cdots = a_{n-1} = \frac{1}{n-1}, \quad a_n = n - 1\]
(or any cyclic permutation).
P 5.58. If \( a_1, a_2, \ldots, a_n \) are nonnegative real numbers so that \( a_1^2 + a_2^2 + \cdots + a_n^2 = n \), then
\[
a_1^3 + a_2^3 + \cdots + a_n^3 \geq \sqrt{n^2 - n + 1 + \left(1 - \frac{1}{n}\right)(a_1^6 + a_2^6 + \cdots + a_n^6)}.
\]

(Vasile C., 2008)

Solution. For \( n = 2 \), the inequality is equivalent to
\[
a_1^6 + a_2^6 + 4a_1^3a_2^3 \geq 6,
\]
\[
(a_1^2 + a_2^2)^3 - 3a_1^2a_2(a_1^2 + a_2^2) + 6a_1^3a_2^3 \geq 6,
\]
\[
2a_1^3a_2^3 - 3a_1^2a_2^2 + 1 \geq 0,
\]
\[
(a_1a_2 - 1)^2(2a_1a_2 + 1) \geq 0.
\]

For \( n \geq 3 \), we apply Corollary 5 for \( k = 3/2 \) and \( m = 3 \): if \( 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \) and
\[
x_1 + x_2 + \cdots + x_n = n, \quad x_1^{3/2} + x_2^{3/2} + \cdots + x_n^{3/2} = \text{constant},
\]
then
\[
S_n = x_1^3 + x_2^3 + \cdots + x_n^3
\]
is maximum for \( x_1 = \cdots = x_{n-1} \leq x_n \). Thus, we only need to prove the homogeneous inequality
\[
(a_1^3 + a_2^3 + \cdots + a_n^3)^2 \geq \frac{n^2 - n + 1}{n^3}(a_1^6 + a_2^6 + \cdots + a_n^6)^3 + \left(1 - \frac{1}{n}\right)(a_1^6 + a_2^6 + \cdots + a_n^6)
\]
for \( a_1 = \cdots = a_{n-1} = 0 \) and for \( a_1 = \cdots = a_{n-1} = 1 \). For the nontrivial case \( a_1 = \cdots = a_{n-1} = 1 \), the inequality becomes
\[
n^3(n-1 + a_n^3)^2 \geq (n^2 - n + 1)(n-1 + a_n^3)^3 + n^2(n-1)(n-1 + a_n^6),
\]
\[
(a_n - 1)^2(a_n - n + 1)(2a_n + n - 1) \geq 0.
\]
The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for
\[
a_1 = \cdots = a_{n-1} = \frac{a_n}{n-1} = \sqrt[3]{\frac{n}{2(n-1)}}
\]
(or any cyclic permutation).

P 5.59. If \( a, b, c \) are positive real numbers so that \( abc = 1 \), then
\[
4\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + \frac{50}{a + b + c} \geq 27.
\]

(Vasile C., 2012)
Solution. According to Corollary 5 (case \(k=0\) and \(m=-1\), if

\[a + b + c = \text{constant}, \quad abc = 1, \quad 0 < a \leq b \leq c,\]

then

\[S_3 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\]

is minimum for \(0 < a = b \leq c\). Thus, we only need to prove that

\[4 \left( \frac{2}{a} + \frac{1}{c} \right) + \frac{50}{2a + c} \geq 27\]

for

\[a^2c = 1, \quad a \leq 1.\]

The inequality is equivalent to

\[8a^6 - 54a^4 - 26a^3 - 27a + 8 \geq 0,\]

\[(2a - 1)^2(2a^4 + 2a^3 - 12a^2 + 5a + 8) \geq 0.\]

It is true for \(a \in (0, 1]\) because

\[2a^4 + 2a^3 - 12a^2 + 5a + 8 > -12a^2 + 4a + 8 = 4(1-a)(2+3a) \geq 0.\]

The equality holds for

\[a = b = \frac{1}{2}, \quad c = 4\]

(or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization:

- If \(a_1, a_2, \ldots, a_n\) are positive real numbers so that \(a_1a_2 \cdots a_n = 1\), then

\[2^n \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right) + \frac{(2^n + n - 1)^2}{a_1 + a_2 + \cdots + a_n} \geq 2n(2^n + 1),\]

with equality for

\[a_1 = \cdots = a_{n-1} = \frac{1}{2}, \quad a_n = 2^{n-1}\]

(or any cyclic permutation).

For

\[a_1 = \cdots = a_{n-1} = a \leq 1, \quad a^{n-1}a_n = 1,\]

the inequality is equivalent to \(f(a) \geq 0\), where

\[f(a) = 2^n \left( \frac{n-1}{a} + a^{n-1} \right) + \frac{(2^n + n - 1)^2 a^{n-1}}{(n-1)a^n + 1} - 2n(2^n + 1).\]
We have
\[
\frac{f'(a)}{n-1} = \frac{2^n(a^n - 1)}{a^2} - \frac{(2^n + n - 1)^2 a^{n-2}(a^n - 1)}{[(n-1)a^n + 1]^2}
\]
\[
= \frac{(a^n - 1)(2^n a^n - 1)[(n-1)^2 a^n - 2^n]}{a^2[(n-1)a^n + 1]^2}.
\]
Since
\[
(n-1)^2 a^n - 2^n \leq (n-1)^2 - 2^n < 0,
\]
it follows that \(f'(a) < 0\) for \(a \in \left(0, \frac{1}{2}\right)\), and \(f'(a) > 0\) for \(a \in \left(\frac{1}{2}, 1\right)\). Therefore, \(f\) is decreasing on \(\left(0, \frac{1}{2}\right]\) and increasing on \(\left[\frac{1}{2}, 1\right)\), hence
\[
f(a) \geq f \left(\frac{1}{2}\right) = 0.
\]

\textbf{P 5.60.} If \(a, b, c\) are positive real numbers so that \(abc = 1\), then
\[
a^3 + b^3 + c^3 + 15 \geq 6 \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).
\]

\textit{(Michael Rozenberg, 2006)}

\textbf{Solution.} Replacing \(a, b, c\) by their reverses \(1/a, 1/b, 1/c\), we need to show that \(abc = 1\) involves
\[
\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + 15 \geq 6(a + b + c).
\]
According to Corollary 5 (case \(k=0\) and \(m = -3\), if \(a + b + c = constant\), \(abc = 1\), \(0 < a \leq b \leq c\),
then
\[
S_3 = \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}
\]
is minimum for \(0 < a = b \leq c\). Thus, we only need to prove that
\[
\frac{2}{a^3} + \frac{1}{c^3} + 15 \geq 6(2a + c)
\]
for \(a^2c = 1\), \(a \leq 1\).

The inequality is equivalent to
\[
\frac{2}{a^3} + a^6 + 15 \geq 6 \left(2a + \frac{1}{a^2}\right),
\]
\[ a^9 - 12a^4 + 15a^3 - 6a + 2 \geq 0, \]
\[ (1 - a)^2(2 - 2a - 6a^2 + 5a^3 + 4a^4 + 3a^5 + 2a^6 + a^7) \geq 0. \]

It suffices to show that

\[ 2 - 2a - 6a^2 + 5a^3 + 3a^4 \geq 0. \]

Indeed, we have

\[ 2(2 - 2a - 6a^2 + 5a^3 + 3a^4) = (2 - 3a)^2 \left( 1 + 2a + \frac{3}{4}a^2 \right) + a^3 \left( 1 - \frac{3}{4}a \right) \geq 0. \]

The equality holds for \( a = b = c = 1. \)

\[ \square \]

**P 5.61.** Let \( a_1, a_2, \ldots, a_n \) be positive numbers so that \( a_1 a_2 \cdots a_n = 1. \) If \( k \geq n - 1, \) then

\[ a_1^k + a_2^k + \cdots + a_n^k + (2k - n)n \geq (2k - n + 1) \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right). \]

(Vasile C., 2008)

**Solution.** For \( n = 2 \) and \( k = 1, \) the inequality is an identity. For \( n = 2 \) and \( k > 1, \) we need to show that \( f(a) \geq 0 \) for \( a > 0, \) where

\[ f(a) = a^k + a^{-k} + 4(k - 1) - (2k - 1)(a + a^{-1}). \]

We have

\[ f'(a) = ka^{k-1} - a^{-k-1} - (2k - 1)(1 - a^{-2}), \]
\[ f''(a) = k[(k - 1)a^{k-2} + (k + 1)a^{-k-2}] - 2(2k - 1)a^{-3}. \]

By the weighted AM-GM inequality, we get

\[ (k - 1)a^{k-2} + (k + 1)a^{-k-2} \geq 2ka \frac{(k-1)(k-2)+1}{2k} = 2ka^{-3}, \]

hence

\[ f''(a) \geq 2k^2a^{-3} - 2(2k - 1)a^{-3} = 2(k - 1)^2a^{-3} > 0, \]

\( f' \) is strictly increasing. Since \( f'(1) = 0, \) it follows that \( f'(a) < 0 \) for \( a < 1 \) and \( f'(a) > 0 \) for \( a > 1, \) \( f \) is decreasing on \( (0, 1] \) and increasing on \( [1, \infty), \) hence \( f(a) \geq f(1) = 0. \)

Consider further that \( n \geq 3. \) Replacing \( a_1, a_2, \ldots, a_n \) by \( 1/a_1, 1/a_2, \ldots, 1/a_n, \) we need to show that \( a_1 a_2 \cdots a_n = 1 \) involves

\[ \frac{1}{a_1^k} + \frac{1}{a_2^k} + \cdots + \frac{1}{a_n^k} + (2k - n)n \geq (2k - n + 1)(a_1 + a_2 + \cdots + a_n). \]
According to Corollary 5, if \( 0 < a_1 \leq a_2 \leq \cdots \leq a_n \) and 
\[
a_1 + a_2 + \cdots + a_n = \text{constant}, \quad a_1 a_2 \cdots a_n = 1,
\]
then
\[
S_n = \frac{1}{a_1^k} + \frac{1}{a_2^k} + \cdots + \frac{1}{a_n^k}
\]
is minimum for \( 0 < a_1 = \cdots = a_{n-1} \leq a_n \). Thus, we only need to prove the original inequality for \( a_1 = \cdots = a_{n-1} \geq 1 \); that is, to show that \( t \geq 1 \) involves \( f(t) \geq 0 \), where
\[
f(t) = (n-1)t^k + \frac{1}{t^{k(n-1)}} + (2k-n)n - (2k-n+1)\left(\frac{n-1}{t} + t^{n-1}\right).
\]
We have
\[
f'(t) = \frac{(n-1)g(t)}{t^{kn-k+1}}, \quad g(t) = k(t^{kn} - 1) - (2k-n+1)t^{k(n-k)}(t^n-1),
\]
g\( (t) = t^{kn-k-2}h(t), \quad h(t) = k^2nt^{k+1}-(2k-n+1)\left[(k+1)(n-1)t^n-kn+k+1\right],
\]
\[
h'(t) = (k+1)nt^{kn-1}\left[k^2t^{k-n+1}-(2k-n+1)(n-1)\right].
\]
If \( k = n-1 \), then \( h(t) = n(n-1)(n-2) > 0 \). If \( k > n-1 \), then
\[
k^2t^{k-n+1}-(2k-n+1)(n-1) \geq k^2 - (2k-n+1)(n-1) = (k-n+1)^2 > 0,
\]
h\( (t) > 0 \) for \( t \geq 1 \), \( h \) is strictly increasing on \( [1, \infty) \), hence
\[
h(t) \geq h(1) = n[(k-1)^2 + n-2] > 0.
\]
From \( h > 0 \), we get \( g' > 0 \), \( g \) is strictly increasing, \( g(t) \geq g(1) = 0 \) for \( t \geq 1 \),
f\( (t) > 0 \) for \( t > 1 \), \( f \) is strictly increasing, \( f(t) \geq f(1) = 0 \) for \( t \geq 1 \).

The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \). If \( n = 2 \) and \( k = 1 \), then the equality holds for \( a_1 a_2 = 1 \).

\[
\square
\]

**P 5.62.** Let \( a_1, a_2, \ldots, a_n \) \( (n \geq 3) \) be nonnegative numbers so that \( a_1 + a_2 + \cdots + a_n = n \), and let \( k \) be an integer satisfying \( 2 \leq k \leq n+2 \). If
\[
r = \left(\frac{n}{n-1}\right)^{k-1} - 1,
\]
then
\[
a_1^k + a_2^k + \cdots + a_n^k - n \geq nr(1 - a_1 a_2 \cdots a_n).
\]

(\( \text{Vasile C., 2005} \))
**Solution.** According to Corollary 4, if \( 0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \) and
\[
a_1 + a_2 + \cdots + a_n = n, \quad a_1^k + a_2^k + \cdots + a_n^k = \text{constant},
\]
then the product
\[
P = a_1 a_2 \cdots a_n
\]
is minimum for either \( a_1 = 0 \) or \( 0 < a_1 \leq a_2 = \cdots = a_n \).

Case 1: \( a_1 = 0 \). We need to show that
\[
a_2^k + \cdots + a_n^k \geq \frac{n^k}{(n-1)^{k-1}}
\]
for \( a_2 + \cdots + a_n = n \). This follows by Jensen’s inequality
\[
a_2^k + \cdots + a_n^k \geq (n-1)\left(\frac{a_2 + \cdots + a_n}{n-1}\right)^k.
\]

Case 2: \( 0 < a_1 \leq a_2 = \cdots = a_n \). Denoting \( a_1 = x \) and \( a_2 = y \) \( (x \leq y) \), we only need to show that
\[
f(x) \geq 0,
\]
where
\[
f(x) = x^k + (n-1)y^k + nrxy^{n-1} - n(r+1), \quad y = \frac{n-x}{n-1}, \quad 0 < x \leq 1 \leq y.
\]
It is easy to check that
\[
f(0) = f(1) = 0.
\]
Since
\[
y' = \frac{-1}{n-1},
\]
we have
\[
f'(x) = k(x^{k-1} - y^{k-1}) + nry^{n-2}(y-x)
\]
\[
= (y-x)[nry^{n-2} - k(y^{k-2} + y^{k-3}x + \cdots + x^{k-2})]
\]
\[
= (y-x)y^{n-2}[nr - kg(x)],
\]
where
\[
g(x) = \frac{1}{y^{n-k}} + \frac{x}{y^{n-k+1}} + \cdots + \frac{x^{k-2}}{y^{n-2}}.
\]
Since the function
\[
y(x) = \frac{n-x}{n-1}
\]
is strictly decreasing, \( g \) is strictly increasing for \( 2 \leq k \leq n \). Also, \( g \) strictly increasing for \( k = n + 1 \), when

\[
g(x) = y + x + \frac{x^2}{y} + \cdots + \frac{x^{n-1}}{y^{n-2}}
\]

and for \( k = n + 2 \), when

\[
g(x) = y^2 + y x + x^2 + \frac{x^3}{y} + \cdots + \frac{x^n}{y^{n-2}}
\]

and for \( k = n + 1 \), when

\[
g(x) = \frac{(n-2)x + n}{n+1} + \frac{x^2}{y} + \cdots + \frac{x^{n-1}}{y^{n-2}}.
\]

Therefore, the function

\[
h(x) = nr - kg(x)
\]

is strictly decreasing for \( x \in (0, 1] \). Using the contradiction method, we will show that

\[
h(0) > 0, \quad h(1) < 0.
\]

If \( h(0) \leq 0 \), then \( h(x) < 0 \) for \( x \in (0, 1) \), \( f'(x) < 0 \) for \( x \in (0, 1) \), \( f \) is strictly decreasing on \([0, 1]\), hence \( f(0) > f(1) \), which contradicts \( f(0) = f(1) \). If \( h(1) \geq 0 \), then \( h(x) > 0 \) for \( x \in (0, 1) \), \( f'(x) > 0 \) for \( x \in (0, 1) \), \( f \) is strictly increasing on \([0, 1]\), hence \( f(0) < f(1) \), which contradicts \( f(0) = f(1) \). Since \( h(0) > 0 \) and \( h(1) < 0 \), there exists \( x_1 \in (0, 1) \) so that \( h(x_1) = 0 \), \( h(x) > 0 \) for \( x \in [0, x_1) \), and \( h(x) < 0 \) for \( x \in (x_1, 1] \). Consequently, \( f \) is strictly increasing on \([0, x_1]\) and strictly decreasing on \([x_1, 1]\). From \( f(0) = f(1) = 0 \), it follows that \( f(x) \geq 0 \) for \( x \in [0, 1] \).

The equality holds for for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for

\[
a_1 = 0, \quad a_2 = \cdots = a_n = \frac{n}{n-1}
\]

(or any cyclic permutation).

Remark. For the particular case \( k = n \), the inequality has been posted in 2004 on Art of Problem Solving website by Gabriel Dospinescu and Calin Popa.

\[\square\]

**P 5.63.** If \( a, b, c \) are positive real numbers so that \( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 3 \), then

\[
4(a^2 + b^2 + c^2) + 9 \geq 21abc.
\]

(Vasile C., 2006)
Solution. Replacing $a, b, c$ by their reverses $1/a, 1/b, 1/c$, we need to show that $a + b + c = 3$ involves

$$4\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) + 9 \geq \frac{21}{abc}.$$ 

According to Corollary 5 (case $k=0$ and $m=-2$), if

$$a + b + c = 3, \quad abc = constant, \quad 0 < a \leq b \leq c,$$

then

$$S_3 = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

is minimum for $0 < a = b \leq c$. Thus, we only need to prove that

$$4\left(\frac{2}{a^2} + \frac{1}{c^2}\right) + 9 \geq \frac{21}{a^2c}$$

for $2a + b = 3$. The inequality is equivalent to

$$(9a^2 + 8)c^2 - 21c + 4a^2 \geq 0,$$

$$4a^4 - 12a^3 + 13a^2 - 6a + 1 \geq 0,$$

$$(a - 1)^2(2a - 1)^2 \geq 0.$$ 

The equality holds for $a = b = c = 1$, and also for

$$a = b = 2, \quad c = \frac{1}{2}$$

(or any cyclic permutation). 

\[\square\]

\textbf{P 5.64.} If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = n$, then

$$a_1 + a_2 + \cdots + a_n - n \leq e_{n-1}(a_1a_2\cdots a_n - 1),$$

where

$$e_{n-1} = \left(1 + \frac{1}{n-1}\right)^{n-1}.$$ 

\textit{(Gabriel Dospinescu and Calin Popa, 2004)}

Solution. For $n = 2$, the inequality is an identity. For $n \geq 3$, replacing $a_1, a_2, \ldots, a_n$ by $1/a_1, 1/a_2, \ldots, 1/a_n$, we need to show that $a_1 + a_2 + \cdots + a_n = n$ involves

$$a_1a_2\cdots a_n \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n + e_{n-1}\right) \leq e_{n-1}.$$
According to Corollary 5 (case $k = 0$ and $m = -1$), if $0 < a_1 \leq a_2 \leq \cdots \leq a_n$ and 
\[ a_1 + a_2 + \cdots + a_n = n, \quad a_1 a_2 \cdots a_n = \text{constant}, \]
then
\[ S_n = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \]
is maximum for $0 < a_1 \leq a_2 = \cdots = a_n$. Using the notation $a_1 = x$ and $a_2 = y$, we only need to show that $f(x) \leq 0$ for 
\[ x + (n-1)y = n, \quad 0 < x \leq 1, \]
where
\[ f(x) = xy^{n-1} \left( \frac{1}{x} + \frac{n-1}{y} - n + e_{n-1} \right) - e_{n-1} \]
\[ = y^{n-1} + (n-1)xy^{n-2} - (n - e_{n-1})xy^{n-1} - e_{n-1}. \]
Since
\[ y' = -\frac{1}{n-1}, \]
we get
\[ \frac{f'(x)}{y^{n-3}} = (y-x)h(x), \]
where
\[ h(x) = n - 2 - (n - e_{n-1})y = n - 2 - (n - e_{n-1}) \frac{n-x}{n-1} \]
is a linear increasing function. Since
\[ h(0) = \frac{n}{n-1} \left( e_{n-1} - 3 + \frac{2}{n} \right) < 0 \]
and
\[ h(1) = e_{n-1} - 2 > 0, \]
there exists $x_1 \in (0, 1)$ so that $h(x_1) = 0$, $h(x) < 0$ for $x \in [0, x_1)$, and $h(x) > 0$ for $x \in (x_1, 1]$. Consequently, $f$ is strictly decreasing on $[0, x_1]$ and strictly increasing on $[x_1, 1]$. From
\[ f(0) = f(1) = 0, \]
it follows that $f(x) \leq 0$ for $x \in [0, 1]$.

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $n = 2$, then the equality holds for $a_1 + a_2 = 2a_1 a_2$. \[\square\]
P 5.65. If \( a_1, a_2, \ldots, a_n \) are positive real numbers, then
\[
\frac{a_1^n + a_2^n + \cdots + a_n^n}{a_1 a_2 \cdots a_n} + n(n-1) \geq (a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).
\]

(Vasile C., 2004)

Solution. For \( n = 2 \), the inequality is an identity. For \( n \geq 3 \), according to Corollary 5 (case \( k = 0 \) and \( m \in \{-1, n\} \)), if \( 0 < a_1 \leq a_2 \leq \cdots \leq a_n \) and
\[
a_1 + a_2 + \cdots + a_n = \text{constant}, \quad a_1 a_2 \cdots a_n = \text{constant},
\]
then the sum \( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \) is maximum and the sum \( a_1^n + a_2^n + \cdots + a_n^n \) is minimum for
\[
0 < a_1 \leq a_2 = \cdots = a_n.
\]
Consequently, we only need to prove the desired homogeneous inequality for \( a_2 = \cdots = a_n = 1 \), when it becomes
\[
a_1^n + (n-2)a_1 \geq (n-1)a_1^2.
\]
Indeed, by the AM-GM inequality, we have
\[
a_1^n + (n-2)a_1 \geq (n-1)\sqrt[n-1]{a_1^n \cdot a_1^{n-2}} = (n-1)a_1^2.
\]
For \( n \geq 3 \), the equality holds when \( a_1 = a_2 = \cdots = a_n \).

\[\square\]

P 5.66. If \( a_1, a_2, \ldots, a_n \) are nonnegative real numbers, then
\[
(n-1)(a_1^n + a_2^n + \cdots + a_n^n) + na_1 a_2 \cdots a_n \geq (a_1 + a_2 + \cdots + a_n)(a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1}).
\]

(Janos Suranyi, MSC-Hungary)

Solution. For \( n = 2 \), the inequality is an identity. For \( n \geq 3 \), according to Corollary 5 (case \( k = n \) and \( m = n-1 \)), if \( 0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \) and
\[
a_1 + a_2 + \cdots + a_n = \text{constant}, \quad a_1^n + a_2^n + \cdots + a_n^n = \text{constant},
\]
then the sum \( a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} \) is maximum and the product \( a_1 a_2 \cdots a_n \) is min-
imum for either \( a_1 = 0 \) or \( 0 < a_1 \leq a_2 = \cdots = a_n \). Consequently, we only need to consider these cases.

Case 1: \( a_1 = 0 \). The inequality reduces to
\[
(n-1)(a_2^n + \cdots + a_n^n) \geq (a_2 + \cdots + a_n)(a_2^{n-1} + \cdots + a_n^{n-1}),
\]
which follows immediately from Chebyshev’s inequality.

Case 2: \(0 < a_1 \leq a_2 = \cdots = a_n\). Due to homogeneity, we may set \(a_2 = \cdots = a_n = 1\), when the inequality becomes

\[
(n-2)a_1^n + a_1 \geq (n-1)a_1^{n-1}.
\]

Indeed, by the AM-GM inequality, we have

\[
(n-2)a_1^n + a_1 \geq (n-1)\sqrt[n-2]{a_1^{n(n-2)}} \cdot a_1 = (n-1)a_1^{n-1}.
\]

For \(n \geq 3\), the equality holds when \(a_1 = a_2 = \cdots = a_n\), and also when

\[
a_1 = 0, \quad a_2 = \cdots = a_n
\]

(or any cyclic permutation). \(\square\)

**P 5.67.** If \(a_1, a_2, \ldots, a_n\) are nonnegative real numbers, then

\[
(n-1)(a_1^{n+1} + a_2^{n+1} + \cdots + a_n^{n+1}) \geq (a_1 + a_2 + \cdots + a_n)(a_1^n + a_2^n + \cdots + a_n^n - a_1a_2\cdots a_n).
\]

*(Vasile C., 2006)*

**Solution.** For \(n = 2\), the inequality is an identity. For \(n \geq 3\), according to Corollary 5 (case \(k = n + 1\) and \(m = n\), if \(0 \leq a_1 \leq a_2 \leq \cdots \leq a_n\) and

\[
a_1 + a_2 + \cdots + a_n = \text{constant}, \quad a_1^{n+1} + a_2^{n+1} + \cdots + a_n^{n+1} = \text{constant},
\]

then the sum \(a_1^n + a_2^n + \cdots + a_n^n\) is maximum and the product \(a_1a_2\cdots a_n\) is minimum for either \(a_1 = 0\) or \(0 < a_1 \leq a_2 = \cdots = a_n\). Consequently, we only need to consider these cases.

Case 1: \(a_1 = 0\). The inequality reduces to

\[
(n-1)(a_2^{n+1} + \cdots + a_n^{n+1}) \geq (a_2 + \cdots + a_n)(a_2^n + \cdots + a_n^n),
\]

which follows immediately from Chebyshev’s inequality.

Case 2: \(0 < a_1 \leq a_2 = \cdots = a_n\). Due to homogeneity, we may set \(a_2 = \cdots = a_n = 1\), when the inequality becomes

\[
(n-2)a_1^{n+1} + a_1^2 \geq (n-1)a_1^n.
\]

Indeed, by the AM-GM inequality, we have

\[
(n-2)a_1^{n+1} + a_1^2 \geq (n-1)\sqrt[n-2]{a_1^{n(n-2)}} \cdot a_1^2 = (n-1)a_1^n.
\]

For \(n \geq 3\), the equality holds when \(a_1 = a_2 = \cdots = a_n\), and also when

\[
a_1 = 0, \quad a_2 = \cdots = a_n
\]

(or any cyclic permutation). \(\square\)
P 5.68. If $a_1, a_2, \ldots, a_n$ are positive real numbers, then

$$(a_1 + a_2 + \cdots + a_n - n) \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n \right) + a_1a_2 \cdots a_n + \frac{1}{a_1a_2 \cdots a_n} \geq 2.$$ 

(Vasile C., 2006)

Solution. For $n = 2$, the inequality reduces to

$$(1 - a_1)^2(1 - a_2)^2 \geq 0.$$ 

Consider further that $n \geq 3$. Since the inequality remains unchanged by replacing each $a_i$ with $1/a_i$, we may consider $a_1a_2 \cdots a_n \geq 1$. By the AM-GM inequality, we get

$$a_1 + a_2 + \cdots + a_n \geq n \sqrt[n]{a_1a_2 \cdots a_n} \geq n.$$ 

According to Corollary 5 (case $k = 0$ and $m = -1$), if $0 < a_1 \leq a_2 \leq \cdots \leq a_n$ and

$$a_1 + a_2 + \cdots + a_n = \text{constant}, \quad a_1a_2 \cdots a_n = \text{constant},$$

then the sum

$$S_n = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

is minimum for $0 < a_1 = a_2 = \cdots = a_{n-1} \leq a_n$. Consequently, we only need to consider

$$a_1 = a_2 = \cdots = a_{n-1} = x, \quad a_n = y, \quad x \leq y.$$ 

The inequality becomes

$$[(n-1)x + y - n]\left( \frac{n-1}{x} + \frac{1}{y} - n \right) + x^{n-1}y + \frac{1}{x^{n-1}y} \geq 2,$$

$$\left( x^{n-1} + \frac{n-1}{x} - n \right) y + \left[ \frac{1}{x^{n-1}} + (n-1)x - n \right] \frac{1}{y} \geq \frac{n(n-1)(x-1)^2}{x}.$$ 

Since

$$x^{n-1} + \frac{n-1}{x} - n = \frac{x-1}{x} \left[ (x^{n-1} - 1) + (x^{n-2} - 1) + \cdots + (x-1) \right]$$

$$= \frac{(x-1)^2}{x} \left[ x^{n-2} + 2x^{n-3} + \cdots + (n-1) \right],$$

and

$$\frac{1}{x^{n-1}} + (n-1)x - n = \frac{(x-1)^2}{x} \left[ \frac{1}{x^{n-2}} + 2x^{n-3} + \cdots + (n-1) \right],$$

it is enough to prove the inequality

$$\left[ x^{n-2} + 2x^{n-3} + \cdots + (n-1) \right] y + \left[ \frac{1}{x^{n-2}} + 2x^{n-3} + \cdots + (n-1) \right] \frac{1}{y} \geq n(n-1),$$
which is equivalent to
\[
\left( x^{n-2}y + \frac{1}{x^{n-2}y} - 2 \right) + 2\left( x^{n-3}y + \frac{1}{x^{n-3}y} - 2 \right) + \cdots + (n-1)\left( y + \frac{1}{y} - 2 \right) \geq 0,
\]
\[
\frac{(x^{n-2}y - 1)^2}{x^{n-2}y} + 2\frac{(x^{n-3}y - 1)^2}{x^{n-3}y} + \cdots + \frac{(n-1)(y-1)^2}{y} \geq 0.
\]
The equality holds if \( n - 1 \) of the numbers \( a_i \) are equal to 1.

\[\square\]

**P 5.69.** If \( a_1, a_2, \ldots, a_n \) are positive real numbers so that \( a_1a_2 \cdots a_n = 1 \), then
\[
\left| \frac{1}{\sqrt{a_1 + a_2 + \cdots + a_n - n}} - \frac{1}{\sqrt{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n}} \right| < 1.
\]

*(Vasile C., 2006)*

**Solution.** Let
\[
A = a_1 + a_2 + \cdots + a_n - n, \quad B = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n.
\]

By the AM-GM inequality, it follows that \( A > 0 \) and \( B > 0 \). According to the preceding P 5.68, the following inequality holds
\[
(a_1 + \cdots + a_{n+1} - n - 1)\left( \frac{1}{a_1} + \cdots + \frac{1}{a_{n+1} - n - 1} \right) + a_1 \cdots a_{n+1} + \frac{1}{a_1 \cdots a_{n+1}} \geq 2,
\]
which is equivalent to
\[
(A - 1 + a_{n+1})\left( B - 1 + \frac{1}{a_{n+1}} \right) + a_{n+1} + \frac{1}{a_{n+1}} \geq 2,
\]
\[
\frac{A}{a_{n+1}} + Ba_{n+1} + AB - A - B \geq 0.
\]

Choosing
\[
a_{n+1} = \sqrt[2]{\frac{A}{B}},
\]
we get
\[
2\sqrt{AB} + AB - A - B \geq 0,
\]
\[
AB \geq \left( \sqrt{A} - \sqrt{B} \right)^2,
\]
\[
1 \geq \left| \frac{1}{\sqrt{A}} - \frac{1}{\sqrt{B}} \right|.
\]

\[\square\]
P 5.70. If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1 a_2 \cdots a_n = 1$, then
\[
a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} + \frac{n^2(n-2)}{a_1 + a_2 + \cdots + a_n} \geq (n-1) \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right).
\]

**Solution.** For $n = 2$, the inequality is an identity. Consider further that $n \geq 3$. According to Corollary 5 (case $k = 0$), if $0 < a_1 \leq a_2 \leq \cdots \leq a_n$ and
\[
a_1 + a_2 + \cdots + a_n = \text{constant}, \quad a_1 a_2 \cdots a_n = 1,
\]
then the sum $a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1}$ is minimum and the sum $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$ is maximum for $0 < a_1 \leq a_2 = \cdots = a_n$. Thus, we only need to prove the homogeneous inequality
\[
a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} + \frac{n^2(n-2)a_1 a_2 \cdots a_n}{a_1 + a_2 + \cdots + a_n} \geq (n-1)a_1 a_2 \cdots a_n \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right)
\]
for $a_2 = \cdots = a_n = 1$; that is, to show that $f(x) \geq 0$ for $x \in [0, 1]$, where
\[
f(x) = x^{n-2} + \frac{n^2(n-2)}{x+n-1} - (n-1)^2,
\]
\[
f'(x) = x^{n-3} - \frac{n^2}{(x+n-1)^2}.
\]
Since $f'$ is increasing, we have $f'(x) \leq f'(1) = 0$ for $x \in [0, 1]$, $f$ is decreasing on $[0, 1]$, hence $f(x) \geq f(1) = 0$.

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $n = 2$, then the equality holds for $a_1 a_2 = 1$.  

\[\square\]

P 5.71. If $a, b, c$ are nonnegative real numbers, then
\[
(a + b + c - 3)^2 \geq \frac{abc - 1}{abc + 1} (a^2 + b^2 + c^2 - 3).
\]

*(Vasile C., 2006)*

**Solution.** For $a = 0$, the inequality reduces to
\[
b^2 + c^2 + bc + 3 \geq 3(b + c),
\]
which is equivalent to
\[
(b - c)^2 + 3(b + c - 2)^2 \geq 0.
\]
For $abc > 0$, according to Corollary 5 (case $k = 0$ and $m = 2$), if

$$a + b + c = \text{constant}, \quad abc = \text{constant},$$

then

$$S_3 = a^2 + b^2 + c^2$$

is minimum and maximum when two of $a, b, c$ are equal. Thus, we only need to prove the desired inequality for $a = b$; that is,

$$(2a + c - 3)^2 \geq \frac{a^2c - 1}{a^2c + 1}(2a^2 + c^2 - 3),$$

which is equivalent to

$$(a - 1)^2[ca^2 + 2c(c - 2)a + c^2 - 3c + 3] \geq 0.$$

For $c \geq 2$, the inequality is clearly true. It is also true for $c \leq 2$, because

$$ca^2 + 2c(c - 2)a + c^2 - 3c + 3 = c(a + c - 2)^2 + (1 - c)^2(3 - c) \geq 0.$$

The equality holds if two of $a, b, c$ are equal to 1.

\[\square\]

**P 5.72.** If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1 + a_2 + \cdots + a_n = n$, then

$$(a_1a_2 \cdots a_n)^\frac{1}{n} (a_1^2 + a_2^2 + \cdots + a_n^2) \leq n.$$  

(Vasile C., 2006)

**Solution.** For $n = 2$, the inequality is equivalent to

$$(a_1a_2 - 1)^2 \geq 0.$$  

For $n \geq 3$, according to Corollary 5 (case $k = 0$, $m = 2$), if $0 < a_1 \leq a_2 \leq \cdots \leq a_n$ and

$$a_1 + a_2 + \cdots + a_n = n, \quad a_1a_2 \cdots a_n = \text{constant},$$

then the sum

$$S_n = a_1^2 + a_2^2 + \cdots + a_n^2$$

is maximum for $a_1 = a_2 = \cdots = a_{n-1}$. Therefore, we only need to prove the homogeneous inequality

$$(a_1a_2 \cdots a_n)^\frac{1}{n} \cdot \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n} \leq \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^{2 + \frac{2}{n-1}}.$$
for $a_1 = a_2 = \cdots = a_{n-1} = 1$. The inequality is equivalent to $f(x) \geq 0$ for $x \geq 1$, where

$$f(x) = \left(2 + \frac{n}{\sqrt{n-1}}\right) \ln \frac{x + n - 1}{n} - \frac{\ln x}{\sqrt{n-1}} - \ln \frac{x^2 + n - 1}{n}.$$ 

Let

$$p = \frac{1}{\sqrt{n-1}}.$$ 

Since

$$f'(x) = \frac{2 + np}{x + n - 1} - \frac{p}{x} - \frac{2x}{x^2 + n - 1} = \frac{(n-1)(x-1)}{x + n - 1} \left(p - \frac{2}{x} \frac{p}{x^2 + n - 1}\right) = \frac{p(n-1)(x-1)(x-\sqrt{n-1})^2}{x(x + n - 1)(x^2 + n - 1)} \geq 0,$$

$f(x)$ is increasing for $x \geq 1$, hence

$$f(x) \geq f(1) = 0.$$ 

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

**Remark.** For $n = 5$, from the homogeneous inequality above, we get the following nice results:

- If $a, b, c, d, e$ are positive real numbers so that $a^2 + b^2 + c^2 + d^2 + e^2 = 5$,

then

(a) $abcde(a^4 + b^4 + c^4 + d^4 + e^4) \leq 5$;

(b) $a + b + c + d + e \geq 5 \sqrt[5]{abcde}$.

\[\square\]

**P 5.73.** If $a_1, a_2, \ldots, a_n$ are positive real numbers so that $a_1^3 + a_2^3 + \cdots + a_n^3 = n$, then

$$a_1 + a_2 + \cdots + a_n \geq n^{\frac{3}{5}} \sqrt[n]{a_1 a_2 \cdots a_n}.$$ 

(Vasile C., 2007)

**Solution.** For $n = 2$, we need to show that $a_1^3 + a_2^3 = 2$ involves $(a_1 + a_2)^3 \geq 8a_1 a_2$.

Let

$$x = a_1 + a_2.$$ 

From

$$2 = a_1^3 + a_2^3 = x^3 - 3a_1 a_2 x,$$
we get 
\[ a_1 a_2 = \frac{x^3 - 2}{3x}. \]

Thus,
\[ (a_1 + a_2)^3 - 8a_1 a_2 = x^3 - \frac{8(x^3 - 2)}{3x} = \frac{(x^2 - 2)(3x^2 + 4x + 4)}{3x} \geq 0. \]

For \( n \geq 3 \), according to Corollary 4, if \( 0 < a_1 \leq a_2 \leq \cdots \leq a_n \) and
\[ a_1 + a_2 + \cdots + a_n = \text{constant}, \quad a_1^3 + a_2^3 + \cdots + a_n^3 = n, \]
then the product
\[ P = a_1 a_2 \cdots a_n \]
is maximum for \( a_1 = a_2 = \cdots = a_{n-1} = 1 \). Therefore, we only need to prove the homogeneous inequality
\[ \left( \frac{a_1 + a_2 + \cdots + a_n}{n} \right)^{n+1} \geq a_1 a_2 \cdots a_n \sqrt[3]{\frac{a_1^3 + a_2^3 + \cdots + a_n^3}{n}} \]
for \( a_1 = a_2 = \cdots = a_{n-1} = 1 \). The inequality is equivalent to \( f(x) \geq 0 \) for \( x \geq 1 \), where
\[ f(x) = (n + 1) \ln \frac{x + n - 1}{n} - \ln x - \frac{1}{3} \ln \frac{x^3 + n - 1}{n}. \]
Since
\[ f'(x) = \frac{n + 1}{x + n - 1} - \frac{1}{x} - \frac{x^2}{x^3 + n - 1} = \frac{(n - 1)(x - 1)(x^3 - x^2 - x + 1)}{x(x + n - 1)(x^3 + n - 1)} \geq \frac{(n - 1)(x - 1)(x^3 - x^2 - x + 1)}{x(x + n - 1)(x^3 + n - 1)} = \frac{(n - 1)(x - 1)^3(x + 1)}{x(x + n - 1)(x^3 + n - 1)}, \]
\( f(x) \) is increasing for \( x \geq 1 \), hence
\[ f(x) \geq f(1) = 0. \]

The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \).
P 5.74. Let \( a, b, c \) be nonnegative real numbers so that \( ab + bc + ca = 3 \). If

\[
k \geq 2 - \frac{\ln 4}{\ln 3} \approx 0.738,
\]

then

\[
a^k + b^k + c^k \geq 3.
\]

\[(Vasile C., 2004)\]

Solution. Let

\[
r = 2 - \frac{\ln 4}{\ln 3}.
\]

By the power mean inequality, we have

\[
\frac{a^k + b^k + c^k}{3} \geq \left( \frac{a^r + b^r + c^r}{3} \right)^{k/r}.
\]

Thus, it suffices to show that

\[
a^r + b^r + c^r \geq 3.
\]

Since

\[
2(ab + bc + ca) = (a + b + c)^2 - (a^2 + b^2 + c^2),
\]

according to Corollary 5 (case \( k = 2, m = r \), if \( a \leq b \leq c \) and

\[
a + b + c = \text{constant}, \quad a^2 + b^2 + c^2 = \text{constant},
\]

then

\[
S_3 = a^r + b^r + c^r
\]

is minimum for either \( a = 0 \) or \( 0 < a \leq b = c \).

Case 1: \( a = 0 \). We need to show that \( bc = 3 \) implies \( b^r + c^r \geq 3 \). Indeed, by the AM-GM inequality, we have

\[
b^r + c^r \geq 2\sqrt{(bc)^r} = 2 \cdot 3^{r/2} = 3.
\]

Case 2: \( 0 < a \leq b = c \). We only need to show that the homogeneous inequality

\[
a^r + b^r + c^r \geq 3 \left( \frac{ab + bc + ca}{3} \right)^{r/2}
\]

holds for \( b = c = 1 \); that is, to show that \( a \in (0, 1] \) involves

\[
a^r + 2 \geq 3 \left( \frac{2a + 1}{3} \right)^{r/2},
\]
which is equivalent to $f(a) \geq 0$, where

$$f(a) = \ln \frac{a^r + 2}{3} - \frac{r}{2} \ln \frac{2a + 1}{3}.$$  

The derivative

$$f'(a) = \frac{ra^{r-1}}{a^r + 2} - \frac{r}{2a + 1} = \frac{rg(a)}{a^{1-r}(a^r + 2)(2a + 1)},$$

where

$$g(a) = a - 2a^{1-r} + 1.$$  

From

$$g'(a) = 1 - \frac{2(1-r)}{a^r},$$  

it follows that $g'(a) < 0$ for $a \in (0, a_1)$, and $g'(a) > 0$ for $a \in (a_1, 1)$, where

$$a_1 = (2 - 2r)^{1/r} \approx 0.416.$$  

Then, $g$ is strictly decreasing on $[0, a_1]$ and strictly increasing on $[a_1, 1]$. Since $g(0) = 1$ and $g(1) = 0$, there exists $a_2 \in (0, 1)$ so that $g(a_2) = 0$, $g(a) > 0$ for $a \in [0, a_2)$, and $g(a) < 0$ for $a \in (a_2, 1]$. Consequently, $f$ is increasing on $[0, a_2]$ and decreasing on $[a_2, 1]$. Since $f(0) = f(1) = 0$, we have $f(a) \geq 0$ for $0 < a \leq 1$.

The equality holds for $a = b = c = 1$. If $k = 2 - \frac{\ln 4}{\ln 3}$, then the equality holds also for

$$a = 0, \quad b = c = \sqrt{3}$$

(or any cyclic permutation).

**Remark.** For $k = 3/4$, we get the following nice results (see P 3.33 in Volume 1):

- Let $a, b, c$ be positive real numbers.
  - (a) If $a^4b^4 + b^4c^4 + c^4a^4 = 3$, then
    $$a^3 + b^3 + c^3 \geq 3.$$  
  - (b) If $a^3 + b^3 + c^3 = 3$, then
    $$a^4b^4 + b^4c^4 + c^4a^4 \leq 3.$$  

\[\square\]

**P 5.75.** Let $a, b, c$ be nonnegative real numbers so that $a + b + c = 3$. If

$$k \geq \frac{\ln 9 - \ln 8}{\ln 3 - \ln 2} \approx 0.29,$$  

then

$$a^k + b^k + c^k \geq ab + bc + ca.$$  

*(Vasile C., 2005)*
**Solution.** For $k \geq 1$, by Jensen’s inequality, we get

$$a^k + b^k + c^k \geq 3 \left(\frac{a + b + c}{3}\right)^k = 3 = \frac{1}{3}(a + b + c)^2 \geq ab + bc + ca.$$ 

Let

$$r = \frac{\ln 9 - \ln 8}{\ln 3 - \ln 2}.$$ 

Assume further that

$$r \leq k < 1,$$

and write the inequality as

$$2(a^k + b^k + c^k) + a^2 + b^2 + c^2 \geq 9.$$ 

By Corollary 5, if $a \leq b \leq c$ and

$$a + b + c = 3, \quad a^2 + b^2 + c^2 = \text{constant},$$

then the sum

$$S_3 = a^k + b^k + c^k$$

is minimum for either $a = 0$ or $0 < a \leq b = c$. Thus, we only need to prove the desired inequality for these cases.

**Case 1:** $a = 0$. We need to show that $b + c = 3$ involves $b^k + c^k \geq bc$. Indeed, by the AM-GM inequality, we have

$$b^k + c^k - bc \geq 2(bc)^{k/2} - bc = (bc)^{k/2} \left[2 - (bc)^{1-k/2}\right] \geq (bc)^{k/2} \left[2 - \left(\frac{b + c}{2}\right)^{2-k}\right] = (bc)^{k/2} \left[2 - \left(\frac{3}{2}\right)^{2-r}\right] = 0.$$ 

**Case 2:** $0 < a \leq b = c$. We only need to show that the homogeneous inequality

$$(a^k + b^k + c^k) \left(\frac{a + b + c}{3}\right)^{2-k} \geq ab + bc + ca$$

holds for $b = c = 1$; that is, to show that $a \in (0, 1]$ involves

$$(a^k + 2) \left(\frac{a + 2}{3}\right)^{2-k} \geq 2a + 1,$$

which is equivalent to $f(a) \geq 0$, where

$$f(a) = \ln(a^k + 2) + (2-k)\ln \frac{a + 2}{3} - \ln(2a + 1).$$
We have
\[ f'(a) = \frac{ka^{k-1}}{a^k + 2} + \frac{2 - k}{2a + 1} = \frac{2g(a)}{a^{1-k}(a^k + 2)(2a + 1)}, \]
where
\[ g(a) = a^2 + (2k - 1)a + k + 2(1 - k)a^{2-k} - (k + 2)a^{1-k}, \]
with
\[ g'(a) = 2a + 2k - 1 + 2(1 - k)(2 - k)a^{1-k} - (k + 2)(1 - k)a^{-k}, \]
\[ g''(a) = 2 + 2(1 - k)^2(2 - k)a^{-k} + k(k + 2)(1 - k)a^{-k-1}. \]
Since \( g'' > 0 \), \( g' \) is strictly increasing. From \( g'(0) = -\infty \) and \( g'(1) = 3(1 - k) + 3k^2 > 0 \), it follows that there exists \( a_1 \in (0,1) \) so that \( g'(a_1) = 0 \), \( g'(a) < 0 \) for \( a \in (0,a_1) \) and \( g'(a) > 0 \) for \( a \in (a_1,1) \). Therefore, \( g \) is strictly decreasing on \([0,a_1]\) and strictly increasing on \([a_1,1]\). Since \( g(0) = k > 0 \) and \( g(1) = 0 \), there exists \( a_2 \in (0,a_1) \) so that \( g(a_2) = 0 \), \( g(a) > 0 \) for \( a \in [0,a_2] \) and \( g(a) < 0 \) for \( a \in (a_2,1] \). Consequently, \( f \) is increasing on \([0,a_2]\) and decreasing on \([a_2,1]\).

Since
\[ f(0) = \ln 2 + (3 - k)\ln \frac{2}{3} \geq \ln 2 + (3 - r)\ln \frac{2}{3} = 0 \]
and \( f(1) = 0 \), we get \( f(a) \geq 0 \) for \( 0 \leq a \leq 1 \).

The equality holds for \( a = b = c = 1 \). If \( k = \frac{\ln 9 - \ln 8}{\ln 3 - \ln 2} \), then the equality holds also for
\[ a = 0, \quad b = c = \frac{3}{2} \]
(or any cyclic permutation).

\[ \Box \]

**P 5.76.** If \( a_1, a_2, \ldots, a_n \ (n \geq 4) \) are nonnegative numbers so that \( a_1 + a_2 + \cdots + a_n = n \), then
\[ \frac{1}{n + 1 - a_2a_3 \cdots a_n} + \frac{1}{n + 1 - a_3a_4 \cdots a_1} + \cdots + \frac{1}{n + 1 - a_1a_2 \cdots a_{n-1}} \leq 1. \]

\[ \text{(Vasile C., 2004)} \]

**Solution.** Let \( a_1 \leq a_2 \leq \cdots \leq a_n \) and
\[ e_{n-1} = \left(1 + \frac{1}{n-1}\right)^{n-1}. \]

By the AM-GM inequality, we have
\[ a_2a_3 \cdots a_n \leq \left( \frac{a_2 + a_3 + \cdots + a_n}{n-1} \right)^{n-1} \leq \left( \frac{a_1 + a_2 + \cdots + a_n}{n-1} \right)^{n-1} = e_{n-1}, \]
hence
\[ n + 1 - a_2 a_3 \cdots a_n \geq n + 1 - e_{n-1} = (n - 2) + (3 - e_{n-1}) > 0. \]
Consider the cases \( a_1 = 0 \) and \( a_1 > 0 \).

**Case 1**: \( a_1 = 0 \). We need to show that \( a_2 + a_3 + \cdots + a_n = n \) involves

\[ \frac{1}{n + 1 - a_2 a_3 \cdots a_n} + \frac{n - 1}{n + 1} \leq 1, \]
which is equivalent to

\[ a_2 a_3 \cdots a_n \leq \frac{n + 1}{2}. \]

Since

\[ a_2 a_3 \cdots a_n \leq \left( \frac{a_2 + a_3 + \cdots + a_n}{n - 1} \right)^{n-1} = e_{n-1}, \]
it suffices to show that

\[ e_{n-1} \leq \frac{n + 1}{2}. \]

For \( n = 4 \), we have

\[ \frac{n + 1}{2} - e_{n-1} = \frac{7}{54} > 0. \]

For \( n \geq 5 \), we get

\[ \frac{n + 1}{2} \geq 3 > e_{n-1}. \]

**Case 2**: \( 0 < a_1 \leq a_2 \leq \cdots \leq a_n \). Denote

\[ a_1 a_2 \cdots a_n = (n + 1)r, \quad r > 0. \]

From \( a_2 a_3 \cdots a_n \leq e_{n-1} \), we get

\[ a_1 \geq a, \quad a = \frac{(n + 1)r}{e_{n-1}} > r. \]

Write the inequality as follows

\[ \frac{a_1}{a_1 - r} + \frac{a_2}{a_2 - r} + \cdots + \frac{a_n}{a_n - r} \leq n + 1, \]

\[ \frac{1}{a_1 - r} + \frac{1}{a_2 - r} + \cdots + \frac{1}{a_n - r} \leq \frac{1}{r}, \]

\[ f(a_1) + f(a_2) + \cdots + f(a_n) + \frac{1}{r} \geq 0, \]

where

\[ f(u) = \frac{-1}{u - r}, \quad u \geq a. \]
We will apply Corollary 3 to the function $f$. We have

$$f'(u) = \frac{1}{(u-r)^2},$$

$$g(x) = f'(\frac{1}{x}) = \frac{x^2}{(1-rx)^2}, \quad g''(x) = \frac{4rx + 2}{(1-rx)^4}.$$

Since $g'' > 0$ for $\frac{1}{x} \geq a$, $g$ is strictly convex on $[0, \frac{1}{a}]$. According to Corollary 3 and Note 5/Note 3, if $a \leq a_1 \leq a_2 \leq \cdots \leq a_n$ and

$$a_1 + a_2 + \cdots + a_n = n, \quad a_1 a_2 \cdots a_n = (n+1)r = \text{constant},$$

then the sum $S_n = f(a_1) + f(a_2) + \cdots + f(a_n)$ is minimum for $a \leq a_1 \leq a_2 = \cdots = a_n$. Thus, we only need to prove the homogeneous inequality

$$\frac{1}{n+1 - \frac{a_2 \cdots a_n}{s^{n-1}}} + \frac{1}{n+1 - \frac{a_3 a_4 \cdots a_1}{s^{n-1}}} + \cdots + \frac{1}{n+1 - \frac{a_1 a_2 \cdots a_{n-1}}{s^{n-1}}} \leq 1$$

for $0 < a_1 \leq a_2 = a_3 = \cdots = a_n = 1$, where

$$s = \frac{a_1 + a_2 + \cdots + a_n}{n},$$

that is,

$$\frac{s^{n-1}}{(n+1)s^{n-1} - 1} + \frac{(n-1)s^{n-1}}{(n+1)s^{n-1} - a_1} \leq 1, \quad s = \frac{a_1 + n - 1}{n},$$

which is equivalent to

$$f(s) \geq 0, \quad s_1 < s \leq 1,$$

where $s_1 = \frac{n-1}{n}$ and

$$f(s) = (n+1)s^{2n-2} - n^2 s^n + (n+1)(n-2)s^{n-1} + ns - n + 1.$$

We have

$$f'(s) = 2(n^2 - 1)s^{2n-3} - n^3 s^{n-1} + (n^2 - 1)(n-2)s^{n-2} + n,$$

$$f''(s) = (n-1)s^{n-3}g(s),$$

where

$$g(s) = 2(2n-3)(n+1)s^{n-1} - n^3 s + (n-2)^2(n+1),$$

$$g'(s) = 2(2n-3)(n^2 - 1)s^{n-2} - n^3.$$

Since

$$g'(s) \geq g'(s_1) = \frac{2n(2n-3)(n+1)}{3} - n^3$$

$$> \frac{2n(2n-3)(n+1)}{3} - n^3 = \frac{n(n^2-2n-6)}{3} > 0,$$
g is increasing. There are two cases to consider: \( g(s_1) \geq 0 \) and \( g(s_1) < 0 \).

**Subcase A:** \( g(s_1) \geq 0 \). Then, \( g(s) \geq 0, f''(s) \geq 0, f' \) is increasing. Since \( f'(1) = 0 \), it follows that \( f'(s) \leq 0 \) for \( s \in [s_1, 1] \), \( f \) is decreasing, hence \( f(s) \geq f(1) = 0 \).

**Subcase B:** \( g(s_1) < 0 \). Then, since \( g(1) = n^2 - 2n + 4 > 0 \), there exists \( s_2 \in (s_1, 1) \) so that \( g(s_2) = 0, g(s) < 0 \) for \( s \in [s_1, s_2) \) and \( g(s) > 0 \) for \( s \in (s_2, 1] \), \( f' \) is decreasing on \( [s_1, s_2] \) and increasing on \( [s_2, 1] \). We see that \( f'(1) = 0 \). If \( f'(s_1) \leq 0 \), then \( f'(s) \leq 0 \) for \( s \in [s_1, 1] \), \( f \) is decreasing, hence \( f(s) \geq f(1) = 0 \). If \( f'(s_1) > 0 \), then there exists \( s_3 \in (s_1, s_2) \) so that \( f'(s_3) = 0, f'(s) > 0 \) for \( s \in [s_1, s_3) \) and \( g(s) < 0 \) for \( s \in (s_3, 1] \), hence \( f \) is increasing on \( [s_1, s_3] \) and decreasing on \( [s_3, 1] \). Since \( f(1) = 0 \), it suffices to show that \( f(s_1) \geq 0 \). This is true since \( s = s_1 \) involves \( a_1 = 0 \), and we have shown that the desired inequality holds for \( a_1 = 0 \).

The equality occurs for \( a_1 = a_2 = \cdots = a_n = 1 \).

\[ \square \]

**P 5.77.** If \( a, b, c \) are nonnegative real numbers so that
\[
a + b + c \geq 2, \quad ab + bc + ca \geq 1,
\]
then
\[
\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} \geq 2.
\]

(Vasile C., 2005)

**Solution.** According to Corollary 5 (case \( k = 2 \) and \( m = 1/3 \)), if \( 0 \leq a \leq b \leq c \) and
\[
a + b + c = \text{constant}, \quad ab + bc + ca = \text{constant},
\]
then the sum \( S_3 = \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} \) is minimum for either \( a = 0 \) or \( 0 < a \leq b = c \).

**Case 1:** \( a = 0 \). The hypothesis \( ab + bc + ca \geq 1 \) implies \( bc \geq 1 \); consequently,
\[
\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} = \sqrt[3]{b} + \sqrt[3]{c} \geq 2\sqrt[3]{bc} \geq 2.
\]

**Case 2:** \( 0 < a \leq b = c \). If \( c \geq 1 \), then
\[
\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} \geq 2\sqrt[3]{c} \geq 2.
\]

If \( c < 1 \), then
\[
\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} \geq a + b + c \geq 2.
\]

The equality holds for \( a = 0, b = c = 1 \)

(or any cyclic permutation).

\[ \square \]
**P 5.78.** If \( a, b, c, d \) are positive real numbers so that \( abcd = 1 \), then

\[
(a + b + c + d)^4 \geq 36\sqrt{3} \left( a^2 + b^2 + c^2 + d^2 \right).
\]

(Vasile C., 2008)

**Solution.** According to Corollary 5 (case \( k = 0 \) and \( m = 2 \)), if \( a \leq b \leq c \leq d \) and

\[
a + b + c + d = \text{constant}, \quad abcd = 1,
\]

then the sum

\[
S_4 = a^2 + b^2 + c^2 + d^2
\]

is maximum for \( a = b = c \leq d \). Thus, we only need to show that

\[
(3a + d)^4 \geq 36\sqrt{3} \left( 3a^2 + d^2 \right)
\]

for \( a^3 d = 1 \). Write this inequality as \( f(a) \geq 0 \), where

\[
f(a) = 4\ln\left( 3a + \frac{1}{a^3} \right) - \ln\left( 3a^2 + \frac{1}{a^6} \right) - \ln 36\sqrt{3}, \quad 0 < a \leq 1.
\]

Since

\[
f'(a) = \frac{12(a^4 - 1)}{a(3a^4 + 1)} - \frac{6(a^8 - 1)}{a(3a^8 + 1)} = \frac{6(a^4 - 1)^2(3a^4 - 1)}{a(3a^8 + 1)(3a^4 + 1)},
\]

\( f \) is decreasing on \([0, 1/\sqrt{3}]\) and increasing on \([1/\sqrt{3}, 1]\); therefore,

\[
f(a) \geq f\left( \frac{1}{\sqrt{3}} \right) = 0.
\]

The equality holds for

\[
a = b = c = \frac{1}{\sqrt{3}}, \quad d = \sqrt{27}
\]

(or any cyclic permutation).

**Remark.** In the same manner, we can prove the following generalization:

- If \( a_1, a_2, \ldots, a_n \) are positive real numbers so that \( a_1a_2\cdots a_n = 1 \), then

\[
(a_1 + a_2 + \cdots + a_n)^4 \geq \frac{16}{n} \sqrt{(n-1)^{3n-2}} \left( a_1^2 + a_2^2 + \cdots + a_n^2 \right),
\]

with equality for

\[
a_1 = a_2 = \cdots = a_{n-1} = \frac{1}{\sqrt{n-1}}, \quad a_n = \sqrt{(n-1)^{n-1}}
\]

(or any cyclic permutation).
**P 5.79.** If $a, b, c, d$ are nonnegative real numbers, then
\[
\left( \sum_{\text{sym}} a b \right) \left( \sum_{\text{sym}} a^2 b^2 \right) \geq 9 \sum a^2 b^2 c^2.
\]

*(Vasile C., 2007)*

**Solution.** Consider $a \leq b \leq c$.

For $a = 0$, the inequality reduces to
\[
(bc + cd + db)(b^2c^2 + c^2d^2 + d^2b^2) \geq 9b^2c^2d^2.
\]

We get this inequality by multiplying the AM-GM inequalities
\[
bc + cd + db \geq 3\sqrt[3]{a^2b^2c^2},
\]
\[
b^2c^2 + c^2d^2 + d^2b^2 \geq 3\sqrt[3]{a^4b^4c^4}.
\]

For $a > 0$, replacing $a, b, c$ by $1/a, 1/b, 1/c$, the inequality becomes
\[
\left( \sum_{\text{sym}} a b \right) \left( \sum_{\text{sym}} a^2 b^2 \right) \geq 9abcd \sum a^2.
\]

Due to homogeneity, we may consider
\[
a^2 + b^2 + c^2 + d^2 = 1.
\]

Since
\[
2 \sum_{\text{sym}} ab = \left( \sum a \right)^2 - \sum a^2 = \left( \sum a \right)^2 - 1
\]
and
\[
2 \sum_{\text{sym}} a^2 b^2 = \left( \sum a^2 \right)^2 - \sum a^4 = 1 - \sum a^4,
\]

the inequality can be written as
\[
\left[ \left( \sum a \right)^2 - 1 \right] \left( 1 - \sum a^4 \right) \geq 9abcd.
\]

By Corollary 5 (case $k = 2$ and $m = 4$), if $0 < a \leq b \leq c \leq d$ and
\[
a + b + c + d = \text{constant}, \quad a^2 + b^2 + c^2 + d^2 = 1,
\]
then the sum $a^4 + b^4 + c^4 + d^4$ is maximum for $a = b = c \leq d$. By Corollary 4, if $0 < a \leq b \leq c \leq d$ and
\[
a + b + c + d = \text{constant}, \quad a^2 + b^2 + c^2 + d^2 = 1,
\]
then the product $abcd$ is maximum for $a = b = c = d$. Thus, we only need to prove the homogeneous inequality for $a = b = c = 1$; that is,

$$(3d + 3)(3d^2 + 3) \geq 9d(d^2 + 3),$$

$$9(d - 1)^2 \geq 0.$$  

The equality holds for $a = b = c = d = 1$, and also for

$$a = 0, \quad b = c = d$$

(or any cyclic permutation).

\[ \Box \]

**P 5.80.** If $a, b, c$ are nonnegative real numbers so that $ab + bc + ca = 1$, then

$$\sqrt{33a^2 + 16} + \sqrt{33b^2 + 16} + \sqrt{33c^2 + 16} \leq 9(a + b + c).$$

\[ (Vasile C., 2006) \]

**Solution.** Write the inequality as

$$f(a) + f(b) + f(c) + 297(a + b + c) \geq 0,$$

where

$$f(u) = \frac{1}{33} \sqrt{33u^2 + 16}, \quad u \geq 0.$$  

We have

$$g(x) = f'(x) = \frac{-x}{\sqrt{33x^2 + 16}},$$

$$g''(x) = \frac{33 \cdot 48x}{(33x^2 + 16)^{5/2}}.$$  

Since $g''(x) > 0$ for $x > 0$, $g$ is strictly convex on $[0, \infty)$. According to Corollary 1, if $0 \leq a \leq b \leq c$ and

$$a + b + c = constant, \quad a^2 + b^2 + c^2 = constant,$$

then the sum

$$S_n = f(a) + f(b) + f(c)$$

is minimum for either $a = 0$ or $0 < a \leq b = c$.

**Case 1:** $a = 0$. We need to show that $bc = 1$ involves

$$\sqrt{33b^2 + 16} + \sqrt{33c^2 + 16} \leq 9(b + c) - 4.$$
We see that
\[9(b + c) - 4 \geq 18\sqrt{bc} - 4 = 14 > 0.\]
By squaring, the inequality becomes
\[\sqrt{528t^2 + 289} \leq 24t^2 - 36t + 25,\]
where
\[t = b + c \geq 2.\]
Since
\[24t^2 - 36t + 25 \geq 6t^2 + 25,\]
it suffices to show that
\[528t^2 + 289 \leq (6t^2 + 25)^2,\]
which is equivalent to
\[(t^2 - 4)(3t^2 - 7) \geq 0.\]

Case 2: \(0 < a \leq b = c\). Write the inequality in the homogeneous form
\[\sum \sqrt{33a^2 + 16(ab + bc + ca)} \leq 9(a + b + c).\]
Without loss of generality, assume that \(b = c = 1\), when the inequality becomes
\[\sqrt{33a^2 + 32a + 16 + 2\sqrt{32a + 49}} \leq 9a + 18.\]
By squaring twice, the inequality turns into
\[\sqrt{(33a^2 + 32a + 16)(32a + 49)} \leq 12a^2 + 41a + 28,\]
\[72a(2a^3 - a^2 - 4a + 3) \geq 0,\]
\[72a(a - 1)^2(2a + 3) \geq 0.\]

The equality holds for \(a = b = c = \frac{1}{\sqrt{3}}\), and also for
\[a = 0, \quad b = c = 1\]
(or any cyclic permutation).

\[
\text{P 5.81. If } a, b, c \text{ are positive real numbers so that } a + b + c = 3, \text{ then} \\
a^2b^2 + b^2c^2 + c^2a^2 \leq \frac{3}{\sqrt{abc}}. \\
(Vasile C., 2006)
\]
**Solution.** Write the inequality in the homogeneous form

\[
\left( \frac{a + b + c}{3} \right)^{15} \geq abc \left( \frac{a^2 b^2 + b^2 c^2 + c^2 a^2}{3} \right)^{\frac{3}{2}}.
\]

Since

\[
a^2 b^2 + b^2 c^2 + c^2 a^2 = (ab + bc + ca)^2 - 2abc(a + b + c)
\]

\[
= \frac{1}{4} (9 - a^2 - b^2 - c^2) - 6abc,
\]

we will apply Corollary 5 (case \(k = 0\) and \(m = 2\)):
- If \(0 \leq a \leq b \leq c\) and

\[
a + b + c = 3, \quad abc = \text{constant},
\]

then the sum

\[
S_3 = a^2 + b^2 + c^2
\]

is minimal for \(0 < a \leq b = c\).

Therefore, we only need to prove the homogeneous inequality for \(0 < a \leq 1\) and \(b = c = 1\). Taking logarithms, we have to show that \(f(a) \geq 0\), where

\[
f(a) = 15 \ln \frac{a + 2}{3} - \ln a - 3 \ln \frac{2a^2 + 1}{3}.
\]

Since the derivative

\[
f'(a) = \frac{15}{a + 2} - \frac{1}{a} - \frac{12a}{2a^2 + 1} = \frac{2(a - 1)(2a - 1)(4a - 1)}{a(a + 2)(2a^2 + 1)}
\]

is negative for \(a \in \left(0, \frac{1}{4}\right) \cup \left(\frac{1}{2}, 1\right)\) and positive for \(a \in \left(\frac{1}{4}, \frac{1}{2}\right)\), \(f\) is decreasing on \(\left(0, \frac{1}{4}\right] \cup \left[\frac{1}{2}, 1\right]\) and increasing on \(\left[\frac{1}{4}, \frac{1}{2}\right]\). Therefore, it suffices to show that \(f\left(\frac{1}{4}\right) \geq 0\) and \(f(1) \geq 0\). Indeed, we have \(f(1) = 0\) and

\[
f\left(\frac{1}{4}\right) = \ln \frac{3^{12}}{2^{19}} > 0.
\]

The equality holds for \(a = b = c = 1\). \(\square\)
P 5.82. If \( a_1, a_2, \ldots, a_n \) \( (n \leq 81) \) are nonnegative real numbers so that
\[
a_1^2 + a_2^2 + \cdots + a_n^2 = a_1^5 + a_2^5 + \cdots + a_n^5,
\]
then
\[
a_1^6 + a_2^6 + \cdots + a_n^6 \leq n.
\]

(Vasile C., 2006)

Solution. Setting \( a_n = 1 \), we obtain the statement for \( n - 1 \) numbers \( a_i \). Consequently, it suffices to prove the inequality for \( n = 81 \). We need to show that the following homogeneous inequality holds:
\[
81(a_1^5 + a_2^5 + \cdots + a_{81}^5)^2 \geq (a_1^6 + a_2^6 + \cdots + a_{81}^6)(a_1^2 + a_2^2 + \cdots + a_{81}^2)^2.
\]
According to Corollary 5 (case \( k = 3 \) and \( m = 5/2 \)), if \( 0 \leq a_1 \leq a_2 \leq \cdots \leq a_{81} \) and
\[
a_1^2 + a_2^2 + \cdots + a_{81}^2 = \text{constant}, \quad a_1^6 + a_2^6 + \cdots + a_{81}^6 = \text{constant},
\]
then the sum \( a_1^5 + a_2^5 + \cdots + a_{81}^5 \) is minimum for \( a_1 = a_2 = \cdots = a_{80} \leq a_{81} \). Therefore, we only need to prove the homogeneous inequality for \( a_1 = a_2 = \cdots = a_{80} = 0 \) and for \( a_1 = a_2 = \cdots = a_{80} = 1 \). The first case is trivial. In the second case, denoting \( a_{81} \) by \( x \), the homogeneous inequality becomes as follows:
\[
81(80 + x^5)^2 \geq (80 + x^6)(80 + x^2)^2,
\]
\[
x^{10} - 2x^8 - 80x^6 + 162x^5 - x^4 - 160x^2 + 80 \geq 0,
\]
\[
(x - 1)^2(x - 2)^2(x^6 + 6x^5 + 21x^4 + 60x^3 + 75x^2 + 60x + 20) \geq 0.
\]
Thus, the proof is completed. The equality holds for \( a_1 = a_2 = \cdots = a_n = 1 \). If \( n = 81 \), then the equality holds also for
\[
a_1 = a_2 = \cdots = a_{80} = \frac{a_{81}}{2} = \sqrt[3]{\frac{3}{4}}
\]
(or any cyclic permutation).

P 5.83. If \( a, b, c \) are nonnegative real numbers so that \( a + b + c = 3 \), then
\[
1 + \sqrt{1 + a^3 + b^3 + c^3} \geq \sqrt{3(a^2 + b^2 + c^2)}.
\]

(Vasile C., 2006)
Solution. Write the inequality as

\[ \sqrt{1 + a^3 + b^3 + c^3} \geq \sqrt{3(a^2 + b^2 + c^2)} - 1. \]

By squaring, we may rewrite the inequality in the homogeneous form

\[ a^3 + b^3 + c^3 + 2\left( \frac{a + b + c}{3} \right)^2 \sqrt{3(a^2 + b^2 + c^2)} \geq (a + b + c)(a^2 + b^2 + c^2). \]

According to Corollary 5 (case \( k = 2 \) and \( m = 3 \), if \( 0 \leq a \leq b \leq c \) and\)

\[ a + b + c = \text{constant}, \quad a^2 + b^2 + c^2 = \text{constant}, \]

then the sum

\[ S_3 = a^3 + b^3 + c^3 \]

is minimum for either \( a = 0 \) or \( 0 < a \leq b = c \). Thus, we only need to prove the homogeneous inequality for \( a = 0 \) and for \( b = c = 1 \).

Case 1: \( a = 0 \). We need to show that

\[ b^3 + c^3 + 2\left( \frac{b + c}{3} \right)^2 \sqrt{3(b^2 + c^2)} \geq (b + c)(b^2 + c^2). \]

Simplifying by \( b + c \), it remains to show that

\[ (b + c)\sqrt{b^2 + c^2} \geq \frac{3\sqrt{3}}{2} bc. \]

Indeed,

\[ (b + c)\sqrt{b^2 + c^2} \geq \left( 2\sqrt{bc} \right) \sqrt{2bc} \geq \frac{3\sqrt{3}}{2} bc. \]

Case 2: \( b = c = 1 \). We need to prove that

\[ (a + 2)^2 \sqrt{3(a^2 + 2)} \geq 9(a^2 + a + 1). \]

By squaring, the inequality becomes

\[ a^6 + 8a^5 - a^4 - 6a^3 - 17a^2 + 10a + 5 \geq 0, \]

\[ (a - 1)^2(a^4 + 10a^3 + 18a^2 + 20a + 5) \geq 0. \]

The equality holds for \( a = b = c = 1 \). \( \square \)
P 5.84. If $a, b, c$ are nonnegative real numbers so that $a + b + c = 3$, then

$$
\sqrt{a + b} + \sqrt{b + c} + \sqrt{c + a} \leq \sqrt{16 + \frac{2}{3}(ab + bc + ca)}.
$$

(Lorian Saceanu, 2017)

Solution. Write the inequality in the form

$$
f(a) + f(b) + f(c) + \sqrt{16 + \frac{2}{3}(ab + bc + ca)} \geq 0,
$$

where

$$
f(u) = -\sqrt{3} - u, \quad 0 \leq u \leq 3.
$$

We have

$$
g(x) = f'(x) = \frac{1}{2\sqrt{3 - x}},
$$

$$
g''(x) = \frac{3}{8}(3 - x)^{-5/2}.
$$

Since $g''(x) > 0$ for $x \in [0, 3)$, $g$ is strictly convex on $[0, 3]$. According to Corollary 1 and Note 5/Note 2, if $0 \leq a \leq b \leq c$ and

$$
a + b + c = 3, \quad ab + bc + ca = \text{constant},
$$

then the sum $S_3 = f(a) + f(b) + f(c)$ is minimum for either $a = 0$ or $0 < a \leq b = c$.

Therefore, we only need to prove the homogeneous inequality

$$
\sqrt{a + b} + \sqrt{b + c} + \sqrt{c + a} \leq \sqrt{16 + \frac{2(ab + bc + ca)}{a + b + c}}
$$

for $a = 0$ and $b = c = 1$.

Case 1: $a = 0$. We need to show that

$$
\sqrt{b} + \sqrt{c} + \sqrt{b + c} \leq \sqrt{16 + \frac{2bc}{b + c}}.
$$

Consider the nontrivial case $b, c > 0$, use the substitution

$$
x = \sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}}, \quad x \geq 2,
$$

and write the inequality as

$$
\sqrt{b + c} + 2\sqrt{bc} + \sqrt{b + c} \leq \sqrt{16 + \frac{2bc}{b + c}}.
$$
\[ \sqrt{x + 2} + \sqrt{x} \leq \sqrt{\frac{16}{3} x + \frac{2}{x}}. \]

By squaring twice, the inequality becomes as follows:

\[ \sqrt{x(x + 2)} \leq \frac{5}{3} x - 1 + \frac{1}{x}, \]

\[ 16x^4 - 48x^3 + 39x^2 - 18x + 9 \geq 0, \]

\[ (x - 2)[16x^2(x - 1) + 7x - 4] + 1 \geq 0. \]

**Case 2:** \( b = c = 1 \). We need to prove that

\[ 2 \sqrt{a + 1} + \sqrt{2} \leq \sqrt{\frac{16}{3}(a + 2) + \frac{2(2a + 1)}{a + 2}}. \]

By squaring twice, the inequality becomes as follows:

\[ 6(a + 2)\sqrt{2(a + 1)} \leq 2a^2 + 17a + 17, \]

\[ 4a^4 - 4a^3 - 3a^2 + 2a + 1 \geq 0, \]

\[ (a - 1)^2(2a + 1)^2 \geq 0. \]

The equality holds for \( a = b = c = 1 \).

\[ \square \]

**P 5.85.** If \( a, b, c \) are positive real numbers so that \( abc = 1 \), then

(a) \[ \frac{a + b + c}{3} \geq \sqrt{\frac{2 + a^2 + b^2 + c^2}{5}}; \]

(b) \[ a^3 + b^3 + c^3 \geq \sqrt{3(a^4 + b^4 + c^4)}. \]

(Vasile C., 2006)

**Solution.** (a) According to Corollary 5 (case \( k = 0 \) and \( m = 2 \)), if \( a \leq b \leq c \) and

\[ a + b + c = \text{constant}, \quad abc = 1, \]

the sum \( S_3 = a^2 + b^2 + c^2 \) is maximum for \( 0 < a = b \leq c \). Thus, we only need to show that \( a^2c = 1 \) involves

\[ \frac{2a + c}{3} \geq \sqrt{\frac{2 + 2a^2 + c^2}{5}}, \]

which is equivalent to

\[ 5 \left(2a + \frac{1}{a^2}\right)^3 \geq 27 \left(2 + 2a^2 + \frac{1}{a^4}\right). \]
\[40a^9 - 54a^8 + 6a^6 + 30a^3 - 27a^2 + 5 \geq 0,\]

\[(a - 1)^2(40a^7 + 26a^6 + 12a^5 + 4a^4 - 4a^3 - 12a^2 + 10a + 5) \geq 0.\]

The inequality is true since

\[12a^5 + 4a^4 - 4a^3 - 12a^2 + 10a + 5 > 2a^5 + 4a^4 - 4a^3 - 12a^2 + 10a\]

\[= 2a(a - 1)^2(a^2 + 4a + 5) \geq 0.\]

The equality holds for \(a = b = c = 1.\)

(b) According to Corollary 5 (case \(k = 0\) and \(m = 4/3\)), if \(a \leq b \leq c\) and

\[a^3 + b^3 + c^3 = \text{constant}, \quad a^3 b^3 c^3 = 1,\]

the sum \(S_3 = a^4 + b^4 + c^4\) is maximum for \(0 < a = b \leq c\). Thus, we only need to show that

\[2a^3 + c^3 \geq \sqrt{3(2a^4 + c^4)}\]

for \(a^2 c = 1, \ a \leq 1\). The inequality is equivalent to

\[\left(2a^3 + \frac{1}{a^6}\right)^2 \geq 3 \left(2a^4 + \frac{1}{a^8}\right).\]

Substituting \(a = 1/t, \ t \geq 1\), the inequality becomes

\[\left(\frac{2}{t^3} + t^6\right)^2 \geq 3 \left(\frac{2}{t^4} + t^8\right),\]

which is equivalent to \(f(t) \geq 0\), where

\[f(t) = t^{18} - 3t^{14} + 4t^9 - 6t^2 + 4.\]

We have

\[f'(t) = 6t g(t), \quad g(t) = 3t^{16} - 7t^{12} + 6t^7 - 2,\]

\[g'(t) = 6t^6 h(t), \quad h(t) = 8t^9 - 14t^5 + 7,\]

\[h'(t) = 2t^4(36t^2 - 35).\]

Since \(h'(t) > 0\) for \(t \geq 1\), \(h\) is increasing, \(h(t) \geq h(1) = 1\) for \(t \geq 1\), \(g\) is increasing, \(g(t) \leq g(1) = 0\) for \(t \geq 1\), \(f\) is increasing, hence \(f(t) \geq f(1) = 0\) for \(t \geq 1\).

The equality holds for \(a = b = c = 1.\)

---

**P 5.86.** If \(a, b, c, d\) are nonnegative real numbers so that \(a^2 + b^2 + c^2 + d^2 = 4\), then

\[(2 - abc)(2 - bcd)(2 - cda)(2 - dab) \geq 1.\]

(Vasile C., 2007)
**Solution.** Assume that $a \leq b \leq c \leq d$. From
\[
4 \geq b^2 + c^2 + d^2 \geq 3(bcd)^{2/3},
\]
we get
\[
bcd \leq \frac{8}{3\sqrt{3}} < 2.
\]
There are two cases to consider: $a = 0$ and $a > 0$.

**Case 1:** $a = 0$. We need to show that
\[
8(2 - bcd) \geq 1,
\]
which is equivalent to
\[
bcd \leq \frac{15}{8}.
\]
This is true because
\[
bcd \leq \frac{8}{3\sqrt{3}} < \sqrt{3} < \frac{15}{8}.
\]

**Case 2:** $0 < a \leq b \leq c \leq d$. Substituting
\[
x = a^2, \quad y = b^2, \quad z = c^2, \quad w = d^2, \quad p = abcd = \sqrt{xyzw}, \quad p \in (0, 1],
\]
we need to show that $x + y + z + w = 4$ involves
\[
\left(2 - \frac{p}{\sqrt{x}}\right)\left(2 - \frac{p}{\sqrt{y}}\right)\left(2 - \frac{p}{\sqrt{z}}\right)\left(2 - \frac{p}{\sqrt{w}}\right) \geq 1,
\]
which is equivalent to
\[
f(x) + f(y) + f(z) + f(w) \geq 0,
\]
where
\[
f(u) = \ln\left(2 - \frac{p}{\sqrt{u}}\right), \quad u > \frac{p^2}{4}.
\]

We have
\[
f'(u) = \frac{p}{2u(2\sqrt{u} - p)},
\]
\[
g(x) = f'\left(\frac{1}{x}\right) = \frac{px\sqrt{x}}{2(2 - p\sqrt{x})},
\]
\[
g''(x) = \frac{p(6 - p\sqrt{x})}{4\sqrt{x}(2 - p\sqrt{x})^3}.
\]
Since $g''(x) > 0$ for $\frac{1}{x} > \frac{p^2}{4}$, $g$ is strictly convex on $\left(0, \frac{4}{p^2}\right)$. According to Corollary 3 and Note 5/Note 1, if $p^2/4 < x \leq y \leq z \leq w$ and
\[
x + y + z + w = 4, \quad xyzw = p^2, \quad p \in (0, 1],
\]
then
\[
\left(2 - \frac{p}{\sqrt{x}}\right)\left(2 - \frac{p}{\sqrt{y}}\right)\left(2 - \frac{p}{\sqrt{z}}\right)\left(2 - \frac{p}{\sqrt{w}}\right) \geq 1.
\]
then the sum \( S_4 = f(x) + f(y) + f(z) + f(w) \) is minimum for \( p^2/4 < x \leq y = z = w \). Thus, we only need to prove the original inequality for \( a \leq b = c = d \); that is, to show that
\[
a^2 + 3b^2 = 4, \quad a \leq 1 \leq b \leq \frac{2}{\sqrt{3}}
\]

involves
\[
(2 - b^3)(2 - ab^2)^3 \geq 1.
\]

Let
\[
h(b) = \ln(2 - b^3) + 3 \ln(2 - ab^2), \quad a = \sqrt{4 - 3b^2}, \quad 1 \leq b \leq \frac{2}{\sqrt{3}}.
\]

Since \( h(1) = 0 \), it suffices to show that \( h'(b) \geq 0 \) for \( 1 \leq b \leq \frac{2}{\sqrt{3}} \). From \( a^2 + 3b^2 = 4 \),

we get
\[
a a' + 3b = 0.
\]

Thus,
\[
\frac{1}{3b} f'(b) = \frac{-b - 2a + a' b}{2 - b^3} - \frac{2a^2 - 3b^2}{2 - ab^2} = \frac{-b}{a(2 - b^3)(2 - ab^2)} - \frac{2a^2 - 3b^2}{a(2 - b^3)(2 - ab^2)}
\]
\[
= \frac{6b^2 - 4a^2 - 2ab - 3b^3(b^2 - a^2)}{a(2 - b^3)(2 - ab^2)}
\]
\[
\geq \frac{5(b^2 - a^2) - 3b^3(b^2 - a^2)}{a(2 - b^3)(2 - ab^2)}
\]
\[
= \frac{(5 - 3b^3)(b^2 - a^2)}{a(2 - b^3)(2 - ab^2)} \geq 0.
\]

The equality holds for \( a = b = c = d = 1 \).

\[\square\]

**P 5.87.** If \( a, b, c, d \) are nonnegative real numbers so that \( a + b + c + d = 4 \), then
\[
(a^2 + b^2 + c^2 + d^2 - 4)(a^2 + b^2 + c^2 + d^2 + 18) \leq 10(a^3 + b^3 + c^3 + d^3 - 4).
\]

*(Vasile Cîrtoaje, 2010)*

**Solution.** Apply Corollary 2 for \( n = 4, k = 2, m = 3 \):

- If \( a, b, c, d \) are real numbers so that \( 0 \leq a \leq b \leq c \leq d \) and
  \[
a + b + c + d = 4, \quad a^2 + b^2 + c^2 + d^2 = constant,
\]

then
\[
S_4 = a^3 + b^3 + c^3 + d^3
\]
is minimum for either $0 < a \leq b = c = d$ or $a = 0$.

Case 1: $0 < a \leq b = c = d$. We need to show that $a + 3d = 4$ involves

$$(a^2 + 3d^2 - 4)(a^2 + 3d^2 + 18) \leq 10(a^3 + 3d^3 - 4).$$

This inequality is equivalent to

$$(1 - d)^2(1 + d)(4 - 3d) \geq 0,$$

$$(1 - d)^2(1 + d)a \geq 0.$$

Case 2: $a = 0$. Let

$$s = b^2 + c^2 + d^2.$$

We need to show that $b + c + d = 4$ involves

$$(s - 4)(s + 18) \leq 10(b^3 + c^3 + d^3 - 4).$$

By the Cauchy-Schwarz inequality, we have

$$s \geq \frac{1}{3}(b + c + d)^2 = \frac{16}{3}$$

and

$$(b + c + d)(b^3 + c^3 + d^3) \geq (b^2 + c^2 + d^2)^2, \quad b^3 + c^3 + d^3 \geq \frac{s^2}{4}.$$ 

Thus, it suffices to show that

$$(s - 4)(s + 18) \leq 10\left(\frac{s^2}{4} - 4\right),$$

which is equivalent to the obvious inequality

$$(s - 4)(3s - 16) \geq 0.$$

The equality holds for $a = b = c = d = 1$, and also for

$$a = 0, \quad b = c = d = \frac{4}{3}$$

(or any cyclic permutation).

\[\square\]

**P 5.88.** If $a_1, a_2, \ldots, a_8$ are nonnegative real numbers, then

$$19(a_1^2 + a_2^2 + \cdots + a_8^2)^2 \geq 12(a_1 + a_2 + \cdots + a_8)(a_1^3 + a_2^3 + \cdots + a_8^3).$$

(Vasile C., 2007)
Solution. By Corollary 5 (case \( n = 8, k = 2, m = 3 \)), if \( 0 \leq a_1 \leq a_2 \leq \cdots \leq a_8 \) and
\[
a_1 + a_2 + \cdots + a_8 = \text{constant}, \quad a_1^2 + a_2^2 + \cdots + a_8^2 = \text{constant},
\]
then the sum
\[
S_8 = a_1^3 + a_2^3 + \cdots + a_8^3
\]
is maximum for \( a_1 = a_2 = \cdots = a_7 \leq a_8 \). Due to homogeneity, we only need to consider the cases \( a_1 = a_2 = \cdots = a_7 = 0 \) and \( a_1 = a_2 = \cdots = a_7 = 1 \). For the second case (nontrivial), we need to show that
\[
19(7 + a_8^2)^2 \geq 12(7 + a_8)(7 + a_8^2),
\]
which is equivalent to
\[
a_8^4 - 12a_8^3 + 38a_8^2 - 12a_8 + 49 \geq 0,
\]
\[
(a_8^2 - 6a_8 + 1)^2 + 48 \geq 0.
\]
The equality holds for \( a_1 = a_2 = \cdots = a_8 = 0 \).

P 5.89. If \( a, b, c \) are nonnegative real numbers so that
\[
5(a^2 + b^2 + c^2) = 17(ab + bc + ca),
\]
then
\[
3\sqrt{\frac{3}{5}} \leq \sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} \leq 1 + \frac{\sqrt{7}}{\sqrt{2}}.
\]
(Vasile C., 2006)

Solution. Due to homogeneity, we may assume that \( a + b + c = 9 \). From the hypothesis \( 5(a^2 + b^2 + c^2) = 17(ab + bc + ca) \), which is equivalent to
\[
27(a^2 + b^2 + c^2) = 17(a + b + c)^2,
\]
we get
\[
a^2 + b^2 + c^2 = 51.
\]
Also, from \( 2(b^2 + c^2) \geq (b + c)^2 \) and
\[
b + c = 9 - a, \quad b^2 + c^2 = 51 - a^2,
\]
we get \( a \leq 7 \). Write the desired inequality in the form
\[
3\sqrt{\frac{3}{5}} \leq f(a) + f(b) + f(c) \leq 1 + \frac{\sqrt{7}}{\sqrt{2}}.
\]
where
\[ f(u) = \sqrt{\frac{u}{9-u}}, \quad 0 \leq u \leq 7. \]

We have
\[ g(x) = f'(x) = \frac{9}{2x^{1/2}(9-x)^{3/2}}, \]
\[ g''(x) = \frac{27(8x^2 - 36x + 81)}{8x^{5/2}(9-x)^{7/2}}. \]

Since \( g''(x) > 0 \) for \( x \in (0, 7] \), \( g \) is strictly convex on \((0, 7]\). According to Corollary 1 and Note 5/Note 2, if \( 0 \leq a \leq b \leq c \) and
\[ a + b + c = 9, \quad a^2 + b^2 + c^2 = 51, \]
then the sum \( S_3 = f(a) + f(b) + f(c) \) is maximum for \( a = b \leq c \), and is minimum for either \( a = 0 \) or \( 0 < a \leq b = c \).

(a) To prove the right inequality, it suffices to consider the case \( a = b \leq c \). From
\[ a + b + c = 9, \quad a^2 + b^2 + c^2 = 51, \]
we get \( a = b = 1 \) and \( c = 7 \), therefore
\[ \sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} = 1 + \frac{\sqrt{7}}{\sqrt{2}}. \]
The original right inequality is an equality for \( a = b = c/7 \) (or any cyclic permutation).

(b) To prove the left inequality, it suffices to consider the cases \( a = 0 \) and \( 0 < a \leq b = c \). For \( a = 0 \), from
\[ a + b + c = 9, \quad a^2 + b^2 + c^2 = 51, \]
we get
\[ \frac{b}{c} + \frac{c}{b} = \frac{17}{5}, \]
therefore
\[ \sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} = \sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}} = \sqrt{\frac{b}{c} + \frac{c}{b} + 2} = 3\sqrt{\frac{3}{5}}. \]
The case \( 0 < a \leq b = c \) is not possible, because from
\[ a + b + c = 9, \quad a^2 + b^2 + c^2 = 51, \]
we get \( a = 7 \) and \( b = c = 1 \), which don’t satisfy the condition \( a \leq b \). The original left inequality is an equality for
\[ a = 0, \quad \frac{b}{c} + \frac{c}{b} = \frac{17}{5} \]
(or any cyclic permutation).
P 5.90. If $a, b, c$ are nonnegative real numbers so that

$$8(a^2 + b^2 + c^2) = 9(ab + bc + ca),$$

then

$$\frac{19}{12} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq \frac{141}{88}.$$

(Vasile C., 2006)

Solution. The proof is similar to the one of the preceding P 5.89. Assume that $a + b + c = 15$, which involves $a^2 + b^2 + c^2 = 81$ and $a \in [3, 7]$, then write the inequality in the form

$$\frac{19}{12} \leq f(a) + f(b) + f(c) \leq \frac{141}{88},$$

where

$$f(u) = \frac{u}{15-u}, \quad 3 \leq u \leq 7.$$  

We have

$$g(x) = f'(x) = \frac{1}{5}(15-x)^2, \quad g''(x) = \frac{90}{(15-x)^3}.$$  

Since $g$ is strictly convex on $[3, 7]$, according to Corollary 1 and Note 5/Note 2, if $0 \leq a \leq b \leq c$ and

$$a + b + c = 15, \quad a^2 + b^2 + c^2 = 81,$$

then the sum $S_3 = f(a) + f(b) + f(c)$ is maximum for $a = b \leq c$, and is minimum for either $a = 0$ or $0 < a \leq b = c$.

(a) To prove the right inequality, it suffices to consider the case $a = b \leq c$, which involves

$$a = b = 4, \quad c = 7,$$

and

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{141}{88}.$$  

The original right inequality is an equality for $2a = b = 4c/7$ (or any cyclic permutation).

(b) To prove the left inequality, it suffices to consider the cases $a = 0$ and $0 < a \leq b = c$. The first case is not possible, while the second case involves

$$a = 3, \quad b = c = 6,$$

and

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{19}{12}.$$  

The original left inequality is an equality for $2a = b = c$ (or any cyclic permutation).
Chapter 6

EV Method for Real Variables

6.1 Theoretical Basis

The Equal Variables Method may be extended to solve some difficult symmetric inequalities in real variables.

**EV-Theorem** (Vasile Cirtoaje, 2010). Let \(a_1, a_2, \ldots, a_n\) (\(n \geq 3\)) be fixed real numbers, and let

\[
0 \leq x_1 \leq x_2 \leq \cdots \leq x_n
\]

so that

\[
x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n, \quad x_1^k + x_2^k + \cdots + x_n^k = a_1^k + a_2^k + \cdots + a_n^k,
\]

where \(k\) is an even positive integer. If \(f\) is a differentiable function on \(\mathbb{R}\) so that the joined function \(g : \mathbb{R} \to \mathbb{R}\) defined by

\[
g(x) = f'(x^{\frac{k}{\sqrt{k}}})
\]

is strictly convex on \(\mathbb{R}\), then the sum

\[
S_n = f(x_1) + f(x_2) + \cdots + f(x_n)
\]

is minimum for \(x_2 = x_3 = \cdots = x_n\), and is maximum for \(x_1 = x_2 = \cdots = x_{n-1}\).

To prove this theorem, we will use EV-Lemma and EV-Proposition below.

**EV-Lemma.** Let \(a, b, c\) be fixed real numbers, not all equal, and let \(x, y, z\) be real numbers satisfying

\[
x \leq y \leq z, \quad x + y + z = a + b + c, \quad x^k + y^k + z^k = a^k + b^k + c^k,
\]

where \(k\) is an even positive integer. Then, there exist two real numbers \(m\) and \(M\) so that \(m < M\) and

\[
(1) \ y \in [m, M];
\]
(2) \( y = m \) if and only if \( x = y \);
(3) \( y = M \) if and only if \( y = z \).

**Proof.** We show first, by contradiction method, that \( x < z \). Indeed, if \( x = z \), then

\[
x = z \Rightarrow x = y = z \Rightarrow x^k + y^k + z^k = 3 \left( \frac{x + y + z}{3} \right)^k
\]

\[
\Rightarrow a^k + b^k + c^k = 3 \left( \frac{a + b + c}{3} \right)^k \Rightarrow a = b = c,
\]

which is false. Notice that the last implication follows from Jensen's inequality

\[
a^k + b^k + c^k \geq 3 \left( \frac{a + b + c}{3} \right)^k,
\]

with equality if and only if \( a = b = c \).

According to the relations

\[
x + z = a + b + c - y, \quad x^k + z^k = a^k + b^k + c^k - y^k,
\]

we may consider \( x \) and \( z \) as functions of \( y \). From

\[
x' + z' = -1, \quad x^{k-1}x' + z^{k-1}z' = -y^{k-1},
\]

we get

\[
x' = \frac{y^{k-1} - z^{k-1}}{z^{k-1} - x^{k-1}}, \quad z' = \frac{y^{k-1} - x^{k-1}}{x^{k-1} - z^{k-1}}.
\]

The two-sided inequality

\[
x(y) \leq y \leq z(y)
\]

is equivalent to the inequalities \( f_1(y) \leq 0 \) and \( f_2(y) \geq 0 \), where

\[
f_1(y) = x(y) - y, \quad f_2(y) = z(y) - y.
\]

Using (*) we get

\[
f_1'(y) = \frac{y^{k-1} - z^{k-1}}{z^{k-1} - x^{k-1}} - 1
\]

and

\[
f_2'(y) = \frac{y^{k-1} - x^{k-1}}{x^{k-1} - z^{k-1}} - 1.
\]

Since \( f_1'(y) \leq -1 \) and \( f_2'(y) \leq -1 \), \( f_1 \) and \( f_2 \) are strictly decreasing. Thus, the inequality \( f_1(y) \leq 0 \) involves \( y \geq m \), where \( m \) is the root of the equation \( x(y) = y \), while the inequality \( f_2(y) \geq 0 \) involves \( y \leq M \), where \( M \) is the root of the equation \( z(y) = y \). Moreover, \( y = m \) if and only if \( x = y \), and \( y = M \) if and only if \( y = z \).

**EV-Proposition.** Let \( a, b, c \) be fixed real numbers, and let \( x, y, z \) be real numbers satisfying

\[
x \leq y \leq z, \quad x + y + z = a + b + c, \quad x^k + y^k + z^k = a^k + b^k + c^k,
\]
where $k$ is an even positive integer. If $f$ is a differentiable function on $\mathbb{R}$ so that the joined function $g : \mathbb{R} \to \mathbb{R}$ defined by
\[
g(x) = f'(\sqrt[k]{x})
\]
is strictly convex on $\mathbb{R}$, then the sum
\[
S = f(x) + f(y) + f(z)
\]
is minimum if and only if $y = z$, and is maximum if and only if $x = y$.

Proof. If $a = b = c$, then
\[
a = b = c \Rightarrow a^k + b^k + c^k = 3\left(\frac{a + b + c}{3}\right)^k
\]
\[
\Rightarrow x^k + y^k + z^k = 3\left(\frac{x + y + z}{3}\right)^k \Rightarrow x = y = z.
\]
Consider further that $a, b, c$ are not all equal. As it is shown in the proof of EV-Lemma, we have $x < z$. According to the relations
\[
x + z = a + b + c - y, \quad x^k + z^k = a^k + b^k + c^k - y^k,
\]
we may consider $x$ and $z$ as functions of $y$. Thus, we have
\[
S = f(x(y)) + f(y) + f(z(y)) := F(y).
\]
According to EV-Lemma, it suffices to show that $F$ is maximum for $y = m$ and is minimum for $y = M$. Using $(*)$, we have
\[
F'(y) = x'f'(x) + f'(y) + z'f'(z)
\]
\[
= \frac{y^{k-1} - x^{k-1}}{z^{k-1} - x^{k-1}} g(x^{k-1}) + \frac{y^{k-1} - x^{k-1}}{x^{k-1} - z^{k-1}} g(z^{k-1}),
\]
which, for $x < y < z$, is equivalent to
\[
F'(y) = \frac{g(x^{k-1})}{(y^{k-1} - x^{k-1})(y^{k-1} - z^{k-1})} + \frac{g(y^{k-1})}{(y^{k-1} - z^{k-1})(y^{k-1} - x^{k-1})} + \frac{g(z^{k-1})}{(z^{k-1} - x^{k-1})(z^{k-1} - y^{k-1})}.
\]
Since $g$ is strictly convex, the right hand side is positive. Moreover, since
\[
(y^{k-1} - x^{k-1})(y^{k-1} - z^{k-1}) < 0,
\]
we have $F'(y) < 0$ for $y \in (m, M)$, hence $F$ is strictly decreasing on $[m, M]$. Therefore, $F$ is maximum for $y = m$ and is minimum for $y = M$. 

**Proof of EV-Theorem.** For \( n = 3 \), EV-Theorem follows immediately from EV-Proposition. Consider next that \( n \geq 4 \). Since \( X = (x_1, x_2, \ldots, x_n) \) is defined in EV-Theorem as a compact set in \( \mathbb{R}^n \), \( S_n \) attains its minimum and maximum values. Using this property and EV-Proposition, we can prove EV-Theorem via contradiction. Thus, for the sake of contradicion, assume that \( S_n \) attains its maximum at \((b_1, b_2, \ldots, b_n)\), where \( b_1 \leq b_2 \leq \cdots \leq b_n \) and \( b_1 < b_{n-1} \). Let \( x_1, x_{n-1} \) and \( x_n \) be real numbers so that

\[
x_1 \leq x_{n-1} \leq x_n, \quad x_1 + x_{n-1} + x_n = b_1 + b_{n-1} + b_n, \quad x_1^k + x_{n-1}^k + x_n^k = b_1^k + b_{n-1}^k + b_n^k.
\]

According to EV-Proposition, the sum \( f(x_1) + f(x_{n-1}) + f(x_n) \) is maximum for \( x_1 = x_{n-1} \), when

\[
f(x_1) + f(x_{n-1}) + f(x_n) > f(b_1) + f(b_{n-1}) + f(b_n).
\]

This result contradicts the assumption that \( S_n \) attains its maximum value at \((b_1, b_2, \ldots, b_n)\) with \( b_1 < b_{n-1} \). Similarly, we can prove that \( S_n \) is minimum for \( x_2 = x_3 = \cdots = x_n \).

Taking \( k = 2 \) in EV-Theorem, we obtain the following corollary.

**Corollary 1.** Let \( a_1, a_2, \ldots, a_n \) (\( n \geq 3 \)) be fixed real numbers, and let \( x_1, x_2, \ldots, x_n \) be real variables so that

\[
x_1 \leq x_2 \leq \cdots \leq x_n,
\]

\[
x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n,
\]

\[
x_1^2 + x_2^2 + \cdots + x_n^2 = a_1^2 + a_2^2 + \cdots + a_n^2.
\]

If \( f \) is a differentiable function on \( \mathbb{R} \) so that the derivative \( f' \) is strictly convex on \( \mathbb{R} \), then the sum

\[
S_n = f(x_1) + f(x_2) + \cdots + f(x_n)
\]

is minimum for \( x_2 = x_3 = \cdots = x_n \), and is maximum for \( x_1 = x_2 = \cdots = x_{n-1} \).

**Corollary 2.** Let \( a_1, a_2, \ldots, a_n \) (\( n \geq 3 \)) be fixed real numbers, and let \( x_1, x_2, \ldots, x_n \) be real variables so that

\[
x_1 \leq x_2 \leq \cdots \leq x_n,
\]

\[
x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n,
\]

\[
x_1^k + x_2^k + \cdots + x_n^k = a_1^k + a_2^k + \cdots + a_n^k,
\]

where \( k \) is an even positive integer. For any positive odd number \( m, m > k \), the power sum

\[
S_n = x_1^m + x_2^m + \cdots + x_n^m
\]

is minimum for \( x_2 = x_3 = \cdots = x_n \), and is maximum for \( x_1 = x_2 = \cdots = x_{n-1} \).

**Proof.** We apply the EV-Theorem the function \( f(u) = u^m \). The joined function

\[
g(x) = f'( \left( \frac{k-1}{k} x \right)) = m \sqrt[k-1]{x^{m-1}}
\]
is strictly convex on $\mathbb{R}$ because its derivative
\[ g'(x) = \frac{m(m-1)}{k-1} \sqrt[1/k]{x^{m-k}} \]
is strictly increasing on $\mathbb{R}$.

**Theorem 1.** Let $a_1, a_2, \ldots, a_n$ ($n \geq 3$) be fixed real numbers, and let $x_1, x_2, \ldots, x_n$ be real variables so that
\[ x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n, \]
\[ x_1^2 + x_2^2 + \cdots + x_n^2 = a_1^2 + a_2^2 + \cdots + a_n^2. \]
The power sum
\[ S_n = x_1^4 + x_2^4 + \cdots + x_n^4 \]
is minimum and maximum when $x_1, x_2, \ldots, x_n$ have at most two distinct values.

To prove this theorem, we will use Proposition 1 below.

**Proposition 1.** Let $a, b, c$ be fixed real numbers, and let $x, y, z$ be real numbers so that
\[ x + y + z = a + b + c, \quad x^2 + y^2 + z^2 = a^2 + b^2 + c^2. \]
The power sum
\[ S = x^4 + y^4 + z^4 \]
is minimum and maximum when two of $x, y, z$ are equal

**Proof.** The proof is based on EV-Lemma. Without loss of generality, assume that $x \leq y \leq z$. For the nontrivial case when $a, b, c$ are not all equal (which involves $x < z$), consider the function of $y$
\[ F(y) = x^4(y) + y^4 + z^4(y). \]
According to (\star), we have
\[ F'(y) = 4x^3x' + 4y^3 + 4z^3z' = 4x^3 \frac{y-z}{z-x} + 4y^3 + 4z^3 \frac{y-x}{x-z} \]
\[ = 4(x+y+z)(y-x)(y-z) = 4(a+b+c)(y-x)(y-z). \]
There are three cases to consider.

**Case 1:** $a + b + c < 0$. Since $F'(y) > 0$ for $x < y < z$, $F$ is strictly increasing on $[m,M]$.

**Case 2:** $a + b + c > 0$. Since $F'(y) < 0$ for $x < y < z$, $F$ is strictly decreasing on $[m,M]$.

**Case 3:** $a + b + c = 0$. Since $F'(y) = 0$, $F$ is constant on $[m,M]$. 
In all cases, \( F \) is monotonic on \([m, M]\). Therefore, \( F \) is minimum and maximum for \( y = m \) or \( y = M \); that is, when \( x = y \) or \( y = z \) (see EV-Lemma). Notice that for \( a + b + c \neq 0 \), \( F \) is strictly monotonic on \([m, M]\), hence \( F \) is minimum and maximum if and only if \( y = m \) or \( y = M \); that is, if and only if \( x = y \) or \( y = z \).

**Proof of Theorem 1.** For \( n = 3 \), Theorem 1 follows from Proposition 1. In order to prove Theorem 1 for any \( n \geq 4 \), we will use the contradiction method. For the sake of contradiction, assume that \((b_1, b_2, \ldots, b_n)\) is an extreme point having at least three distinct components; let us say \( b_1 < b_2 < b_3 \). Let \( x_1, x_2 \) and \( x_3 \) be real numbers so that

\[
x_1 \leq x_2 \leq x_3, \quad x_1 + x_2 + x_3 = b_1 + b_2 + b_3 \quad x_1^2 + x_2^2 + x_3^2 = b_1^2 + b_2^2 + b_3^2.
\]

We need to consider two cases.

**Case 1:** \( b_1 + b_2 + b_3 \neq 0 \). According to Proposition 1, the sum \( x_1^4 + x_2^4 + x_3^4 \) is extreme only when two of \( x_1, x_2, x_3 \) are equal, which contradicts the assumption that the sum \( x_1^4 + x_2^4 + \cdots + x_n^4 \) attains its extreme value at \((b_1, b_2, \ldots, b_n)\) with \( b_1 < b_2 < b_3 \).

**Case 2:** \( b_1 + b_2 + b_3 = 0 \). There exist three real numbers \( x_1, x_2, x_3 \) so that \( x_1 = x_2 \) and

\[
x_1 + x_2 + x_3 = b_1 + b_2 + b_3 = 0, \quad x_1^2 + x_2^2 + x_3^2 = b_1^2 + b_2^2 + b_3^2.
\]

Letting \( x_1 = x_2 := x \) and \( x_3 := y \), we have \( 2x + y = 0, \ x \neq y \). According to Proposition 1, the sum \( x_1^4 + x_2^4 + x_3^4 \) is constant (equal to \( b_1^4 + b_2^4 + b_3^4 \)). Thus, \((x, x, y, b_4, \ldots, b_n)\) is also an extreme point. According to our hypothesis, this extreme point has at least three distinct components. Therefore, among the numbers \( b_4, \ldots, b_n \) there is one, let us say \( b_4 \), so that \( x, y \) and \( b_4 \) are distinct. Since

\[
x + y + b_4 = -x + b_4 \neq 0,
\]

we have a case similar to Case 1, which leads to a contradiction.

**Theorem 2.** Let \( a_1, a_2, \ldots, a_n \ (n \geq 3) \) be fixed real numbers, and let \( x_1, x_2, \ldots, x_n \) be real variables so that

\[
x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n,
\]

\[
x_1^2 + x_2^2 + \cdots + x_n^2 = a_1^2 + a_2^2 + \cdots + a_n^2.
\]

For \( m \in \{6, 8\} \), the power sum

\[
S_n = x_1^m + x_2^m + \cdots + x_n^m
\]

is maximum when \( x_1, x_2, \ldots, x_n \) have at most two distinct values.

Theorem 2 can be proved using Proposition 2 below, in a similar way as the EV-Theorem.
Proposition 2. Let \(a, b, c\) be fixed real numbers, let \(x, y, z\) be real numbers so that
\[
x + y + z = a + b + c, \quad x^2 + y^2 + z^2 = a^2 + b^2 + c^2.
\]
For \(m \in \{6, 8\}\), the power sum
\[
S_m = x^m + y^m + z^m
\]
is maximum if and only if two of \(x, y, z\) are equal.

Proof. Consider the nontrivial case where \(a, b, c\) are not all equal. Let
\[
p = a + b + c, \quad q = ab + bc + ca, \quad r = xyz.
\]
Since \(x + y + z = p\) and \(xy + yz + zx = q\), from
\[
(x - y)^2(y - z)^2(z - x)^2 \geq 0,
\]
which is equivalent to
\[
27r^2 + 2(2p^3 - 9pq)r - p^2q^2 + 4q^3 \leq 0,
\]
we get \(r \in [r_1, r_2]\), where
\[
\begin{align*}
    r_1 &= \frac{9pq - 2p^3 - 2(p^2 - 3q) \sqrt{p^2 - 3q}}{27}, \\
    r_2 &= \frac{9pq - 2p^3 + 2(p^2 - 3q) \sqrt{p^2 - 3q}}{27}.
\end{align*}
\]
From
\[
-27(r - r_1)(r - r_2) = (x - y)^2(y - z)^2(z - x)^2 \geq 0,
\]
it follows that the product \(r = xyz\) attains its minimum value \(r_1\) and its maximum value \(r_2\) only when two of \(x, y, z\) are equal. For fixed \(p\) and \(q\), we have
\[
S_6 = 3r^2 + f_6(p, q)r + h_6(p, q) := g_6(r),
\]
\[
S_8 = 4(3p^2 - 2q)r^2 + f_8(p, q)r + h_8(p, q) := g_8(r).
\]
Since
\[
3p^2 - 2q = \frac{7}{3}p^2 + \frac{2}{3}(p^2 - 3q) > 0,
\]
the functions \(g_6\) and \(g_8\) are strictly convex, hence are maximum only for \(r = r_1\) or \(r = r_2\); that is, only when two of \(x, y, z\) are equal.

Open problem. Theorem 2 is valid for any integer number \(m \geq 3\).

Note. The EV-Theorem for real variables and Corollary 1 are also valid under the conditions in Note 5 from the preceding chapter 5, where \(a, b \in \mathbb{R}\).
6.2 Applications

6.1. If \( a, b, c, d \) are real numbers so that \( a + b + c + d = 4 \), then
\[
\left( a^2 + b^2 + c^2 + d^2 + \frac{8}{3} \right)^2 \geq 4 \left( a^3 + b^3 + c^3 + d^3 + \frac{64}{9} \right).
\]

6.2. If \( a, b, c, d \) are real numbers so that \( a + b + c + d = 4 \), then
\[
(a^2 + b^2 + c^2 + d^2 - 4) \left( a^2 + b^2 + c^2 + d^2 + \frac{76}{3} \right) \geq 8(a^3 + b^3 + c^3 + d^3 - 4).
\]

6.3. If \( a, b, c \) are real numbers so that \( a + b + c = 3 \), then
\[
(a^2 + b^2 + c^2 - 3)(a^2 + b^2 + c^2 + 93) \geq 24(a^3 + b^3 + c^3 - 3).
\]

6.4. If \( a, b, c, d \) are real numbers so that \( a + b + c + d = 4 \), then
\[
(a^2 + b^2 + c^2 + d^2 - 4)(a^2 + b^2 + c^2 + d^2 + 116) \geq 24(a^3 + b^3 + c^3 + d^3 - 4).
\]

6.5. Let \( a, b, c, d \) be real numbers so that \( a + b + c + d = 4 \), and let
\[
E = a^2 + b^2 + c^2 + d^2 - 4, \quad F = a^3 + b^3 + c^3 + d^3 - 4.
\]
Prove that
\[
E \left( \frac{\sqrt{E}}{3} + 3 \right) \geq F.
\]

6.6. Let \( a_1, a_2, \ldots, a_n \) be real numbers so that
\[a_1 + a_2 + \cdots + a_n = 0, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n(n-1).\]
If \( m \) is an odd number \((m \geq 3)\), then
\[n - 1 - (n-1)^m \leq a_1^m + a_2^m + \cdots + a_n^m \leq (n-1)^m - n + 1.\]
6.7. Let \( a_1, a_2, \ldots, a_n \) be real numbers so that
\[
a_1 + a_2 + \cdots + a_n = 1, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n^2 + n - 1.
\]
If \( m \) is an odd number \((m \geq 3)\), then
\[
(n - 1) \left( 1 + \frac{2}{n} \right)^m - (n - 2) \left( n - \frac{2}{n} \right)^m \leq a_1^m + a_2^m + \cdots + a_n^m \leq (n - 1) \left( 1 - \frac{2}{n} \right)^m - (n - 1) \left( 1 + \frac{2}{n} \right)^m.
\]

6.8. Let \( a_1, a_2, \ldots, a_n \) be real numbers so that
\[
a_1 + a_2 + \cdots + a_n = 1, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n^2 - 3n + 3.
\]
If \( m \) is an odd number \((m \geq 3)\), then
\[
n - 1 - (n - 2)^m \leq a_1^m + a_2^m + \cdots + a_n^m \leq \left( n - 2 + \frac{2}{n} \right)^m - (n - 1) \left( 1 - \frac{2}{n} \right)^m.
\]

6.9. Let \( a_1, a_2, \ldots, a_n \) be real numbers so that
\[
a_1 + a_2 + \cdots + a_n = 1, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n - 1.
\]
If \( m \) is an odd number \((m \geq 3)\), then
\[
n - 1 \leq a_1^m + a_2^m + \cdots + a_n^m \leq (n - 1) \left( 1 - \frac{2}{n} \right)^m + \left( 2 - \frac{2}{n} \right)^m.
\]

6.10. Let \( a_1, a_2, \ldots, a_n \) be real numbers so that
\[
a_1 + a_2 + \cdots + a_n = n + 1, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n + 3.
\]
If \( m \) is an odd number \((m \geq 3)\), then
\[
\left( \frac{2}{n} \right)^m + (n - 1) \left( 1 + \frac{2}{n} \right)^m \leq a_1^m + a_2^m + \cdots + a_n^m \leq 2^m + n - 1.
\]

6.11. If \( a_1, a_2, \ldots, a_n \) are real numbers so that
\[
a_1 + a_2 + \cdots + a_n = 1, \quad a_1^4 + a_2^4 + \cdots + a_n^4 = n - 1,
\]
then
\[
a_1^5 + a_2^5 + \cdots + a_n^5 \geq n - 1.
\]
6.12. If \( a, b, c \) are real numbers so that \( a^2 + b^2 + c^2 = 3 \), then

\[
a^3 + b^3 + c^3 + 3 \geq 2(a + b + c).
\]

6.13. If \( a_1, a_2, \ldots, a_n \) are real numbers so that

\[
a_1 + a_2 + \cdots + a_n = 0, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n(n - 1),
\]

then

\[
a_1^4 + a_2^4 + \cdots + a_n^4 \leq n(n - 1)(n^2 - 3n + 3).
\]

6.14. If \( a_1, a_2, \ldots, a_n \) are real numbers so that

\[
a_1 + a_2 + \cdots + a_n = n + 1, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = 4n^2 + n - 1,
\]

then

\[
a_1^4 + a_2^4 + \cdots + a_n^4 \leq 16n^4 + n - 1.
\]

6.15. If \( n \) is an odd number and \( a_1, a_2, \ldots, a_n \) are real numbers so that

\[
a_1 + a_2 + \cdots + a_n = 0, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n(n^2 - 1),
\]

then

\[
a_1^4 + a_2^4 + \cdots + a_n^4 \geq n(n^2 - 1)(n^2 + 1).
\]

6.16. If \( a_1, a_2, \ldots, a_n \) are real numbers so that

\[
a_1 + a_2 + \cdots + a_n = n^2 - n - 1, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n^3 + 2n^2 - n - 1,
\]

then

\[
a_1^4 + a_2^4 + \cdots + a_n^4 \geq n^4 + (n - 1)(n + 1)^4.
\]

6.17. If \( a_1, a_2, \ldots, a_n \) are real numbers so that

\[
a_1 + a_2 + \cdots + a_n = n^2 - 2n - 1, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n^3 + 2n + 1,
\]

then

\[
a_1^4 + a_2^4 + \cdots + a_n^4 \geq (n + 1)^4 + (n - 1)n^4.
\]
6.18. If \( a_1, a_2, \ldots, a_n \) are real numbers so that 
\[
a_1 + a_2 + \cdots + a_n = n^2 - 3n - 2, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n^3 + 2n^2 - 3n - 2,
\]
then
\[
a_1^4 + a_2^4 + \cdots + a_n^4 \geq 2n^4 + (n-2)(n+1)^4.
\]

6.19. If \( a, b, c, d \) are real numbers so that \( a + b + c + d = 4 \), then
\[
(a^2 + b^2 + c^2 + d^2 - 4)(a^2 + b^2 + c^2 + d^2 + 36) \leq 12(a^4 + b^4 + c^4 + d^4 - 4).
\]

6.20. If \( a_1, a_2, \ldots, a_n \) are real numbers so that 
\[
a_1 + a_2 + \cdots + a_n = 0, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n(n-1),
\]
then
\[
a_1^6 + a_2^6 + \cdots + a_n^6 \leq (n-1)^6 + n - 1.
\]

6.21. If \( a_1, a_2, \ldots, a_n \) are real numbers so that 
\[
a_1 + a_2 + \cdots + a_n = 1, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n^2 + n - 1,
\]
then
\[
a_1^6 + a_2^6 + \cdots + a_n^6 \leq n^6 + n - 1.
\]

6.22. If \( a_1, a_2, \ldots, a_n \) are real numbers so that 
\[
a_1 + a_2 + \cdots + a_n = 0, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n(n-1),
\]
then
\[
a_1^8 + a_2^8 + \cdots + a_n^8 \leq (n-1)^8 + n - 1.
\]

6.23. If \( a_1, a_2, \ldots, a_n \) are real numbers so that 
\[
a_1 + a_2 + \cdots + a_n = 1, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n^2 + n - 1,
\]
then
\[
a_1^8 + a_2^8 + \cdots + a_n^8 \leq n^8 + n - 1.
\]
6.24. Let \(a_1, a_2, \ldots, a_n\) \((n \geq 2)\) be real numbers (not all equal), and let
\[
A = \frac{a_1 + a_2 + \cdots + a_n}{n}, \quad B = \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}, \quad C = \frac{a_1^3 + a_2^3 + \cdots + a_n^3}{n}.
\]
Then,
\[
\frac{1}{4} \left( 1 - \sqrt{1 + \frac{2n^2}{n-1}} \right) \leq \frac{B^2 - AC}{B^2 - A^4} \leq \frac{1}{4} \left( 1 + \sqrt{1 + \frac{2n^2}{n-1}} \right).
\]

6.25. If \(a, b, c, d\) are real numbers so that
\[a + b + c + d = 2,
\]
then
\[a^4 + b^4 + c^4 + d^4 \leq 40 + \frac{3}{4}(a^2 + b^2 + c^2 + d^2)^2.
\]

6.26. If \(a, b, c, d, e\) are real numbers, then
\[a^4 + b^4 + c^4 + d^4 + e^4 \leq \frac{31 + 18\sqrt{3}}{8}(a + b + c + d + e)^4 + \frac{3}{4}(a^2 + b^2 + c^2 + d^2 + e^2)^2.
\]

6.27. Let \(a, b, c, d, e \neq \frac{-5}{4}\) be real numbers so that \(a + b + c + d + e = 5\). Then,
\[
\frac{a(a-1)}{(4a+5)^2} + \frac{b(b-1)}{(4b+5)^2} + \frac{c(c-1)}{(4c+5)^2} + \frac{d(d-1)}{(4d+5)^2} + \frac{e(e-1)}{(4e+5)^2} \geq 0.
\]

6.28. If \(a, b, c\) are real numbers so that
\[a + b + c = 9, \quad ab + bc + ca = 15,
\]
then
\[
\frac{19}{175} \leq \frac{1}{b^2 + bc + c^2} + \frac{1}{c^2 + ca + a^2} + \frac{1}{a^2 + ab + b^2} \leq \frac{7}{19}.
\]

6.29. If \(a, b, c\) are real numbers so that
\[8(a^2 + b^2 + c^2) = 9(ab + bc + ca),
\]
then
\[
\frac{419}{175} \leq \frac{a^2}{b^2 + bc + c^2} + \frac{b^2}{c^2 + ca + a^2} + \frac{c^2}{a^2 + ab + b^2} \leq \frac{311}{19}.
\]
6.3 Solutions

P 6.1. If $a, b, c, d$ are real numbers so that $a + b + c + d = 4$, then
\[ \left( a^2 + b^2 + c^2 + d^2 + \frac{8}{3} \right)^2 \geq \frac{4}{9} \left( a^3 + b^3 + c^3 + d^3 + \frac{64}{9} \right). \]

(Vasile Cîrtoaje, 2010)

Solution. Apply Corollary 2 for $n = 4, k = 2, m = 3$:

- If $a, b, c, d$ are real numbers so that $a \leq b \leq c \leq d$ and
  \[ a + b + c + d = 4, \quad a^2 + b^2 + c^2 + d^2 = \text{constant}, \]

then
\[ S_4 = a^3 + b^3 + c^3 + d^3 \]

is maximum for $a = b = c \leq d$.

Thus, we only need to show that $3a + d = 4$ involves
\[ \left( 3a^2 + d^2 + \frac{8}{3} \right)^2 \geq \frac{4}{9} \left( 3a^3 + d^3 + \frac{64}{9} \right). \]

This inequality is equivalent to
\[ (a - 1)^2(3a - 2)^2 \geq 0. \]

The equality holds for $a = b = c = d = 1$, and also for
\[ a = b = c = \frac{2}{3}, \quad d = 2 \]

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- If $a_1, a_2, \ldots, a_n$ are real numbers so that
  \[ a_1 + a_2 + \cdots + a_n = n, \]

then
\[ \left( a_1^2 + a_2^2 + \cdots + a_n^2 + \frac{n^3}{8n - 8} \right)^2 \geq n \left( a_1^3 + a_2^3 + \cdots + a_n^3 \right) + \frac{n^4(n^2 + 16n - 16)}{64(n - 1)^2}, \]

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for
\[ a_1 = a_2 = \cdots = a_{n-1} = \frac{n}{2n - 2}, \quad a_n = \frac{n}{2} \]

(or any cyclic permutation).
P 6.2. If \(a, b, c, d\) are real numbers so that \(a + b + c + d = 4\), then
\[
(a^2 + b^2 + c^2 + d^2 - 4)
\left(a^2 + b^2 + c^2 + d^2 + \frac{76}{3}\right) \geq 8(a^3 + b^3 + c^3 + d^3 - 4).
\]

(Vasile Cîrtoaje, 2010)

Solution. As shown in the preceding P 6.1, we only need to show that
\[
3a + d = 4
\]

involves
\[
(3a^2 + d^2 - 4)
\left(3a^2 + d^2 + \frac{76}{3}\right) \geq 8(3a^3 + d^3 - 4).
\]

This inequality is equivalent to
\[
(a - 1)^2(3a - 1)^2 \geq 0.
\]

The equality holds for \(a = b = c = d = 1\), and also for
\[
a = b = c = \frac{1}{3}, \quad d = 3
\]

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:
- If \(a_1, a_2, \ldots, a_n\) are real numbers so that
\[
a_1 + a_2 + \cdots + a_n = n,
\]

then
\[
(a_1^2 + \cdots + a_n^2 - n)
\left[a_1^2 + \cdots + a_n^2 + \frac{n(n^2 + n - 1)}{n - 1}\right] \geq 2n(a_1^3 + \cdots + a_n^3 - n),
\]

with equality for \(a_1 = a_2 = \cdots = a_n = 1\), and also for
\[
a_1 = a_2 = \cdots = a_{n-1} = \frac{1}{n - 1}, \quad a_n = n - 1
\]

(or any cyclic permutation).

P 6.3. If \(a, b, c\) are real numbers so that \(a + b + c = 3\), then
\[
(a^2 + b^2 + c^2 - 3)(a^2 + b^2 + c^2 + 93) \geq 24(a^3 + b^3 + c^3 - 3).
\]

(Vasile Cîrtoaje, 2010)
Solution. As shown in the proof of P 6.1, we only need to show that

$$2a + c = 3$$

involves

$$(2a^2 + c^2 - 3)(2a^2 + c^2 + 93) \geq 24(2a^3 + c^3 - 3).$$

This inequality is equivalent to

$$(a^2 - 1)^2 \geq 0.$$ 

The equality holds for $a = b = c = 1$, and also for

$$a = b = -1, \quad c = 5$$

(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- Let $a, b, c$ be real numbers so that $a + b + c = 3$. For any real $k$, the following inequality holds

$$(a^2 + b^2 + c^2 - 3)(a^2 + b^2 + c^2 + 6k^2 + 36k - 3) \geq 12k(a^3 + b^3 + c^3 - 3),$$

with equality for $a = b = c = 1$, and also for

$$a = b = 1 - k, \quad c = 1 + 2k$$

(or any cyclic permutation).

P 6.4. If $a, b, c, d$ are real numbers so that $a + b + c + d = 4$, then

$$(a^2 + b^2 + c^2 + d^2 - 4)(a^2 + b^2 + c^2 + d^2 + 116) \geq 24(a^3 + b^3 + c^3 + d^3 - 4).$$

(Vasile Cîrtoaje, 2010)

Solution. As shown in the proof of P 6.1, we only need to show that

$$3a + d = 4$$

involves

$$(3a^2 + d^2 - 4)(3a^2 + d^2 + 116) \geq 24(3a^3 + d^3 - 4).$$

This inequality is equivalent to

$$(a^2 - 1)^2 \geq 0.$$
The equality holds for \( a = b = c = d = 1 \), and also for

\[
a = b = c = -1, \quad d = 7
\]
(or any cyclic permutation).

**Remark.** Similarly, we can prove the following generalization:

- Let \( a_1, a_2, \ldots, a_n \) be real numbers so that

\[
a_1 + a_2 + \cdots + a_n = n.
\]

If \( k \) is a real number, then

\[
\frac{k(a_1^3 + \cdots + a_n^3 - n)}{a_1^2 + \cdots + a_n^2 - n} \leq \frac{a_1^2 + \cdots + a_n^2 + n(n-1)(n-2)k^2 + 6n(n-1)k - n}{2n(n-1)},
\]

with equality for

\[
a_1 = \cdots = a_{n-1} = 1 - (n-2)k, \quad a_n = 1 + (n-1)(n-2)k
\]
(or any cyclic permutation).

For \( k = \frac{-6}{n-2} \), we get the following nice inequality

\[
(a_1^2 + a_2^2 + \cdots + a_n^2 - n)^2 + \frac{12n(n-1)}{n-2} (a_1^3 + a_2^3 + \cdots + a_n^3 - n) \geq 0,
\]
with equality for \( a_1 = a_2 = \cdots = a_n = 1 \), and also for

\[
a_1 = \cdots = a_{n-1} = 7, \quad a_n = 7 - 6n
\]
(or any cyclic permutation).

\[\square\]

**P 6.5.** Let \( a, b, c, d \) be real numbers so that \( a + b + c + d = 4 \), and let

\[
E = a^2 + b^2 + c^2 + d^2 - 4, \quad F = a^3 + b^3 + c^3 + d^3 - 4.
\]

Prove that

\[
E \left( \sqrt[3]{E} + 3 \right) \geq F.
\]

(Vasile Cîrtoaje, 2016)
**Solution.** As shown in the proof of P 6.1, we only need to prove the desired inequality for \(3a + d = 4\) and

\[
E = 3a^2 + d^2 - 4, \quad F = 3a^3 + d^3 - 4.
\]

Since

\[
E = 12(1 - a)^2, \quad F = 12(5 - 2a)(1 - a)^2,
\]

we get

\[
E \left( \sqrt[3]{\frac{E}{3} + 3} \right) - F = 12(1 - a)^2(2|1 - a| + 3) - 12(5 - 2a)(1 - a)^2
\]

\[
= 24(1 - a)^2[|1 - a| - (1 - a)] \geq 0.
\]

The equality holds for

\[
a = b = c = \frac{4 - d}{3} \leq 1,
\]

(or any cyclic permutation).

**Remark.** Similarly, we can prove the following generalization:

- Let \(a_1, a_2, \ldots, a_n\) be real numbers so that \(a_1 + a_2 + \cdots + a_n = n\), and let

\[
E = a_1^2 + a_2^2 + \cdots + a_n^2 - n, \quad F = a_1^3 + a_2^3 + \cdots + a_n^3 - n.
\]

Then,

\[
E \left( \frac{E}{n(n-1)} + 3 \right) \geq F,
\]

with equality for

\[
a_1 = \cdots = a_{n-1} = \frac{n - a_n}{n - 1} \leq 1
\]

(or any cyclic permutation).

\[\square\]

**P 6.6.** Let \(a_1, a_2, \ldots, a_n\) be real numbers so that

\[
a_1 + a_2 + \cdots + a_n = 0, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n(n - 1).
\]

If \(m\) is an odd number \((m \geq 3)\), then

\[
n - 1 - (n - 1)^m \leq a_1^m + a_2^m + \cdots + a_n^m \leq (n - 1)^m - n + 1.
\]

*(Vasile Cîrtoaje, 2010)*
Solution. Without loss of generality, assume that

\[ a_1 \leq a_2 \leq \cdots \leq a_n. \]

(a) Consider the right inequality. For \( n = 2 \), we need to show that

\[ a_1 + a_2 = 0, \quad a_1^2 + a_2^2 = 2 \]

implies

\[ a_1^m + a_2^m \leq 0. \]

We have

\[ a_1 = -1, \quad a_2 = 1, \]

therefore \( a_1^m + a_2^m = 0 \). Assume now that \( n \geq 3 \). According to Corollary 2, the sum

\[ S_n = a_1^m + a_2^m + \cdots + a_n^m \]

is maximum for \( a_1 = a_2 = \cdots = a_{n-1} \). Thus, we only need to show that

\[ (n-1)a + b = 0, \quad (n-1)a^2 + b^2 = n(n-1), \quad a \leq b \]

involve

\[ (n-1)a^m + b^m \leq (n-1)^m - n + 1. \]

From the equations above, we get

\[ a = -1, \quad b = n - 1; \]

therefore,

\[ (n-1)a^m + b^m = (n-1)(-1)^m + (n-1)^m = (n-1)^m - n + 1. \]

The equality holds for

\[ a_1 = \cdots = a_{n-1} = -1, \quad a_n = n - 1 \]

(or any cyclic permutation).

(b) The left inequality follows from the right inequality by replacing \( a_1, a_2, \ldots, a_n \) with \(-a_1, -a_2, \ldots, -a_n\), respectively. The equality holds for

\[ a_1 = -n + 1, \quad a_2 = a_3 = \cdots = a_n = 1 \]

(or any cyclic permutation).
P 6.7. Let \( a_1, a_2, \ldots, a_n \) be real numbers so that
\[
    a_1 + a_2 + \cdots + a_n = 1, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n^2 + n - 1.
\]
If \( m \) is an odd number (\( m \geq 3 \)), then
\[
    (n-1)
    \left(1 + \frac{2}{n}\right)^m
    - \left(n - \frac{2}{n}\right)^m
    \leq a_1^m + a_2^m + \cdots + a_n^m
    \leq n^m - n + 1.
\]

(\text{Vasile \c{C}irtoaje, 2010})

\textbf{Solution.} Without loss of generality, assume that
\[
    a_1 \leq a_2 \leq \cdots \leq a_n.
\]
For \( n = 2 \), we need to show that
\[
    a_1 + a_2 = 1, \quad a_1^2 + a_2^2 = 5,
\]
implies
\[
    2^m - 1 \leq a_1^m + a_2^m \leq 2^m - 1.
\]
We have
\[
    a_1 = -1, \quad a_2 = 2,
\]
for which \( a_1^m + a_2^m = 2^m - 1 \). Assume now that \( n \geq 3 \).

(a) Consider the right inequality. According to Corollary 2, the sum
\[
    S_n = a_1^m + a_2^m + \cdots + a_n^m
\]
is maximum for \( a_1 = a_2 = \cdots = a_{n-1} \). Thus, we only need to show that
\[
    (n-1)a + b = 1, \quad (n-1)a^2 + b^2 = n^2 + n - 1, \quad a \leq b
\]
involve
\[
    (n-1)a^m + b^m \leq n^m - n + 1.
\]
From the equations above, we get
\[
    a = -1, \quad b = n;
\]
therefore,
\[
    (n-1)a^m + b^m = (n-1)(-1)^m + n^m = n^m - n + 1.
\]
The equality holds for
\[
    a_1 = a_2 = \cdots = a_{n-1} = -1, \quad a_n = n
\]
(or any cyclic permutation).
(b) Consider the left inequality. According to Corollary 2, the sum
\[ S_n = a_1^m + a_2^m + \cdots + a_n^m \]
is minimum for \( a_2 = a_3 = \cdots = a_n \). Thus, we only need to show that
\[ a + (n - 1)b = 1, \quad a^2 + (n - 1)b^2 = n^2 + n - 1, \quad a \leq b \]
involve
\[ a^m + (n - 1)b^m \geq (n - 1)\left(1 + \frac{2}{n}\right)^m - \left(n - \frac{2}{n}\right)^m. \]

From the equations above, we get
\[ a = -n + \frac{2}{n}, \quad b = 1 + \frac{2}{n}; \]

therefore,
\[ a^m + (n - 1)b^m = \left(-n + \frac{2}{n}\right)^m + (n - 1)\left(1 + \frac{2}{n}\right)^m \]
\[ = (n - 1)\left(1 + \frac{2}{n}\right)^m - \left(n - \frac{2}{n}\right)^m. \]

The equality holds for
\[ a_1 = -n + \frac{2}{n}, \quad a_2 = a_3 = \cdots = a_n = 1 + \frac{2}{n} \]
(or any cyclic permutation).

\[ \square \]

**P 6.8.** Let \( a_1, a_2, \ldots, a_n \) be real numbers so that
\[ a_1 + a_2 + \cdots + a_n = 1, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n^2 - 3n + 3. \]

If \( m \) is an odd number \((m \geq 3)\), then
\[ n - 1 - (n - 2)^m \leq a_1^m + a_2^m + \cdots + a_n^m \leq \left(n - 2 + \frac{2}{n}\right)^m - (n - 1)\left(1 - \frac{2}{n}\right)^m. \]

*(Vasile Cîrtoaje, 2010)*

**Solution.** Without loss of generality, assume that
\[ a_1 \leq a_2 \leq \cdots \leq a_n. \]

For \( n = 2 \), we need to show that
\[ a_1 + a_2 = 1, \quad a_1^2 + a_2^2 = 1, \]
implies
\[ 1 \leq a_1^m + a_2^m \leq 1. \]

We have
\[ a_1 = 0, \quad a_2 = 1, \]
when \( a_1^m + a_2^m = 1 \). Assume now that \( n \geq 3 \).

(a) Consider the left inequality. According to Corollary 2, the sum
\[ S_n = a_1^m + a_2^m + \cdots + a_n^m \]
is minimum for \( a_2 = a_3 = \cdots = a_n \). Thus, we only need to show that
\[ a + (n - 1)b = 1, \quad a^2 + (n - 1)b^2 = n^2 - 3n + 3, \quad a \leq b \]
involve
\[ a^n + (n - 1)b^m \leq n - 1 - (n - 2)^m. \]
From the equations above, we get
\[ a = 2 - n, \quad b = 1; \]
therefore,
\[ a^m + (n - 1)b^m = (2 - n)^m + n - 1 = n - 1 - (n - 2)^m. \]
The equality holds for
\[ a_1 = 2 - n, \quad a_2 = a_3 = \cdots = a_n = 1 \]
(or any cyclic permutation).

(b) Consider the right inequality. According to Corollary 2, the sum
\[ S_n = a_1^m + a_2^m + \cdots + a_n^m \]
is maximum for \( a_1 = a_2 = \cdots = a_{n-1} \). Thus, we only need to show that
\[ (n - 1)a + b = 1, \quad (n - 1)a^2 + b^2 = n^2 - 3n + 3, \quad a \leq b \]
involve
\[ (n - 1)a^m + b^m \leq \left( n - 2 + \frac{2}{n} \right)^m - (n - 1) \left( 1 - \frac{2}{n} \right)^m. \]
From the equations above, we get
\[ a = -1 + \frac{2}{n}, \quad b = n - 2 + \frac{2}{n}; \]
therefore,
\[
(n-1)a^m + b^m = (n-1)\left(-1 + \frac{2}{n}\right)^m + \left(n-2 + \frac{2}{n}\right)^m
= \left(n-2 + \frac{2}{n}\right)^m - (n-1)\left(1 - \frac{2}{n}\right)^m.
\]

The equality holds for
\[
a_1 = \cdots = a_{n-1} = -1 + \frac{2}{n}, \quad a_n = n - 2 + \frac{2}{n}
\]
(or any cyclic permutation).

\[\square\]

**P 6.9.** Let \(a_1, a_2, \ldots, a_n\) be real numbers so that
\[
a_1 + a_2 + \cdots + a_n = a_1^2 + a_2^2 + \cdots + a_n^2 = n - 1.
\]

If \(m\) is an odd number \((m \geq 3)\), then
\[
n - 1 \leq a_1^m + a_2^m + \cdots + a_n^m \leq (n-1)\left(1 - \frac{2}{n}\right)^m + \left(2 - \frac{2}{n}\right)^m.
\]

*(Vasile Cîrtoaje, 2010)*

**Solution.** Without loss of generality, assume that
\[
a_1 \leq a_2 \leq \cdots \leq a_n.
\]

For \(n = 2\), we need to show that
\[
a_1 + a_2 = 1, \quad a_1^2 + a_2^2 = 1,
\]
implies
\[
1 \leq a_1^m + a_2^m \leq 1.
\]

The above equations involve
\[
a_1 = 0, \quad a_2 = 1,
\]
hence \(a_1^m + a_2^m = 1\). Assume now that \(n \geq 3\).

(a) Consider the left inequality. According to Corollary 2, the sum
\[
S_n = a_1^m + a_2^m + \cdots + a_n^m
\]
is minimum for \(a_2 = a_3 = \cdots = a_n\). Thus, we only need to show that
\[
a + (n-1)b = n - 1, \quad a^2 + (n-1)b^2 = n - 1, \quad a \leq b
\]
involve
\[ a^m + (n-1)b^m \geq n-1. \]
From the equations above, we get
\[ a = 0, \quad b = 1; \]
therefore,
\[ a^m + (n-1)b^m = n-1. \]
The equality holds for
\[ a_1 = 0, \quad a_2 = \cdots = a_n = 1 \]
(or any cyclic permutation).

(b) Consider the right inequality. According to Corollary 2, the sum
\[ S_n = a_1^m + a_2^m + \cdots + a_n^m \]
is maximum for \( a_1 = a_2 = \cdots = a_{n-1} \). Thus, we only need to show that
\[ (n-1)a + b = n-1, \quad (n-1)a^2 + b^2 = n-1, \quad a \leq b \]
involve
\[ (n-1)a^m + b^m \leq (n-1)\left(1 - \frac{2}{n}\right)^m + \left(2 - \frac{2}{n}\right)^m. \]
From the equations above, we get
\[ a = 1 - \frac{2}{n}, \quad b = 2 - \frac{2}{n}, \]
when
\[ (n-1)a^m + b^m = (n-1)\left(1 - \frac{2}{n}\right)^m + \left(2 - \frac{2}{n}\right)^m. \]
The equality holds for
\[ a_1 = a_2 = \cdots = a_{n-1} = 1 - \frac{2}{n}, \quad a_n = 2 - \frac{2}{n} \]
(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- Let \( a_1, a_2, \ldots, a_n \) be real numbers so that
\[ a_1 + a_2 + \cdots + a_n = k, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n^2 + (2k-1)n + k(k-2), \]
where \( k \) is a real number, \( k \geq -n \). If \( m \) is an odd number \( (m \geq 3) \), then
\[ \left(\frac{2k}{n} + 1 - n - k\right)^m + (n-1)\left(\frac{2k}{n} + 1\right)^m \leq a_1^m + a_2^m + \cdots + a_n^m \leq (n+k-1)^m - n + 1. \]
The left inequality is an equality for
\[ a_1 = \frac{2k}{n} + 1 - n - k, \quad a_2 = \cdots = a_n = \frac{2k}{n} + 1 \]
(or any cyclic permutation). The right inequality is an equality for
\[ a_1 = \cdots = a_{n-1} = -1, \quad a_n = n + k - 1 \]
(or any cyclic permutation).

For \( k = 0 \) and \( k = 1 \), we get the inequalities in P 6.6 and P 6.7, respectively. For \( k = -1 \) and \( k = n + 1 \), by replacing \( k \) with \( -k \) and \( a_1, a_2, \ldots, a_n \) with \( -a_1, -a_2, \ldots, -a_n \), we get the inequalities in P 6.8 and P 6.9, respectively.

**P 6.10.** Let \( a_1, a_2, \ldots, a_n \) be real numbers so that
\[ a_1 + a_2 + \cdots + a_n = n + 1, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n + 3. \]
If \( m \) is an odd number \((m \geq 3)\), then
\[ \left( \frac{2}{n} \right)^m + (n-1) \left( 1 + \frac{2}{n} \right)^m \leq a_1^m + a_2^m + \cdots + a_n^m \leq 2^m + n - 1. \]

*(Vasile Cîrtoaje, 2010)*

**Solution.** Without loss of generality, assume that
\[ a_1 \leq a_2 \leq \cdots \leq a_n. \]

For \( n = 2 \), we need to show that
\[ a_1 + a_2 = 3, \quad a_1^2 + a_2^2 = 5, \]
implies
\[ 2^m + 1 \leq a_1^m + a_2^m \leq 2^m + 1. \]
We get
\[ a_1 = 1, \quad a_2 = 2, \]
when \( a_1^m + a_2^m = 2^m + 1 \). Assume now that \( n \geq 3 \).

(a) Consider the left inequality. According to Corollary 2, the sum
\[ S_n = a_1^m + a_2^m + \cdots + a_n^m \]
is minimum for \( a_2 = a_3 = \cdots = a_n \). Thus, we only need to show that
\[ a + (n-1)b = n + 1, \quad a^2 + (n-1)b^2 = n + 3, \quad a \leq b \]
involve
\[ a^m + (n - 1)b^m \geq \left( \frac{2}{n} \right)^m + (n - 1) \left( \frac{1 + \frac{2}{n}}{n} \right)^m. \]

From the equations
\[ a + (n - 1)b = n + 1, \quad a^2 + (n - 1)b^2 = n + 3, \]
we get
\[ a = \frac{2}{n}, \quad b = 1 + \frac{2}{n}; \]
therefore,
\[ a^m + (n - 1)b^m = \left( \frac{2}{n} \right)^m + (n - 1) \left( \frac{1 + \frac{2}{n}}{n} \right)^m. \]
The equality holds for
\[ a_1 = \frac{2}{n}, \quad a_2 = \cdots = a_n = 1 + \frac{2}{n} \]
(or any cyclic permutation).

(b) Consider the right inequality. According to Corollary 2, the sum
\[ S_n = a_1^m + a_2^m + \cdots + a_n^m \]
is maximum for \( a_1 = a_2 = \cdots = a_{n-1} \). Thus, we only need to show that
\[ (n - 1)a + b = n + 1, \quad (n - 1)a^2 + b^2 = n + 3, \quad a \leq b \]
involve
\[ (n - 1)a^m + b^m \leq 2^m + n - 1. \]

From the equations
\[ (n - 1)a + b = n + 1, \quad (n - 1)a^2 + b^2 = n + 3, \]
we get
\[ a = 1, \quad b = 2; \]
therefore,
\[ (n - 1)a^m + b^m = n - 1 + 2^m. \]
The equality holds for
\[ a_1 = \cdots = a_{n-1} = 1, \quad a_n = 2 \]
(or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:
• Let \( a_1, a_2, \ldots, a_n \) be real numbers so that
\[
a_1 + a_2 + \cdots + a_n = k, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n^2 - (2k + 1)n + k(k + 2),
\]
where \( k \) is a positive number, \( k > n \). If \( m \) is an odd number \((m \geq 3)\), then
\[
\left(\frac{2k}{n} - 1 + n - k\right)^{m} + (n-1)\left(\frac{2k}{n} - 1\right)^{m} \leq a_1^m + a_2^m + \cdots + a_n^m \leq (k-n+1)^m + n - 1.
\]
The left inequality is an equality for
\[
a_1 = \frac{2k}{n} - 1 + n - k, \quad a_2 = \cdots = a_n = \frac{2k}{n} - 1
\]
(or any cyclic permutation). The right inequality is an equality for
\[
a_1 = \cdots = a_{n-1} = 1, \quad a_n = k - n + 1
\]
(or any cyclic permutation).

For \( k = n + 1 \), we get the inequalities in P 6.10.

\[\square\]

**P 6.11.** If \( a_1, a_2, \ldots, a_n \) are real numbers so that
\[
a_1 + a_2 + \cdots + a_n = a_1^4 + a_2^4 + \cdots + a_n^4 = n - 1,
\]
then
\[
a_1^5 + a_2^5 + \cdots + a_n^5 \geq n - 1.
\]
*(Vasile Cîrtoaje, 2010)*

**Solution.** For \( n = 2 \), we need to show that
\[
a_1 + a_2 = 1, \quad a_1^4 + a_2^4 = 1,
\]
implies
\[
a_1^5 + a_2^5 \geq 1.
\]
We have
\[
a_1 = 0, \quad a_2 = 1,
\]
or
\[
a_1 = 1, \quad a_2 = 0.
\]
For each of these cases, the inequality is an equality. Assume now that \( n \geq 3 \) and
\[
a_1 \leq a_2 \leq \cdots \leq a_n.
\]
According to Corollary 2, the sum
\[ S_n = a_1^5 + a_2^5 + \cdots + a_n^5 \]
is minimum for \( a_2 = a_3 = \cdots = a_n \). Thus, we only need to show that
\[ a + (n-1)b = a^4 + (n-1)b^4 = n-1, \quad a \leq b \]
involve
\[ a^5 + (n-1)b^5 \geq n-1. \]
The equations
\[ a + (n-1)b = n-1, \quad a^4 + (n-1)b^4 = n-1, \]
are equivalent to
\[ (1-b)[(n-1)^3(1-b)^3 - 1 - b - b^2 - b^3] = 0, \quad a = (n-1)(1-b); \]
that is,
\[ b = 1, \quad a = 0, \]
and
\[ a^3 = 1 + b + b^2 + b^3, \quad a = (n-1)(1-b). \]
For the second case, the condition \( a \leq b \) involves
\[ b^3 \geq 1 + b + b^2 + b^3, \]
which is not possible. Therefore, it suffices to show that
\[ a^5 + (n-1)b^5 \geq n-1 \]
for \( a = 0 \) and \( b = 1 \), that is clearly true. Thus, the proof is completed. The equality holds for
\[ a_1 = 0, \quad a_2 = \cdots = a_n = 1 \]
(or any cyclic permutation).

\[ \square \]

**P 6.12.** If \( a, b, c \) are real numbers so that
\[ a^2 + b^2 + c^2 = 3, \]
then
\[ a^3 + b^3 + c^3 + 3 \geq 2(a + b + c). \]

*(Vasile Cîrtoaje, 2010)*
Solution. Assume that 

\[ a \leq b \leq c. \]

According to Corollary 2, for \( a \leq b \leq c \) and

\[ a + b + c = \text{constant}, \quad a^2 + b^2 + c^2 = 3, \]

the sum

\[ S_3 = a^3 + b^3 + c^3 \]

is minimum for \( a = b = c \). Thus, we only need to show that

\[ a^2 + 2b^2 = 3, \quad a \leq b, \]

involves

\[ a^3 + 2b^3 + 3 \geq 2(a + b). \]

We will show this by two methods. From \( a^2 + 2b^2 = 3 \) and \( a \leq b \), it follows that

\[ -\sqrt{3} \leq a \leq 1, \quad -\frac{3}{2} < b \leq \frac{3}{2}. \]

**Method 1.** Write the desired inequality as

\[ a^3 + b(3 - a^2) + 3 \geq 2(a + 2b), \]

\[ a^3 - 2a + 3 \geq b(a^2 + 1). \]

For \( a \geq 0 \), we have

\[ a^3 - 2a + 3 \geq -2a + 3 > 0, \]

and for \( a \leq 0 \), we have

\[ a^3 - 2a + 3 = a(a^2 - 3) + a + 3 = -2ab^2 + a + 3 \geq a + 3 > 0. \]

Thus, it suffices to show that

\[ (a^3 - 2a + 3)^2 \geq b^2(a^2 + 1)^2, \]

which is equivalent to

\[ 2(a^3 - 2a + 3)^2 \geq (3 - a^2)(a^2 + 1)^2, \]

\[ (a - 1)^2 f(a) \geq 0, \]

where

\[ f(a) = a^4 + 2a^3 + 2a + 5. \]

We need to prove that \( f(a) \geq 0 \). For \( a \geq -1 \), we have

\[ f(a) = (a + 2)(a^3 + 2) + 1 > 0. \]
For $a \leq -1$, we have
\[ f(a) = (a + 1)^2(a + 2)^2 + g(a), \quad g(a) = -4a^3 - 13a^2 - 10a + 1. \]
It suffices to show that $g(a) \geq 0$. Since
\[ g(a) = -(a + 1)\left(2a + \frac{7}{2}\right)^2 + 5h(a), \quad h(a) = a^2 + \frac{13}{4}a + \frac{53}{20} \]
and
\[ h(a) = \left(a + \frac{13}{8}\right)^2 + \frac{3}{320} > 0, \]
the conclusion follows. The equality holds for $a = b = c = 1$.

**Method 2.** Write the desired inequality as follows:
\[
\begin{align*}
2(a^3 - 2a + 1) + 4(b^3 - 2b + 1) &\geq 0, \\
2(a^3 - 2a + 1) + 4(b^3 - 2b + 1) &\geq a^2 + 2b^2 - 3, \\
(2a^3 - a^2 - 4a + 3) + 2(b^3 - b^2 - 4b + 3) &\geq 0, \\
(a - 1)^2(2a + 3) + 2(b - 1)^2(2b + 3) &\geq 0.
\end{align*}
\]
Since $2b + 3 > 0$, the inequality is true for $a \geq -3/2$. Consider further that
\[-\sqrt{3} \leq a \leq -\frac{3}{2},
\]
and rewrite the desired inequality as follows:
\[
\begin{align*}
2(a^3 - 2a + 1) + 4(b^3 - 2b + 1) + 4(a^2 + 2b^2 - 3) &\geq 0, \\
(2a^3 + 4a^2 - 4a - 2) + 2(2b^3 + 4b^2 - 4b - 2) &\geq 0, \\
\left(2a^3 + 4a^2 - 4a - \frac{33}{4}\right) + \left(4b^3 + 8b^2 - 8b + \frac{9}{4}\right) &\geq 0, \\
(2a + 3)\left(a^2 + \frac{1}{2}a - \frac{11}{4}\right) + f(b) &\geq 0,
\end{align*}
\]
where
\[ f(b) = 4b^3 + 8b^2 - 8b + \frac{9}{4}. \]
Since $2a + 3 \leq 0$ and
\[ a^2 + \frac{1}{2}a - \frac{11}{4} \leq 3 + \frac{1}{2}a - \frac{11}{4} = \frac{1}{4}(2a + 1) < 0, \]
it suffices to show that $f(b) \geq 0$. For $b \geq 0$, we have
\[ f(b) > 8b^2 - 8b + 2 = 2(2b - 1)^2 \geq 0, \]
and for $b \leq 0$, we have
\[ f(b) > 4b^3 + 8b^2 = 4b^2(b + 2) \geq 0. \]
\[ \square \]
P 6.13. If \( a_1, a_2, \ldots, a_n \) are real numbers so that
\[
a_1 + a_2 + \cdots + a_n = 0, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n(n-1),
\]
then
\[
a_1^4 + a_2^4 + \cdots + a_n^4 \leq n(n-1)(n^2 - 3n + 3). \tag{Vasile Cîrtoaje, 2010}
\]

**Solution.** For \( n = 2 \), we need to show that
\[
a_1 + a_2 = 0, \quad a_1^2 + a_2^2 = 2,
\]
implies
\[
a_1^4 + a_2^4 \leq 2.
\]
We have
\[
a_1 = -1, \quad a_2 = 1,
\]
or
\[
a_1 = 1, \quad a_2 = -1.
\]
For each of these cases, the desired inequality is an equality. Assume now that \( n \geq 3 \). According to Theorem 1, the sum
\[
S_n = a_1^4 + a_2^4 + \cdots + a_n^4
\]
is maximum for
\[
a_1 = \cdots = a_j, \quad a_{j+1} = \cdots = a_n,
\]
where \( j \in \{1, 2, \ldots, n-1\} \). Thus, we only need to show that
\[
ja_1 + (n-j)a_n = 0, \quad ja_1^2 + (n-j)a_n^2 = n(n-1)
\]
involve
\[
ja_1^4 + (n-j)a_n^4 \leq n(n-1)(n^2 - 3n + 3).
\]
From the equations above, we get
\[
a_1^2 = \frac{(n-j)(n-1)}{j}, \quad a_n^2 = \frac{j(n-1)}{n-j};
\]
therefore,
\[
ja_1^4 + (n-j)a_n^4 = \frac{(n-j)^3 + j^3}{j(n-j)}(n-1)^2 = \left[ \frac{n^2}{j(n-j)} - 3 \right] n(n-1)^2.
\]
Since
\[
j(n-j) - (n-1) = (j-1)(n-j-1) \geq 0,
\]

\[
ja_1^4 + (n-j)a_n^4 \leq n(n-1)(n^2 - 3n + 3).
\]
we get

\[ ja_1^4 + (n-j)a_n^4 \leq \left[ \frac{n^2}{n-1} - 3 \right] n(n-1)^2 = n(n-1)(n^2-3n+3). \]

The equality holds for

\[ a_1 = -n + 1, \quad a_2 = \cdots = a_n = 1 \]

and for

\[ a_1 = n - 1, \quad a_2 = \cdots = a_n = -1 \]

(or any cyclic permutation).

\[ \square \]

**P 6.14.** If \( a_1, a_2, \ldots, a_n \) are real numbers so that

\[ a_1 + a_2 + \cdots + a_n = n + 1, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = 4n^2 + n - 1, \]

then

\[ a_1^4 + a_2^4 + \cdots + a_n^4 \leq 16n^4 + n - 1. \]

(Vasile Cîrtoaje, 2010)

**Solution.** Replacing \( n \) by \( 2n + 1 \) in the preceding P 6.13, we get the following statement:

- If \( a_1, a_2, \ldots, a_{2n+1} \) are real numbers so that

\[ a_1 + a_2 + \cdots + a_{2n+1} = 0, \quad a_1^2 + a_2^2 + \cdots + a_{2n+1}^2 = 2n(2n + 1), \]

then

\[ a_1^4 + a_2^4 + \cdots + a_{2n+1}^4 \leq 2n(2n + 1)(4n^2 - 2n + 1), \]

with equality for

\[ a_1 = -2n, \quad a_2 = \cdots = a_{2n+1} = 1 \]

and for

\[ a_1 = 2n, \quad a_2 = \cdots = a_{2n+1} = -1 \]

(or any cyclic permutation).

Putting

\[ a_{n+1} = \cdots = a_{2n+1} = -1, \]

it follows that

\[ a_1 + a_2 + \cdots + a_n - n - 1 = 0, \quad a_1^2 + a_2^2 + \cdots + a_n^2 + n + 1 = 2n(2n + 1) \]
involve
\[ a_1^4 + a_2^4 + \cdots + a_n^4 + n + 1 \leq 2n(2n+1)(4n^2 - 2n + 1). \]
This is equivalent to the desired statement. The equality holds for
\[ a_1 = 2n, \quad a_2 = \cdots = a_n = -1 \]
(or any cyclic permutation).

\[ \square \]

**P 6.15.** If $n$ is an odd number and $a_1, a_2, \ldots, a_n$ are real numbers so that
\[ a_1 + a_2 + \cdots + a_n = 0, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n(n^2 - 1), \]
then
\[ a_1^4 + a_2^4 + \cdots + a_n^4 \geq n(n^2 - 1)(n^2 + 3). \]

*(Vasile Cîrtoaje, 2010)*

**Solution.** According to Theorem 1, the sum
\[ S_n = a_1^4 + a_2^4 + \cdots + a_n^4 \]
is minimum for
\[ a_1 = \cdots = a_j, \quad a_{j+1} = \cdots = a_n, \]
where $j \in \{1, 2, \ldots, n-1\}$. Thus, we only need to show that
\[ ja_1 + (n-j)a_n = 0, \quad ja_1^2 + (n-j)a_n^2 = n(n^2 - 1) \]
involve
\[ ja_1^4 + (n-j)a_n^4 \leq n(n^2 - 1)(n^2 + 3). \]
From the equations above, we get
\[ a_1^2 = \frac{(n-j)(n^2 - 1)}{j}, \quad a_n^2 = \frac{j(n^2 - 1)}{n-j}; \]
therefore,
\[ ja_1^4 + (n-j)a_n^4 = \frac{(n-j)^3 + j^3}{j(n-j)}(n^2 - 1)^2 = \left[ \frac{n^2}{j(n-j)} - 3 \right] n(n^2 - 1)^2. \]
Since
\[ \frac{n^2-1}{4} - j(n-j) = \frac{(n-2j)^2 - 1}{4} \geq 0, \]
we get
\[ ja_1^4 + (n-j)a_n^4 \geq \left( \frac{4n^2}{n^2-1} - 3 \right) n(n^2 - 1)^2 = n(n^2 - 1)(n^2 + 3). \]
The equality holds when \( \frac{n - 1}{2} \) of \( a_1, a_2, \ldots, a_n \) are equal to \(-n - 1\) and the other \( \frac{n + 1}{2} \) are equal to \( n - 1 \), and also when \( \frac{n - 1}{2} \) of \( a_1, a_2, \ldots, a_n \) are equal to \( n + 1 \) and the other \( \frac{n + 1}{2} \) are equal to \(-n + 1\).

\[ EV \text{ Method for Real Variables} \]

**P 6.16.** If \( a_1, a_2, \ldots, a_n \) are real numbers so that
\[
a_1 + a_2 + \cdots + a_n = n^2 - n - 1, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n^3 + 2n^2 - n - 1,
\]
then
\[
a_1^4 + a_2^4 + \cdots + a_n^4 \geq n^4 + (n - 1)(n + 1)^4.
\]

\[(\text{Vasile Cîrtoaje, 2010)}\]

**Solution.** Replacing \( a_1, a_2, \ldots, a_n \) by \( 2a_1, 2a_2, \ldots, 2a_n \) and then \( n \) by \( 2n + 1 \), the preceding P 6.15 becomes as follows:

- If \( a_1, a_2, \ldots, a_{2n+1} \) are real numbers so that
  \[
a_1 + a_2 + \cdots + a_{2n+1} = 0, \quad a_1^2 + a_2^2 + \cdots + a_{2n+1}^2 = n(n + 1)(2n + 1),
\]
then
  \[
a_1^4 + a_2^4 + \cdots + a_{2n+1}^4 \geq n(n + 1)(2n + 1)(n^2 + n + 1),
\]
with equality when \( n \) of \( a_1, a_2, \ldots, a_{2n+1} \) are equal to \(-n - 1\) and the other \( n + 1 \) are equal to \( n \), and also when \( n \) of \( a_1, a_2, \ldots, a_{2n+1} \) are equal to \( n + 1 \) and the other \( n + 1 \) are equal to \(-n\).

Putting
\[
a_{n+1} = \cdots = a_{2n} = -n, \quad a_{2n+1} = n + 1,
\]
it follows that
\[
a_1 + a_2 + \cdots + a_n + n(-n) + (n + 1) = 0
\]
and
\[
a_1^2 + a_2^2 + \cdots + a_n^2 + n(-n)^2 + (n + 1)^2 = n(n + 1)(2n + 1)
\]
involve
\[
a_1^4 + a_2^4 + \cdots + a_n^4 + n(-n)^4 + (n + 1)^4 \leq n(n + 1)(2n + 1)(n^2 + n + 1).
\]
This is equivalent to the desired statement. The equality holds for
\[
a_1 = \cdots = a_{n-1} = n + 1, \quad a_n = -n
\]
(or any cyclic permutation).

\[ \square \]
P 6.17. If \(a_1, a_2, \ldots, a_n\) are real numbers so that
\[
a_1 + a_2 + \cdots + a_n = n^2 - 2n - 1, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n^3 + 2n + 1,
\]
then
\[
a_1^4 + a_2^4 + \cdots + a_n^4 \geq (n + 1)^4 + (n - 1)n^4.
\]
(Vasile Cîrtoaje, 2010)

**Solution.** As shown in the proof of the preceding P 6.16, the following statement holds:

- If \(a_1, a_2, \ldots, a_{2n+1}\) are real numbers so that
\[
a_1 + a_2 + \cdots + a_{2n+1} = 0, \quad a_1^2 + a_2^2 + \cdots + a_{2n+1}^2 = n(n + 1)(2n + 1),
\]
then
\[
a_1^4 + a_2^4 + \cdots + a_{2n+1}^4 \geq n(n + 1)(2n + 1)(n^2 + n + 1),
\]
with equality when \(n\) of \(a_1, a_2, \ldots, a_{2n+1}\) are equal to \(-n - 1\) and the other \(n + 1\) are equal to \(n\), and also when \(n\) of \(a_1, a_2, \ldots, a_{2n+1}\) are equal to \(n + 1\) and the other \(n + 1\) are equal to \(-n\).

Putting
\[
a_{n+1} = \cdots = a_{2n-1} = -n - 1, \quad a_{2n} = a_{2n+1} = n,
\]
it follows that
\[
a_1 + a_2 + \cdots + a_n + (n - 1)(-n - 1) + 2n = 0
\]
and
\[
a_1^2 + a_2^2 + \cdots + a_n^2 + (n - 1)(-n - 1)^2 + 2n^2 = n(n + 1)(2n + 1)
\]
involve
\[
a_1^4 + a_2^4 + \cdots + a_n^4 + (n - 1)(-n - 1)^4 + 2n^4 \leq n(n + 1)(2n + 1)(n^2 + n + 1),
\]
which is equivalent to the desired statement. The equality holds for
\[
a_1 = n - 1, \quad a_2 = \cdots = a_n = n
\]
(or any cyclic permutation).

\[\square\]

P 6.18. If \(a_1, a_2, \ldots, a_n\) are real numbers so that
\[
a_1 + a_2 + \cdots + a_n = n^2 - 3n - 2, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n^3 + 2n^2 - 3n - 2,
\]
then
\[
a_1^4 + a_2^4 + \cdots + a_n^4 \geq 2n^4 + (n - 2)(n + 1)^4.
\]
(Vasile Cîrtoaje, 2010)
Solution. As shown in the proof of P 6.16, the following statement holds:

- If \( a_1, a_2, \ldots, a_{2n+1} \) are real numbers so that
  \[
  a_1 + a_2 + \cdots + a_{2n+1} = 0, \quad a_1^2 + a_2^2 + \cdots + a_{2n+1}^2 = n(n+1)(2n+1),
  \]
  then
  \[
  a_1^3 + a_2^3 + \cdots + a_{2n+1}^3 \geq n(n+1)(2n+1)(n^2 + n + 1),
  \]
  with equality when \( n \) of \( a_1, a_2, \ldots, a_{2n+1} \) are equal to \(-n-1\) and the other \( n+1 \) are equal to \( n \), and also when \( n \) of \( a_1, a_2, \ldots, a_{2n+1} \) are equal to \( n+1 \) and the other \( n+1 \) are equal to \(-n\).

Putting
\[
a_{n+1} = \cdots = a_{2n-1} = -n, \quad a_{2n} = a_{2n+1} = n+1,
\]
it follows that
\[
a_1 + a_2 + \cdots + a_n + (n-1)(-n) + 2(n+1) = 0
\]
and
\[
a_1^2 + a_2^2 + \cdots + a_n^2 + (n-1)(-n)^2 + 2(n+1)^2 = n(n+1)(2n+1)
\]
involve
\[
a_1^4 + a_2^4 + \cdots + a_n^4 + (n-1)(-n)^4 + 2(n+1)^4 \leq n(n+1)(2n+1)(n^2 + n + 1),
\]
which is equivalent to the desired statement. The equality holds for
\[
a_1 = a_2 = -n, \quad a_3 = \cdots = a_n = n+1
\]
(or any permutation).

\[\square\]

P 6.19. If \( a, b, c, d \) are real numbers so that \( a + b + c + d = 4 \), then
\[
(a^2 + b^2 + c^2 + d^2 - 4)(a^2 + b^2 + c^2 + d^2 + 36) \leq 12(a^4 + b^4 + c^4 + d^4 - 4).
\]

(Vasile Cîrtoaje, 2010)

Solution. By Theorem 1, for \( a + b + c + d = 4 \) and \( a^2 + b^2 + c^2 + d^2 = constant \), the sum \( a^4 + b^4 + c^4 + d^4 \) is maximum when \( a, b, c, d \) have at most two distinct values. Therefore, it suffices to consider the following two cases.

Case 1: \( a = b \) and \( c = d \). We need to show that \( a + c = 2 \) involves
\[
(a^2 + c^2 - 2)(a^2 + c^2 + 18) \leq 6(a^4 + c^4 - 2).
\]
Since
\[ a^2 + c^2 - 2 = (a + c)^2 - 2ac - 2 = 2(1 - ac), \quad a^2 + c^2 + 18 = 2(11 - ac), \]
\[ a^4 + c^4 - 2 = (a^2 + c^2)^2 - 2a^2c^2 - 2 = 2(1 - ac)(7 - ac), \]
the inequality becomes
\[ (1 - ac)(11 - ac) \leq 3(1 - ac)(7 - ac), \]
\[ (1 - ac)(5 - ac) \geq 0. \]
It is true because
\[ ac \leq \frac{1}{4}(a + c)^2 = 1. \]

**Case 2:** \( b = c = d \). We need to show that \( a + 3b = 4 \) involves
\[ (a^2 + 3b^2 - 4)(a^2 + 3b^2 + 36) \leq 12(a^4 + 3b^4 - 4). \]
Since
\[ a^2 + 3b^2 - 4 = 12(b - 1)^2, \quad a^2 + 3b^2 + 36 = 4(3b^2 - 6b + 13), \]
\[ a^4 + 3b^4 - 4 = (4 - 3b)^2 + 3b^4 - 4 = 12(b - 1)^2(7b^2 - 22b + 21), \]
the inequality becomes
\[ (b - 1)^2[(3b^2 - 6b + 13) \leq 3(b - 1)^2(7b^2 - 22b + 21), \]
\[ (b - 1)^2(3b - 5)^2 \geq 0. \]
The equality holds for \( a = b = c = d = 1 \), and also for
\[ a = -1, \quad b = c = d = \frac{5}{3} \]
(or any cyclic permutation).

**P 6.20.** If \( a_1, a_2, \ldots, a_n \) are real numbers so that
\[ a_1 + a_2 + \cdots + a_n = 0, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n(n - 1), \]
then
\[ a_1^6 + a_2^6 + \cdots + a_n^6 \leq (n - 1)^6 + n - 1. \]

*(Vasile Cîrtoaje, 2010)*
**Solution.** For \( n = 2 \), we need to show that

\[
a_1 + a_2 = 0, \quad a_1^2 + a_2^2 = 2,
\]

implies

\[
a_1^6 + a_2^6 \leq 2.
\]

We have

\[
a_1 = -1, \quad a_2 = 1,
\]

or

\[
a_1 = 1, \quad a_2 = -1.
\]

For each of these cases, the desired inequality is an equality. According to Theorem 2, the sum

\[
S_n = a_1^6 + a_2^6 + \cdots + a_n^6
\]

is maximum for

\[
a_1 = \cdots = a_j, \quad a_{j+1} = \cdots = a_n,
\]

where \( j \in \{1, 2, \ldots, n-1\} \). Thus, we only need to show that

\[
ja_1 + (n-j)a_n = 0, \quad ja_1^2 + (n-j)a_n^2 = n(n-1)
\]

involve

\[
ja_1^6 + (n-j)a_n^6 \leq (n-1)^6 + n - 1.
\]

From the equations above, we get

\[
a_1^2 = \frac{(n-j)(n-1)}{j}, \quad a_n^2 = \frac{j(n-1)}{n-j}.
\]

Thus, the desired inequality becomes

\[
\frac{(n-j)^5 + j^5}{j^2(n-j)^2} \leq \frac{(n-1)^5 + 1}{(n-1)^2},
\]

\[
\frac{(n-j)^4 - (n-j)^3 j + (n-j)^2 j^2 - (n-j)j^3 + j^4}{j^2(n-j)^2} \leq \frac{(n-1)^4 - (n-1)^3 + (n-1)^2 - (n-1) + 1}{(n-1)^2},
\]

\[
\frac{(n-j)^2}{j^2} - \frac{n-j}{j} - \frac{j}{n-j} + \frac{j^2}{n-j} \leq (n-1)^2 - (n-1) - \frac{1}{n-1} + \frac{1}{(n-1)^2},
\]

which can be written as

\[
f(a) \geq f(b),
\]

where

\[
f(x) = x^2 - x - \frac{1}{x} + \frac{1}{x^2},
\]
\[ a = n - 1, \quad b = \frac{n}{j} - 1. \]

Since \( a \geq b \) and

\[ ab - 1 = (n-1)\left(\frac{n}{j} - 1\right) - 1 = n\left(\frac{n-1}{j} - 1\right) \geq 0, \]

we have

\[ f(a) - f(b) = (a - b)\left(a + b - 1 + \frac{1}{ab} - \frac{a + b}{ab}a^2b^2\right) \]
\[ = (a - b)\left(1 - \frac{1}{ab}\right)\left[(a + b)\left(1 + \frac{1}{ab}\right) - 1\right] \geq 0. \]

The equality holds for

\[ a_1 = -n + 1, \quad a_2 = \cdots = a_n = 1, \]

and for

\[ a_1 = n - 1, \quad a_2 = \cdots = a_n = -1 \]

(or any cyclic permutation).

\[ \square \]

**P 6.21.** If \( a_1, a_2, \ldots, a_n \) are real numbers so that

\[ a_1 + a_2 + \cdots + a_n = 1, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n^2 + n - 1, \]

then

\[ a_1^6 + a_2^6 + \cdots + a_n^6 \leq n^6 + n - 1. \]

*(Vasile Cîrtoaje, 2010)*

**Solution.** The inequality follows from the preceding P 6.20 by replacing \( n \) with \( n + 1 \), and then making \( a_{n+1} = -1 \). The equality holds for

\[ a_1 = n, \quad a_2 = \cdots = a_n = -1 \]

(or any cyclic permutation).

\[ \square \]

**P 6.22.** If \( a_1, a_2, \ldots, a_n \) are real numbers so that

\[ a_1 + a_2 + \cdots + a_n = 0, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n(n-1), \]

then

\[ a_1^8 + a_2^8 + \cdots + a_n^8 \leq (n-1)^8 + n - 1. \]

*(Vasile Cîrtoaje, 2010)*
Solution. For $n = 2$, we need to show that

$$a_1 + a_2 = 0, \quad a_1^2 + a_2^2 = 2,$$

implies

$$a_1^8 + a_2^8 \leq 2.$$

We have

$$a_1 = -1, \quad a_2 = 1,$$

or

$$a_1 = 1, \quad a_2 = -1.$$

For each of these cases, the desired inequality is an equality. According to Theorem 2, the sum

$$S_n = a_1^8 + a_2^8 + \cdots + a_n^8$$

is maximum for

$$a_1 = \cdots = a_j, \quad a_{j+1} = \cdots = a_n,$$

where $j \in \{1, 2, \ldots, n-1\}$. Thus, we only need to show that

$$ja_1 + (n-j)a_n = 0, \quad ja_1^2 + (n-j)a_n^2 = n(n-1)$$

involve

$$ja_1^8 + (n-j)a_n^8 \leq (n-1)^8 + n - 1.$$

From the equations above, we get

$$a_1^2 = \frac{(n-j)(n-1)}{j}, \quad a_2^2 = \frac{j(n-1)}{n-j}.$$

Thus, the desired inequality becomes

$$\frac{(n-j)^2 + j^2}{j^3(n-j)^3} \leq \frac{(n-1)^7 + 1}{(n-1)^4},$$

$$\frac{(n-j)^3}{j^3} - \frac{(n-j)^2}{j^2} + \frac{n-j}{j} + \frac{j}{n-j} - \frac{j^2}{(n-j)^2} + \frac{j^3}{(n-j)^3} \leq$$

$$\leq (n-1)^3 - (n-1)^2 + (n-1) + \frac{1}{n-1} - \frac{1}{(n-1)^2} + \frac{1}{(n-1)^3},$$

$$f(a) \geq f(b),$$

where

$$a = n-1, \quad b = \frac{n}{j} - 1,$$

$$f(x) = x^3 - x^2 + x + \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3}, \quad x > 0.$$
Since
\[ f(x) = (t - 1)(t^2 - 2), \quad t = x + \frac{1}{x} \geq 2, \]
it suffices to show that
\[ a + \frac{1}{a} \geq b + \frac{1}{b}. \]
We have \( a \geq b, \)
\[ ab - 1 = (n - 1)\left(\frac{n}{j} - 1\right) - 1 = n\left(\frac{n-1}{j} - 1\right) \geq 0, \]
therefore
\[ a + \frac{1}{a} - b - \frac{1}{b} = (a - b)\left(1 - \frac{1}{ab}\right) \geq 0. \]
The equality holds for
\[ a_1 = -n + 1, \quad a_2 = \cdots = a_n = 1 \]
and for
\[ a_1 = n - 1, \quad a_2 = \cdots = a_n = -1 \]
(or any cyclic permutation).

\[ \square \]

**P 6.23.** If \( a_1, a_2, \ldots, a_n \) are real numbers so that
\[ a_1 + a_2 + \cdots + a_n = 1, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = n^2 + n - 1, \]
then
\[ a_1^8 + a_2^8 + \cdots + a_n^8 \leq n^8 + n - 1. \]

*(Vasile Cîrtoaje, 2010)*

**Solution.** The inequality follows from the preceding P 6.22 by replacing \( n \) with \( n + 1 \), and making \( a_{n+1} = -1 \). The equality holds for
\[ a_1 = n, \quad a_2 = \cdots = a_n = -1 \]
(or any cyclic permutation).

\[ \square \]
P 6.24. Let \( a_1, a_2, \ldots, a_n \) \((n \geq 2)\) be real numbers (not all equal), and let
\[
A = \frac{a_1 + a_2 + \cdots + a_n}{n}, \quad B = \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}, \quad C = \frac{a_1^3 + a_2^3 + \cdots + a_n^3}{n}.
\]
Then,
\[
\frac{1}{4} \left( 1 - \sqrt{1 + \frac{2n^2}{n-1}} \right) \leq \frac{B^2 - AC}{B^2 - A^4} \leq \frac{1}{4} \left( 1 + \sqrt{1 + \frac{2n^2}{n-1}} \right).
\]

Solution. It is well-known that \( B > A^2 \), hence \( B^2 > A^4 \).

(a) For \( n = 2 \), the right inequality reduces to \((a_1^2 - a_2^2)^2 \geq 0\). Consider further that \( n \geq 3 \). Since the right inequality remains unchanged by replacing \( a_1, a_2, \ldots, a_n \) with \(-a_1, -a_2, \ldots, -a_n\), we may suppose that \( A \geq 0 \). Assuming that \( A = \text{constant}, B = \text{constant} \), we only need to consider the case when \( C \) is minimum. Thus, according to Corollary 2, it suffices to prove the required inequality for \( a_1 < a_2 = a_3 = \cdots = a_n \). Setting
\[
a_1 := a, \quad a_2 = a_3 = \cdots = a_n := b, \quad a < b,
\]
the inequality becomes
\[
\frac{\left[ \frac{a^2 + (n-1)b^2}{n} \right]^2 - \frac{a + (n-1)b}{n} \cdot \frac{a^3 + (n-1)b^3}{n}}{\left[ \frac{a^2 + (n-1)b^2}{n} \right]^2 - \left[ \frac{a + (n-1)b}{n} \right]^4} \leq \frac{1}{4} \left( 1 + \sqrt{1 + \frac{2n^2}{n-1}} \right),
\]
After dividing the numerator and denominator of the left fraction by \((a - b)^2\), the inequality reduces to
\[
\frac{-4n^2ab}{(n+1)a^2 + 2(n-1)ab + (2n^2 - 3n + 1)b} \leq 1 + \sqrt{\frac{2n^2}{n-1}},
\]
\[
\frac{-2ab}{(n+1)a^2 + 2(n-1)ab + (2n^2 - 3n + 1)b} \leq \frac{1}{\sqrt{(n^2 - 1)(2n-1) - n + 1}},
\]
\[
\left( a + \sqrt{\frac{2n^2 - 3n + 1}{n + 1} b} \right)^2 \geq 0.
\]
The equality holds for
\[
-a = a_1 = a_2 = \cdots = a_n
\]
(or any cyclic permutation).

(b) For $n = 2$, the left inequality reduces to $(a_1 - a_2)^4 \geq 0$. For $n \geq 3$, the proof is similar to the one of the right inequality. The equality holds for

$$\sqrt{\frac{n + 1}{(n - 1)(2n - 1)}} a_1 = a_2 = \cdots = a_n$$

(or any cyclic permutation).

\[ \square \]

**P 6.25.** If $a, b, c, d$ are real numbers so that

$$a + b + c + d = 2,$$

then

$$a^4 + b^4 + c^4 + d^4 \leq 40 + \frac{3}{4}(a^2 + b^2 + c^2 + d^2)^2.$$  

*(Vasile Cîrtoaje, 2010)*

**Solution.** Write the inequality in the homogeneous form

$$10(a + b + c + d)^4 + 3(a^2 + b^2 + c^2 + d^2)^2 \geq 4(a^4 + b^4 + c^4 + d^4).$$

By Theorem 1, for $a + b + c + d = constant$ and $a^2 + b^2 + c^2 + d^2 = constant$, the sum $a^4 + b^4 + c^4 + d^4$ is maximum when $a, b, c, d$ have at most two distinct values. Therefore, it suffices to consider the following two cases.

**Case 1:** $a = b$ and $c = d$. The inequality reduces to

$$41(a^2 + c^2)^2 + 160ac(a^2 + c^2) + 164a^2c^2 \geq 0,$$

which can be written in the obvious form

$$(a^2 + c^2)^2 + 40(a^2 + c^2 + 2ac)^2 + 4a^2c^2 \geq 0.$$

**Case 2:** $b = c = d$. The inequality reduces to the obvious form

$$(a + 5b)^2(3a^2 + 10ab + 11b^2) \geq 0.$$  

Since the homogeneous inequality becomes an equality for

$$\frac{-a}{5} = b = c = d$$

(or any cyclic permutation), the original inequality is an equality for

$$a = 5, \quad b = c = d = -1$$

(or any cyclic permutation).  

\[ \square \]
P 6.26. If \(a, b, c, d, e\) are real numbers, then
\[
a^4 + b^4 + c^4 + d^4 + e^4 \leq \frac{31 + 18\sqrt{3}}{8} (a + b + c + d + e)^4 + \frac{3}{4}(a^2 + b^2 + c^2 + d^2 + e^2)^2.
\]

(Vasile Cîrtoaje, 2010)

**Solution.** We proceed as in the proof of the preceding P 6.25. Taking into account Theorem 1, it suffices to consider the cases \(b = c = d = e\), and \(a = b\) and \(c = d = e\).

**Case 1:** \(b = c = d = e\). Due to homogeneity, we may consider \(b = c = d = e = 0\) and \(b = c = d = e = 1\). The first case is trivial. In the second case, the inequality becomes
\[
a^4 + 4 \leq \frac{31 + 18\sqrt{3}}{8} (a + 4)^4 + \frac{3}{4}(a^2 + 4)^2,
\]
\[
(a + 2 + 2\sqrt{3})^2 [f(a) + 2\sqrt{3} g(a)] \geq 0,
\]
where
\[
f(a) = 29a^2 + 164a + 272, \quad g(a) = 9a^2 + 50a + 76.
\]
It suffices to show that \(f(a) \geq 0\) and \(g(a) \geq 0\). Indeed, we have
\[
f(a) > 25a^2 + 164a + 269 = \left(5a + \frac{82}{5}\right)^2 + \frac{1}{25} > 0,
\]
\[
g(a) > 9a^2 + 50a + 70 = \left(3a + \frac{25}{3}\right)^2 + \frac{5}{9} > 0.
\]

**Case 2:** \(a = b\) and \(c = d = e\). It suffices to show that
\[
a^4 + b^4 + c^4 + d^4 + e^4 \leq \frac{3}{4}(a^2 + b^2 + c^2 + d^2 + e^2)^2,
\]
which reduces to
\[
2a^4 + 3c^4 \leq \frac{3}{4}(2a^2 + 3c^2)^2,
\]
\[
3(2a^2 + 3c^2)^2 \geq 4(2a^4 + 3c^4),
\]
\[
4a^4 + 36a^2c^2 + 15c^4 \geq 0.
\]

The equality holds for
\[
\frac{-a}{2(1 + \sqrt{3})} = b = c = d = e
\]
(or any cyclic permutation). \(\square\)
P 6.27. Let \( a, b, c, d, e \neq -\frac{5}{4} \) be real numbers so that \( a + b + c + d + e = 5 \). Then,

\[
\frac{a(a - 1)}{(4a + 5)^2} + \frac{b(b - 1)}{(4b + 5)^2} + \frac{c(c - 1)}{(4c + 5)^2} + \frac{d(d - 1)}{(4d + 5)^2} + \frac{e(e - 1)}{(4e + 5)^2} \geq 0.
\]

(Vasile Cîrtoaje, 2010)

**Solution.** Write the inequality as

\[
\sum \left[ \frac{180a(a - 1)}{(4a + 5)^2} + 1 \right] \geq 5,
\]

\[
\sum \left( \frac{14a - 5}{4a + 5} \right)^2 \geq 5.
\]

By the Cauchy-Schwarz inequality, we have

\[
\sum \left( \frac{14a - 5}{4a + 5} \right)^2 \geq \frac{\left( \sum (4a + 5)(14a - 5) \right)^2}{\sum (4a + 5)^4}.
\]

Therefore, it suffices to show that

\[
\left( 56 \sum a^2 + 125 \right)^2 \geq 5 \sum (4a + 5)^4.
\]

Using the substitution

\[
a_1 = \frac{4a + 5}{9}, a_2 = \frac{4b + 5}{9}, \ldots, a_5 = \frac{4e + 5}{9},
\]

we need to prove that \( a_1 + a_2 + a_3 + a_4 + a_5 = 5 \) involves

\[
\left( 7 \sum_{i=1}^{5} a_i^2 - 25 \right)^2 \geq 20 \sum_{i=1}^{5} a_i^4.
\]

Rewrite this inequality in the homogeneous form

\[
\left[ 7 \sum_{i=1}^{5} a_i^2 - \left( \sum_{i=1}^{5} a_i \right)^2 \right]^2 \geq 20 \sum_{i=1}^{5} a_i^4.
\]

By Theorem 1, for \( a_1 + a_2 + a_3 + a_4 + a_5 = 5 \) and \( a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 = \text{constant} \), the sum \( a_1^3 + a_2^3 + a_3^3 + a_4^3 + a_5^3 \) is maximum when \( a_1, a_2, a_3, a_4, a_5 \) have at most two distinct values. Therefore, we need to consider the following two cases.

**Case 1:** \( a_1 = x \) and \( a_2 = a_3 = a_4 = a_5 = y \). The homogeneous inequality reduces to

\[
(3x^2 + 6y^2 - 4xy)^2 \geq 5(x^4 + 4y^4),
\]
which is equivalent to the obvious inequality

\[(x - y)^2(x - 2y)^2 \geq 0.\]

Case 2: \(a_1 = a_2 = x\) and \(a_3 = a_4 = a_5 = y\). The homogeneous inequality becomes

\[(5x^2 + 6y^2 - 6xy)^2 \geq 5(2x^4 + 3y^4),\]

which is equivalent to the obvious inequality

\[(x - y)^2[5(x - y)^2 + 2y^2] \geq 0.\]

The equality holds for \(a = b = c = d = e = 1\), and also for

\[a = \frac{5}{2}, \quad b = c = d = e = \frac{5}{8}\]

(or any cyclic permutation).

**Remark.** Similarly, we can prove the following generalization.

- Let \(x_1, x_2, \ldots, x_n \neq -k\) be real numbers so that \(x_1 + x_2 + \cdots + x_n = n\), where

\[k \geq \frac{n}{2\sqrt{n-1}}.

Then,

\[\frac{x_1(x_1 - 1)}{(x_1 + k)^2} + \frac{x_2(x_2 - 1)}{(x_2 + k)^2} + \cdots + \frac{x_n(x_n - 1)}{(x_n + k)^2} \geq 0,

with equality for \(x_1 = x_2 = \cdots = x_n = 1\). If \(k = \frac{n}{2\sqrt{n-1}}\), then the equality holds also for

\[x_1 = \frac{n}{2}, \quad x_2 = \cdots = x_n = \frac{n}{2(n-1)}\]

(or any cyclic permutation).

\[\square\]

**P 6.28.** If \(a, b, c\) are real numbers so that

\[a + b + c = 9, \quad ab + bc + ca = 15,

then

\[\frac{19}{175} \leq \frac{1}{b^2 + bc + c^2} + \frac{1}{c^2 + ca + a^2} + \frac{1}{a^2 + ab + b^2} \leq \frac{7}{19}.

(Vasile C., 2011)
**Solution.** From

\[(b + c)^2 \geq 4bc\]

and

\[b + c = 9 - a, \quad bc = 15 - a(b + c) = 15 - a(9 - a) = a^2 - 9a + 15,\]

we get \(a \leq 7\). Since

\[b^2 + bc + c^2 = (a + b + c)(b + c) - (ab + bc + ca) = 9(9 - a) - 15 = 3(22 - 3a),\]

we may write the inequality in the form

\[
\frac{57}{175} \leq f(a) + f(b) + f(c) \leq \frac{21}{19}.
\]

where

\[f(u) = \frac{1}{22 - 3u}, \quad u \leq 7.\]

We have

\[g(x) = f'(x) = \frac{3}{(22 - 3x)^2},\]

\[g''(x) = \frac{162}{(22 - 3x)^4}.\]

Since \(g''(x) > 0\) for \(x \leq 7\), \(g\) is strictly convex on \((-\infty, 7]\). According to Corollary 1, if \(a \leq b \leq c\) and

\[a + b + c = 9, \quad a^2 + b^2 + c^2 = 51,\]

then the sum \(S_3 = f(a) + f(b) + f(c)\) is maximum for \(a = b \leq c\), and is minimum for \(a \leq b = c\).

(a) To prove the right inequality, it suffices to consider the case \(a = b \leq c\). From

\[a + b + c = 9, \quad ab + bc + ca = 15,\]

we get \(a = b = 1\) and \(c = 7\), therefore

\[
\frac{1}{b^2 + bc + c^2} + \frac{1}{c^2 + ca + a^2} + \frac{1}{a^2 + ab + b^2} = \frac{7}{19}.
\]

The original right inequality is an equality for \(a = b = 1\) and \(c = 7\) (or any cyclic permutation).

(b) To prove the left inequality, it suffices to consider the case \(a \leq b = c\), which involves \(a = -1\) and \(b = c = 5\), hence

\[
\frac{1}{b^2 + bc + c^2} + \frac{1}{c^2 + ca + a^2} + \frac{1}{a^2 + ab + b^2} = \frac{19}{175}.
\]

The original left inequality is an equality for \(a = -1\) and \(b = c = 5\) (or any cyclic permutation).
P 6.29. If \( a, b, c \) are real numbers so that
\[
8(a^2 + b^2 + c^2) = 9(ab + bc + ca),
\]
then
\[
\frac{419}{175} \leq \frac{a^2}{b^2 + bc + c^2} + \frac{b^2}{c^2 + ca + a^2} + \frac{c^2}{a^2 + ab + b^2} \leq \frac{311}{19}.
\]
(Vasile C., 2011)

Solution. Due to homogeneity, we may assume that
\[a + b + c = 9, \quad a^2 + b^2 + c^2 = 51.\]

Next, the proof is similar to the one of the preceding P 6.28. Write the inequality in the form
\[
\frac{1257}{175} \leq f(a) + f(b) + f(c) \leq \frac{933}{19},
\]
where
\[f(u) = \frac{u^2}{22 - 3u}, \quad u \leq 7.\]

We have
\[g(x) = f'(x) = \frac{-3x^2 + 44x}{(22 - 3x)^2}, \quad g''(x) = \frac{8712}{(22 - 3x)^4}.\]

Since \( g \) is strictly convex on \((-\infty, 7]\), according to Corollary 1, the sum \( S_3 = f(a) + f(b) + f(c) \) is maximum for \( a = b \leq c \), and is minimum for \( a \leq b = c \).

(a) To prove the right inequality, it suffices to consider the case \( a = b \leq c \), which involves
\[a = b = 1, \quad c = 7,
\]
and
\[
\frac{a^2}{b^2 + bc + c^2} + \frac{b^2}{c^2 + ca + a^2} + \frac{c^2}{a^2 + ab + b^2} = \frac{311}{19}.
\]
The original right inequality is an equality for \( a = b = c/7 \) (or any cyclic permutation).

(b) To prove the left inequality, it suffices to consider the case \( a \leq b = c \), which involves \( a = -1 \) and \( b = c = 5 \), hence
\[
\frac{a^2}{b^2 + bc + c^2} + \frac{b^2}{c^2 + ca + a^2} + \frac{c^2}{a^2 + ab + b^2} = \frac{419}{175}.
\]
The original left inequality is an equality for \(-5a = b = c \) (or any cyclic permutation).
Appendix A

Glosar

1. AM-GM (ARITHMETIC MEAN-GEOMETRIC MEAN) INEQUALITY

If \( a_1, a_2, \ldots, a_n \) are nonnegative real numbers, then

\[
a_1 + a_2 + \cdots + a_n \geq n \sqrt[n]{a_1 a_2 \cdots a_n},
\]

with equality if and only if \( a_1 = a_2 = \cdots = a_n \).

2. WEIGHTED AM-GM INEQUALITY

Let \( p_1, p_2, \ldots, p_n \) be positive real numbers satisfying

\[
p_1 + p_2 + \cdots + p_n = 1.
\]

If \( a_1, a_2, \ldots, a_n \) are nonnegative real numbers, then

\[
p_1 a_1 + p_2 a_2 + \cdots + p_n a_n \geq a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n},
\]

with equality if and only if \( a_1 = a_2 = \cdots = a_n \).

3. AM-HM (ARITHMETIC MEAN-HARMONIC MEAN) INEQUALITY

If \( a_1, a_2, \ldots, a_n \) are positive real numbers, then

\[
(a_1 + a_2 + \cdots + a_n) \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right) \geq n^2,
\]

with equality if and only if \( a_1 = a_2 = \cdots = a_n \).
4. POWER MEAN INEQUALITY

The power mean of order $k$ of positive real numbers $a_1, a_2, \ldots, a_n$,

$$M_k = \begin{cases} \left( \frac{a_1^k + a_2^k + \cdots + a_n^k}{n} \right)^{\frac{1}{k}}, & k \neq 0 \\ \sqrt[n]{a_1 a_2 \cdots a_n}, & k = 0 \end{cases},$$

is an increasing function with respect to $k \in \mathbb{R}$. For instance, $M_2 \geq M_1 \geq M_0 \geq M_{-1}$ is equivalent to

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{1}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}.$$

5. BERNOULLI’S INEQUALITY

For any real number $x \geq -1$, we have

a) $(1 + x)^r \geq 1 + rx$ for $r \geq 1$ and $r \leq 0$;

b) $(1 + x)^r \leq 1 + rx$ for $0 \leq r \leq 1$.

If $a_1, a_2, \ldots, a_n$ are real numbers such that either $a_1, a_2, \ldots, a_n \geq 0$ or

$$-1 \leq a_1, a_2, \ldots, a_n \leq 0,$$

then

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geq 1 + a_1 + a_2 + \cdots + a_n.$$

6. SCHUR’S INEQUALITY

For any nonnegative real numbers $a, b, c$ and any positive number $k$, the inequality holds

$$a^k(a - b)(a - c) + b^k(b - c)(b - a) + c^k(c - a)(c - b) \geq 0,$$

with equality for $a = b = c$, and for $a = 0$ and $b = c$ (or any cyclic permutation). For $k = 1$, we get the third degree Schur’s inequality, which can be rewritten as follows

$$a^3 + b^3 + c^3 + 3abc \geq ab(a + b) + bc(b + c) + ca(c + a),$$

$$(a + b + c)^3 + 9abc \geq 4(a + b + c)(ab + bc + ca),$$

$$a^2 + b^2 + c^2 + \frac{9abc}{a + b + c} \geq 2(ab + bc + ca),$$
\[(b - c)^2(b + c - a) + (c - a)^2(c + a - b) + (a - b)^2(a + b - c) \geq 0.\]

For \(k = 2\), we get the fourth degree Schur's inequality, which holds for any real numbers \(a, b, c\), and can be rewritten as follows

\[a^4 + b^4 + c^4 + abc(a + b + c) \geq ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2),\]
\[a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2 \geq (ab + bc + ca)(a^2 + b^2 + c^2 - ab - bc - ca),\]
\[(b - c)^2(b + c - a)^2 + (c - a)^2(c + a - b)^2 + (a - b)^2(a + b - c)^2 \geq 0,\]
\[6abc \geq (p^2 - q)(4q - p^2), \quad p = a + b + c, \quad q = ab + bc + ca.\]

A generalization of the fourth degree Schur's inequality, which holds for any real numbers \(a, b, c\) and any real number \(m\), is the following (Vasile Cirtoaje, 2004)

\[\sum (a - mb)(a - mc)(a - c) \geq 0,\]
with equality for \(a = b = c\), and also for \(a/m = b = c\) (or any cyclic permutation). This inequality is equivalent to

\[\sum a^4 + m(m + 2) \sum a^2b^2 + (1 - m^2)abc \sum a \geq (m + 1) \sum ab(a^2 + b^2),\]
\[\sum (b - c)^2(b + c - a - ma)^2 \geq 0.\]

7. **CAUCHY-SCHWARZ INEQUALITY**

If \(a_1, a_2, \ldots, a_n\) and \(b_1, b_2, \ldots, b_n\) are real numbers, then

\[(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2,\]

with equality for \(\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}\).

Notice that the equality conditions are also valid for \(a_i = b_i = 0\), where \(1 \leq i \leq n\).

8. **HÖLDER’S INEQUALITY**

If \(x_{ij} (i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)\) are nonnegative real numbers, then

\[\prod_{i=1}^{m} \left( \sum_{j=1}^{n} x_{ij} \right) \geq \left( \sum_{j=1}^{n} \sqrt[m]{\prod_{i=1}^{m} x_{ij}} \right)^m.\]
9. CHEBYSHEV’S INEQUALITY
Let \( a_1 \geq a_2 \geq \cdots \geq a_n \) be real numbers.

a) If \( b_1 \geq b_2 \geq \cdots \geq b_n \), then
\[
\frac{n}{n} \sum_{i=1}^{n} a_i b_i \geq \left( \sum_{i=1}^{n} a_i \right) \left( \sum_{i=1}^{n} b_i \right);
\]

b) If \( b_1 \leq b_2 \leq \cdots \leq b_n \), then
\[
\frac{n}{n} \sum_{i=1}^{n} a_i b_i \leq \left( \sum_{i=1}^{n} a_i \right) \left( \sum_{i=1}^{n} b_i \right).
\]

10. REARRANGEMENT INEQUALITY
(1) If \( (a_1, a_2, \ldots, a_n) \) and \( (b_1, b_2, \ldots, b_n) \) are two increasing (or decreasing) real sequences, and \( (i_1, i_2, \ldots, i_n) \) is an arbitrary permutation of \((1, 2, \cdots, n)\), then
\[
a_1 b_{i_1} + a_2 b_{i_2} + \cdots + a_n b_{i_n} \geq a_1 b_1 + a_2 b_2 + \cdots + a_n b_n
\]
and
\[
n(a_1 b_{i_1} + a_2 b_{i_2} + \cdots + a_n b_{i_n}) \geq (a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n).
\]
(2) If \( (a_1, a_2, \ldots, a_n) \) is decreasing and \( (b_1, b_2, \ldots, b_n) \) is increasing, then
\[
a_1 b_{i_1} + a_2 b_{i_2} + \cdots + a_n b_{i_n} \leq a_1 b_1 + a_2 b_2 + \cdots + a_n b_n
\]
and
\[
n(a_1 b_{i_1} + a_2 b_{i_2} + \cdots + a_n b_{i_n}) \leq (a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n).
\]
(3) Let \( b_1, b_2, \ldots, b_n \) and \( (c_1, c_2, \ldots, c_n) \) be two real sequences such that
\[
b_1 + \cdots + b_i \geq c_1 + \cdots + c_i, \quad i = 1, 2, \cdots, n.
\]
If \( a_1 \geq a_2 \geq \cdots \geq a_n \geq 0 \), then
\[
a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \geq a_1 c_1 + a_2 c_2 + \cdots + a_n c_n.
\]
Notice that all these inequalities follow immediately from the identity
\[
\sum_{i=1}^{n} a_i (b_i - c_i) = \sum_{i=1}^{n} (a_i - a_{i+1}) \left( \sum_{j=1}^{i} b_j - \sum_{j=1}^{i} c_j \right), \quad a_{n+1} = 0.
\]
11. SQUARE PRODUCT INEQUALITY

Let $a, b, c$ be real numbers, and let

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc,$$

$$s = \sqrt{p^2 - 3q} = \sqrt{a^2 + b^2 + c^2 - ab - bc - ca}.$$

From the identity

$$(a - b)^2(b - c)^2(c - a)^2 = -27r^2 + 2(9pq - 2p^3)r + p^2q^2 - 4q^3,$$

it follows that

$$-2p^3 + 9pq - 2(p^2 - 3q)\sqrt{p^2 - 3q} \leq r \leq -2p^3 + 9pq + 2(p^2 - 3q)\sqrt{p^2 - 3q},$$

which is equivalent to

$$\frac{p^3 - 3ps^2 - 2s^3}{27} \leq r \leq \frac{p^3 - 3ps^2 + 2s^3}{27}.$$

Therefore, for constant $p$ and $q$, the product $r$ is minimum and maximum when two of $a, b, c$ are equal.

12. KARAMATA’S MAJORIZATION INEQUALITY

Let $f$ be a convex function on a real interval $I$. If a decreasingly ordered sequence

$$A = (a_1, a_2, \ldots, a_n), \quad a_i \in I,$$

majorizes a decreasingly ordered sequence

$$B = (b_1, b_2, \ldots, b_n), \quad b_i \in I,$$

then

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq f(b_1) + f(b_2) + \cdots + f(b_n).$$

We say that a sequence $A = (a_1, a_2, \ldots, a_n)$ with $a_1 \geq a_2 \geq \cdots \geq a_n$ majorizes a sequence $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \geq b_2 \geq \cdots \geq b_n$, and write it as

$$A \succ B,$$

if

$$a_1 \geq b_1,$$

$$a_1 + a_2 \geq b_1 + b_2,$$

$\ldots$ $\ldots$ $\ldots$ $\ldots$ $\ldots$

$$a_1 + a_2 + \cdots + a_{n-1} \geq b_1 + b_2 + \cdots + b_{n-1},$$

$$a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n.$$
13. CONVEX FUNCTIONS

A function $f$ defined on a real interval $\mathbb{I}$ is said to be convex if

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$$

for all $x, y \in \mathbb{I}$ and any $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. If the inequality is reversed, then $f$ is said to be concave.

If $f$ is differentiable on $\mathbb{I}$, then $f$ is (strictly) convex if and only if the derivative $f'$ is (strictly) increasing. If $f'' \geq 0$ on $\mathbb{I}$, then $f$ is convex on $\mathbb{I}$. Also, if $f'' \geq 0$ on $(a, b)$ and $f$ is continuous on $[a, b]$, then $f$ is convex on $[a, b]$.

**Jensen's inequality.** Let $p_1, p_2, \ldots, p_n$ be positive real numbers. If $f$ is a convex function on a real interval $\mathbb{I}$, then for any $a_1, a_2, \ldots, a_n \in \mathbb{I}$, the inequality holds

$$\frac{p_1 f(a_1) + p_2 f(a_2) + \cdots + p_n f(a_n)}{p_1 + p_2 + \cdots + p_n} \geq f\left(\frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_1 + p_2 + \cdots + p_n}\right).$$

For $p_1 = p_2 = \cdots = p_n$, Jensen's inequality becomes

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right).$$

**Right Half Convex Function Theorem** (Vasile Cîrtoaje, 2004). Let $f$ be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\geq s}$, where $s \in \text{int}(\mathbb{I})$. The inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying

$$a_1 + a_2 + \cdots + a_n = ns$$

if and only if

$$f(x) + (n-1)f(y) \geq nf(s)$$

for all $x, y \in \mathbb{I}$ such that $x \leq s \leq y$ and $x + (n-1)y = ns$.

**Left Half Convex Function Theorem** (Vasile Cîrtoaje, 2004). Let $f$ be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\leq s}$, where $s \in \text{int}(\mathbb{I})$. The inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying

$$a_1 + a_2 + \cdots + a_n = ns$$

if and only if

$$f(x) + (n-1)f(y) \geq nf(s)$$
for all \(x, y \in I\) such that \(x \geq s \geq y\) and \(x + (n - 1)y = ns\).

**Left Convex-Right Concave Function Theorem** (Vasile Cîrtoaje, 2004). Let \(a \leq c\) be real numbers, let \(f\) be a continuous function defined on \(I = [a, \infty)\), strictly convex on \([a, c]\) and strictly concave on \([c, \infty)\), and let

\[
E(a_1, a_2, \ldots, a_n) = f(a_1) + f(a_2) + \cdots + f(a_n).
\]

If \(a_1, a_2, \ldots, a_n \in I\) such that

\[
a_1 + a_2 + \cdots + a_n = S = \text{constant},
\]

then

(a) \(E\) is minimum for \(a_1 = a_2 = \cdots = a_{n-1} \leq a_n\);

(b) \(E\) is maximum for either \(a_1 = a \) or \(a < a_1 \leq a_2 = \cdots = a_n\).

**Right Half Convex Function Theorem for Ordered Variables** (Vasile Cîrtoaje, 2008). Let \(f\) be a real function defined on an interval \(I\) and convex on \(I \geq s\), where \(s \in \text{int}(I)\). The inequality

\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf \left( \frac{a_1 + a_2 + \cdots + a_n}{n} \right)
\]

holds for all \(a_1, a_2, \ldots, a_n \in I\) satisfying

\[
a_1 + a_2 + \cdots + a_n = ns
\]

and

\[
a_1 \leq a_2 \leq \cdots \leq a_m \leq s, \quad m \in \{1, 2, \ldots, n-1\},
\]

if and only if

\[
f(x) + (n-m)f(y) \geq (1+n-m)f(s)
\]

for all \(x, y \in I\) such that

\[
x \leq s \leq y, \quad x + (n-m)y = (1+n-m)s.
\]

**Left Half Convex Function Theorem for Ordered Variables** (Vasile Cîrtoaje, 2008). Let \(f\) be a real function defined on an interval \(I\) and convex on \(I \leq s\), where \(s \in \text{int}(I)\). The inequality

\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf \left( \frac{a_1 + a_2 + \cdots + a_n}{n} \right)
\]

holds for all \(a_1, a_2, \ldots, a_n \in I\) satisfying

\[
a_1 + a_2 + \cdots + a_n = ns
\]

and

\[
a_1 \geq a_2 \geq \cdots \geq a_m \geq s, \quad m \in \{1, 2, \ldots, n-1\},
\]
if and only if
\[ f(x) + (n - m)f(y) \geq (1 + n - m)f(s) \]
for all \( x, y \in \mathbb{I} \) such that
\[ x \geq s \geq y, \quad x + (n - m)y = (1 + n - m)s. \]

**Right Partially Convex Function Theorem** (Vasile Cîrtoaje, 2012). Let \( f \) be a real function defined on an interval \( \mathbb{I} \) and convex on \([s, s_0]\), where \( s, s_0 \in \mathbb{I}, s < s_0 \). In addition, \( f \) is decreasing on \( \mathbb{I}_{\leq s_0} \) and \( f(u) \geq f(s_0) \) for \( u \in \mathbb{I} \). The inequality
\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \]
holds for all \( a_1, a_2, \ldots, a_n \in \mathbb{I} \) satisfying
\[ a_1 + a_2 + \cdots + a_n = ns \]
if and only if
\[ f(x) + (n - 1)f(y) \geq nf(s) \]
for all \( x, y \in \mathbb{I} \) such that \( x \leq s \leq y \) and \( x + (n - 1)y = ns \).

**Left Partially Convex Function Theorem** (Vasile Cîrtoaje, 2012). Let \( f \) be a real function defined on an interval \( \mathbb{I} \) and convex on \([s_0, s]\), where \( s_0, s \in \mathbb{I}, s_0 < s \). In addition, \( f \) is increasing on \( \mathbb{I}_{\geq s_0} \) and \( f(u) \geq f(s_0) \) for \( u \in \mathbb{I} \). The inequality
\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \]
holds for all \( a_1, a_2, \ldots, a_n \in \mathbb{I} \) satisfying
\[ a_1 + a_2 + \cdots + a_n = ns \]
if and only if
\[ f(x) + (n - 1)f(y) \geq nf(s) \]
for all \( x, y \in \mathbb{I} \) such that \( x \geq s \geq y \) and \( x + (n - 1)y = ns \).

**Right Partially Convex Function Theorem for Ordered Variables** (Vasile Cirtoaje, 2014). Let \( f \) be a real function defined on an interval \( \mathbb{I} \) and convex on \([s, s_0]\), where \( s, s_0 \in \mathbb{I}, s < s_0 \). In addition, \( f \) is decreasing on \( \mathbb{I}_{\leq s_0} \) and \( f(u) \geq f(s_0) \) for \( u \in \mathbb{I} \). The inequality
\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \]
holds for all \( a_1, a_2, \ldots, a_n \in \mathbb{I} \) satisfying
\[ a_1 + a_2 + \cdots + a_n = ns \]
and
\[ a_1 \leq a_2 \leq \cdots \leq a_m \leq s, \quad m \in \{1, 2, \ldots, n-1\}, \]
if and only if
\[ f(x) + (n-m)f(y) \geq (1+n-m)f(s) \]
for all \( x, y \in \mathbb{I} \) such that \( x \leq s \leq y \) and \( x + (n-m)y = (1+n-m)s \).

**Left Partially Convex Function Theorem for Ordered Variables** (Vasile Cirtoaje, 2014). Let \( f \) be a real function defined on an interval \( \mathbb{I} \) and convex on \([s_0, s]\), where \( s_0, s \in \mathbb{I}, s_0 < s \). In addition, \( f \) is increasing on \( \mathbb{I} \geq s \) and \( f(u) \geq f(s_0) \) for \( u \in \mathbb{I} \). The inequality
\[ f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \]
holds for all \( a_1, a_2, \ldots, a_n \in \mathbb{I} \) satisfying
\[ a_1 + a_2 + \cdots + a_n = ns \]
and
\[ a_1 \geq a_2 \geq \cdots \geq a_m \geq s, \quad m \in \{1, 2, \ldots, n-1\}, \]
if and only if
\[ f(x) + (n-m)f(y) \geq (1+n-m)f(s) \]
for all \( x, y \in \mathbb{I} \) such that \( x \geq s \geq y \) and \( x + (n-m)y = (1+n-m)s \).

**Equal Variables Theorem for Nonnegative Variables** (Vasile Cirtoaje, 2005). Let \( a_1, a_2, \ldots, a_n \ (n \geq 3) \) be fixed nonnegative real numbers, and let
\[ 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \]
such that
\[ x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n, \quad x_1^k + x_2^k + \cdots + x_n^k = a_1^k + a_2^k + \cdots + a_n^k, \]
where \( k \) is a real number \((k \neq 1)\); for \( k = 0 \), assume that
\[ x_1x_2\cdots x_n = a_1a_2\cdots a_n. \]
Let \( f \) be a real-valued function, continuous on \([0, \infty)\) and differentiable on \((0, \infty)\), such that the associated function
\[ g(x) = f'(x^{\frac{1}{n}}) \]
is strictly convex on \((0, \infty)\). Then, the sum
\[ S_n = f(x_1) + f(x_2) + \cdots + f(x_n) \]
is maximum for
\[ x_1 = x_2 = \cdots = x_{n-1} \leq x_n, \]
and is minimum for
\[ 0 < x_1 \leq x_2 = x_3 = \cdots = x_n \]
or
\[ 0 = x_1 = \cdots = x_j \leq x_{j+1} \leq x_{j+2} = \cdots = x_n, \quad j \in \{1, 2, \ldots, n-1\}. \]

**Equal Variables Theorem for Real Variables** (Vasile Cirtoaje, 2010). Let \( a_1, a_2, \ldots, a_n \) \((n \geq 3)\) be fixed real numbers, and let
\[ 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \]
such that
\[ x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n, \quad x_1^k + x_2^k + \cdots + x_n^k = a_1^k + a_2^k + \cdots + a_n^k, \]
where \( k \) is an even positive integer. If \( f \) is a differentiable function on \( \mathbb{R} \) such that the associated function \( g : \mathbb{R} \to \mathbb{R} \) defined by
\[ g(x) = f'(\sqrt[k]{x}) \]
is strictly convex on \( \mathbb{R} \), then the sum
\[ S_n = f(x_1) + f(x_2) + \cdots + f(x_n) \]
is minimum for \( x_2 = x_3 = \cdots = x_n \), and is maximum for \( x_1 = x_2 = \cdots = x_{n-1} \).

**Best Upper Bound of Jensen’s Difference Theorem** (Vasile Cirtoaje, 1990). Let \( p_1, p_2, \ldots, p_n \) \((n \geq 3)\) be fixed positive real numbers, and let \( f \) be a convex function on \( \mathbb{I} = [a, b] \). If \( a_1, a_2, \ldots, a_n \in \mathbb{I} \), then Jensen’s difference
\[ \frac{p_1 f(a_1) + p_2 f(a_2) + \cdots + p_n f(a_n)}{p_1 + p_2 + \cdots + p_n} - f\left(\frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_1 + p_2 + \cdots + p_n}\right) \]
is maximum when all \( a_i \in \{a, b\} \).
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\textbf{U}

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